

Fluid-inertia torques from particle-shape symmetry

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Numerical simulation of particle motion in fluids at low particle Reynolds numbers is often based on empirical force and torque models obtained by fitting force and torque from *ab initio* computations for simple particle shapes such as spheres, spheroids, or cylindrical disks and fibers. To do the same for more complex particles shapes, one needs to first know how particle shape constrains the dependence of force and torque on flow velocity, its gradient, and on particle orientation. Here we use symmetry analysis and perturbation theory to determine the form of the hydrodynamic torque on a particle settling in a quiescent fluid at low but nonzero particle Reynolds numbers for particle shapes with different point-group symmetries. The symmetry conclusions are verified by comparing with explicit calculations for nearly spherical particles.

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I. INTRODUCTION

Consider a small particle moving in a fluid. How does particle shape constrain the hydrodynamic force and torque experienced by the particle? This is an important question, because numerical studies of particle motion in complex flows tend to rely on models for force and torque, which allow for efficient computer simulation. A common way to derive such models is to use parametrizations of force and torque obtained by *ab initio* simulations [1–7], requiring an *ansatz* for how force and torque depend on the flow velocity and its gradients. Any such *ansatz*, in turn, depends on particle shape. In Stokes flow, it is known how particle-shape symmetries determine force and torque on small particles in a viscous fluid [8–15]. In many applications, however, the particles are too large or move too rapidly for the Stokes approximation to apply, because it neglects corrections to hydrodynamic force and torque that arise at nonzero particle Reynolds number Re_p . Examples include ice crystals in cold clouds [16–20], and other atmospheric particles [21–25]. Atmospheric particles are frequently modelled using highly symmetric shapes such as spheroids [24], but often atmospheric particles are asymmetric, meaning that certain rotation and reflection symmetries are broken [16,26].

At small but nonzero particle Reynolds number Re_p , hydrodynamic forces and torques on axisymmetric particles (such as straight fibers and flat disks) settling in a quiescent fluid have been calculated in perturbation theory [27–29], *ab initio* computer simulations [30], and verified in laboratory experiments [31–33]. This has led to reliable parametrizations of inertial forces and

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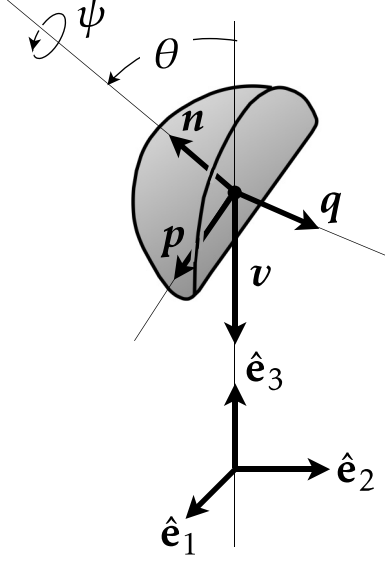


FIG. 1. Particle with C_{2v} shape symmetry moving through a quiescent fluid with velocity $\mathbf{v}$. The particle has the shape of a bent disk [14]. It has one C_2 -rotation axis $\mathbf{n}$ and two reflection symmetries $\mathbf{p} \rightarrow -\mathbf{p}$ and $\mathbf{q} \rightarrow -\mathbf{q}$. Here $\mathbf{p}, \mathbf{q}, \mathbf{n}$ are the basis vectors of a coordinate system that rotates with the particle. The basis vectors of the laboratory coordinate system are denoted by $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$, and the velocity is $\mathbf{v} = -v\hat{\mathbf{e}}_3$. Also shown are the Euler angles θ, ψ that specify particle orientation. We use the $Z_\phi Y_\theta Z_\psi$ convention, as described in the Supplemental Material of Ref. [25].

torques for such particles. Brenner and Cox [34] and Cox [35] derived general expressions for the steady force on a particle of arbitrary shape settling in a quiescent fluid, in terms of tensors and pseudotensors of rank-2 and rank-3. They also considered how shape symmetries (reflections and $\frac{\pi}{2}$ as well as π rotations) constrain certain contributions to force and torque to vanish. Here we analyze how particle-shape symmetries constrain the general form of the nonzero contributions to the hydrodynamic torque on a particle with arbitrary shape settling in a quiescent fluid and determine in particular how many parameters are needed to fully parameterize the torque to order Re_p . The general conclusion is as follows: The more shape symmetries are broken, the more parameters are needed. We verify our symmetry considerations by comparing with explicit calculations for nearly spherical particles [36,37] with different point-group symmetries and discuss the implications of our results for steady settling of small asymmetrical particles in a quiescent fluid. Last, we generalize the results of Cox [35] to unsteady motion of small asymmetrical particles at nonzero Re_p , indicating how history effects may change the transient angular dynamics for such particles.

Our calculations are motivated in part by recent experiments, numerical simulations, and calculations that show how breaking of fore-aft symmetry changes the fluid-inertia torque [38–40]. Another motivation comes from recent experimental results for the fluid-inertia torque on curved atmospheric fibers settling in a quiescent fluid [25,41] which indicate that breaking a reflection symmetry can have a significant effect on how the fluid-inertia torque depends on the orientation of an asymmetric particle (Fig. 1) that settles in a quiescent fluid at small but nonzero particle Reynolds numbers.

