

Nonlocal eddy viscosity for Reynolds stress and passive vector flux in turbulence

Fujihiro Hamba ^{*}

Institute of Industrial Science, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8505, Japan



(Received 23 September 2025; accepted 12 November 2025; published 3 December 2025)

Local eddy viscosity and diffusivity models are widely used to understand and predict turbulent flows. However, local approximations in space and time are not always valid for actual turbulent flows. Recently, a nonlocal expression for the turbulent scalar flux was verified using the direct numerical simulation (DNS) of homogeneous isotropic turbulence with an inhomogeneous mean scalar. A model for the nonlocal eddy diffusivity was proposed and validated using the DNS data. In this study, using a similar approach, a nonlocal expression for the Reynolds stress and the passive vector flux was investigated using the DNS of homogeneous isotropic turbulence with an inhomogeneous passive vector. The Green's function for the passive vector fluctuation was evaluated to obtain the nonlocal eddy viscosity. The nonlocal expression for the passive vector flux agreed with the value directly obtained from the DNS. The nonlocal effects accounted for the overestimation of the passive vector flux by the local expression as well as the phenomenon of the countergradient diffusion appearing in some regions. A model for the nonlocal eddy viscosity was proposed systematically in a customary manner in the statistical theory of turbulence. The nonlocal eddy viscosity obtained from the model agreed well with the DNS values, including its temporal behavior, and the nonlocal model reproduced the passive vector flux. Because it was already known that the nonlocal eddy viscosity for the passive vector flux can also be applied to the Reynolds stress, these results indicate that the nonlocal eddy viscosity model is useful for gaining insight into momentum transport in turbulence.

DOI: [10.1103/lxjd-g9cl](https://doi.org/10.1103/lxjd-g9cl)

I. INTRODUCTION

Eddy viscosity and diffusivity models that are widely used in turbulence simulations are local in space and time. For example, in the eddy viscosity model, the turbulent shear stress at a point is assumed to be proportional to the mean velocity gradient at the same point. The local approximation requires the characteristic scale of the transport mechanism to be smaller than the distance over which the mean gradient of the transported property changes appreciably [1]. However, the condition for the local approximation does not necessarily hold for actual turbulent flows. A typical example is the scalar transport in the atmospheric boundary layer, where convective eddies driven by buoyancy are as large as the height of the boundary layer [2]. In addition to local eddy diffusivity models, several attempts have been made to develop nonlocal models [2–5]. Nonlocal boundary

^{*}Contact author: hamba@iis.u-tokyo.ac.jp

layer parametrization is also important for oceanic vertical mixing [6]. Berkowicz and Prahm [7] proposed a generalization of eddy diffusivity, in which the turbulent scalar flux is expressed by a spatial integral of the mean scalar gradient. Romanof [8] studied space–time nonlocal models for turbulent diffusions, and Romanof [9] applied them to diffusion in atmospheric calm. Nonlocal models have also been developed for the Reynolds stress. Nakayama and Vengadesan [10] proposed a nonlocal eddy viscosity model for the Reynolds stress. Egolf [11] developed a nonlocal model for the Reynolds stress, called the difference-quotient turbulence model.

Recently, Mani and Park [12] developed the macroscopic forcing method to reveal the differential operators associated with turbulence closures. Using this method, Shirian and Mani [13] computed the scale-dependent eddy diffusivity characterizing scalar and momentum transport and demonstrated that a nonlocal operator captures the eddy diffusivity behavior. Liu *et al.* [14] developed a systematic and cost-effective approach for modeling the nonlocal eddy diffusivity using matched moment inverse operators. Lavacot *et al.* [15] investigated the nonlocality of scalar transport in Rayleigh–Taylor instability using the macroscopic forcing method. By contrast, by conditionally averaging the passive tracer equation, Souza *et al.* [16] analyzed stochastic advection problems to investigate the nonlocality of the turbulent flux. Using the two-dimensional turbulence, Flierl and Souza [17] calculated and examined the structure of the functional, which is nonlocal in both space and time, to understand when nonlocality needs to be considered.

Fractional calculus is another important method for describing the nonlocal transport of turbulence. Fractional derivatives involve both differential and integral operators and can describe nonlocal properties [18]. Nonlocal models for subgrid-scale viscosity and diffusivity in large eddy simulations have been proposed using fractional operators [19–23]. Di Leoni *et al.* [20] showed that a nonlocal eddy viscosity model based on the fractional derivative accounts for the correlation between the filtered strain rate and the subfilter stress. Seyedi and Zayernouri [22] proposed a nonlocal subgrid-scale model for homogeneous isotropic turbulence using the fractional Laplacian operator. Reynold-averaged Navier–Stokes (RANS) models have also been investigated using fractional derivatives. Fang *et al.* [24] applied the fractional derivative for the velocity gradient in neural network models to represent the nonlocal property of the Reynolds stress. Mehta [25] proposed fractional models for RANS equations by formulating a stress–strain relationship with variable-order fractional derivative.

Nonlocal expressions for momentum and scalar transport have also been investigated using the conventional statistical theory of turbulence. Using the direct interaction approximation (DIA) developed by Kraichnan [26], Roberts [27] studied turbulence diffusion to derive the probability distributions of the positions of fluid elements corresponding to nonlocal eddy diffusivity. Kraichnan [28] demonstrated that nonlocal eddy diffusivity can be approximated in terms of the averaged Green’s function and velocity correlation. Moreover, using Green’s functions for velocity and scalar fluctuations, Kraichnan [29] derived implicit exact nonlocal expressions for the Reynolds stress and the turbulent scalar flux. By modifying the Green’s function for scalar fluctuations, Hamba [30] derived an explicit exact nonlocal expression for the turbulent scalar flux and evaluated the nonlocal eddy diffusivity in the atmospheric convective boundary layer.

Recently, we examined a nonlocal expression for the turbulent scalar flux in detail using the DNS of homogeneous isotropic turbulence with an inhomogeneous mean scalar [31]. We proposed a model for the nonlocal eddy diffusivity proportional to the two-point velocity correlation in a customary manner in the statistical theory of turbulence [26,32,33]. In this model, the two-point velocity correlation was expressed in terms of the energy spectrum in the wave number space, which is convenient for homogeneous turbulence but not for inhomogeneous turbulence. Hamba [34] improved the model for the nonlocal eddy diffusivity by using the scale-space energy density developed by Hamba [35] instead of the energy spectrum. Furthermore, by incorporating the effects of turbulence anisotropy, inhomogeneity, and wall boundaries, Hamba [36] improved the model for the nonlocal eddy diffusivity and applied it to turbulent channel flow.

Because the scalar equation is linear, the formulation of the nonlocal expression for the turbulent scalar flux is straightforward. By contrast, the Navier–Stokes equation for the velocity field is

nonlinear, and the formulation of the nonlocal expression for the Reynolds stress becomes complicated. By introducing a passive vector field in turbulence, Hamba [37] showed that the Green's function for the passive vector fluctuation can be used to evaluate the nonlocal eddy viscosity for the Reynolds stress. The nonlocal expression for the Reynolds stress was verified using the DNS data of turbulent channel flow. However, a model for the nonlocal eddy viscosity was not developed because of its complex profiles. In this study, we investigate the nonlocal expression for the Reynolds stress in detail using the DNS data of homogeneous isotropic turbulence because this turbulent field is suitable as a first step of analysis and modeling. In addition to the velocity field, we introduce an inhomogeneous passive vector field and calculate its time evolution. Such a passive vector field has also been used to explore the differences and similarities in the statistical properties between the velocity vector and the passive scalar [38] and to examine momentum transport by the macroscopic forcing method [12]. We evaluate the Green's function for the passive vector fluctuation to obtain the nonlocal eddy viscosity. We examine the nonlocal expression for the passive vector flux using the DNS data. We also propose a model for the nonlocal eddy viscosity in a customary manner in the statistical theory of turbulence.

The remainder of this paper is organized as follows: In Sec. II we describe an exact nonlocal expression for the Reynolds stress, in which the nonlocal eddy viscosity is expressed in terms of the velocity fluctuation and the Green's function for the passive vector fluctuation. In Sec. III we discuss the DNS conducted for homogeneous isotropic turbulence with an inhomogeneous passive vector. Using the DNS data, we evaluate the profile of the nonlocal eddy viscosity to verify the nonlocal expression for the passive vector flux. By comparing the profiles with those obtained from the local expression, we discuss nonlocal effects on the passive vector flux. In Sec. IV we propose a model for the nonlocal eddy viscosity and compare its profiles with the DNS values. We also examine the profiles of the passive vector flux obtained from the model. Finally, we conclude the paper in Sec. V.

