

Linear corner-mode instability of magnetohydrodynamic Rayleigh-Bénard convection in a rectangular domain

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Rayleigh-Bénard convection in a horizontal fluid layer is sustained only above a finite critical Rayleigh number. When the fluid is electrically conducting, a vertical magnetic field increases the critical Rayleigh number for the instability of the purely heat-conducting state in the bulk of the layer. When electrically insulating side walls are taken into account, the least stable mode becomes localized near these walls. Previous works studied the linear stability of such a wall-attached convection mode in circular containers or along a straight periodic wall. The present work considers a rectangular container, where induced eddy currents are presumably even more obstructed at the corners between insulating side walls. The linear stability problem is formulated using the quasistatic approximation of the induction equation. It is discretized and solved numerically for free-slip container walls by means of a two-dimensional spectral collocation method. The localization of the least stable mode at the corner is confirmed, and its properties are characterized by various geometric quantities. The effect of the magnetic field strength quantified by the Hartmann number is also examined, and similarities and differences between the wall-attached convection mode on a straight wall and in a corner are pointed out.

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I. INTRODUCTION

Heat transfer by thermal convection is an important aspect of many fluid flows in nature and in technological processes or devices. The behavior of thermal convection in the classical Rayleigh-Bénard system, i.e., a horizontal fluid layer confined between a hot and a cold plate, has intrigued researchers for more than a century and has led to numerous experimental, theoretical, and numerical studies. The onset of Rayleigh-Bénard convection (RBC) as a linear instability of the basic state with pure heat conduction has been studied in various settings [1]. The laminar cellular flow patterns of RBC have also been explored thoroughly [2,3]. They occur at higher temperature differences between the top and the bottom wall than for the onset of RBC. These temperature differences are represented by the Rayleigh number Ra as a nondimensional driving parameter.

The focus of current research on RBC is mostly on turbulent dynamics and transport properties, which can nowadays be studied in detail thanks to novel techniques of optical flow measurement and highly resolved direct numerical simulations (DNS) [4,5]. The effects of additional forces,

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especially Coriolis and Lorentz forces, are also of current interest, e.g., in geo- and astrophysics. These forces typically act anisotropically and tend to suppress fluid motion. In particular, the critical Rayleigh number Ra_c of linear instability is raised as a consequence [1,6,7]. In laboratory experiments on RBC, lateral boundaries of the fluid container also typically increase the resistance to fluid motion and thereby increase Ra_c for narrow containers. In a wide container, these wall effects become less important. With increasing aspect ratio of the container, the critical value Ra_c then tends to a value for a laterally unbounded layer, where horizontal periodicity and isotropy can be assumed [8]. Remarkably, this need not be the case for rotating RBC or RBC in a homogeneous vertical magnetic field. The lateral walls can then reduce the damping effects of Coriolis and Lorentz forces on the flow. In the latter case, the lateral walls must be poor conductors of electricity. As a result, Ra_c still increases with the rotation rate or magnetic field strength when side walls are present, but this increase is weaker than for an unbounded layer. The associated convective motions occur only near the wall.

Work on these side-wall modes was first performed for rotating RBC. It was found that they appear as traveling waves, i.e., with an intrinsic time dependence [9–11]. The robustness of these modes as nonlinear states has also been demonstrated recently [12]. By contrast, the wall-attached mode in magnetic RBC is steady. A first investigation of the linear stability of RBC in a cylinder with an axial magnetic field was performed numerically and semianalytically by Houchens *et al.* [13]. A few years later, Busse [14] simplified the problem considerably and obtained an asymptotic analytical solution for a straight side wall. He also found an asymptotic power law between the critical Rayleigh number $Ra_{c,w}$ for the wall-attached mode and the Hartmann number Ha measuring the magnetic field strength. The scaling exponent for the wall-attached mode is smaller than that for the relation between critical Rayleigh number $Ra_{c,b}$ and Ha for the bulk mode [1].

Since then, several studies have reported DNS of magnetic RBC that covered Rayleigh numbers below $Ra_{c,b}$. In all cases, the side walls are electrically insulating, and the thermal Prandtl number is close to or equal to $Pr = 0.025$. Liu *et al.* [15] considered convection in a box with aspect ratio $\Gamma = 4$ and measured the properties of steady wall-attached convection at a fixed $Ra = 10^7$ for Hartmann numbers up to $Ha = 2000$, i.e., down to $Ra \approx 0.25Ra_{c,b}$. McCormack *et al.* [16] also considered a box but with $\Gamma = 1$ and varied Ra systematically for several fixed Ha . They also found steady wall-attached convection below $Ra_{c,b}$ and studied the transition to time-dependent flow upon increasing Ra . Wu *et al.* [17] also studied square domains with $\Gamma = 2$ and $\Gamma = 4$ at fixed $Ha = 200$ and $Ha = 1000$. They paid particular attention to the scaling of the integral heat transport and integral velocity as well as the spreading of the wall-attached convective state away from the wall into the bulk region upon increasing Ra .

In cylindrical geometries, Akhmedagayev *et al.* [18] reported DNS results for $\Gamma = 1$ with wall-attached structures resembling those of Liu *et al.* [15] at Ra slightly larger than $Ra_{c,b}$. In contrast to the box studies, these structures drift along the curved wall. Recently, Xu *et al.* [19] considered aspect ratios $\Gamma = 1$ and $\Gamma = 2$ and performed both DNS and experiments with liquid Gallium. They confirmed that convection is present for $Ra < Ra_{c,b}$. In contrast to Ref. [18], they did not find drifting wall-attached structures near the onset of this convection mode. Large aspect ratios were not systematically considered in a uniform magnetic field. DNS with a spatially varying magnetic field in a wide rectangular domain were performed by Bhattacharya *et al.* [20]. The setup is shown in Fig. 1(a). The two-dimensional magnetic field in the paper [20] corresponds to that in the uniform gap between two wide rectangular bars with uniform magnetization. With increasing distance from the edge of magnetic bars, the field distribution becomes uniform and vertical inside the gap. The field was strong enough ($Ha_{\max} > 100$) to raise $Ra_{c,b}$ above the imposed $Ra = 10^5$. It was found in Ref. [20] that convection became suppressed inside the gap but persisted near the corners of the side wall that was placed farthest into the gap. Figure 1(b), which is reproduced from Ref. [20], shows a representative case. This observation raises the question if RBC in a homogeneous field is least damped near a corner. The critical Rayleigh number should then be lower than for a straight wall. Reference [20] does not answer this question since the magnetic field is still somewhat nonuniform and not strictly vertical inside the gap between the magnets. The DNS in a uniform magnetic field

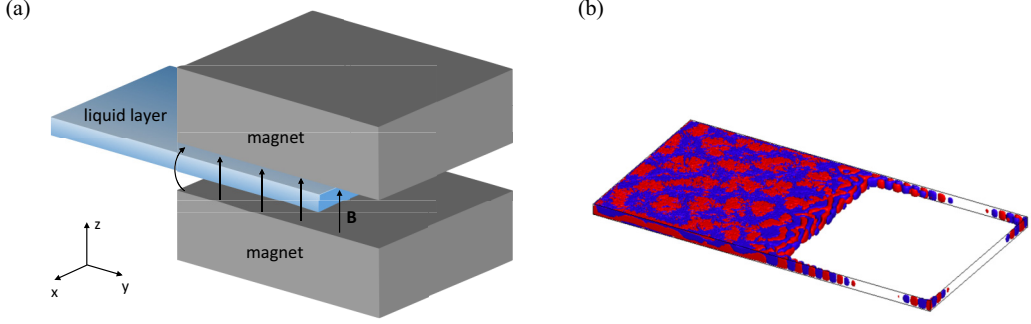


FIG. 1. (a) Sketch of the configuration studied in Ref. [20]. The magnets are unbounded in the x -direction and in the positive y -direction. The liquid layer has the aspect ratio $16 \times 32 \times 1$. One-half of the layer remains outside the magnets. (b) Figure 13(a) from Ref. [20]: Isosurfaces of the vertical velocity $u_z = \pm 0.01 U_f$ (red/blue) for the case when the two gaps between magnets and liquid are equal to the height of the liquid. In the regions with strong magnetic fields, convective motions are confined near the sidewalls. U_f denotes the free-fall velocity.

[15–19] do not provide a clear answer because either the aspect ratio is too small for convection to become localized in the corners (cases with $\Gamma \leq 4$) and/or because Ha was not increased further.

The present work addresses the existence of a distinct corner mode and its properties by a linear stability analysis of magnetohydrodynamic RBC in a rectangular box with electrically and thermally insulating side walls and large aspect ratios. For simplicity, the approach chosen by Busse [14] is adopted, whereby the eigenvalue problem can be reduced to two spatial dimensions. The following sections describe the mathematical model, its discretization, and the testing of the stability code. The main section is devoted to the spectra, spatial features of the eigenmodes, and the influence of Ha on them. The paper ends with a brief summary and some suggestions for further work.

II. PROBLEM FORMULATION

The system under consideration is a viscous liquid in a closed, rectangular box with dimensions $L_x \times L_y \times H$ and a coordinate system as shown in Fig. 2. The electrically conducting liquid with

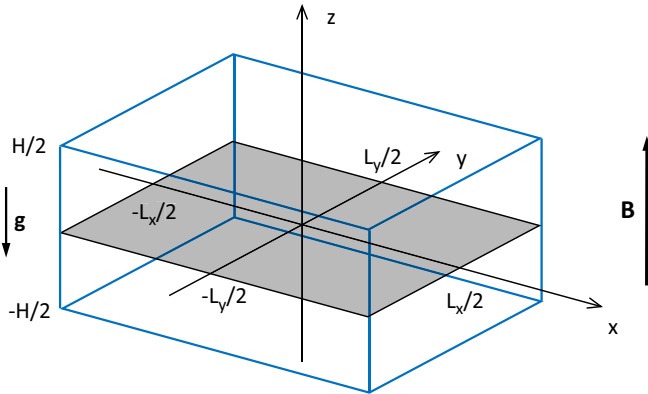


FIG. 2. Sketch of the hexahedral convection cell and the coordinate system. The horizontal midplane $z = 0$ is highlighted. The homogeneous applied magnetic field $\mathbf{B}$ is vertical. The walls are electrically insulating and stress-free. The top and bottom walls are isothermal. The side walls are thermally insulating.

conductivity σ is subject to a steady and homogeneous vertical magnetic field $\mathbf{B}$ parallel to the z axis. The bounding surfaces are stress-free and electrically insulating. The side walls are adiabatic, and the top and bottom surfaces $z = \pm H/2$ are isothermal at temperatures T_1 and $T_2 > T_1$. The mathematical model for the thermal driving of the flow by the Rayleigh-Bénard-instability is the Boussinesq approximation, i.e., the density ρ depends linearly on temperature T and its variation with temperature is taken into account only in the gravity force term. Heat transport in the system by pure conduction leads to a linear temperature profile $T_b(z) = T_2 - \Delta T(z/H + 1/2)$, where $\Delta T = T_2 - T_1 > 0$. The deviation from this temperature distribution is denoted by θ , i.e., $\theta(x, y, z, t) = T(x, y, z, t) - T_b(z)$.