The remainder of this article is organized as follows. In Sec. II, we first recall results in the Stokes limit [8,10]. Then we describe the general form of fluid-inertia corrections to Stokes torque for small but nonzero values of Re_p , for particles of arbitrary shape in a uniform flow. In Sec. III we derive the form of the fluid-inertia torque for the particle shapes illustrated in Fig. 2, to order Re_p . In Sec. IV we summarize explicit results for nearly spherical yet asymmetrical particles that confirm

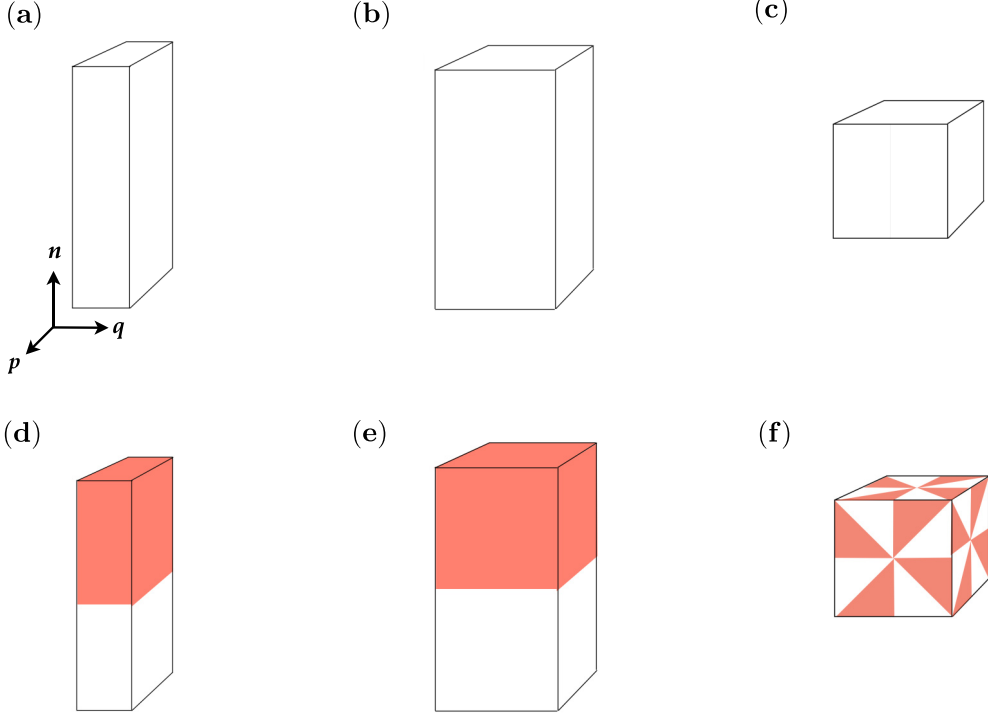


FIG. 2. Particle-shape symmetries considered in Sec. III (see that section for the names of the six point groups). In panel (a), the orientation of the body-centered coordinate system $\mathbf{p}, \mathbf{q}, \mathbf{n}$ is shown (same for other panels). Point-group symmetries represented as in Figs. 7.2 and 7.3 in Ref. [42]. (a) D_{2h} (cuboid, ellipsoid); (b) D_{nh} with $n > 2$ (prism, spheroid); (c) O_h (cube, sphere); (d) C_{2v} (particle in Fig. 1); (e) C_{nv} with $n > 2$ (axisymmetric fibre O (isotropic helicoid) w/o fore-aft symmetry); and (f) O (isotropic helicoid).

the symmetry analysis. We discuss our findings in Sec. V, and relate them to known results for nearly spherical [35], slender [28], and ellipsoidal [43] particles. Section VI contains our conclusions and a brief discussion of open questions, as well as possible applications.

II. BACKGROUND

Consider a particle moving with velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ in a quiescent fluid (Fig. 1). Hydrodynamic force and torque upon a particle in a fluid are determined by integrals of components of the stress tensor over the particle surface $\mathcal{S}$:

$$\mathbf{F} = \int_{\mathcal{S}} \boldsymbol{\sigma} \cdot d\mathbf{s} \quad \text{and} \quad \mathbf{T} = \int_{\mathcal{S}} \mathbf{r} \wedge (\boldsymbol{\sigma} \cdot d\mathbf{s}). \quad (1)$$

Here $\mathbf{r}$ is the vector from the chosen reference point to a point on the particle surface (we take the center-of-mass as the reference). The elements of the stress tensor $\boldsymbol{\sigma}$ are given by $\sigma_{mn} = -p\delta_{mn} + 2\mu S_{mn}$, where μ denotes the viscosity, and S_{mn} are the elements of the strain-rate matrix, the symmetric part of the matrix of disturbance-velocity gradients $\partial w_i / \partial x_j$. In the frame translating with the particle, the disturbance flow $\mathbf{w}$ satisfies the Navier-Stokes equations:

$$\text{Re}_p \text{Sl} \frac{\partial \mathbf{w}}{\partial t} - \text{Re}_p \mathbf{v} \cdot \nabla \mathbf{w} + \text{Re}_p \mathbf{w} \cdot \nabla \mathbf{w} = -\nabla p + \Delta \mathbf{w}, \quad \nabla \cdot \mathbf{w} = 0. \quad (2)$$