II. NONLOCAL EXPRESSION FOR REYNOLDS STRESS

We first describe an exact nonlocal expression for the Reynolds stress [37]. The velocity u_i^* is divided into mean and fluctuating parts as

$$u_i^* = U_i + u_i, \quad U_i = \langle u_i^* \rangle, \quad (1)$$

where $\langle \rangle$ denotes the ensemble averaging. As one of prevalent local models for the Reynolds stress $\langle u_i u_j \rangle$, the nonlinear eddy viscosity model is given by

$$\langle u_i u_j \rangle - \frac{1}{3} \langle u_k^2 \rangle \delta_{ij} = -2\nu_T S_{ij} + \zeta_1 (S_{ik} S_{kj} - \frac{1}{3} S_{km}^2 \delta_{ij}) + \zeta_2 (S_{ik} \Omega_{kj} + \Omega_{ik} S_{kj}), \quad (2)$$

where δ_{ij} is the Kronecker delta symbol, $S_{ij} = (\partial U_i / \partial x_j + \partial U_j / \partial x_i) / 2$ is the mean strain-rate tensor, $\Omega_{ij} = (\partial U_i / \partial x_j - \partial U_j / \partial x_i) / 2$ is the mean vorticity tensor, ν_T is the eddy viscosity, ζ_1 and ζ_2 are transport coefficients for the second-order terms, and the summation convention is used for repeated indices [33]. In general, local eddy viscosity models can be written as

$$\langle u_i u_j \rangle - \frac{1}{3} \langle u_k^2 \rangle \delta_{ij} = -\nu_{Tijk m} \frac{\partial U_k}{\partial x_m}, \quad (3)$$

where $\nu_{Tijk m}$ is the anisotropic eddy viscosity tensor that can explicitly depend on S_{ij} and Ω_{ij} . The eddy viscosity models are local in space and time, in the sense that the Reynolds stress at a point and time is expressed in terms of physical quantities at the same point and time. This local approximation is only valid if the turbulence length and time scales are significantly smaller than the length and time scales of the mean-field variation [1]. However, this condition does not always hold for actual turbulent flows.

A nonlocal expression for the Reynolds stress can be written as

$$\langle u_i u_j \rangle(\mathbf{x}, t) = - \int d\mathbf{x}' \int_{-\infty}^t dt' \nu_{NLijk m}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial}{\partial x'_m} U_k(\mathbf{x}', t'), \quad (4)$$

where $\int d\mathbf{x} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz$ [37]. Here $\nu_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t')$ is the nonlocal eddy viscosity, representing the nonlocal effect of the mean velocity gradient at $(\mathbf{x}', t')$ on the Reynolds stress at $(\mathbf{x}, t)$. By modifying the analysis by Kraichnan [28,29], Hamba [37] derived an exact expression for the nonlocal eddy viscosity using the Green's function as follows: The transport equation for the velocity fluctuation u_i is given by

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j^* u_i - \langle u_j^* u_i \rangle) + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = -u_j \frac{\partial U_i}{\partial x_j}, \quad (5)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (6)$$

where p is the pressure fluctuation, and ν is the molecular viscosity. By considering the right-hand side of Eq. (5) as a forcing term for u_i , we introduce the Green's function $g_{ijk}(\mathbf{x}, t; \mathbf{x}', t')$ satisfying the following equation:

$$\frac{\partial g_{ijk}}{\partial t} + \frac{\partial}{\partial x_m} (u_m^* g_{ijk} - \langle u_m^* g_{ijk} \rangle) + \frac{\partial p_{Gjk}}{\partial x_i} - \nu \frac{\partial^2 g_{ijk}}{\partial x_m \partial x_m} = u_k \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (7)$$

$$\frac{\partial g_{ijk}}{\partial x_i} = 0, \quad (8)$$

where p_{Gjk} is a tensor that plays a similar role to the pressure and guarantees the solenoidal condition for g_{ijk} given by Eq. (8). Note that the convection velocity u_j^* is the same between Eqs. (5) and (7), and the velocity fluctuation $u_k(\mathbf{x}', t')$ is included on the right-hand side of Eq. (7). The Green's function $g_{ijk}(\mathbf{x}, t; \mathbf{x}', t')$ represents the i th component of a velocity field at $(\mathbf{x}, t)$ driven by the j th component of a force at $(\mathbf{x}', t')$ whose value is proportional to $u_k(\mathbf{x}', t')$. Using the Green's function, a formal solution of Eq. (5) can be written as

$$u_i(\mathbf{x}, t) = - \int d\mathbf{x}' \int_{-\infty}^t dt' g_{ijk}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial}{\partial x'_k} U_j(\mathbf{x}', t'). \quad (9)$$

Thus, the Reynolds stress can be written as

$$\langle u_i u_j \rangle(\mathbf{x}, t) = - \int d\mathbf{x}' \int_{-\infty}^t dt' \langle u_i(\mathbf{x}, t) g_{jkm}(\mathbf{x}, t; \mathbf{x}', t') \rangle \frac{\partial}{\partial x'_m} U_k(\mathbf{x}', t'), \quad (10)$$

which yields

$$\nu_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t') = \langle u_i(\mathbf{x}, t) g_{jkm}(\mathbf{x}, t; \mathbf{x}', t') \rangle. \quad (11)$$

Therefore, the nonlocal eddy viscosity is given by the correlation between the velocity fluctuation and the Green's function. Note that Eq. (10) is not a model but an exact expression for the Reynolds stress. Hamba [37] evaluated the Green's function using the DNS data of turbulent channel flow to verify the nonlocal expression given by Eq. (10).

Strictly speaking, $g_{ijk}(\mathbf{x}, t; \mathbf{x}', t')$ satisfying Eq. (7) is not the ordinary Green's function for the u_i equation given by Eq. (5) because Eq. (5) is not linear with respect to u_i . To provide a clear explanation, we introduce a passive vector field v_i^* that satisfies the following equations:

$$\frac{\partial v_i^*}{\partial t} + \frac{\partial}{\partial x_j} (u_j^* v_i^*) + \frac{\partial p_V^*}{\partial x_i} - \nu \frac{\partial^2 v_i^*}{\partial x_j \partial x_j} = 0, \quad \frac{\partial v_i^*}{\partial x_i} = 0. \quad (12)$$

Because u_j^* is adopted as the convection velocity rather than v_j^* , Eq. (12) is linear with respect to v_i^* . Equation (12) was also used to analyze momentum transport by the macroscopic forcing method [12]. The passive vector v_i^* is divided into mean and fluctuating parts as

$$v_i^* = V_i + v_i, \quad V_i = \langle v_i^* \rangle, \quad (13)$$

and the equations for the passive vector fluctuation v_i are given by

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j^* v_i - \langle u_j^* v_i \rangle) + \frac{\partial p_V}{\partial x_i} - \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} = -u_j \frac{\partial V_i}{\partial x_j}, \quad \frac{\partial v_i}{\partial x_i} = 0. \quad (14)$$

Because Eq. (14) is linear with respect to v_i , the Green's function $g_{ijk}(\mathbf{x}, t; \mathbf{x}', t')$ can be used straightforwardly to obtain a formal solution of Eq. (14) as

$$v_i(\mathbf{x}, t) = - \int d\mathbf{x}' \int_{-\infty}^t dt' g_{ijk}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial}{\partial x'_k} V_j(\mathbf{x}', t'). \quad (15)$$

Therefore, $g_{ijk}(\mathbf{x}, t; \mathbf{x}', t')$ is indeed the Green's function for the v_i equation given by Eq. (14). It represents the i th component of a passive vector field at $(\mathbf{x}, t)$ driven by the j th component of a force at $(\mathbf{x}', t')$ whose value is proportional to $u_k(\mathbf{x}', t')$. A nonlocal expression for the passive vector flux $\langle u_i v_j \rangle$ can be written as

$$\langle u_i v_j \rangle(\mathbf{x}, t) = - \int d\mathbf{x}' \int_{-\infty}^t dt' v_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial}{\partial x'_m} V_k(\mathbf{x}', t'), \quad (16)$$

where $v_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t')$ is given by Eq. (11). By replacing v_j by u_j and V_k by U_k in Eq. (16), we obtain the nonlocal expression for the Reynolds stress given by Eq. (4). Therefore, the same nonlocal eddy viscosity $v_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t')$ can be used in the nonlocal expressions (4) and (16) for the Reynolds stress and the passive vector flux, respectively. By examining the nonlocal eddy viscosity for the passive vector flux, we can also obtain information about the nonlocal expression for the Reynolds stress.

Let us consider the relationship between the nonlocal expression and the local approximation. It is expected that the nonlocal eddy viscosity $v_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t')$ has a nonzero value if the distance $|\mathbf{x} - \mathbf{x}'|$ and the time difference $t - t'$ are comparable to or less than the turbulence length and time scales, respectively. If the mean gradients $\partial U_k / \partial x_m$ and $\partial V_k / \partial x_m$ are nearly constant in this region in terms of space and time, then the Reynolds stress and the passive vector flux can be approximated as

$$\langle u_i u_j \rangle(\mathbf{x}, t) = -v_{Lijkm}(\mathbf{x}, t) \frac{\partial U_k}{\partial x_m}, \quad (17)$$

$$\langle u_i v_j \rangle(\mathbf{x}, t) = -v_{Lijkm}(\mathbf{x}, t) \frac{\partial V_k}{\partial x_m}, \quad (18)$$

respectively, where $v_{Lijkm}(\mathbf{x}, t)$ is the local eddy viscosity defined as

$$v_{Lijkm}(\mathbf{x}, t) = \int d\mathbf{x}' \int_{-\infty}^t dt' v_{NLijkm}(\mathbf{x}, t; \mathbf{x}', t'). \quad (19)$$

Conversely, if the mean gradients change appreciably in the region, the local approximation is invalid, and the nonlocal expression should be used to predict the Reynolds stress and the passive vector flux.