It is further assumed that the magnetic Reynolds number Re_m is small, which is usually satisfied in laboratory experiments with liquid metals. Such liquids have a very small magnetic Prandtl number (typically of order 10^{-6}) while their thermal Prandtl number is of order 10^{-2} . The timescale of magnetic diffusion is therefore much shorter than the other characteristic timescales. In these circumstances, one can apply the quasistatic approximation, i.e., an expansion of the induced electromagnetic quantities up to first power in Re_m in the induction equation [21,22]. The induced electric field in the quasistatic approximation is then curl-free, and the induced magnetic field is negligible by comparison with the imposed one. As a result, Ohm's law for the current density $\mathbf{j}$ in a moving conductor simplifies to $\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B})$, where the negative gradient of the electric potential ϕ represents the induced electric field and $\mathbf{u}$ is the velocity. The electric potential is determined by the condition $\nabla \cdot \mathbf{j} = 0$ from Ampère's law. Consequently, the induced currents become a linear function of the velocity and have no independent dynamics as would be required, e.g., for Alfvén waves to occur. Moreover, the magnetic Prandtl number vanishes from the governing equations.

Using H , H^2/κ , ΔT , B , and $\sigma\kappa B/H$ as the scales for length, time, temperature, magnetic field, and electric potential, the nondimensional governing equations read

$$\frac{1}{Pr} \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \nabla^2 \mathbf{u} + Ra \theta \mathbf{e}_z + Ha^2 (\mathbf{j} \times \mathbf{e}_z), \quad (1a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = \nabla^2 \theta + u_z, \quad (1b)$$

$$\mathbf{j} = -\nabla \phi + \mathbf{u} \times \mathbf{e}_z, \quad (1c)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (1d)$$

$$\nabla \cdot \mathbf{j} = 0. \quad (1e)$$

Note that the DNS studies of wall-attached magnetoconvection mentioned before [15–19] also utilize these equations although they may look different owing to a different choice of reference units. In Eqs. (1a)–(1e), the Prandtl number $Pr = \nu/\kappa$ is the ratio of kinematic viscosity and thermal diffusivity. The other parameters are the Rayleigh and Hartmann numbers

$$Ra = \frac{\alpha g \Delta T H^3}{\nu \kappa}, \quad Ha = B H \sqrt{\frac{\sigma}{\rho \nu}}. \quad (2)$$

The quantity α is the volumetric expansion coefficient of the liquid and g the acceleration due to gravity.

The instability is assumed to be monotonous, i.e., the critical Rayleigh number Ra_c corresponds to a steady solution of the linearized equations (1). Equations (1a) and (1b) therefore simplify to

$$\nabla^2 \mathbf{u} + Ra \theta \mathbf{e}_z + Ha^2 (\mathbf{j} \times \mathbf{e}_z) = \nabla p, \quad (3a)$$

$$\nabla^2 \theta + u_z = 0. \quad (3b)$$

Equations (1c)–(1e) remain unchanged. Together with Eqs. (3), they represent an eigenvalue problem for Ra , which depends on Ha and the aspect ratios $l_x = L_x/H$, $l_y = L_y/H$ of the domain.

III. SPECTRAL DISCRETIZATION AND NUMERICAL SOLUTION

A. Pressure-free formulation of the neutral stability equations

An explicit computation of the pressure field can be avoided by representing the velocity field as the curl of a vector stream function $\boldsymbol{\psi}$. The incompressibility constraint is then always satisfied. For the uniqueness of $\boldsymbol{\psi}$, it is necessary to impose a gauge condition, which is the Coulomb gauge in the present work. The velocity field is therefore given by

$$\mathbf{u} = \nabla \times \boldsymbol{\psi}, \quad \nabla \cdot \boldsymbol{\psi} = 0. \quad (4)$$

By that, the vector stream function is defined up to a gradient of a harmonic function, i.e., one must impose a scalar boundary condition on $\boldsymbol{\psi}$ in addition to the mechanical boundary conditions for $\mathbf{u}$. The pressure is eliminated by taking the curl of Eq. (3a). Before this step, it is useful to consider the vertical structure of $\mathbf{u}$, θ , and $\mathbf{j}$, as well as the consequences for $\boldsymbol{\psi}$ and ϕ . The stress-free boundary conditions should permit a separation ansatz with pure sine/cosine dependence on z of the velocity components as in [14]. The zero normal velocity on the top and bottom walls $z = \pm 1/2$ implies the representation

$$u_z = \hat{u}_z(x, y) \cos(\pi z) = \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}. \quad (5)$$

The vertical wave number π is the smallest admissible choice and should lead to the lowest values of Ra_c . It is apparent from Eq. (5) that both ψ_x and ψ_y must have the same cosine dependence on z as u_z . The gauge condition

$$\nabla \cdot \boldsymbol{\psi} = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z} = 0, \quad (6)$$

imposes that $\psi_z = \hat{\psi}_z(x, y) \sin(\pi z)$. One can therefore represent $\hat{\psi}_z$ via the gauge condition as

$$\hat{\psi}_z = -\frac{1}{\pi} \left(\frac{\partial \hat{\psi}_x}{\partial x} + \frac{\partial \hat{\psi}_y}{\partial y} \right). \quad (7)$$

As a result, the horizontal velocity components u_i are given by $u_i = \hat{u}_i \sin(\pi z)$. In summary, the velocity field components depend on x and y according to the following representations:

$$\hat{u}_x = -\frac{1}{\pi} \left(\frac{\partial^2 \hat{\psi}_x}{\partial y \partial x} + \frac{\partial^2 \hat{\psi}_y}{\partial x^2} - \pi^2 \hat{\psi}_y \right), \quad (8a)$$

$$\hat{u}_y = \frac{1}{\pi} \left(\frac{\partial^2 \hat{\psi}_y}{\partial y \partial x} + \frac{\partial^2 \hat{\psi}_x}{\partial x^2} - \pi^2 \hat{\psi}_x \right), \quad (8b)$$

$$\hat{u}_z = \frac{\partial \hat{\psi}_y}{\partial x} - \frac{\partial \hat{\psi}_x}{\partial y}. \quad (8c)$$

It is therefore sufficient to derive equations for $\hat{\psi}_x$ and $\hat{\psi}_y$ from the momentum equation to obtain the velocity field.

The separation ansatz for θ is analogous to u_z , i.e., $\theta = \hat{\theta} \cos(\pi z)$. It is consistent with Eq. (3b). The vertical current component does not contain u_z and vanishes at $z = \pm 1/2$. This implies that $\phi = \hat{\phi}(x, y) \sin(\pi z)$. This representation is also consistent with Eqs. (1c) and (1e). The representation of the components of the vorticity $\boldsymbol{\omega}$ is analogous to that for the vector stream function $\boldsymbol{\psi}$ because $\boldsymbol{\omega} = -\nabla^2 \boldsymbol{\psi}$ and $\nabla \cdot \boldsymbol{\omega} = 0$.

The pressure-free formulation of the equations is obtained by substitution of the representations for $\mathbf{u}$, ϕ , and θ into the curl of the momentum equation (3a) and Eqs. (3b) and (1c)–(1e). This system

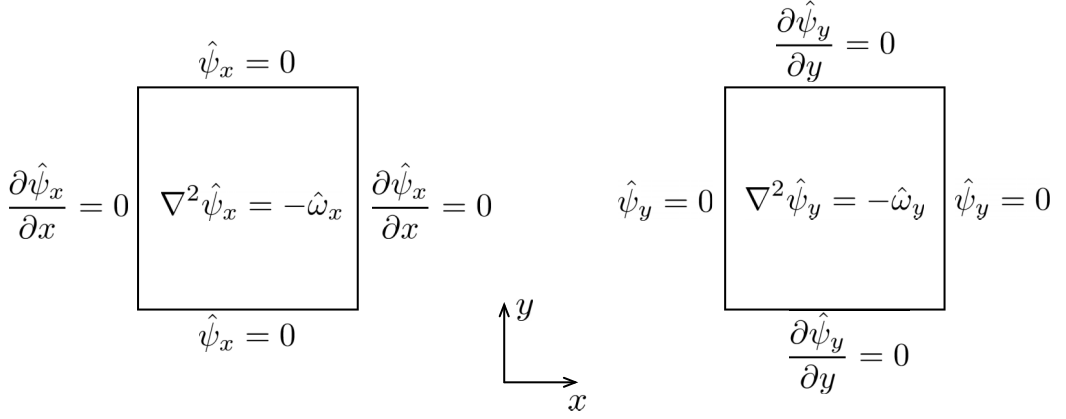


FIG. 3. Sketch of boundary conditions for Eqs. (9a) and (9b).

reads

$$\nabla^2 \hat{\psi}_x + \hat{\omega}_x = 0, \quad (9a)$$

$$\nabla^2 \hat{\psi}_y + \hat{\omega}_y = 0, \quad (9b)$$

$$\nabla^2 \hat{\omega}_x + Ha^2 \pi \left(-\frac{\partial \hat{\phi}}{\partial x} + \hat{u}_y \right) = -Ra \frac{\partial \hat{\theta}}{\partial y}, \quad (9c)$$

$$\nabla^2 \hat{\omega}_y + Ha^2 \pi \left(-\frac{\partial \hat{\phi}}{\partial y} - \hat{u}_x \right) = +Ra \frac{\partial \hat{\theta}}{\partial x}, \quad (9d)$$

$$\nabla^2 \hat{\theta} + \frac{\partial \hat{\psi}_y}{\partial x} - \frac{\partial \hat{\psi}_x}{\partial y} = 0, \quad (9e)$$

$$\nabla^2 \hat{\phi} + \frac{1}{\pi} \left(\frac{\partial \hat{\omega}_x}{\partial x} + \frac{\partial \hat{\omega}_y}{\partial y} \right) = 0. \quad (9f)$$

The velocity components $\hat{u}_x, \hat{u}_y$ are given by Eqs. (8a) and (8b).