We impose no-slip boundary conditions on the particle surface, $\mathbf{w} = \mathbf{v} + \boldsymbol{\omega} \wedge \mathbf{r}$, and require that $\mathbf{w} \rightarrow \mathbf{0}$ as $|\mathbf{r}| \rightarrow \infty$. Equation (2) was nondimensionalized using an estimate v_0 of the particle velocity, the particle size a , the timescale τ_0 at which the particle boundary conditions change, the angular-velocity scale $\omega_0 = v_0/a$, and the pressure scale $\mu v_0/a$. Force and torque are nondimensionalized as

$$\mathbf{F}/(\mu a v_0) \rightarrow \mathbf{F} \quad \text{and} \quad \mathbf{T}/(\mu a^2 v_0) \rightarrow \mathbf{T}. \quad (3)$$

From here on, we quote all expressions in nondimensional variables. The nondimensional numbers in Eq.(2) are the particle Reynolds number Re_p and the Strouhal number Sl (a nondimensional measure of the timescale at which the particle boundary conditions change):

$$\text{Re}_p = a v_0 / \nu, \quad \text{Sl} = a / (v_0 \tau_0). \quad (4)$$

For $\text{Re}_p = 0$, the left-hand side of Eq. (2) vanishes, and one obtains the steady Stokes approximation to the disturbance flow. Integrating the resulting $\phi^{(0)}$ over the particle surface as specified in Eqs. (1) yields Stokes force $\mathbf{F}^{(0)}$ and torque $\mathbf{T}^{(0)}$. Both are linearly related to the particle velocity $\mathbf{v}$ and its angular velocity $\boldsymbol{\omega}$. In a quiescent fluid, this relation is written as [44]

$$\begin{bmatrix} \mathbf{F}^{(0)} \\ \mathbf{T}^{(0)} \end{bmatrix} = - \begin{bmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & \mathbb{C} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}. \quad (5)$$

Here $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are Stokes resistance tensors which depend on particle orientation, on particle shape, but not on particle size a . The superscript T denotes the transpose.

Happel and Brenner [10] analyzed how the resistance tensors in Eq. (5) depend on particle shape (for corresponding results in shear flow, see Refs. [8,11,45,46]). Using the fact that the energy dissipation is a scalar, they determined how the Stokes resistance tensors transform under orthogonal coordinate transformations, and concluded the following: if the boundary conditions imposed by the particle upon the flow are invariant under an orthogonal coordinate transformation $\mathbb{O}$, this constrains the elements of the Stokes tensors as follows [10]:

$$A_{ij} = O_{mi} A_{mn} O_{nj}, \quad (6a)$$

$$B_{ij} = \det[\mathbb{O}] O_{mi} B_{mn} O_{nj}, \quad (6b)$$

$$C_{ij} = O_{mi} C_{mn} O_{nj}. \quad (6c)$$

Alternatively, these transformation rules can be derived from the covariance of Eq. (5). To this end we transform $\mathbf{F}(\mathbf{v}, \boldsymbol{\omega})$ and $\mathbf{T}(\mathbf{v}, \boldsymbol{\omega})$ as $\mathbf{F}' = \mathbb{O} \cdot \mathbf{F}$ and $\mathbf{T}' = \det[\mathbb{O}] \mathbb{O} \cdot \mathbf{T}$, where $\mathbf{F}' = \mathbf{F}(\mathbb{O} \cdot \mathbf{v}, \det[\mathbb{O}] \mathbb{O} \cdot \boldsymbol{\omega})$ and $\mathbf{T}' = \mathbf{T}(\mathbb{O} \cdot \mathbf{v}, \det[\mathbb{O}] \mathbb{O} \cdot \boldsymbol{\omega})$, and $\mathbb{O}$ is a particle symmetry transformation [11]. Equation (6b) shows that the translation-rotation coupling tensor $\mathbb{B}$ transforms in a different way from $\mathbb{A}$ and $\mathbb{C}$. The factor $\det[\mathbb{O}]$ ensures that $\mathbb{B} = 0$ for particles with sufficient numbers of reflection symmetries (spheroid, ellipsoid).

For nonzero particle Reynolds number Re_p , the left side of Eq. (2) matters. To first order in Re_p , general expressions for the steady hydrodynamic force and torque ($\text{Sl} = 0$) were derived in Refs. [27,34,35]. Lovalenti and Brady [47] derived the first-order force corrections in Re_p for unsteady flow. Here we obtain the corresponding expressions for the unsteady torque, for particles of arbitrary shapes. In the steady limit, our results simplify to those in Refs. [27,34,35].