III. DNS OF ISOTROPIC TURBULENCE AND INHOMOGENEOUS PASSIVE VECTOR

To verify the nonlocal expression for the Reynolds stress given by Eqs. (4) and (11), Hamba [37] examined the DNS data of turbulent channel flow. However, the nonlocal eddy viscosity showed complex profiles because of the anisotropy and inhomogeneity of turbulence. It was therefore difficult to propose a model for the nonlocal eddy viscosity that agreed with such complex profiles. Recently, to evaluate the nonlocal eddy diffusivity for the turbulent scalar flux and to assess its model, Hamba [31] examined the DNS data of homogeneous isotropic turbulence. This turbulent field was suitable as a first step of analysis and modeling. By incorporating anisotropy and inhomogeneity, Hamba [36] then improved the model for nonlocal diffusivity and validated it using

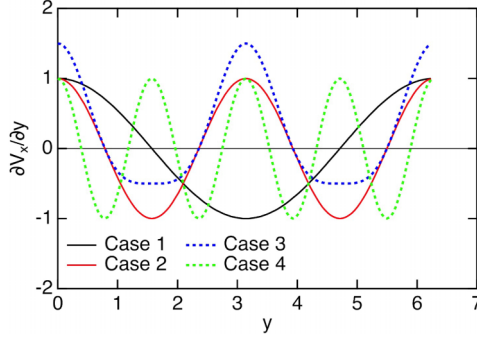


FIG. 1. Profiles of the mean passive vector gradient $\partial V_x/\partial y$ as functions of y for four cases.

the DNS data of turbulent channel flow. Similar to Hamba [31], in this study we examine the DNS data of homogeneous isotropic turbulence with an inhomogeneous passive vector to evaluate the nonlocal eddy viscosity and to assess its model. Because the nonlocal eddy viscosity is determined solely by the velocity fluctuation, its profile is homogeneous and isotropic, making it suitable as a first step. Using the DNS data, we obtained the profile of the nonlocal eddy viscosity and verified the nonlocal expression for the passive vector flux.

The DNS was performed as follows: Using a pseudospectral method, numerical solutions for the velocity were obtained from the Navier–Stokes and continuity equations:

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_j}(u_j u_i) - \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i, \quad \frac{\partial u_i}{\partial x_i} = 0, \quad (20)$$

where f_i is the external force. The mean velocity U_i was set to zero. The size of the computational domain was $L_x \times L_y \times L_z = 2\pi \times 2\pi \times 2\pi$, and the number of grid points was 512^3 . The velocity was normalized such that the initial velocity variance $\langle u_i^2 \rangle$ was unity. Hereafter, the physical quantities are nondimensionalized by the initial turbulence intensity $\langle u_i^2 \rangle^{1/2}$ and the length scale $L_x/2\pi$. The initial velocity spectrum was set to $E(k) \propto k^4 \exp[-2(k/k_0)^2]$, where $k_0 = 3.5$. The external forcing of negative viscosity [39,40] was applied to low wave numbers around $k = k_0$ to maintain constant turbulent kinetic energy over time. The viscosity was set to $\nu = 6 \times 10^{-4}$, and the Taylor microscale Reynolds number $R_\lambda (= \langle u_x^2 \rangle^{1/2} \lambda / \nu)$ was 122.

In addition to the velocity field u_i , we solved the equations for the passive vector fluctuation given by

$$\frac{\partial v_i}{\partial t} = -\frac{\partial}{\partial x_j}(u_j v_i - \langle u_j v_i \rangle) - \frac{\partial p_v}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - u_y \frac{\partial V_x}{\partial y}, \quad \frac{\partial v_i}{\partial x_i} = 0, \quad (21)$$

where a fixed one-dimensional profile of the mean passive vector $(V_x(y), 0, 0)$ was used such that the passive vector fluctuation v_i was inhomogeneous in the y direction and homogeneous in the x and z directions. We adopted four functions of the mean passive vector gradient given by

$$\frac{\partial V_x}{\partial y} = \begin{cases} \cos y, & \text{Case 1,} \\ \cos 2y, & \text{Case 2,} \\ \cos 2y + (1 + \cos 4y)/4, & \text{Case 3,} \\ \cos 4y, & \text{Case 4.} \end{cases} \quad (22)$$

Their profiles are shown in Fig. 1. The length scale of the mean field variation is longest in Case 1 and shortest in Case 4. In Cases 1, 2, and 4, the magnitudes of the positive and negative peaks are equal, whereas the positive peaks are higher than the negative peaks in Case 3.

Because the mean velocity vanishes and the turbulent field is isotropic, we could not examine the nonlocal expression for the Reynolds stress $\langle u_y u_x \rangle$. Instead, we examined the nonlocal expression

for the passive vector flux $\langle u_y v_x \rangle$, which is nonzero owing to the inhomogeneous passive vector. Because the velocity field is statistically steady and homogeneous in the x and z directions, the nonlocal expression for the passive vector flux $\langle u_y v_x \rangle$ given by Eq. (16) can be rewritten as

$$\langle u_y v_x \rangle_{NL}(y) = - \int_{-\infty}^{\infty} dy' v_{NLyxy}(y; y') \frac{\partial V_x}{\partial y'}, \quad (23)$$

where

$$v_{NLyxy}(y; y') = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \int_{-\infty}^t dt' \langle u_y(\mathbf{x}, t) g_{xy}(\mathbf{x}, t; \mathbf{x}', t') \rangle. \quad (24)$$

The nonlocal eddy viscosity $v_{NLyxy}(y; y')$ appearing in Eq. (23) is a function of y and y' only. To evaluate it, we must calculate the integrals with respect to x' , z' , and t' in Eq. (24); it would incur considerable computational costs. To reduce computational costs, we introduce another Green's function $g_{ijk}(\mathbf{x}, t; y')$ defined as

$$g_{ijk}(\mathbf{x}, t; y') = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \int_{-\infty}^t dt' g_{ijk}(\mathbf{x}, t; \mathbf{x}', t'), \quad (25)$$

which satisfies the following equations:

$$\frac{\partial g_{ijk}}{\partial t} + \frac{\partial}{\partial x_m} (u_m g_{ijk} - \langle u_m g_{ijk} \rangle) + \frac{\partial p_{Gjk}}{\partial x_i} - \nu \frac{\partial^2 g_{ijk}}{\partial x_m \partial x_m} = u_k \delta_{ij} \delta(y - y'), \quad (26)$$

$$\frac{\partial g_{ijk}}{\partial x_i} = 0. \quad (27)$$

In contrast to Eq. (7), only a delta function of $y - y'$ is included on the right-hand side of Eq. (26). The Green's function $g_{ijk}(\mathbf{x}, t; y')$ represents the i th component of a passive vector field at $(\mathbf{x}, t)$ driven by the j th component of a statistically steady force on the x - z plane at $y = y'$ whose value is proportional to $u_k(\mathbf{x}', t')$ at each point and time. We numerically solved Eq. (26) instead of Eq. (7) to obtain the Green's function $g_{xy}(\mathbf{x}, t; y')$. We can then evaluate the nonlocal eddy viscosity as

$$v_{NLyxy}(y; y') = \langle u_y(\mathbf{x}, t) g_{xy}(\mathbf{x}, t; y') \rangle, \quad (28)$$

without any additional integrals. Because the velocity field is also homogeneous in the y direction, the nonlocal eddy viscosity $v_{NLyxy}(y; y')$ is only a function of the separation $y - y'$, and the local eddy viscosity defined as

$$v_{Lyxy}(y) = \int_{-\infty}^{\infty} dy' v_{NLyxy}(y; y') \quad (29)$$

is constant in y . Consequently, the nonlocal and local expressions for the passive vector flux can be written as

$$\langle u_y v_x \rangle_{NL}(y) = - \int_{-\infty}^{\infty} dy' v_{NLyxy}(y - y') \frac{\partial V_x}{\partial y'}, \quad (30)$$

$$\langle u_y v_x \rangle_L(y) = - v_{Lyxy} \frac{\partial V_x}{\partial y}, \quad (31)$$

respectively. In the present DNS, to assess Eqs. (30) and (31), the profiles of $v_{NLyxy}(y - y')$ and $\langle u_y v_x \rangle$ were obtained by averaging over the x - z plane and over a time period of 5 normalized by $L_x / (2\pi \langle u_i^2 \rangle^{1/2})$.

The profile of the nonlocal eddy viscosity $v_{NLyxy}(y - y')$ is shown as a black line in Fig. 2. It represents the nonlocal effect of the mean passive vector gradient $\partial V_x / \partial y'$ at y' on the passive vector flux $\langle u_y v_x \rangle$ at y . The separation $y - y'$ is normalized by the integral length scale $L (= \int_0^\infty dr_x \langle u_x(\mathbf{x}) u_x(\mathbf{x} + r_x \mathbf{e}_x) \rangle / \langle u_x^2 \rangle)$, which is 0.465 in the present DNS. As the profile is symmetric with respect to $(y - y')/L = 0$, it is plotted only in the positive region at $(y - y')/L \geq 0$. It exhibits a

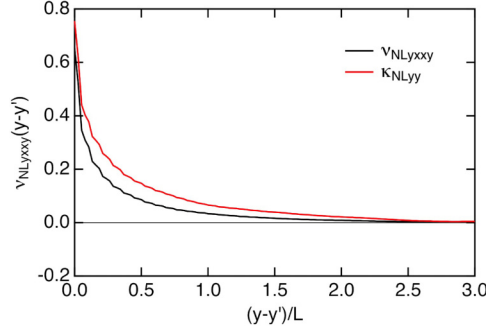


FIG. 2. Profiles of the nonlocal eddy viscosity $\nu_{NLy_{xxy}}(y-y')$ and the nonlocal eddy diffusivity $\kappa_{NLy_{yy}}(y-y')$ as functions of $(y-y')/L$ obtained from DNS. The separation $y-y'$ was normalized by the integral length scale $L(=0.465)$.

sharp peak at $(y-y')/L = 0$ and decays quickly as $(y-y')/L$ increases. The value at $(y-y')/L = 1$ is 5% of the peak value. This profile implies that the value of $\partial V_x/\partial y'$ at $-L < y-y' < L$ mainly affects the passive vector flux at y in Eq. (30). Because the profile of $\nu_{NLy_{xxy}}(y-y')$ is also applied to the nonlocal expression for the Reynolds stress, the result suggests that the Reynolds stress at a point is affected nonlocally by the mean velocity gradient in adjacent regions within the integral length scale.