B. Lateral boundary conditions

Boundary conditions for Eqs. (9a) and (9b) are obtained from the nonpenetration condition (zero normal velocity u_n) and a second, scalar condition to determine $\hat{\psi}$ uniquely. An advantageous choice is to set the tangential component of $\hat{\psi}$ to zero on the walls, i.e., $\hat{\psi}_x = 0$ on $y = \pm l_y/2$ as well as $\hat{\psi}_y = 0$ on $x = \pm l_x/2$. With the definitions (8a) and (8b), it follows that the normal derivatives of either $\hat{\psi}_x$ or $\hat{\psi}_y$ vanish on the other set of walls. Figure 3 shows the assignment of these conditions to Eqs. (9a) and (9b).

The Poisson equations [(9c), (9d)] also require a boundary condition on each of the edges of the rectangular domain. It turns out that the boundary conditions established for the components $\hat{\psi}_x, \hat{\psi}_y$ in Fig. 3 apply equally to the vorticity components $\hat{\omega}_x, \hat{\omega}_y$. This can be seen from the following argument. The vertical vorticity $\hat{\omega}_z$ is zero on each lateral wall because of the free-slip condition on the in-plane velocity component and the zero normal velocity $\hat{u}_n = 0$. The free-slip condition on the vertical velocity component and $\hat{u}_n = 0$ imply that the tangential in-plane component of the vorticity is zero. It therefore follows from $\nabla \cdot \hat{\omega} = 0$ that the normal derivative of the normal vorticity component $\hat{\omega}_n$ is zero on each wall.

The boundary condition on the lateral walls for Eq. (9e) is the homogeneous Neumann condition. The lateral boundaries are electrically insulating, i.e., the Poisson equation (9f) for the electric potential has the boundary condition $j_n = 0$. This amounts to

$$\frac{\partial \hat{\phi}}{\partial x} = \hat{u}_y \text{ on } x = \pm l_x/2, \quad \frac{\partial \hat{\phi}}{\partial y} = -\hat{u}_x \text{ on } y = \pm l_y/2. \quad (10)$$

C. Spatial discretization in horizontal directions

A suitable discretization of this problem is obtained by a spectral collocation method with Chebyshev polynomials $T_n(z) = \cos\{n \arccos(z)\}$ defined over $[-1, 1]$. The scalar fields such as $\hat{\theta}$ are expanded as a finite double sum, i.e.,

$$f(x, y) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} \tilde{f}_{i,j} T_i(2x/l_x) T_j(2y/l_y), \quad (11)$$

where f denotes either $\hat{\omega}_x$, $\hat{\omega}_y$, $\hat{\theta}$, or $\hat{\phi}$. Equations (9c)–(9f) and the corresponding boundary conditions are enforced pointwise at the $(N_x + 1)(N_y + 1)$ Gauss-Lobatto collocation points x_i and y_j defined by

$$x_i = \frac{l_x}{2} \cos(i\pi/N_x), \quad y_j = \frac{l_y}{2} \cos(j\pi/N_y). \quad (12)$$

Each collocation point provides one scalar equation for the expansion coefficients of $\hat{\omega}_x$, $\hat{\omega}_y$, $\hat{\theta}$, or $\hat{\phi}$ [23]. The corners of the rectangular domain require special consideration when normal derivatives are specified along the boundaries. In these cases, the differential equation itself is imposed at a corner. In the present case, this is done for Eqs. (9e) and (9f).

As a result, one obtains a generalized linear eigenvalue problem of the type

$$\mathbf{A}\mathbf{X} = Ra \mathbf{B}\mathbf{X}, \quad (13)$$

where the vector $\mathbf{X}$ contains the unknown expansion coefficients of $\hat{\omega}_x$, $\hat{\omega}_y$, $\hat{\theta}$, and $\hat{\phi}$, i.e., $\mathbf{X}$ has the size $4(N_x + 1)(N_y + 1)$.

The vector stream function components $\hat{\psi}_x$, $\hat{\psi}_y$ in Eqs. (9c)–(9f) are represented as linear functions of $\hat{\omega}_x$ or $\hat{\omega}_y$ since they are given by Eqs. (9a) and (9b) with the corresponding boundary conditions shown in Fig. 3. The discrete representation of these quantities is also obtained through spectral collocation. However, this requires more collocation points because $\hat{\psi}_x$, $\hat{\psi}_y$ do not only depend on the inhomogeneity but also on the boundary data. The number of expansion coefficients for $\hat{\psi}_x$, $\hat{\psi}_y$ is therefore $(N_x + 3)(N_y + 3)$ in the representation (11) and in Eq. (12). By that, we obtain an invertible linear system between the expansion coefficients of either $\hat{\omega}_x$ or $\hat{\omega}_y$ augmented by the zero boundary data and the expansion coefficients of $\hat{\psi}_x$, $\hat{\psi}_y$. The two inverse matrices are computed and stored before the matrices of the generalized eigenvalue problem (13) are assembled. The procedure was detailed for a similar eigenvalue problem in the Appendix of Ref. [24].

Problem (13) was coded in MATLAB [25] and solved with the `eig` routine to find all eigenvalues and eigenvectors. The routine also works with a matrix $\mathbf{B}$ whose rank is smaller than the rank of $\mathbf{A}$ [as is the case for (13)]. It associates the spurious solutions that stem from equations not containing the eigenvalue Ra with infinite eigenvalues. In the following discussion, it is assumed that the set of finite positive eigenvalues is sorted in ascending order, i.e., the smallest one will be referred to as the first eigenvalue etc. In addition, the stability results with only one set of lateral walls will be required for comparison. For this case, the stability equations reduce to a one-dimensional problem. The details and solution method are given in Appendix A.

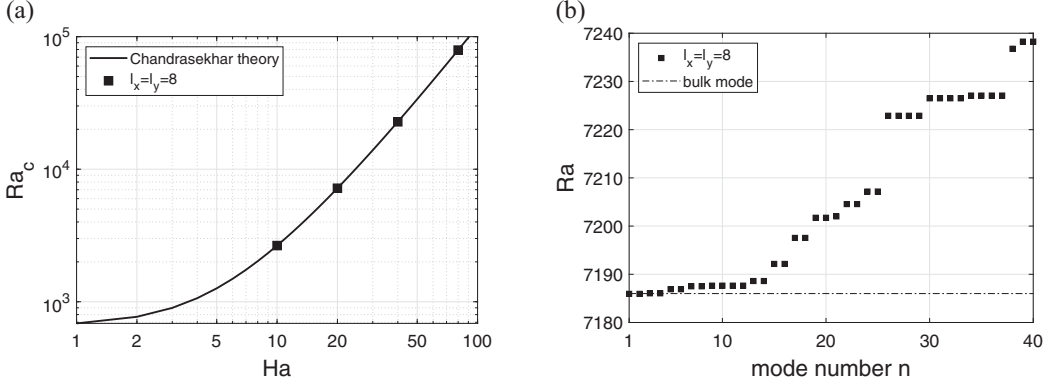


FIG. 4. (a) Smallest eigenvalue from the stability code for $l_x = l_y = 8$ and conducting sidewalls as a function of Ha . (b) Spectrum of eigenvalues Ra for $l_x = l_y = 8$ and $Ha = 20$.

D. Code validation for perfectly conducting lateral walls

When the lateral walls are perfectly conducting, i.e., the condition $\hat{\phi} = 0$ applies, the eigenvalue problem has an analytical solution. The eigenmodes can then be assumed to be products of a sine/cosine in x and a sine/cosine in y for the respective scalar quantities. Specifically, the vertical component of the vorticity can be represented as

$$\hat{\omega}_z = \Omega \cos(\alpha x) \cos(\beta y), \quad (14a)$$

$$\hat{\omega}_z = \Omega \cos(\alpha x) \sin(\beta y), \quad (14b)$$

$$\hat{\omega}_z = \Omega \sin(\alpha x) \cos(\beta y), \quad (14c)$$

$$\hat{\omega}_z = \Omega \sin(\alpha x) \sin(\beta y), \quad (14d)$$

where α and β are wave numbers and Ω is a constant. These functions are odd or even with respect to x and y . The boundary conditions for ω_z are zero on the walls $x = \pm l_x/2$, $y = \pm l_y/2$, which implies $\alpha l_x/2 = (p + 1/2)\pi$ for a mode that is even in x and $\alpha l_x/2 = p\pi$ for a mode that is odd in x ($p \geq 0$ is an integer). The same conditions apply in the coordinate y , i.e., $\beta l_y/2 = (q + 1/2)\pi$ for even parity and $\beta l_y/2 = q\pi$ for the odd parity ($q > 0$ is an integer).

To each of the possible selections [(14a)–(14d)] for ω_z , there is a corresponding one for all the other scalar fields of the eigenmode, which also take the form specified by (14). The correct one can be found by inspection from the definition of the vector stream function and Eqs. (9a)–(9f).

The analytical solution for the eigenvalue Ra can be computed from the eigenmode by eliminating $\hat{\phi}$ from Eqs. (9c) and (9d) and representing $\hat{\theta}$ in terms of $\hat{u}_z$. It is

$$Ra(Ha, \alpha, \beta) = \frac{\pi^2 + \alpha^2 + \beta^2}{\alpha^2 + \beta^2} ((\pi^2 + \alpha^2 + \beta^2)^2 + \pi^2 Ha^2), \quad (15)$$

and simplifies to Chandrasekhar's result $Ra(Ha, k)$ [1] for the unbounded layer, i.e., the bulk case, upon setting $k^2 = \alpha^2 + \beta^2$, where k is the magnitude of the wave vector.