To compute the effects of weak fluid inertia upon hydrodynamic force and torque, one must account for the fact that the perturbation problem is singular, due to the slow spatial decay of the disturbance flow [48,49]. We use matched asymptotic expansions [48,49] to deal with this problem. In this approach, an inner solution valid near the particle is matched to an outer solution valid far from the particle. In the following, it is assumed that $1 \gg \text{Re}_p \gg \text{Re}_p \text{Sl}$. In this case, the nonlinear term $\text{Re}_p \mathbf{w} \cdot \nabla \mathbf{w}$ is of higher order in the outer problem. It can therefore be neglected (see Ref. [50] for details of the matching procedure and the associated scaling). Furthermore, far from the particle, its boundary condition is replaced by a point-source forcing. As a consequence, the outer version of

Eq. (2) becomes

$$\text{Re}_p \text{Sl} \frac{\partial \mathbf{w}_{\text{out}}}{\partial t} - \text{Re}_p \mathbf{v} \cdot \nabla \mathbf{w}_{\text{out}} = -\nabla p_{\text{out}} + \Delta \mathbf{w}_{\text{out}} + \mathbf{f}^{(0)} \delta(\mathbf{r}). \quad (7)$$

Here $\mathbf{f}^{(0)} = -\mathbf{F}^{(0)}$ denotes the force exerted by the particle on the fluid in the Stokes limit. This outer problem can be solved by Fourier transform. The outer solution in Fourier space can be expanded as a series in the small parameter $\sqrt{\text{Re}_p \text{Sl}}$, keeping in mind that this expansion must be carried out in the sense of distributions [50,51]. The two terms on the left-hand side of Eq. (7) have different prefactors. Since we choose to expand in the small parameter $\sqrt{\text{Re}_p \text{Sl}}$, we write $\text{Re}_p = \sqrt{\text{Re}_p \text{Sl}} \sqrt{\text{Re}_p / \text{Sl}}$. The factor $\sqrt{\text{Re}_p / \text{Sl}}$ quantifies the competition between convective and unsteady fluid inertia. According to the assumptions stated above, convective inertia dominates. In configuration space, the outer series expansion takes the form

$$\mathbf{w}_{\text{out}} = \frac{1}{8\pi} \left(\frac{\mathbb{1}}{r} + \frac{\mathbf{r} \otimes \mathbf{r}}{r^3} \right) \cdot \mathbf{f}^{(0)} - \sqrt{\text{Re}_p \text{Sl}} \left\{ \sqrt{\frac{\text{Re}_p}{\text{Sl}}} \mathbf{v} \cdot \nabla \left[\frac{3r}{32\pi} \left(\mathbb{1} - \frac{1}{3} \frac{\mathbf{r} \otimes \mathbf{r}}{r^2} \right) \cdot \mathbf{f}^{(0)} \right] + \mathcal{U} \right\}. \quad (8)$$

The term $\mathcal{U}$ corresponds to a uniform flow and originates from a singular contribution to the expansion of the outer solution in Fourier space, namely, a term proportional to the Dirac distribution $\delta(\mathbf{k})$. This uniform contribution takes the form

$$\mathcal{U} = \int_0^t d\tau \mathbb{K}^{(1)}(t, \tau) \cdot \frac{d\mathbf{f}^{(0)}(\tau)}{d\tau} + \sqrt{\frac{\text{Re}_p}{\text{Sl}}} \int_0^t d\tau \mathbb{K}^{(2)}(t, \tau) \cdot \mathbf{f}^{(0)}(\tau). \quad (9)$$

The kernels $\mathbb{K}^{(1)}(t, \tau)$ and $\mathbb{K}^{(2)}(t, \tau)$ are given in Appendix A. They are equivalent to the kernel involved in the Re_p correction to the force in unsteady flow, derived by Lovalenti and Brady [47]. The kernels also appear in the work of Redaelli *et al.* [50], except that there the expressions were simplified using the fact that the force $\mathbf{f}^{(0)}$ and the velocity $\mathbf{v}$ are collinear for a spherical particle.

In the inner region, disturbance velocity and pressure are expanded as

$$\mathbf{w}_{\text{in}} = \mathbf{w}_{\text{in}}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} \mathbf{w}_{\text{in}}^{(1)} + \dots, \quad p_{\text{in}} = p_{\text{in}}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} p_{\text{in}}^{(1)} + \dots. \quad (10)$$

These inner fields are then matched order by order with their counterparts in the outer solution. The leading-order term $\mathbf{w}_{\text{in}}^{(0)}$ is the solution of the steady Stokes problem. The term $\text{Re}_p \text{Sl} \partial_t \mathbf{w}_{\text{in}}^{(0)}$ is of higher order in the regular inner expansion. This leading-order (Stokes) solution matches the first term of the outer expansion (8). The next correction $\mathbf{w}_{\text{in}}^{(1)}$ satisfies the inhomogeneous Stokes problem [50],

$$\sqrt{\frac{\text{Re}_p}{\text{Sl}}} [-\mathbf{v} \cdot \nabla \mathbf{w}_{\text{in}}^{(0)} + \mathbf{w}_{\text{in}}^{(0)} \cdot \nabla \mathbf{w}_{\text{in}}^{(0)}] = -\nabla p^{(1)} + \Delta \mathbf{w}_{\text{in}}^{(1)}. \quad (11)$$