Hamba [31] examined a nonlocal expression for the turbulent scalar flux using the DNS in a similar manner. The turbulent scalar flux $\langle u_y \theta \rangle$ can be written as

$$\langle u_y \theta \rangle_{NL}(y) = - \int_{-\infty}^{\infty} dy' \kappa_{NLy_{yy}}(y-y') \frac{\partial \Theta}{\partial y'}, \quad (32)$$

where Θ and θ are the mean and fluctuating parts of a passive scalar, respectively. The nonlocal eddy diffusivity $\kappa_{NLy_{yy}}(y-y')$ was evaluated as

$$\kappa_{NLy_{yy}}(y-y') = \langle u_y(\mathbf{x}, t) g_y(\mathbf{x}, t; y') \rangle, \quad (33)$$

where $g_y(\mathbf{x}, t; y')$ is the Green's function for a scalar, which satisfies the following equation:

$$\frac{\partial g_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j g_i - \langle u_j g_i \rangle) - \kappa \frac{\partial^2 g_i}{\partial x_j \partial x_j} = u_i \delta(y-y'). \quad (34)$$

The scalar flux $\langle u_y \theta \rangle$ given by Eq. (32) corresponds to the passive vector flux given by Eq. (30). The nonlocal eddy diffusivity $\kappa_{NLy_{yy}}(y-y')$ plays a similar role to the nonlocal eddy viscosity $\nu_{NLy_{xxy}}(y-y')$. The profile of the nonlocal eddy diffusivity $\kappa_{NLy_{yy}}(y-y')$ is shown as a red line in Fig. 2. The overall profile of $\kappa_{NLy_{yy}}(y-y')$ is similar to that of $\nu_{NLy_{xxy}}(y-y')$; it exhibits a peak at $(y-y')/L = 0$ and decays as $(y-y')/L$ increases. The value of $\kappa_{NLy_{yy}}(y-y')$ at $(y-y')/L = 1$ is 9% of the peak value; its width is slightly larger than $\nu_{NLy_{xxy}}(y-y')$. The magnitude of $\kappa_{NLy_{yy}}(y-y')$ is greater than that of $\nu_{NLy_{xxy}}(y-y')$; the value of the local eddy viscosity $\nu_{Ly_{xxy}}$ defined as Eq. (29) is 0.14 whereas, that of the local eddy diffusivity $\kappa_{Ly_{yy}}$ defined similarly is 0.23.

The difference between the eddy viscosity and diffusivity has already been investigated [41, 42]. For example, in the center region of turbulent channel flow, the turbulent Prandtl number $Pr_T (= \nu_T / \kappa_T)$ is 0.7–0.9 and the eddy diffusivity κ_T is greater than the eddy viscosity ν_T [42]. In this paper the reason for the difference between $\nu_{NLy_{xxy}}(y-y')$ and $\kappa_{NLy_{yy}}(y-y')$ is discussed qualitatively as follows: Eqs. (28) and (33) represent that $\nu_{NLy_{xxy}}(y-y')$ and $\kappa_{NLy_{yy}}(y-y')$ are the fluxes of $g_{xxy}(\mathbf{x}, t; y')$ and $g_y(\mathbf{x}, t; y')$ convected by u_y , respectively. The difference can then be attributed to that in $g_{xxy}(\mathbf{x}, t; y')$ and $g_y(\mathbf{x}, t; y')$. In their transport equations given by Eqs. (26) and (34), the molecular viscosity and diffusivity are equal because the Prandtl number $Pr (= \nu/\kappa)$ is set to unity.

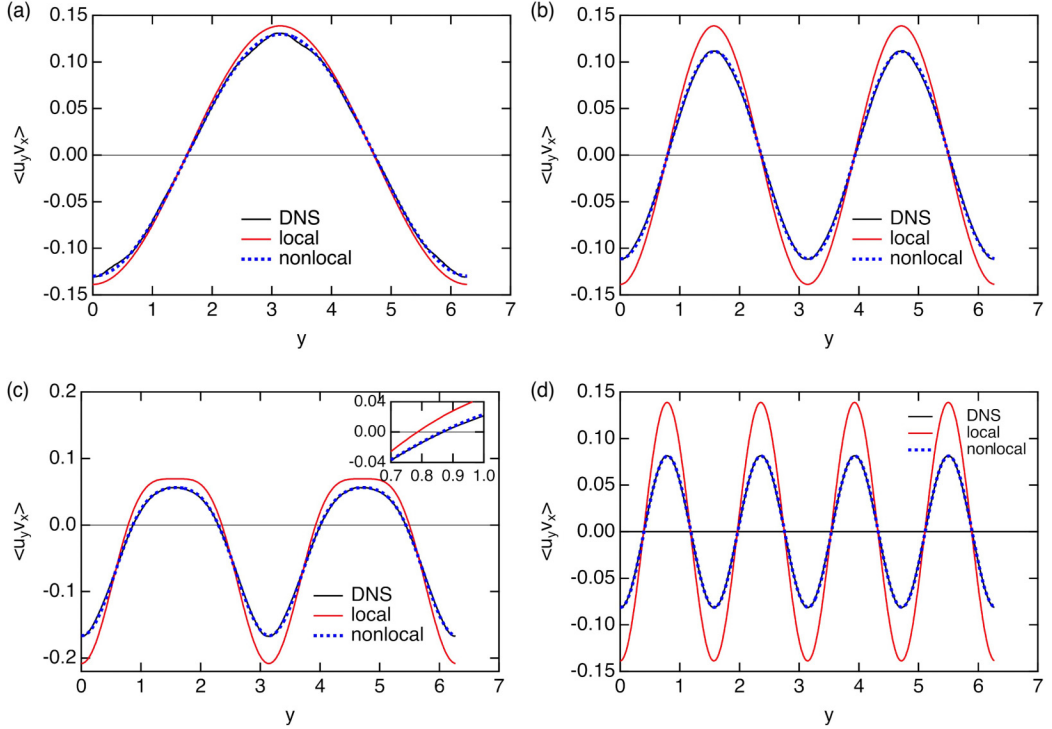


FIG. 3. Profiles of the passive vector fluxes $\langle u_y v_x \rangle$, $\langle u_y v_x \rangle_L$, and $\langle u_y v_x \rangle_{NL}$ as functions of y for (a) Case 1, (b) Case 2, (c) Case 3, and (d) Case 4. Inset shows enlarged profiles near the first zero point.

The forcing and source terms are both $u_y \delta(y - y')$. The sole difference is the pressure gradient term; it appears in the equation for $g_{xy}(\mathbf{x}, t; y')$, whereas it does not appear in that for $g_y(\mathbf{x}, t; y')$. In Eq. (34), $g_y(\mathbf{x}, t; y')$ is first generated by the source term and then transported by the convection and diffusion effects. By contrast, in Eq. (26), $g_{xy}(\mathbf{x}, t; y')$ is first generated by the forcing term and then transported by the convection and diffusion effects but always affected by the pressure gradient. This is because $g_{xy}(\mathbf{x}, t; y')$ is the x component of a solenoidal vector field (g_{xy}, g_{yx}, g_{zz}) . The component $g_{xy}(\mathbf{x}, t; y')$ decreases owing to the pressure gradient such that the solenoidal condition is satisfied. From the energy perspective, the variance of streamwise component g_{xy}^2 is redistributed to the other components g_{yx}^2 and g_{zz}^2 because of the pressure effect. Thus, the magnitude of g_{xy} can be smaller than that of g_y for which there is no pressure effect. Therefore, the corresponding flux v_{NLyxy} is less than κ_{NLy} . The dissimilarity between the nonlocal eddy viscosity and diffusivity in turbulent channel flow with control was also investigated in detail by Fang [43].

Next, we present our investigation of the nonlocal expression for the passive vector flux. Figure 3 shows the profiles of the passive vector fluxes as functions of y for four cases. Here “DNS” denotes $\langle u_y v_x \rangle$ evaluated directly, “local” denotes $\langle u_y v_x \rangle_L$ given by Eq. (31), and “nonlocal” denotes $\langle u_y v_x \rangle_{NL}$ given by Eq. (30). Note that the same profile of $v_{NLyxy}(y - y')$ was used to evaluate $\langle u_y v_x \rangle_{NL}$ for all cases because $v_{NLyxy}(y - y')$ was determined solely by the velocity field that is common in four cases. The profiles of $\langle u_y v_x \rangle_{NL}$ plotted as blue dotted lines agreed with the DNS values plotted as black lines for all cases. This agreement verifies the nonlocal expression for the passive vector flux given by Eq. (16). By contrast, the profiles of $\langle u_y v_x \rangle_L$ plotted as red lines overpredicted the DNS values. For example, the positive peak values of $\langle u_y v_x \rangle_L$ in Cases 1, 2, and 4 shown in Figs. 3(a), 3(b), and 3(d) were all 0.14 because they were determined locally by the values of $v_{Lyxy} = 0.14$ and $\partial V_x / \partial y = -1$ at the peaks. However, the peak values of DNS were clearly less than 0.14 in Cases

1, 2, and 4. The small peak values can be accounted for by the nonlocal effect; the passive vector flux at a peak was affected nonlocally by $\partial V_x/\partial y$ in adjacent regions where its magnitude was less than unity. The nonlocal effect was most significant in Case 4. This is because in Case 4, the length scale of the mean passive vector was shortest and the profile did not satisfy the condition for local approximation discussed in Sec. II. This result suggests that local expression is not accurate in the case where the length scale of mean field is as short as the turbulence length scale and that nonlocal expression is necessary to reproduce the DNS value.