In principle, one can compute all discrete eigenvalues for a given set of (l_x, l_y) and Ha from Eq. (15) by varying p and q . They should be reproduced by the numerical code. For simplicity, only the smallest Ra , i.e., the critical value Ra_c , will be compared between the numerical solution of equations (9a)–(9f) and Eq. (15). One can derive the minimum $Ra_c(Ha)$ and the corresponding wave number k_c from Eq. (15) numerically. Figure 4(a) shows a comparison of the bulk case with the smallest eigenvalue from the numerical solution obtained with the stability code for a box with $l_x = l_y = 8$ and different values of Ha . In each case, the values of Ra_c agree at least to five digits between the numerical stability code and the result from Eq. (15). Figure 4(b) shows a spectrum of

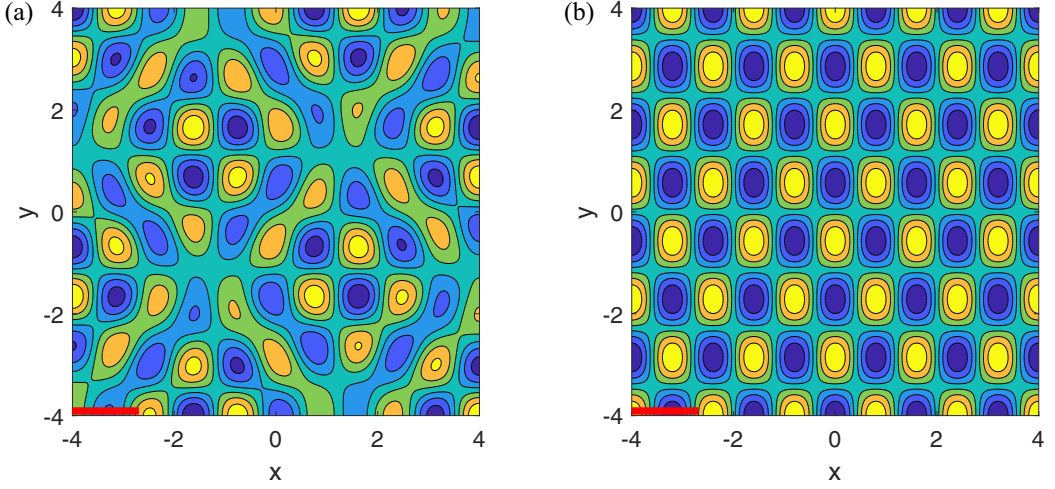


FIG. 5. The subfigures show isocontours of temperature perturbation θ of the first eigenmode in the plane $z = 0$ for $Ha = 20$ and side length $l_x = 8$. In (a), $l_y = l_x$, whereas in (b), $l_y = l_x (1 + \varepsilon)$ with $\varepsilon = 10^{-4}$. The mean value of θ is zero. The thick red line shows the wavelength $2\pi/k$ of the bulk mode according to Eq. (15) for comparison.

the lowest eigenvalues for the case $Ha = 20$. It is fairly dense, which is needed to obtain a good approximation of Ra_c .

The spatial structure of the mode with the smallest Ra for $Ha = 20$ is shown in Fig. 5. When one uses $l_x = l_y$, certain eigenvalues become degenerate since one can swap the variables x and y in the representations (14) without changing Ra , and the boundary conditions remain satisfied. The eigenvalue solver cannot separate such eigenmodes into different parities, as is apparent in Fig. 5(a). It is therefore advantageous to change l_y slightly, whereby the degenerate eigenvalues are separated. This provides eigenmodes with a specific parity in both x and y as shown in Fig. 5(b). The two wavelengths of the eigenmode in Fig. 5(b) are given by $l_x/5 = 2\pi/\alpha$ and $2l_y/7 = 2\pi/\beta$. The corresponding length of the wave vector is $k = 4.7933$, which is very close to the exact minimum $k = 4.7942$ of Chandrasekhar's function $Ra(Ha = 20, k)$, i.e., Eq. (15).

IV. RESULTS

Following the successful code validation, the problem with insulating side walls was investigated numerically. The domain was essentially square, i.e., $l_y = l_x(1 + \varepsilon)$, $\varepsilon = 10^{-4}$. The computations were performed with Matlab [25] on different single nodes of a parallel cluster with Intel Xeon processors (4th generation, 32 cores).

A. Spectrum of eigenvalues

The first series of computations examined the effect of the side length l_x on the eigenvalues at two fixed Hartmann numbers $Ha = 20$ and $Ha = 80$. The convergence of the first eigenvalue upon increasing the resolution was systematically tested for $l_x = 8$. The results are given in Table I.

The spectra of eigenvalues are shown in Figs. 6(a) and 6(b). It is apparent that the spectrum begins at lower values of Ra than $Ra_{c,b}$ for the bulk mode, which is expected on account of the wall modes. However, the spectrum shows that the lowest four values of Ra lie below the critical value $Ra_{c,w}$ of the wall mode with periodicity along the wall (see Appendix A for the computation of $Ra_{c,w}$ with periodic boundary conditions in y). The lowest four eigenvalues remain essentially constant upon increasing the side length l_x once l_x exceeds a value of about $l_x = 4$. Moreover, they become

TABLE I. First eigenvalue for $l_x = 8$ and insulating walls for different number of modes $N_x = N_y$. The rapid convergence with N_x is apparent in both cases. Run times vary with the computational load on the parallel cluster. Tabulated values correspond to the best case at the indicated resolution.

$N_x + 2$	$Ra_1 (Ha = 20)$	$Ra_1 (Ha = 80)$	Run time (hours)	RAM size (GB)
50	$3.420272699 \times 10^{+3}$		2.3	3.6
60	$3.420276980 \times 10^{+3}$		5.8	7.8
70	$3.420277353 \times 10^{+3}$	$1.936468089 \times 10^{+4}$	15.0	13.4
80	$3.420277382 \times 10^{+3}$	$1.936790418 \times 10^{+4}$	20.9	27.4
90		$1.936857781 \times 10^{+4}$	43.8	38.5
100		$1.936873446 \times 10^{+4}$	90.4	62.4

essentially equal, cf. Table II. This suggests that the corresponding eigenmodes are localized in the four corners on the rectangular domain and that they appear in the four different parities as in the case of perfectly conducting boundaries, see Eqs. (14).

Another interesting feature of Figs. 6(a) and 6(b) is that there is a distinct change in the growth of the eigenvalue Ra with the mode number n between the interval $Ra_{c,w} < Ra < Ra_{c,b}$ and above $Ra_{c,b}$. The eigenvalues grow more rapidly with n for $Ra_{c,w} < Ra < Ra_{c,b}$ but much less so for $Ra > Ra_{c,b}$. This effect can be attributed to the confinement of the wall modes to the near-wall region. The number of excitable wavelengths and modes at a given Ra then grows in proportion to the perimeter length of the domain. Figure 7(a) shows such a linear trend for the larger values of l_x for both $Ha = 20$ and $Ha = 80$.

The growth of the difference $Ra - Ra_{c,b}$ with n above $Ra_{c,b}$ is shown in Fig. 7(b) for $Ha = 20$ for perfectly conducting and insulating walls. In both cases, these shifted eigenvalue spectra are plotted vs the relative index $n - n_b$ commencing at $Ra_{c,b}$. As $n - n_b$ grows, both curves approach one another. They also agree with the asymptotic growth rate

$$Ra = \frac{16\pi^2 n^2}{l_x^4}, \quad (16)$$

that can be derived for perfectly conducting walls (see Appendix B). The estimate is independent of Ha . The deviations between the asymptotic growth estimate and the numerical results for large n are caused by the finite numerical resolution and can be improved by increasing N_x, N_y .

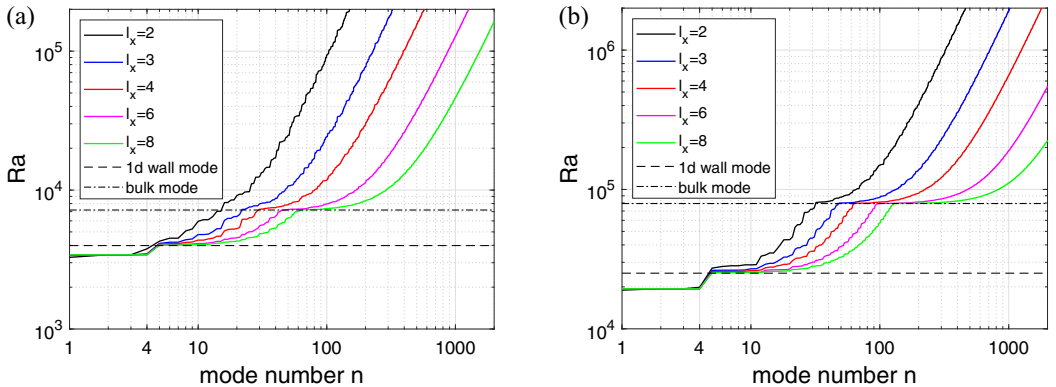


FIG. 6. Spectra for insulating sidewalls: eigenvalues Ra as a function of the mode number n at various l_x for $Ha = 20$ (a) and $Ha = 80$ (b). The horizontal lines correspond to the stability limits for the periodic wall mode and the bulk mode.

TABLE II. First four eigenvalues at $Ha = 20$ for various l_x (insulating walls).

l_x	Ra_1	Ra_2	Ra_3	Ra_4	$N_x + 2$
2	$3.278492182 \times 10^{+3}$	$3.422825316 \times 10^{+3}$	$3.422864828 \times 10^{+3}$	$3.775505771 \times 10^{+3}$	50
3	$3.381915658 \times 10^{+3}$	$3.419958599 \times 10^{+3}$	$3.419983777 \times 10^{+3}$	$3.475513183 \times 10^{+3}$	60
4	$3.410798506 \times 10^{+3}$	$3.420225550 \times 10^{+3}$	$3.420237622 \times 10^{+3}$	$3.431086076 \times 10^{+3}$	70
6	$3.419883137 \times 10^{+3}$	$3.420278137 \times 10^{+3}$	$3.420280023 \times 10^{+3}$	$3.420681107 \times 10^{+3}$	70
8	$3.420277382 \times 10^{+3}$	$3.420279347 \times 10^{+3}$	$3.420279569 \times 10^{+3}$	$3.420281545 \times 10^{+3}$	80

B. Eigenmodes

The first visualization of the eigenmodes uses the temperature perturbation in the largest domain. Figure 8 demonstrates that the essentially equal eigenvalues in Table II at $Ha = 20$ correspond to eigenmodes with different parities but essentially the same structure and localization in the corners. It seems appropriate to call these modes *corner modes*. They also occur at other values of Ha , e.g., at $Ha = 80$.