The function $\mathbf{w}_{\text{in}}^{(1)}$ satisfies no-slip condition upon the particle surface [i.e., $\mathbf{w}_{\text{in}}^{(1)} = \mathbf{0}$] and matches the last two terms of the outer expansion (8). The solution of the inhomogeneous problem Eq. (11) is sought in the form $\mathbf{w}_{\text{in}}^{(1)} = (\mathbf{w}_p + \mathcal{U}) + \mathbf{w}_h$, that is, as the sum of a particular (forced) solution $\mathbf{w}_p + \mathcal{U}$ and a homogeneous solution $\mathbf{w}_h$. Here $\mathbf{w}_p$ satisfies (11), while $\mathbf{w}_h$ satisfies the Stokes equations together with the boundary condition $\mathbf{w}_h = -(\mathbf{w}_p + \mathcal{U})$ on the particle surface, ensuring that $\mathbf{w}_{\text{in}}^{(1)}$ fulfills the no-slip condition. The particular solution $\mathbf{w}_p$ depends quadratically on $\mathbf{v}$ and $\boldsymbol{\omega}$, due to the quadratic convective term in (11). Similarly, the component of $\mathbf{w}_h$ linked to $\mathbf{w}_p$ inherits this quadratic dependence because of the boundary condition imposed on the homogeneous part. Consequently, the total hydrodynamical force and torque are

$$\mathbf{F} = \mathbf{F}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} \left(\sqrt{\frac{\text{Re}_p}{\text{Sl}}} \mathbf{F}_{\text{reg}}^{(1)} + \mathbf{F}_{\text{sing}}^{(1)} \right), \quad \mathbf{T} = \mathbf{T}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} \left(\sqrt{\frac{\text{Re}_p}{\text{Sl}}} \mathbf{T}_{\text{reg}}^{(1)} + \mathbf{T}_{\text{sing}}^{(1)} \right), \quad (12)$$

where so-called singular contributions come from the singular part of the outer expansion,

$$\mathbf{F}_{\text{sing}}^{(1)} = \mathbb{A} \cdot \mathbf{U}, \quad \mathbf{T}_{\text{sing}}^{(1)} = \mathbb{B} \cdot \mathbf{U}. \quad (13)$$

The “regular” contributions are proportional to Re_p , with

$$\mathbf{F}_{\text{reg}}^{(1)} = \mathbb{M}^{(vv)} : \mathbf{v} \otimes \mathbf{v} + \mathbb{M}^{(\omega\omega)} : \boldsymbol{\omega} \otimes \boldsymbol{\omega} + \mathbb{M}^{(v\omega)} : \mathbf{v} \otimes \boldsymbol{\omega}, \quad (14a)$$

$$\mathbf{T}_{\text{reg}}^{(1)} = \mathbb{N}^{(vv)} : \mathbf{v} \otimes \mathbf{v} + \mathbb{N}^{(\omega\omega)} : \boldsymbol{\omega} \otimes \boldsymbol{\omega} + \mathbb{N}^{(v\omega)} : \mathbf{v} \otimes \boldsymbol{\omega}. \quad (14b)$$

Here the colon denotes double contraction (the elements of $\mathbb{M} : \mathbb{S}$ with the second order tensor $\mathbb{S}$ are $\sum_{jk} M_{ijk} S_{jk}$). We remark that terms of the form $\mathbb{M}^{(\omega v)} : \boldsymbol{\omega} \otimes \mathbf{v}$ and $\mathbb{N}^{(\omega v)} : \boldsymbol{\omega} \otimes \mathbf{v}$ can be omitted since they have the same structure and transformation rules as $\mathbb{M}^{(v\omega)} : \mathbf{v} \otimes \boldsymbol{\omega}$ and $\mathbb{N}^{(v\omega)} : \mathbf{v} \otimes \boldsymbol{\omega}$. Because the contributions (14) are determined by the inner solution, they are valid instantaneously, also for nonzero angular velocities. Such quadratic terms were obtained in Refs. [27,34,35] using a reformulation of the reciprocal theorem.

Equation (9) simplifies in limiting cases. For example, in order to obtain a steady disturbance flow, $\mathbf{v}$, $\boldsymbol{\omega}$, and $\mathbf{f}^{(0)}$ must be independent of time (or at least vary very slowly). Equation (5) relates these three vectors: $\mathbf{f}^{(0)} = -\mathbf{F}^{(0)} = \mathbb{A} \cdot \mathbf{v} + \mathbb{B}^T \cdot \boldsymbol{\omega}$. Accounting for a possible time dependence of $\mathbb{A}$, one finds:

$$\frac{d\mathbf{f}^{(0)}}{dt} = \boldsymbol{\omega} \wedge \mathbf{f}^{(0)} - \mathbb{A} \cdot (\boldsymbol{\omega} \wedge \mathbf{v}) + \mathbb{A} \cdot \frac{d\mathbf{v}}{dt} + \mathbb{B}^T \cdot \frac{d\boldsymbol{\omega}}{dt}. \quad (15)$$

We conclude that it is not possible in general to simultaneously satisfy $\frac{d}{dt} \mathbf{v} = \frac{d}{dt} \boldsymbol{\omega} = \frac{d}{dt} \mathbf{f}^{(0)} = \mathbf{0}$, at least when $\mathbb{B} \neq \mathbf{0}$. An exception to this rule is the following example: An axisymmetric pear-shaped particle that rotates around its symmetry axis with constant angular velocity $\boldsymbol{\omega}$, so that $\mathbf{f}^{(0)}$, $\boldsymbol{\omega}$, and $\mathbf{v}$ are collinear. A second example is a particle moving at constant velocity with zero angular velocity. In this case, Eq. (15) allows for both $\mathbf{f}^{(0)}$ and $\mathbf{v}$ to be constant.