In addition to the small peak values, another nonlocal effect was observed in Case 3. In Fig. 3(c) the locations of zero points were different between the local expression $\langle u_y v_x \rangle_L$ and the DNS value $\langle u_y v_x \rangle$. For example, as clearly seen in the inset of Fig. 3(c), the zero point of the local expression was located at $y = 0.79 (= \pi/4)$, where $\partial V_x/\partial y = 0$ as plotted in Fig. 1, whereas the zero point of the DNS value shifted to $y = 0.87$. This difference in the zero-point location led to the countergradient diffusion phenomenon; that is, the DNS values of $\partial V_x/\partial y$ and $\langle u_y v_x \rangle$ were both negative at $0.79 < y < 0.87$, which is inconsistent with the local eddy viscosity model. This countergradient diffusion of the passive vector flux can also be accounted for by the nonlocal effect. For example, at $y = 0.79$, $\langle u_y v_x \rangle_L$ vanished because of $\partial V_x/\partial y = 0$, whereas $\langle u_y v_x \rangle_{NL}$ had a nonzero value because of the contribution of $\partial V_x/\partial y$ in adjacent regions. Because the magnitude of $\partial V_x/\partial y$ at $y < 0.79$ was greater than that at $y > 0.79$ in Case 3, as shown in Fig. 1, the positive $\partial V_x/\partial y$ at $y < 0.79$ affected $\langle u_y v_x \rangle$ at $y = 0.79$ more significantly than the negative $\partial V_x/\partial y$ at $y > 0.79$ did, which led to a negative value of $\langle u_y v_x \rangle$ at $y = 0.79$. Such an asymmetric nonlocal contribution leads to the countergradient diffusion of the passive vector flux. A similar countergradient diffusion due to the nonlocal effect was discussed for the turbulent scalar flux in Hamba [31] and Flierl and Souza [17].

In summary, because of the nonlocal effects, the peak value of the passive vector flux obtained from the DNS was less than the value estimated by the local expression, and the countergradient diffusion of the passive vector flux appeared near the zero points. These behaviors were reproduced by the nonlocal expression given by Eq. (30). The results indicate the importance of the nonlocal expression for the passive vector flux given by Eq. (16). Because the nonlocal eddy viscosity is also applied to Eq. (4), the results suggest that the nonlocal effects are also important for the Reynolds stress.

IV. MODELING THE NONLOCAL EDDY VISCOSITY

As discussed in Sec. III, the nonlocal expression for the passive vector flux given by Eqs. (28) and (30) was verified using the DNS data of homogeneous isotropic turbulence with an inhomogeneous passive vector. The profile of the nonlocal eddy viscosity $\nu_{NLyxy}(y - y')$ shown in Fig. 2 was obtained by solving the transport equation for the Green's function $g_{ijk}(\mathbf{x}, t; y')$ given by Eqs. (26) and (27). The exact expression for the nonlocal eddy viscosity given by Eq. (28) is important and useful for understanding the turbulence transport. However, the Green's function cannot be easily evaluated for complicated turbulent flows because of its computational cost. Therefore, some models for the nonlocal eddy viscosity are required for practical simulations. Moreover, model expressions can aid in the interpretation of the physical meaning of the nonlocal eddy viscosity. In this section we propose a model for the nonlocal eddy viscosity as a first step toward developing a practical nonlocal model for the Reynolds stress.

We considered a case in which the mean passive vector has a general profile varying in space and time, whereas the velocity field is given by homogeneous isotropic turbulence. We start with the original nonlocal expression for the passive vector flux given by Eq. (16) rather than Eq. (30) and the nonlocal eddy viscosity $\nu_{NLijk}(\mathbf{x}, t; \mathbf{x}', t')$ rather than $\nu_{NLijk}(y - y')$. Because the velocity field is statistically steady and homogeneous, the nonlocal eddy viscosity is a function of the separation $\mathbf{r} (= \mathbf{x} - \mathbf{x}')$ and the time difference $\tau (= t - t')$ as

$$\nu_{NLijk}(\mathbf{x}, t; \mathbf{x}', t') = \nu_{NLijk}(\mathbf{r}, \tau). \quad (35)$$

We investigated the profile of $v_{NLijkm}(\mathbf{r}, \tau)$ and propose its model in a customary manner in the statistical theory of turbulence such as the DIA and TSDIA. A similar approach was taken for modeling the nonlocal eddy diffusivity $\kappa_{NLij}(\mathbf{r}, \tau)$ for the turbulent scalar flux in Hamba [31,34]. We followed the method of Hamba [31,34] to obtain a model for $v_{NLijkm}(\mathbf{r}, \tau)$.

By replacing u_k with unity on the right-hand side of Eq. (7), we can obtain the transport equation for the ordinary Green's function $g_{ij}(\mathbf{x}, t; \mathbf{x}', t')$ for the passive vector fluctuation. Here we assumed the following relationship:

$$g_{ijk}(\mathbf{x}, t; \mathbf{x}', t') = g_{ij}(\mathbf{x}, t; \mathbf{x}', t')u_k(\mathbf{x}', t'). \quad (36)$$

The nonlocal eddy viscosity can then be written as

$$v_{NLijkm}(\mathbf{r}, \tau) = \langle u_i(\mathbf{x}, t)g_{jk}(\mathbf{x}, t; \mathbf{x}', t')u_m(\mathbf{x}', t') \rangle. \quad (37)$$

A similar expression for the eddy viscosity was theoretically derived by the DIA [28,29]. By factoring Eq. (37) into the product of two averages in a manner similar to the DIA, we obtained the approximation

$$v_{NLijkm}(\mathbf{r}, \tau) = G_{jk}(\mathbf{r}, \tau)Q_{im}(\mathbf{r}, \tau), \quad (38)$$

where $Q_{im}(\mathbf{r}, \tau) (= \langle u_i(\mathbf{x}, t)u_m(\mathbf{x}', t') \rangle)$ denotes the two-point two-time velocity correlation. Here $G_{jk}(\mathbf{r}, \tau)$ is simply a proportional coefficient between $v_{NLijkm}(\mathbf{r}, \tau)$ and $Q_{im}(\mathbf{r}, \tau)$ and is not necessarily equal to the mean Green's function $\langle g_{jk}^*(\mathbf{x}, t; \mathbf{x}', t') \rangle$. However, we expected $G_{jk}(\mathbf{r}, \tau)$ to play a similar role to the mean Green's function. In modeling the nonlocal eddy diffusivity, $Q_{im}(\mathbf{r}, \tau)$ was approximated by the one-time correlation $Q_{im}(\mathbf{r}) (= \langle u_i(\mathbf{x}, t)u_m(\mathbf{x}', t) \rangle)$ because as τ increased, $Q_{im}(\mathbf{r}, \tau)$ decayed more slowly than $G(\mathbf{r}, \tau)$ did [31]. We also adopted this assumption to obtain

$$v_{NLijkm}(\mathbf{r}, \tau) = G_{jk}(\mathbf{r}, \tau)Q_{im}(\mathbf{r}), \quad (39)$$

where the dependence on τ is represented only by $G_{jk}(\mathbf{r}, \tau)$. Because $v_{NLijkm}(\mathbf{r}, \tau)$ is determined solely by the velocity fluctuation that is statistically isotropic, we assumed simple isotropic forms for $G_{jk}(\mathbf{r}, \tau)$ and $Q_{im}(\mathbf{r})$ as follows:

$$G_{jk}(\mathbf{r}, \tau) = G(r, \tau)\delta_{jk}, \quad (40)$$

$$Q_{im}(\mathbf{r}) = Q(r)\delta_{im}, \quad (41)$$

where $G(r, \tau) = G_{ii}(\mathbf{r}, \tau)/3$, $Q(r) = Q_{ii}(\mathbf{r})/3$, and $r = |\mathbf{r}|$. In general, $Q_{im}(\mathbf{r})$ can depend not only on δ_{im} but also on $r_i r_m / r^2$, and the same holds for $G_{jk}(\mathbf{r}, \tau)$. In this study we restricted ourselves to these forms for simplicity. As a result, we obtained the following expression:

$$v_{NLijkm}(\mathbf{r}, \tau) = G(r, \tau)Q(r)\delta_{jk}\delta_{im}. \quad (42)$$

We need to obtain model expressions for $G(r, \tau)$ and $Q(r)$, respectively.