The different combinations for the temperature perturbation are even in x , even in y (e/e) in Fig. 8(a), odd in x , even in y (o/e) in Fig. 8(b), and (e/o), (o/o) in Figs. 8(c) and 8(d). The color range for the parity (e/e) is different from the others since only maxima appear at the corners in this case. The maxima and minima are therefore not distributed symmetrically with respect to zero. The seemingly less rapid decay of $\hat{\theta}$ toward the middle of the edges in Fig. 8(a) can also be attributed to this feature.

As the length l_x becomes shorter, the opposite corners/walls begin to influence the corner modes. The increasing confinement modifies their shape somewhat and can cause mode crossing, i.e., a mode of a given parity may swap its place with another mode in the ascending sequence Ra_n . These effects were seen in the data for smaller l_x but were not examined further since the focus of the present work is on clearly separated corner modes.

The other fields of the first eigenmode with (e/e) parity for the temperature perturbation are shown in Fig. 9. The parity of the vertical velocity coincides with that of the temperature. The other quantities have different parities that are easily identified. They are (o/e) for $\hat{u}_x$, (e/o) for $\hat{u}_y$, and (o/o) for the electric potential $\hat{\phi}$ in agreement with the discussion in Sec. III D.

When the corner modes are localized, they can be regarded as isolated. Their properties should then no longer depend on the particular parity. For this reason, the velocity and electric current density distributions will be examined only near the corner ($l_x/2, l_y/2$).

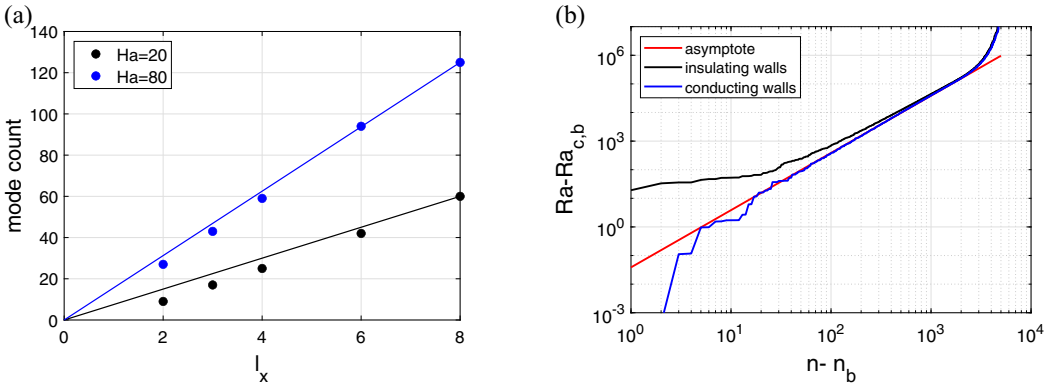


FIG. 7. (a) Number of eigenvalues in the interval $Ra_{c,w} \leq Ra \leq Ra_{c,b}$ for $Ha = 20$, $Ha = 80$ and insulating walls. (b) Spectrum above $Ra_{c,b}$ for perfectly conducting and insulating walls in the case $Ha = 20$, $l_x = 8$. n_b denotes the mode number of the largest mode below $Ra_{c,b}$.

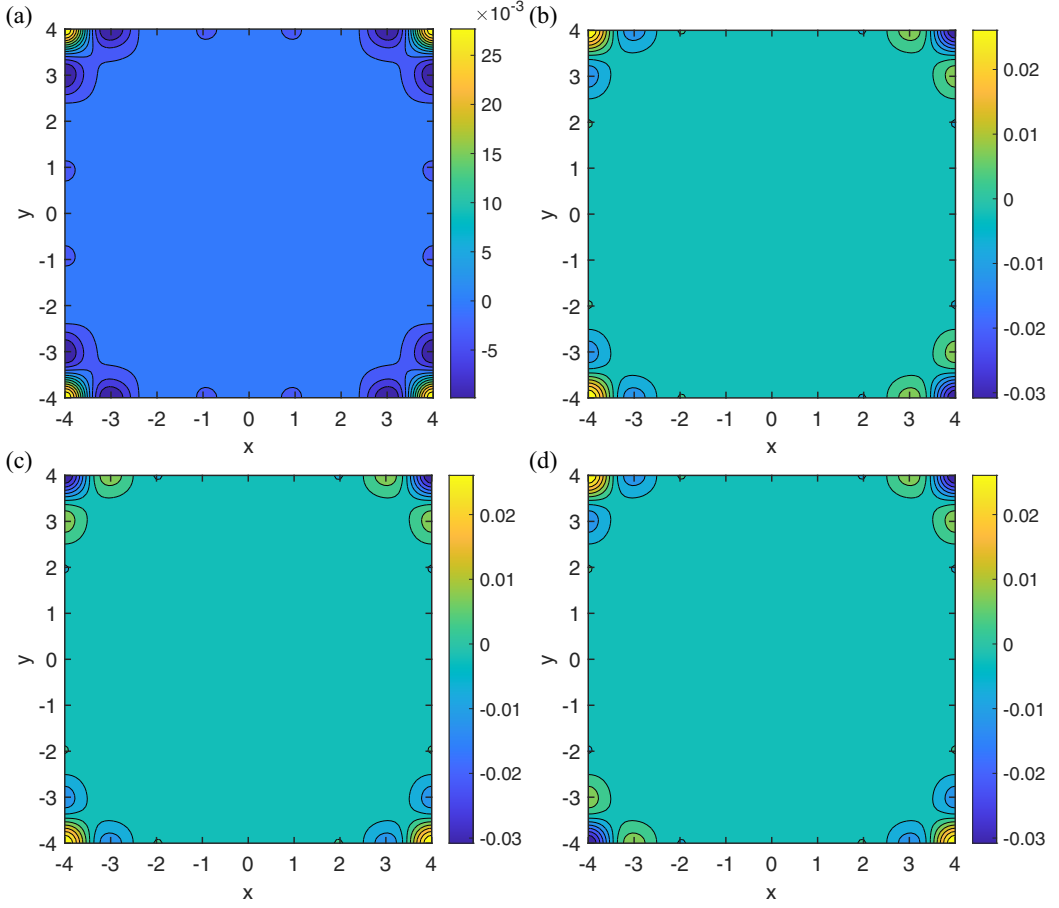


FIG. 8. Isocontours of temperature perturbation $\hat{\theta}$ of the first four eigenmodes in the plane $z = 0$ for $Ha = 20$ and side length $l_x = 8$ in ascending order.

Figure 10 shows streamlines of the velocity field [Figs. 10(a) and 10(c)] and the current density field [Figs. 10(b) and 10(d)] at $Ha = 20$ from the top and in a three-dimensional view for the same mode as in Fig. 9. It is apparent that the streamlines have a mirror-symmetry about the midplane $z = 0$ since the velocity and current density components have a corresponding vertical structure. The streamlines of the velocity show a flow that is concentrated near the corner. Visually, it appears as a curved convection roll whose axis is perpendicular to each side wall [blue streamlines in Figs. 10(a) and 10(c)]. Farther away, the streamlines (in red) suggest the presence of further convection rolls emanating from the side walls, which are bent to connect with the inner corner flow.

By contrast to the velocity field, the current density streamlines seem to organize in loops around the diagonal near the corner. Farther away, they form loops that surround the region of strong vertical flow, e.g., near $x = 4, y = 3$. These loops can even reach the opposite wall when they pass very close to the wall, as can be seen by the complex spiralling streamline in Fig. 10(d). Note that the seed points of the streamlines are different between the current density and the velocity fields.

It is instructive to compare the corner mode with a typical wall mode. Figure 11 shows the vertical velocity and electric potential of the first mode that is not localized in the corner. These quantities display a regular cellular pattern along the perimeter with a well-defined wavelength and a rather small width, i.e., there is essentially no flow in the bulk. The vertical velocity is close to zero in the

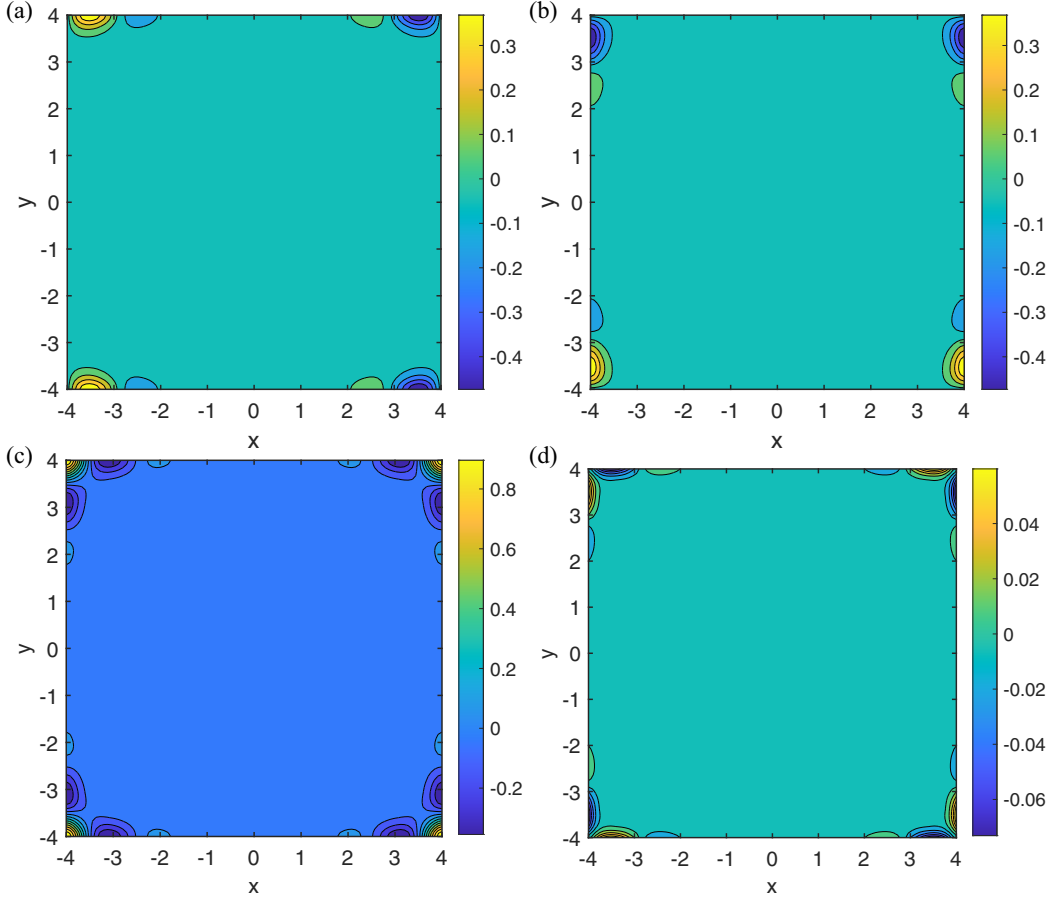


FIG. 9. Isocontours of $\hat{u}_x$, $\hat{u}_y$, $\hat{u}_z$, and $\hat{\phi}$ in the order (a)–(d) for the first eigenmode at $Ha = 20$ and side length $l_x = 8$.

corners, where the pattern also shows some distortion. The different parities of $\hat{u}_z$ and $\hat{\phi}$ are also apparent.