When $\mathbf{f}^{(0)}$ varies slowly (so that its timescale of variation is much larger than a^2/ν , or equivalently $\text{Sl} \rightarrow 0$), Eq. (9) simplifies. The first integral on the right-hand side of Eq. (9) vanishes, and $\mathbf{f}^{(0)}$ can be pulled out from the second integral. Upon performing the remaining integral, one obtains

$$\mathbf{U} = -\sqrt{\frac{\text{Re}_p}{\text{Sl}}} \frac{3v}{32\pi} \left(\mathbb{I} - \frac{1}{3} \hat{\mathbf{v}} \otimes \hat{\mathbf{v}} \right) \cdot \mathbf{f}^{(0)}, \quad (16)$$

where $\hat{\mathbf{v}}$ denotes the unit vector in the direction of $\mathbf{v}$ and $\mathbb{I}$ is the identity matrix. Since $\mathbf{U}$ corresponds to a uniform flow at infinity, the singular contributions simplify to

$$\begin{aligned} \mathbf{F}_{\text{sing}}^{(1)} &= -\sqrt{\frac{\text{Re}_p}{\text{Sl}}} \frac{3v}{32\pi} \mathbb{A} \cdot \left[\mathbb{A} \cdot \mathbf{v} - \frac{1}{3} (\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}) \cdot \mathbb{A} \cdot \mathbf{v} \right], \\ \mathbf{T}_{\text{sing}}^{(1)} &= -\sqrt{\frac{\text{Re}_p}{\text{Sl}}} \frac{3v}{32\pi} \mathbb{B} \cdot \left[\mathbb{A} \cdot \mathbf{v} - \frac{1}{3} (\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}) \cdot \mathbb{A} \cdot \mathbf{v} \right]. \end{aligned} \quad (17)$$

These contributions to the steady force and torque approximation were first obtained in Refs. [27,34,35] using the reformulation of the reciprocal theorem mentioned above. Cox [35] includes terms with $\boldsymbol{\omega} \neq \mathbf{0}$ in his Re_p contributions for force and torque. As pointed out above, this is consistent in certain special cases, but not in general.

Asymptotic matching, employed here, offers additional insights. First, it gives a clear interpretation of where the different terms to the fluid-inertia forces and torques come from. The contributions (17) originate from the far field and may contain transient, time-dependent terms, while the terms (14) are instantaneous contributions deriving from the inner solution. Second, asymptotic matching allows us to compute the transient, time-dependent terms (9), so-called history forces and torques that affect the transient dynamics of the particle. Third, the derivation of Eqs. (17) from the unsteady $\mathbf{U}$ highlights that Eqs. (17) assume that the particle velocity is steady. Upon a change of center of

reference at nonzero ω this assumption may fail. Therefore one needs to go back to Eqs. (9) and (13) in order to see that the particle dynamics does not depend on the arbitrary choice of the center of reference.

Now consider how particle-shape symmetries constrain force and torque to order Re_p . The tensors $\mathbb{A}$ and $\mathbb{B}$ in Eq. (13) are defined in the Stokes limit, and their components must obey Eqs. (6). The same is true for $\mathbf{F}_{\text{reg}}^{(1)}$ and $\mathbf{T}_{\text{reg}}^{(1)}$, which constrains the components of the tensors $\mathbb{M}$ and $\mathbb{N}$ as follows:

$$N_{ijk}^{(vv)} = \det[\mathbb{O}] O_{mi} N_{mnl}^{(vv)} O_{nj} O_{lk} \quad \text{and} \quad M_{ijk}^{(vv)} = O_{mi} M_{mnl}^{(vv)} O_{nj} O_{lk}, \quad (18a)$$

$$N_{ijk}^{(\omega\omega)} = \det[\mathbb{O}] O_{mi} N_{mnl}^{(\omega\omega)} O_{nj} O_{lk} \quad \text{and} \quad M_{ijk}^{(\omega\omega)} = O_{mi} M_{mnl}^{(\omega\omega)} O_{nj} O_{lk}, \quad (18b)$$

$$N_{ijk}^{(v\omega)} = O_{mi} N_{mnl}^{(v\omega)} O_{nj} O_{lk}, \quad \text{and} \quad M_{ijk}^{(v\omega)} = \det[\mathbb{O}] O_{mi} M_{mnl}^{(v\omega)} O_{nj} O_{lk}. \quad (18c)$$

Note that the rank-3 tensors $\mathbb{M}$ and $\mathbb{N}$ arise in the first order of a systematic expansion in Re_p . So their elements are determined by Stokes solutions, but the tensors depend on particle shape in a more complex way than solely through the rank-2 Stokes resistance tensors $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$. Equation (18) implies that $\mathbb{N}^{(vv)}$, $\mathbb{N}^{(\omega\omega)}$, and $\mathbb{M}^{(v\omega)}$ transform as the Stokes tensor $\mathbb{H}$ that couples $\mathbf{T}^{(0)}$ to the strain $\mathbb{S}$ [8,11,45,46]. Likewise, $\mathbb{M}^{(vv)}$, $\mathbb{M}^{(\omega\omega)}$, and $\mathbb{N}^{(v\omega)}$ share the same transformation properties.