The time-dependent part $G(r, \tau)$ appearing in Eq. (42), which corresponds to the mean Green's function, is expressed as

$$G(r, \tau) = \frac{1}{(4\pi)^{3/2}(C_{\omega G}u_0\tau)^3} \exp\left[-\frac{r^2}{4(C_{\omega G}u_0\tau)^2}\right], \quad (43)$$

where $u_0 = \langle u_i^2 \rangle^{1/2} = (2K)^{1/2}$, $K (= \langle u_i^2 \rangle / 2)$ is the turbulent kinetic energy, and $C_{\omega G}$ is a model constant [31]. This expression is a solution of a three-dimensional isotropic diffusion problem with a time-dependent diffusivity [44]. In this case, the diffusion is driven by the turbulent velocity fluctuations and the effective diffusivity is given by $C_{\omega G}^2 u_0^2 \tau$. As the value of model constant $C_{\omega G}$ increases, the effective diffusivity increases, and the profile diffuses more quickly. Gaussian functions similar to Eq. (43) were actually used to study turbulent diffusion processes [8,9,27].

For the velocity correlation $Q(r)$, we used the energy density in the scale space instead of the energy spectrum in the wave number space [34,35]. The velocity correlation is expressed as the

following integral with respect to the scale s :

$$\mathcal{Q}(r) \left(\equiv \frac{1}{3} \mathcal{Q}_{ii}(\mathbf{r}) \right) = \frac{1}{3} \int_0^\infty ds \hat{Q}_{ii}(s) \exp\left(-\frac{r^2}{4s}\right). \quad (44)$$

The velocity correlation is decomposed into modes in the scale space. The integrand represents the amount of velocity correlation at scale s and is expressed as a simple Gaussian function. The scale-space energy density $\hat{Q}_{ii}(s)$ appearing in Eq. (44) is formulated using the filtered velocities and represents the amount of kinetic energy at scale s [35]. It can be modeled as the following simple form:

$$\hat{Q}_{ii}(s) = \begin{cases} \nu^{-1} \varepsilon, & s < s_d, \\ C_s \varepsilon^{2/3} s^{-2/3}, & s_d \leq s \leq s_c, \\ C_s \varepsilon^{2/3} s_c^{11/6} s^{-5/2}, & s > s_c, \end{cases} \quad (45)$$

where C_s is a model constant, and two interface scales are introduced: $s_d = C_s^{3/2} \nu^{3/2} \varepsilon^{-1/2}$ in the dissipation range and $s_c = (6/11)^3 C_s^{-3} K^3 \varepsilon^{-2} (1 + C_s^{3/2} \nu^{1/2} K^{-1} \varepsilon^{1/2})^3$ in the energy-containing range. The function $C_s \varepsilon^{2/3} s^{-2/3}$ for the inertial range $s_d \leq s \leq s_c$ in Eq. (45) corresponds to the Kolmogorov energy spectrum $E(k) = C_K \varepsilon^{2/3} k^{-5/3}$. The form for $s < s_d$ is set because of the constraint that $\hat{Q}_{ii}(s)$ tends to $\nu^{-1} \varepsilon$ as s decreases to zero in the formulation of the scale-space energy density. The form for $s > s_c$ is also set because of the requirement that $\hat{Q}_{ii}(s)$ is proportional to $s^{-5/2}$ as s increases much greater than the integral length scale.

Finally, by substituting Eqs. (43)–(45) into Eq. (42), we obtained a model for the nonlocal eddy viscosity $\nu_{NLijkm}(\mathbf{r}, \tau)$. It is expressed in terms of the turbulent kinetic energy K , its dissipation rate ε , the molecular viscosity ν , and the nondimensional model constants C_s and $C_{\omega G}$. The present model for the nonlocal eddy viscosity $\nu_{NLijkm}(\mathbf{r}, \tau)$ is similar to that for the nonlocal eddy diffusivity $\kappa_{NLij}(\mathbf{r}, \tau)$ for the turbulent scalar flux [34]. The values of model constants C_s and $C_{\omega G}$ can be different between the eddy viscosity and diffusivity.

If the mean passive vector is constant in time as is the case of Sec. III, the time-integrated eddy viscosity $\nu_{NLijkm}(\mathbf{r})$ defined as

$$\nu_{NLijkm}(\mathbf{r}) = \int_0^\infty d\tau \nu_{NLijkm}(\mathbf{r}, \tau), \quad (46)$$

can be used for the nonlocal expression for the passive vector flux. Its model is given by

$$\nu_{NLijkm}(\mathbf{r}) = G(r) \mathcal{Q}(r) \delta_{jk} \delta_{im}, \quad (47)$$

where

$$G(r) \left(\equiv \int_0^\infty d\tau G(r, \tau) \right) = \frac{1}{4\pi^{3/2} C_{\omega G} u_0 r^2}. \quad (48)$$

The model for the three-dimensional nonlocal eddy viscosity $\nu_{NLijkm}(\mathbf{r})$ is a function of r only. In Sec. III we evaluated $\nu_{NLijkm}(r_y)$ given by

$$\nu_{NLijkm}(r_y) = \int_{-\infty}^\infty dr_x \int_{-\infty}^\infty dr_z \nu_{NLijkm}(r), \quad (49)$$

which is a one-dimensional function of r_y . To convert a model of $\nu_{NLijkm}(r)$ into $\nu_{NLijkm}(r_y)$ in Eq. (49), we can use the following formula:

$$\nu_{NLijkm}(r_y) = \int_{r_y}^\infty dr 2\pi r \nu_{NLijkm}(r). \quad (50)$$

By substituting Eqs. (44), (45), and (48) into Eq. (47) and using Eq. (50), we can evaluate the model for the nonlocal eddy viscosity $\nu_{NLijkm}(r_y)$.

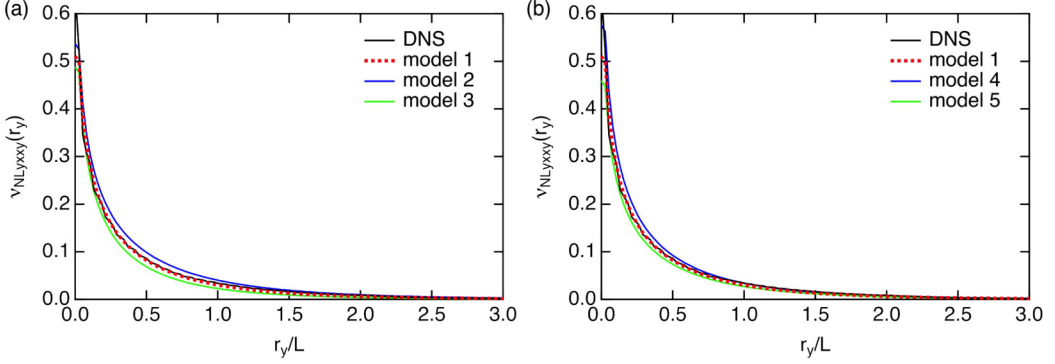


FIG. 4. Profiles of the nonlocal eddy viscosity $\nu_{NL\gamma xy}(r_y)$ as functions of r_y/L obtained from DNS and the model: (a) models 1, 2, and 3 and (b) models 1, 4, and 5.

Figure 4 shows the profiles of the nonlocal eddy viscosity $\nu_{NL\gamma xy}(r_y)$ as functions of r_y/L obtained from the DNS and the model proposed above. The values of the two model constants are given in Table I. The values optimized by trial and error are shown as model 1. To check how sensitive the model is to the constants, we varied the value of C_s in models 2 and 3 and the value of $C_{\omega G}$ in models 4 and 5. The profile of model 1 agreed well with the DNS value in Fig. 4. The constant C_s is included in the scale of the energy-containing range $s_c \simeq (6/11)^3 C_s^{-3} K^3 \varepsilon^{-2}$, and a larger (smaller) value of C_s leads to a smaller (larger) length scale of the velocity correlation $Q(r)$, resulting a smaller (larger) value of $\nu_{NL\gamma xy}(r_y)$ as shown in Fig. 4(a). Similarly, $C_{\omega G}$ is included in Eq. (48) for $G(r)$, and a larger (smaller) value of $C_{\omega G}$ leads to a smaller (larger) value of $\nu_{NL\gamma xy}(r_y)$ as shown in Fig. 4(b). By optimizing the values of two model constants, we obtained a good profile of $\nu_{NL\gamma xy}(r_y)$ in Fig. 4. A similar model for the nonlocal eddy diffusivity $\kappa_{NLij}(\mathbf{r}, \tau)$ was obtained by Hamba [34], and the two constants were optimized as $C_s = 1.3$ and $C_{\omega G} = 0.46$. These values are smaller than the present values because the nonlocal eddy diffusivity is greater than the nonlocal eddy viscosity as shown in Fig. 2.

If the mean passive vector varies in time, we need to use the original nonlocal eddy viscosity $\nu_{NLijkm}(r_y, \tau)$ rather than $\nu_{NLijkm}(r_y)$ to model the passive vector flux. The nonlocal eddy viscosity $\nu_{NLijkm}(r_y)$ is decomposed into $\nu_{NLijkm}(r_y, \tau)$ as

$$\nu_{NLijkm}(r_y) = \int_0^\infty d\tau \nu_{NLijkm}(r_y, \tau). \quad (51)$$

The nonlocal eddy viscosity $\nu_{NLijkm}(r_y, \tau)$ can be obtained from $\nu_{NLijkm}(r, \tau)$ given by Eq. (42) using the same formula as Eq. (50). It depends not only on r_y but also on τ and represents the spatial and temporal nonlocal effect of the mean passive vector gradient at $(y \pm r_y, t - \tau)$ on the passive vector flux at (y, t) . Figures 5 and 6 show the profiles of $\nu_{NL\gamma xy}(r_y, \tau)$ as functions of r_y/L at $\tau = 0.2, 0.4$, and 0.6 for models 1, 2, and 3 and for models 1, 4, and 5, respectively. The three

TABLE I. Values of model constants for the nonlocal eddy viscosity.