The streamlines of velocity and electric current density of this wall mode are shown in Fig. 12 for the central part of the edge at $y = l_y/2$. They share certain features with those of the corner mode. The velocity streamlines are relatively flat near the wall. In the up/downflow zones near the wall, some streamlines are relatively flat with respect to the $x - z$ plane. As one moves away from the wall in the regions of mostly horizontal flow, the streamlines are increasingly bent and eventually no longer connect two neighboring up- and downflow regions close to the wall. The current density streamlines are separate loops around the up/downflow regions near the wall that are symmetric with respect to the midplane $z = 0$ and flattened with respect to the $x - y$ -plane. Away from this region, the current density streamlines turn into loops that cross the plane $z = 0$. They are also flattened in the $y - z$ -plane in the regions of mostly horizontal flow, which is expected by the direction of the electromotive force $\vec{u} \times \vec{e}_z$.

In conclusion, the streamline pattern of the corner mode seems to resemble what one would obtain by folding the streamline structure in Fig. 12(c) around a center that corresponds to one vertical up- or downflow line on the wall. The same applies for the electric current density streamlines in Fig. 12(d).

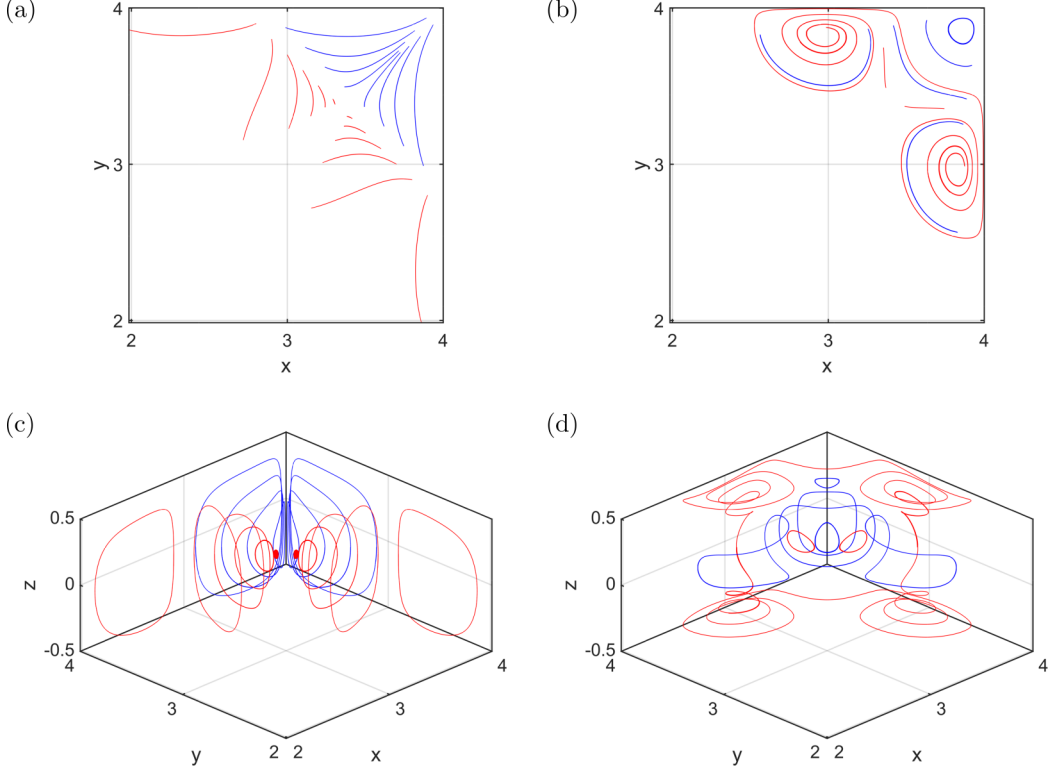


FIG. 10. Streamlines of velocity [(a) and (c)] and electric current density [(b) and (d)] of the first eigenmode at $Ha = 20$ and $l_x = 8$. Different colors are used to improve the perception of the three-dimensional structure of the streamlines.

C. Influence of the Hartmann number

The resolution requirements for the number of expansion coefficients, i.e., N_x , N_y , limit the accessible range of Hartmann numbers for large side lengths l_x , l_y . The numerical effort can be reduced when only one of the four types of parity is admitted. It is then sufficient to consider only a single quadrant of the original domain. The parity type shown in Fig. 9 is convenient for this simplification. Neither the flow nor the heat flux crosses the lines $x = 0$, $y = 0$ in this case. It is only necessary to modify the boundary conditions for the electric potential to $\hat{\phi} = 0$ on the lines $x = 0$, $y = 0$. The other boundary conditions remain unchanged. This modified configuration will be called the *reduced domain* in what follows. The new positions of the boundaries of the original upper right quadrant in Figs. 8(a), 9 will be shifted by $-l_x/2$ in x , $-l_y/2$ in y in order to keep the expansions (11). The lengths l_x , l_y in the expansions (11) for the reduced domain are equal to one-half of the side length $l_x = 8$ in the original domain (cf. the Sec. IV B), i.e., $l_x = l_y = 4$ in the reduced domain.

On the reduced domain, the Hartmann number could be increased to $Ha = 200$ while keeping N_x , N_y in the same range as in Table I. Changes in the corner mode between $Ha = 20$ and $Ha = 200$ are apparent from the isocontours of $\hat{\theta}$ and $\hat{u}_z$ in Fig. 13. Owing to the small-scale features, only the top quadrant of the reduced domain is shown. While the flow near the corner keeps its overall structure described in the previous Sec. IV B, its overall size is reduced at $Ha = 200$. This concerns both the width of the up- and downwelling regions along the walls as well as the thickness in the direction transverse to each side wall. The width seems to shrink in equal measure for $\hat{\theta}$ and $\hat{u}_z$. However, the vertical velocity distribution seems much thinner at $Ha = 200$ than the distribution of the temperature perturbation.

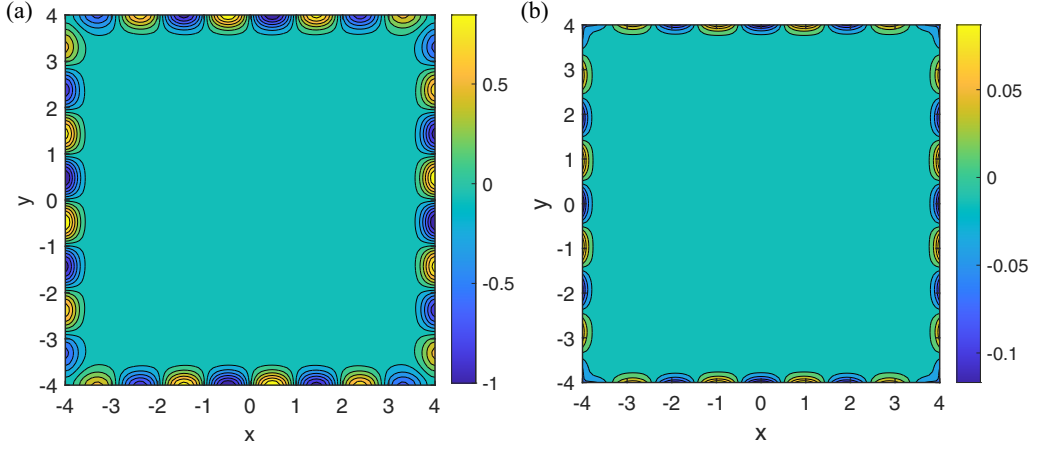


FIG. 11. Isocontours of $\hat{u}_z$ (a) and $\hat{\phi}$ (b) of the fifth eigenmode for $Ha = 20$ and side length $l_x = 8$. The eigenvalue is $Ra_5 \approx 4.013 \times 10^4$.

A more detailed comparison can be made with profiles of $\hat{\theta}$ and $\hat{u}_z$ along the top wall shown in Fig. 14. Both quantities exhibit a more rapid decay and a shorter width upon increasing Ha from $Ha = 20$ to $Ha = 200$ in roughly geometric progression. Generally, the profile of $\hat{\theta}$ has a slightly larger width than that of $\hat{u}_z$ at a given Ha , and $\hat{u}_z$ decays more rapidly near $x = 2$ than $\hat{\theta}$. The differences for a given quantity become less significant between the successive values of Ha .

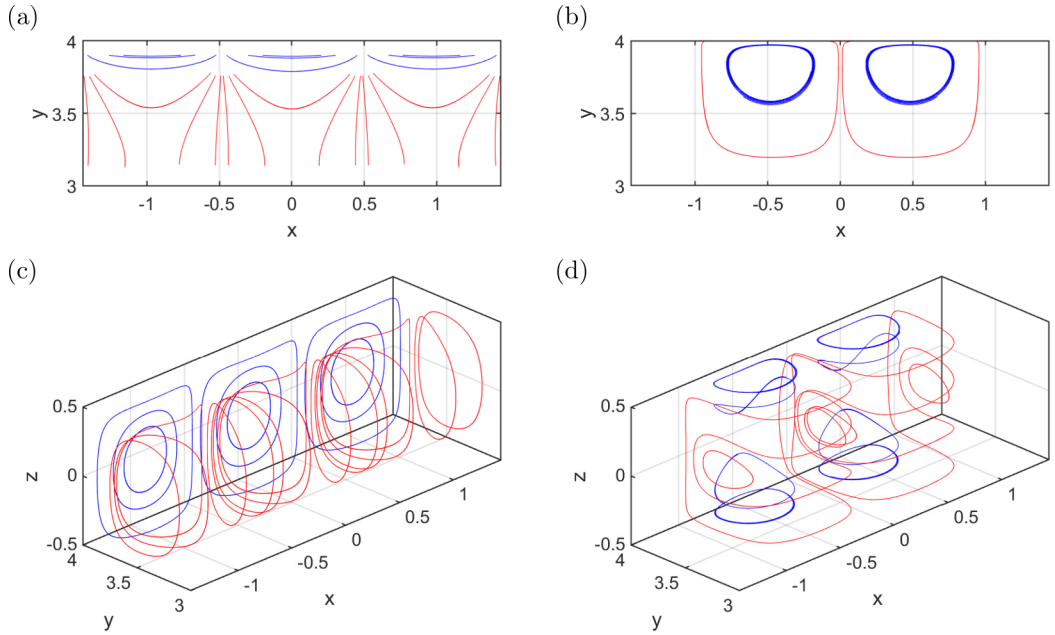


FIG. 12. Streamlines of velocity [(a) and (c)] and electric current density [(b) and (d)] of the fifth eigenmode at $Ha = 20$ and $l_x = 8$. Different colors are used to improve the perception of the three-dimensional structure of the streamlines.