Brenner and Cox [34] and Cox [35] determined under which circumstances the components of the vectors $\mathbb{M}^{(vv)} : \mathbf{v} \otimes \mathbf{v}$ and $\mathbb{N}^{(vv)} : \mathbf{v} \otimes \mathbf{v}$ (and the corresponding contributions for the other rank-3 tensors) vanish by considering $\frac{\pi}{2}$ and π rotations as well as reflections. In general, these contributions do not vanish. Here we use Eqs. (6), (14), (17), and (18) to derive constraints on the nonzero components of the fluid-inertia torque for asymmetric particles, obtained from symmetric shapes by breaking rotation and/or reflection symmetries. This allows us to understand similarities and differences in the dynamics of asymmetric particles settling in a quiescent fluid [14,25,38,39,52].

III. SYMMETRY CONSTRAINTS ON TORQUE

Figure 2 illustrates particles with different point-group symmetries. The first row shows particles with reflection and discrete rotation symmetries, such as cuboid, prism, and cube. In the limit where the rotation symmetry becomes continuous, these generalize to ellipsoid, spheroid, and sphere. The second row shows shapes with broken reflection symmetries, namely broken fore-aft symmetry (point-group symmetry C_{2v}), and the isotropic helicoid [53] (point-group symmetry O).

A. Prismatic symmetry D_{2h}

The cuboid shown in Fig. 2(a) has three reflection symmetries in the planes orthogonal to the three principal axes (point group symmetry D_{2h}). The three orthogonal reflections constrain the three Stokes resistance tensors to be $\mathbb{A} = \text{diag}[A_{pp}, A_{qq}, A_{nn}]$, $\mathbb{C} = \text{diag}[C_{pp}, C_{qq}, C_{nn}]$, and $\mathbb{B} = 0$. Further, the reflections constrain the inertial rank-3 tensors to $\mathbb{N}^{(v\omega)} = 0$ and

$$\mathbb{N}_{p::}^{(vv)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & N_{pqn}^{(vv)} \\ 0 & N_{pnq}^{(vv)} & 0 \end{bmatrix}, \quad \mathbb{N}_{q::}^{(vv)} = \begin{bmatrix} 0 & 0 & N_{qpn}^{(vv)} \\ 0 & 0 & 0 \\ N_{qnp}^{(vv)} & 0 & 0 \end{bmatrix}, \quad \mathbb{N}_{n::}^{(vv)} = \begin{bmatrix} 0 & N_{npq}^{(vv)} & 0 \\ N_{nqp}^{(vv)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (19)$$

It follows from the discussion in Sec. II that the tensors $\mathbb{N}^{(\omega\omega)}$ and $\mathbb{M}^{(v\omega)}$ have the same form as $\mathbb{N}^{(vv)}$ but with independent parameters. The tensors $\mathbb{M}^{(vv)}$ and $\mathbb{M}^{(\omega\omega)}$ vanish. The above constraints yield the following expressions for the Re_p corrections to the

torque:

$$T_p^{(1)} = 2N_{p(nq)}^{(vv)}v_nv_q + 2N_{p(nq)}^{(\omega\omega)}\omega_n\omega_q, \quad (20a)$$

$$T_q^{(1)} = 2N_{q(pn)}^{(vv)}v_pv_n + 2N_{q(pn)}^{(\omega\omega)}\omega_p\omega_n, \quad (20b)$$

$$T_n^{(1)} = 2N_{n(pq)}^{(vv)}v_pv_q + 2N_{n(pq)}^{(\omega\omega)}\omega_p\omega_q, \quad (20c)$$

in the basis $\mathbf{p}, \mathbf{q}, \mathbf{n}$ (Fig. 2). Here $N_{p(nq)}^{(vv)} = \frac{1}{2}(N_{pnq}^{(vv)} + N_{qp n}^{(vv)})$, $\dots$, are symmetrized coupling coefficients. Since $\mathbb{B} = 0$, the singular contributions to the inertial torque, Eq. (13), vanish. Thus, only the instantaneous terms from the inner solution contribute, meaning that Eqs. (20) are valid in the unsteady case, in particular for nonzero values of ω .