	C_s	$C_{\omega G}$
Model 1	1.7	0.53
Model 2	1.5	0.53
Model 3	1.9	0.53
Model 4	1.7	0.47
Model 5	1.7	0.59

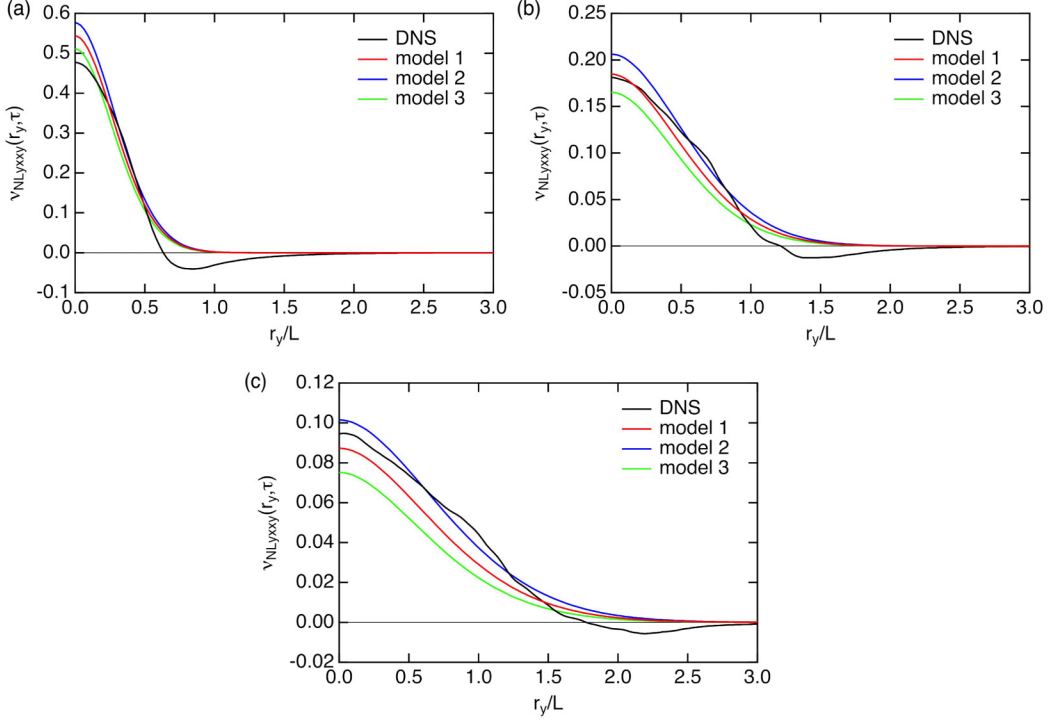


FIG. 5. Profiles of the nonlocal eddy viscosity $v_{NLyxy}(r_y, \tau)$ as functions of r_y/L obtained from DNS and models 1, 2, and 3 at (a) $\tau = 0.2$, (b) $\tau = 0.4$, and (c) $\tau = 0.6$.

time differences were short compared with the turbulence timescale $T(=K/\varepsilon) = 2.7$ in the present DNS. The black lines represent the DNS values; they were obtained by averaging over x - z plane and 160 samples. In each figure, the profile exhibits a peak at $r_y/L = 0$ and decays to zero as r_y/L increases. Unlike the time-integrated viscosity $v_{NLyxy}(r_y)$ plotted in Fig. 4, the profiles show a small negative value before tending to zero in Figs. 5 and 6. As the time difference τ increased, the peak value at $r_y/L = 0$ decayed rapidly and the width increased gradually. The spatial region in which the mean passive vector gradient nonlocally affects the passive vector flux was narrow for small τ and became wider as τ increased. The colored lines represent the values obtained from the model described above. The profiles of model 1 approximately reproduced the behavior of the DNS values. The profiles of models 2–5 near the peak show the same trends as in Fig. 4. A larger (smaller) value of C_s and $C_{\omega G}$ leads to a smaller (larger) value of $v_{NLyxy}(r_y, \tau)$ near the peak in Figs. 5 and 6. By contrast, the profiles far from the peak show the opposite trend between C_s and $C_{\omega G}$. In Fig. 5, a larger value of C_s also leads to a smaller value of $v_{NLyxy}(r_y, \tau)$ far from the peak because the length scale of $Q(r)$ decreases as C_s increases. On the other hand, in Fig. 6, a larger value of $C_{\omega G}$ leads to a larger value of $v_{NLyxy}(r_y, \tau)$ far from the peak because the increase of $C_{\omega G}$ in $G(r, \tau)$ represents a rapid diffusion process. In Figs. 5 and 6, the small negative value was not reproduced in the model profiles. One of the reasons is that the velocity correlation $Q(r)$ is expressed in terms of a simple Gaussian function $\exp(-r^2/4s)$ at each scale s in Eq. (44). It should be modified to improve the behavior of $v_{NLijkm}(r_y, \tau)$ in future work. Nevertheless, the temporal behavior of the nonlocal eddy viscosity was approximately reproduced by the model given by Eqs. (42)–(45).

The profiles shown in Figs. 5 and 6 suggest the physical meaning of the nonlocal eddy viscosity given by Eqs. (42)–(45) as follows: As the time difference increases, the nonlocal contribution becomes weak but wide in space. This temporal behavior is expressed as the turbulent diffusion represented by $G(r, \tau)$ given by Eq. (43). Because $G(r, \tau)$ is a normalized function whose spatial

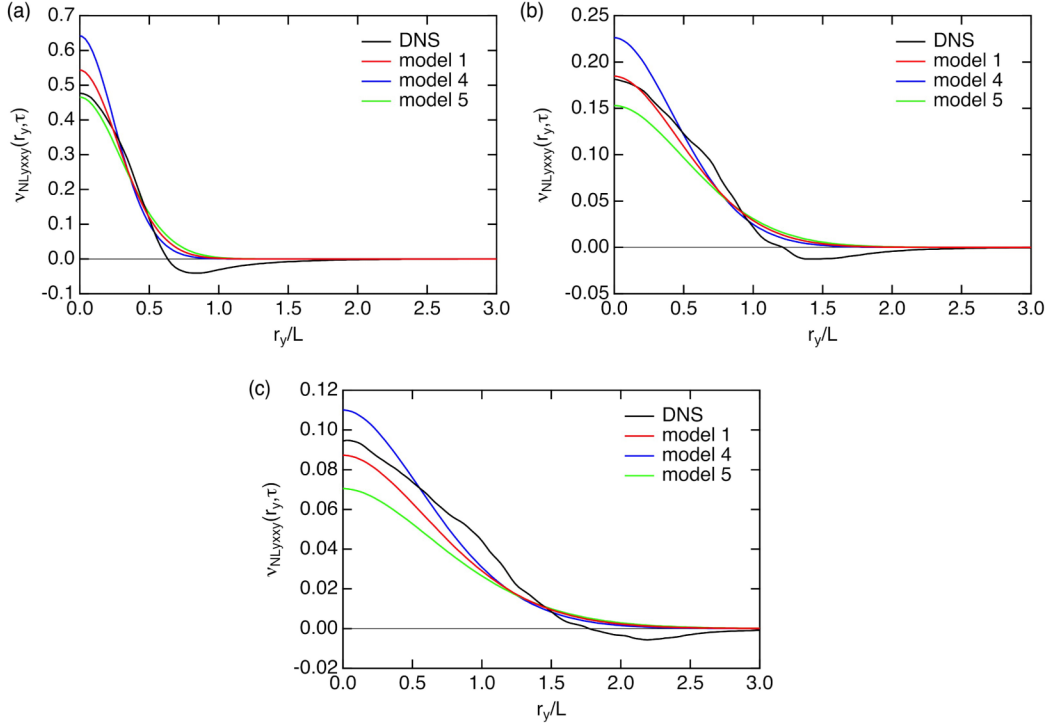


FIG. 6. Profiles of the nonlocal eddy viscosity $\nu_{NLyxy}(r_y, \tau)$ as functions of r_y/L obtained from DNS and models 1, 4, and 5 at (a) $\tau = 0.2$, (b) $\tau = 0.4$, and (c) $\tau = 0.6$.

integral is unity, the magnitude of the nonlocal eddy viscosity is expressed by $Q(r)$ given by Eqs. (44) and (45). Another role of $Q(r)$ is to limit the width of the profile of the nonlocal eddy viscosity. In general, the solution of a diffusion problem exhibits a profile spreading infinitely over time. However, the width of the nonlocal eddy viscosity is limited by the turbulence length scale, which is expressed by $Q(r)$.