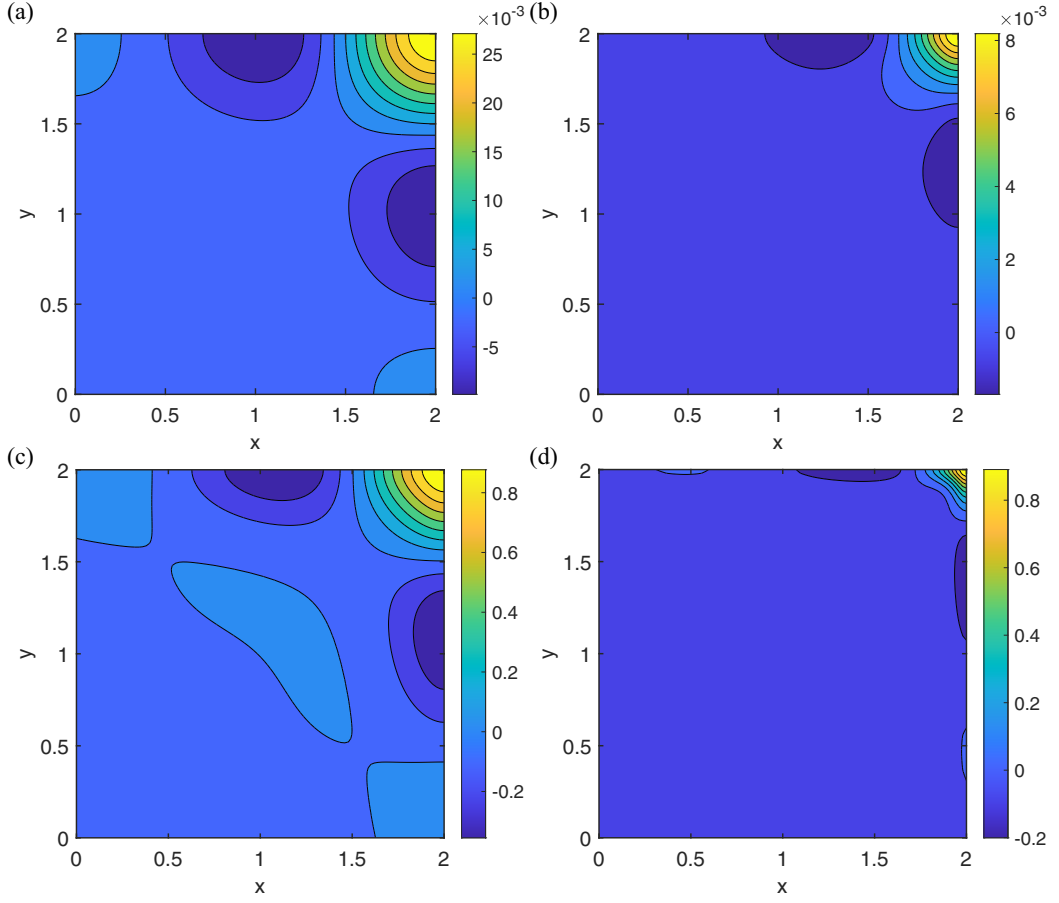


FIG. 13. First eigenmode at $Ha = 20$ (a), (c) and $Ha = 200$ (b), (d) in the upper right part of the reduced domain. (a), (b) Isocontours of temperature perturbation $\hat{\theta}$. (c), (d) Isocontours of $\hat{u}_z$.

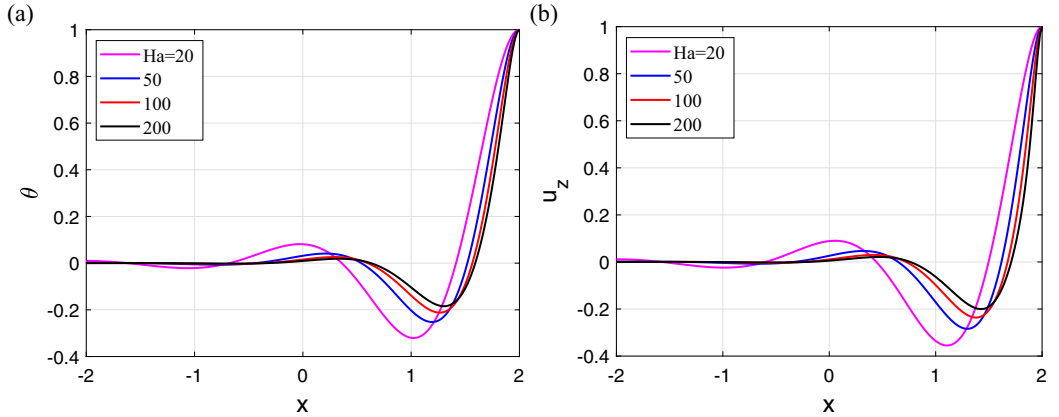


FIG. 14. Normalized profiles of $\hat{\theta}$ (a) and $\hat{u}_z$ (b) for the corner mode along the top wall $y = 2$ in the reduced domain.

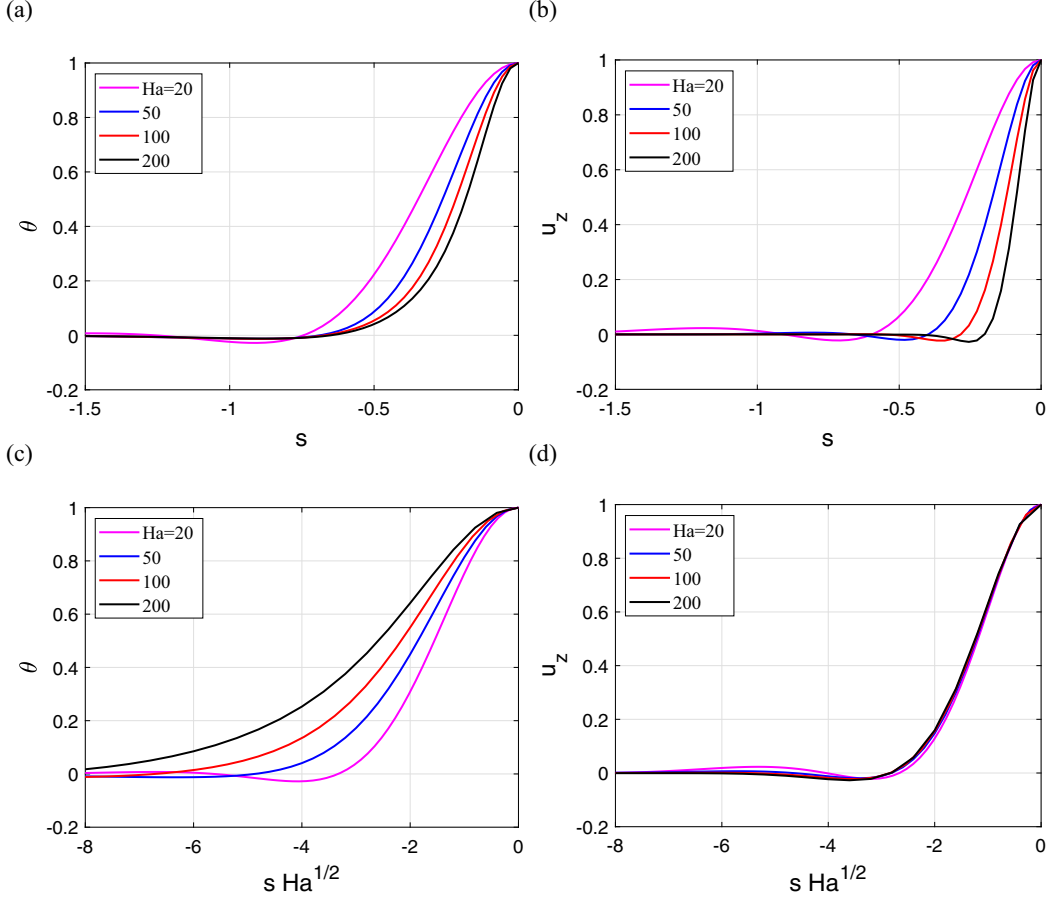


FIG. 15. Normalized profiles of $\hat{\theta}$ (a),(c) and $\hat{u}_z$ (b),(d) for the corner mode along the diagonal line $x = y$ in the reduced domain. A rescaled coordinate is used in (c),(d).

To characterize the thickness of the distributions of temperature perturbation and vertical velocity, one can compare profiles of these quantities along the diagonal line $x = y$. The results are shown in Fig. 15. The coordinate s measures the distance from the corner point $(x_c, y_c) = (2, 2)$ along the diagonal. More precisely, $s = \sqrt{2}(x - x_c)$ for the point $(x, y = x)$, i.e., s takes negative values in the reduced domain.

It is apparent from Fig. 15(a) that $\hat{\theta}$ behaves analogously to Fig. 14(a), i.e., differences between successive Ha diminish as Ha grows. However, there is much more rapid decay in amplitude than along the wall. By contrast, the profiles of $\hat{u}_z$ in Fig. 15(b) show a clear reduction in thickness from a given Ha to the next. Otherwise, the profiles appear to have a comparable shape that is indicative of a scaling behavior. Guided by the scaling proposed by Busse for the wall mode [14], it can be expected that the thickness should decrease as $Ha^{-1/2}$. This is applied in Fig. 15(d), which shows a satisfactory collapse of the rescaled vertical velocity profiles for different values of Ha . The scaling does not hold for $\hat{\theta}$, which is demonstrated by the separation between profiles in Fig. 15(c).

Finally, it remains to examine the dependence of $Ra_c = Ra_1$ on Ha for the corner mode. It is shown in Fig. 16(a) together with the second eigenvalue Ra_2 for the first wall mode in the reduced domain (numerical values are given in Table III). The results for the periodic wall mode (see Appendix A) and the asymptotic estimate $Ra \sim Ha^{3/2}$ from [14] are also shown for comparison. As can be expected, the asymptotic estimate exceeds Ra_2 but Ra_2 approaches a very similar scaling

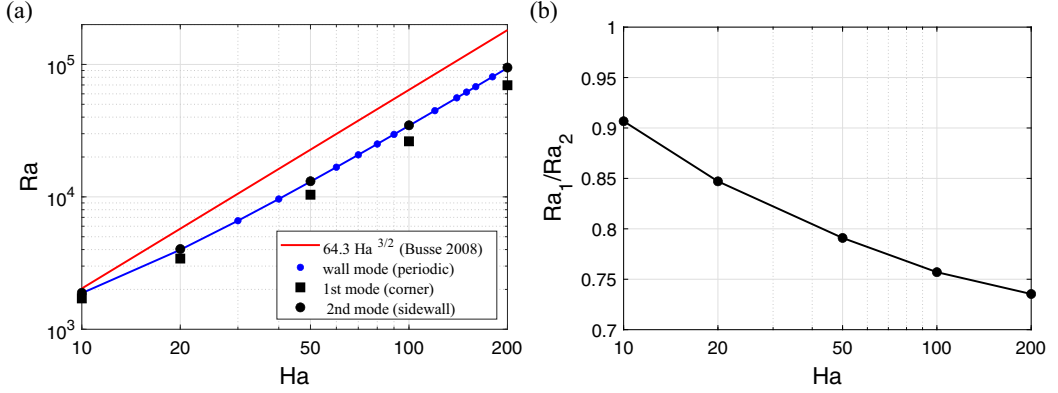


FIG. 16. (a) Eigenvalues in reduced domain ($l_x = 4$) as functions of Ha . The results for a straight periodic wall and the asymptotic relation from [14] are shown for comparison. (b) Ratio of first to second eigenvalue in reduced domain as a function of Ha .

on Ha . There is also a very good agreement between Ra_2 and $Ra_{c,w}$ for the periodic wall mode (cf. Appendix A). The value Ra_1 for the corner mode is systematically smaller than Ra_2 , and the relative deviation increases with Ha as shown in Fig. 16(b). However, as the slope in Fig. 16(b) decreases with Ha , it seems possible that the ratio Ra_1/Ra_2 tends to some finite limit. The corner mode would then have the same asymptotic scaling exponent as the wall mode.

V. CONCLUSIONS AND FUTURE WORK

The present work addressed the classical stability problem of magnetohydrodynamic Rayleigh-Bénard convection with a vertical uniform field in a rectangular container. It was hypothesized that the corners present the regions where the effect of eddy current damping is least effective when the side walls are insulating, i.e., the critical eigenmode should be localized there. The mathematical model was formulated using the quasistatic limit of the induction problem and the assumption of free-slip walls. The vertical dependence of the different fields could then be represented by a single sine or cosine function, and the remaining eigenvalue problem became two-dimensional. It was discretized and solved in Matlab with a spectral Chebyshev-Chebyshev collocation method. The expectation about the nature of the leading eigenmode was confirmed by the computations. The spatial structure of the corner mode and its dependence on Ha were studied up to $Ha = 200$ on a square domain. Apart from the localization, it shares many properties of the wall mode with one periodic direction.

The simplified boundary conditions in the model present a significant limitation that should be surmounted in future work. Previous DNS with a nonuniform magnetic field [20] indicate that a corner mode also occurs with no-slip boundaries, but it remains to be shown by linear stability analysis. By contrast to the present case, this stability problem can most likely not be reduced to

TABLE III. First two eigenvalues at $l_x = l_y = 4$ in the reduced domain (insulating walls).

Ha	Ra_1	Ra_2	$N_x + 2$
20	3.420×10^3	4.038×10^3	50
50	1.038×10^4	1.312×10^4	60
100	2.628×10^4	3.471×10^4	70
200	6.967×10^4	9.474×10^4	100

two dimensions. Further numerical work could address the nonlinear behavior of the corner mode by DNS. The works published so far [15–17] are limited to a relatively small aspect ratio. This parameter would have to be increased considerably.

Finally, it must be mentioned that a clean wall mode in magnetic RBC is difficult to detect in an experiment since it is inherently weak. Recently, Zürner *et al.* [26] could confirm its presence through measurements of radial velocity profiles in a cylindrical cell with $\Gamma = 1$ and $H = 180$ mm filled with GaInSn alloy. Accomplishing this task for a corner mode presumably presents an even harder challenge.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [27]; embargo periods may apply.

APPENDIX A: STABILITY PROBLEM WITH ONE UNBOUNDED HORIZONTAL DIRECTION

The y -direction is chosen as unbounded. The linear stability problem is then homogeneous with respect to this coordinate, and one can represent the dependence on y by a Fourier mode $\exp(i\beta y)$ with an arbitrary real wave number β . The basic formulation of the equations is unchanged, i.e., $\hat{\psi}_x$ and $\hat{\psi}_y$ are used to represent the velocity field. The eigenvalue problem (9) and the expressions (8) for the velocity components apply as well. Derivatives with respect to y become multiplications by $i\beta$. As a result, it becomes advantageous to eliminate $\hat{\omega}_x$ and $\hat{\omega}_y$ from the system (9) and work with $\hat{\psi}_x$, $\hat{\psi}_y$, $\hat{\theta}$ and $\hat{\phi}$ only. Equations (9c) and (9d) then contain the biharmonic operator. The simplified system of ordinary differential equations reads

$$\left(\frac{\partial^2}{\partial x^2} - \beta^2 - \pi^2\right)^2 \hat{\psi}_x - Ha^2 \left(\frac{\partial^2}{\partial x^2} - \pi^2\right) \hat{\psi}_x - i\beta Ha^2 \frac{\partial \hat{\psi}_y}{\partial x} + Ha^2 \pi \frac{\partial \hat{\phi}}{\partial x} = Ra i\beta \hat{\theta}, \quad (\text{A1a})$$

$$\left(\frac{\partial^2}{\partial x^2} - \beta^2 - \pi^2\right)^2 \hat{\psi}_y + Ha^2 (\pi^2 + \beta^2) \hat{\psi}_y - i\beta Ha^2 \frac{\partial \hat{\psi}_x}{\partial x} + iHa^2 \pi \beta \hat{\phi} = -Ra \frac{\partial \hat{\theta}}{\partial x}, \quad (\text{A1b})$$

$$\left(\frac{\partial^2}{\partial x^2} - \beta^2 - \pi^2\right) \hat{\theta} + \frac{\partial \hat{\psi}_y}{\partial x} - i\beta \hat{\psi}_x = 0, \quad (\text{A1c})$$

$$\left(\frac{\partial^2}{\partial x^2} - \beta^2 - \pi^2\right) \hat{\phi} - \frac{1}{\pi} \left(\frac{\partial^2}{\partial x^2} - \beta^2 - \pi^2\right) \left(\frac{\partial \hat{\psi}_x}{\partial x} + i\beta \hat{\psi}_y\right) = 0. \quad (\text{A1d})$$

The spectral discretization using Chebyshev polynomials is only required for the dependence on x , i.e.,

$$f(x) = \sum_{k=0}^{N_x} \tilde{f}_k T_k(2x/l_x). \quad (\text{A2})$$

The number of unknowns in the eigenvalue problem [(A1a)–(A1d)] is $4(N_x + 1)$. The boundary conditions at $x = \pm l_x/2$ are $\hat{\psi}_y = \partial_x^2 \hat{\psi}_y = \partial_x \hat{\psi}_x = \partial_x^3 \hat{\psi}_x = \partial_x \hat{\theta} = 0$ and $\partial_x \hat{\phi} = \hat{u}_x$. Table IV lists results for several Ha and $l_x = 8$ obtained with $N_x = 150$. The second eigenvalue Ra_2 is essentially

TABLE IV. Minima Ra_i of the first and third eigenvalues and the corresponding wave numbers β_i from solving the system (A1). The wave number k corresponds to the minimum of Ra from Eq. (15).

Ha	Ra_1	Ra_3	β_1	β_3	k
20	$3.985 \times 10^{+3}$	$7.186 \times 10^{+3}$	3.18	4.77	4.794
50	$1.303 \times 10^{+4}$	$3.376 \times 10^{+4}$	3.65	6.70	6.720
100	$3.445 \times 10^{+4}$	$1.198 \times 10^{+5}$	3.91	8.59	8.606
200	$9.411 \times 10^{+4}$	$4.455 \times 10^{+5}$	4.09	10.95	10.962

identical to Ra_1 since the modes on opposite walls do not interact for large l_x . Ra_3 agrees with Ra_c from Eq. (15) to the accuracy shown in the table.

APPENDIX B: ASYMPTOTIC GROWTH RATE OF Ra WITH MODE NUMBER

The derivation is based on the solution with perfectly conducting walls, i.e., Eqs. (14) on a square domain ($l = l_x = l_y$). The wave numbers must satisfy either $\alpha l/2 = (p + 1/2)\pi$ or $\alpha l/2 = p\pi$ and likewise either $\beta l/2 = (q + 1/2)\pi$ or $\beta l/2 = q\pi$. The admissible wave number pairs are therefore the intersection points of corresponding grid lines in the (α, β) -plane. With $\alpha_0 = \pi/l$ as a basic wave number, the density of intersection points ρ is therefore four per area with boundaries $p\alpha_0 \leq \alpha < (p+1)\alpha_0$ and $q\alpha_0 \leq \beta < (q+1)\alpha_0$, i.e., $\rho = 4/\alpha_0^2$. A circular shell of thickness dk and radius k in the upper right quadrant of the (α, β) -plane therefore contains

$$dn = \rho \frac{\pi k}{2} dk,$$

modes in the limit of large k . Upon integration, it follows that

$$k^2 = \frac{4\pi n}{l^2}. \quad (\text{B1})$$

Upon substitution of Eq. (B1) into Eq. (15) with $k^2 = \alpha^2 + \beta^2 = 4\pi n/l^2$, one can easily verify that Eq. (16) is the correct estimate for Ra as a function of the mode number n in the limit of large n .

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