Finally, we examined the nonlocal expression for the passive vector flux again. Using the nonlocal eddy viscosity $\nu_{NLyxy}(r_y)$ obtained from the model shown in Fig. 4, we evaluated the passive vector flux $\langle u_y v_x \rangle_{NL}$ given by Eq. (30). Figures 7 and 8 show the profile of the passive vector flux $\langle u_y v_x \rangle$ as functions of y for four cases for models 1, 2, and 3 and for models 1, 4, and 5, respectively. The profiles of model 1 agreed well with the DNS values. The peak values were accurately evaluated in all four cases, and the zero points in Case 3 were reproduced well, as shown in the inset of Figs. 7(c) and 8(c). Note that the same profile of $\nu_{NLyxy}(r_y)$ was used for all cases because the nonlocal eddy viscosity was determined solely by the velocity field, which was common to the four cases. The profiles of models 2–5 in Figs. 7 and 8 show the same trends as in Fig. 4. A larger (smaller) value of C_s and $C_{\omega G}$ leads to a smaller (larger) value of $\langle u_y v_x \rangle$. The sensitivity of the model is slightly different between C_s and $C_{\omega G}$. The sensitivity to C_s is greater in Case 1 in Fig. 7(a) than in Case 4 in Fig. 7(d), whereas the sensitivity to $C_{\omega G}$ is nearly equal in Cases 1–4 in Fig. 8.

Modeling the passive vector flux includes two steps. The first step is the nonlocal expression for the passive vector flux given by Eq. (30). This was verified by the “nonlocal” profiles in Fig. 3, where the nonlocal eddy viscosity was numerically evaluated by Eq. (28). The second step is the model for the nonlocal eddy viscosity given by Eqs. (42)–(45). It was shown in Fig. 4 that the model profile of $\nu_{NLyxy}(r_y)$ agreed well with the DNS value. By combining the two steps, that is, the nonlocal expression for the passive vector flux and the model for the nonlocal eddy viscosity, we obtained accurate profiles of the passive vector flux in Figs. 7 and 8. We should note that the good profile of

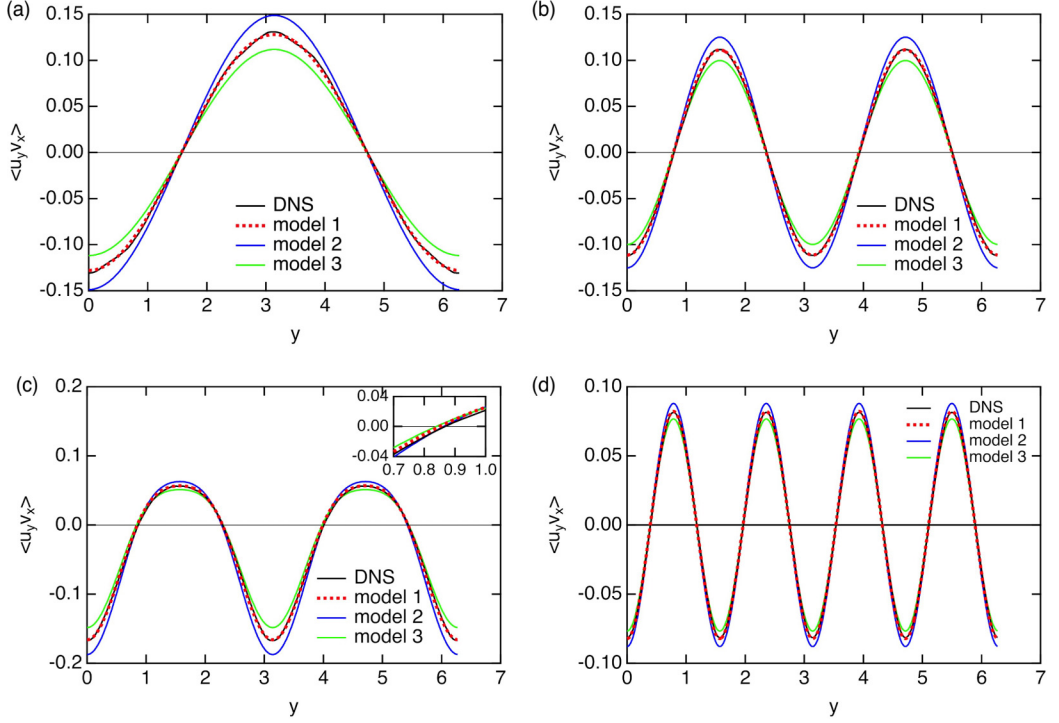


FIG. 7. Profiles of the passive vector flux $\langle u_y v_x \rangle$ obtained from DNS and models 1, 2, and 3 for (a) Case 1, (b) Case 2, (c) Case 3, and (d) Case 4.

the nonlocal eddy viscosity was obtained by optimizing the values of model constants C_s and $C_{\omega G}$ for the present turbulent field. It is not clear whether the same values of model constants can be used for other turbulent fields with different external forcing or at different Reynolds numbers; such cases need to be examined in future work. Nevertheless, the present results suggest the potential of the nonlocal modeling of the passive vector flux and the Reynolds stress.

V. CONCLUSIONS

A nonlocal expression for the Reynolds stress and the passive vector flux was investigated using the DNS data of homogeneous isotropic turbulence with an inhomogeneous passive vector. In addition to the velocity field, the time evolution of the passive vector fluctuation was calculated for four cases with different mean passive vector gradients. The Green's function for the passive vector fluctuation was also evaluated to obtain the nonlocal eddy viscosity. The profile of the nonlocal eddy viscosity decayed as the separation increased, with its width approximately equal to the integral length scale. This profile was similar to that of the nonlocal eddy diffusivity for the turbulent scalar flux obtained previously, but the magnitude was smaller, corresponding to turbulent Prandtl number less than unity. It was verified that the nonlocal expression for the passive vector flux agreed with the DNS values for all four cases. By contrast, the local expression overpredicted the peak values of the passive vector flux. This is because the passive vector flux at a peak is affected nonlocally by the mean passive vector gradient in adjacent regions, an effect the local expression cannot capture. The countergradient diffusion phenomenon, where the passive vector flux and the mean passive vector gradient have the same sign, appeared near the zero points and was also accounted for by an asymmetric nonlocal effect.

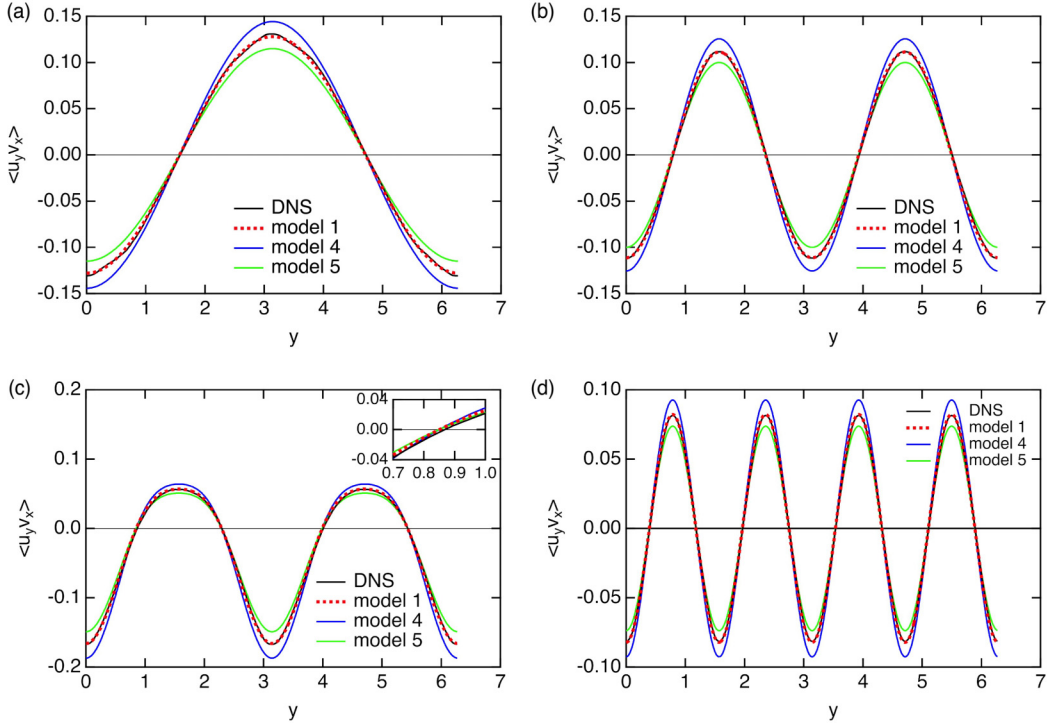


FIG. 8. Profiles of the passive vector flux $\langle u_y v_x \rangle$ obtained from DNS and models 1, 4, and 5 for (a) Case 1, (b) Case 2, (c) Case 3, and (d) Case 4.

A model for the nonlocal eddy viscosity was proposed systematically in a customary manner in the statistical theory of turbulence. Following the DIA, we first assumed the nonlocal eddy viscosity was proportional to the two-point velocity correlation and the time-dependent part corresponding the mean Green's function. The two-point velocity correlation was then expressed in terms of the scale-space energy density, and the time-dependent part was expressed using the solution of the diffusion problem with turbulent motions. The profile of the nonlocal eddy viscosity agreed well with the DNS value by optimizing the two model constants. The profiles of the nonlocal eddy viscosity at three time differences were also evaluated using the DNS data. As the time difference increased, the spatial region widened in which the mean passive vector gradient nonlocally affected the passive vector flux. This behavior was approximately reproduced by the model. The passive vector flux obtained from the model also agreed with the DNS value for four cases. Because it is already known that the nonlocal eddy viscosity for the passive vector flux can also be applied to the Reynolds stress, the present results indicate the potential of the nonlocal eddy viscosity model for the Reynolds stress. We expect that the analysis and modeling of the nonlocal eddy viscosity will provide further insight into momentum transport in turbulence.

ACKNOWLEDGMENT

This work was supported by JSPS KAKENHI Grant No. JP23K03670.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data

would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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