B. Prismatic symmetry (D_{nh} for $n > 2$)

Straight columns with a regular n -gon cross sections with $n > 2$ (*prismatic symmetry* D_{nh} with $n > 2$) possess discrete n -fold rotational symmetry (C_n) about the $\mathbf{n}$ axis. The case $n = 4$ is illustrated in Fig. 2(b). Another example is cloud-ice crystals, often possessing hexagonal ($n = 6$) cross sections [54,55]. Imposing C_n -rotation symmetry with $n > 2$ around $\mathbf{n}$ imposes additional constraints on the nonzero coefficients in Eq. (20) compared with $n = 2$:

$$A_{pp} = A_{qq} = A_{\perp}, \quad A_{nn} = A_{\parallel}, \quad C_{pp} = C_{qq} = C_{\perp}, \quad C_{nn} = C_{\parallel}, \quad (21a)$$

$$N_{pqn}^{(vv)} = -N_{qpn}^{(vv)} = N_1^{(vv)}, \quad N_{pnq}^{(vv)} = -N_{qnp}^{(vv)} = N_2^{(vv)}, \quad N_{npq}^{(vv)} = -N_{nqp}^{(vv)} = N_3^{(vv)}. \quad (21b)$$

The elements of $\mathbb{N}^{(\omega\omega)}$ are constrained in the same way. This leaves six undetermined parameters in the $\mathbb{N}$ -tensors. The Re_p contributions to the torque simplify to

$$T_p^{(1)} = (N_1^{(vv)} + N_2^{(vv)})v_qv_n + (N_1^{(\omega\omega)} + N_2^{(\omega\omega)})\omega_q\omega_n, \quad (22a)$$

$$T_q^{(1)} = -(N_1^{(vv)} + N_2^{(vv)})v_pv_n - (N_1^{(\omega\omega)} + N_2^{(\omega\omega)})\omega_p\omega_n, \quad (22b)$$

$$T_n^{(1)} = 0, \quad (22c)$$

in the basis $\mathbf{p}, \mathbf{q}, \mathbf{n}$ (Fig. 2). Similarly to Eqs. (20), these expressions are valid in the unsteady case. The rotational symmetry around $\mathbf{n}$ causes the torque T_n to vanish. This follows because $\mathbb{N}_{n::}^{(vv)}$ and $\mathbb{N}_{n::}^{(\omega\omega)}$ are antisymmetric and contracted with the symmetric $\mathbf{v} \otimes \mathbf{v}$ and $\boldsymbol{\omega} \otimes \boldsymbol{\omega}$, respectively. If $\boldsymbol{\omega} = 0$, then the torque is orthogonal to $\mathbf{n}$ and proportional to the velocity component v_n : $\mathbf{T} = (N_1^{(vv)} + N_2^{(vv)})(\mathbf{v} \cdot \mathbf{n})(\mathbf{v} \wedge \mathbf{n})$, consistent with Refs. [28–30].

C. Full octahedral symmetry (O_h)

Now consider particles with *full octahedral symmetry* O_h [Fig. 2(c)]. Examples are the cube and also the sphere with continuous rotation symmetry. For both examples, rotation and reflection symmetries further constrain Eqs. (21). The Stokes tensors $\mathbb{A}$ and $\mathbb{C}$ become proportional to the identity matrix, and the tensors $\mathbb{N}^{(vv)}$ and $\mathbb{N}^{(\omega\omega)}$ become proportional to the antisymmetric Levi-Civita tensor $\boldsymbol{\epsilon}$: $A_{\perp} = A_{\parallel} = A$, $C_{\perp} = C_{\parallel} = C$, $\mathbb{N}^{(vv)} = N^{(vv)}\boldsymbol{\epsilon}$, and $\mathbb{N}^{(\omega\omega)} = N^{(\omega\omega)}\boldsymbol{\epsilon}$. The latter conditions cause the torque (22) to vanish.

As discussed in Sec. II, $\mathbb{M}^{(v\omega)}$ is constrained in the same way as $\mathbb{N}^{(vv)}$, so $\mathbb{M}^{(v\omega)} = M^{(v\omega)}\boldsymbol{\epsilon}$, yielding a contribution proportional to $\mathbf{v} \wedge \boldsymbol{\omega}$, the Rubinow-Keller lift force for a spherical particle (Eq. (71) in Ref. [56]), and its generalization to particles with O_h symmetry. See Appendix B for details.

Using that $\mathbb{A} = A\mathbb{I}$ and that $\hat{\mathbf{v}}$ is a unit vector, the singular contribution to the force (17) is isotropic, proportional to $\frac{2}{3}A^2v^2\hat{\mathbf{v}}$ [34,35]. The theory to linear order in Re_p is isotropic but does not rule out the existence of anisotropic terms at higher orders in the Reynolds number. It is instructive to compare this conclusion with a more general symmetry argument that assumes only that force and torque to order Re_p are third-order polynomials in the components of $\hat{\mathbf{v}}$ with coefficients that are constrained by symmetry. Requiring invariance of force and torque under particle-shape symmetry

transformations yields results consistent with the above conclusions, but they constrain force and torque less because the relation between force and torque encoded in Eq. (13) is not accounted for.

D. Orthorhombic pyramidal symmetry (C_{2v})

Figure 2(d) shows a particle that has the same symmetry as D_{2h} but with broken reflection symmetry in the $\mathbf{n}$ -direction (orthorhombic pyramidal symmetry C_{2v}). Examples of such particles are curved planar fibers with constant radius of curvature [25], and symmetrically bent disks [14] (Fig. 1). The tensors $\mathbb{A}$, $\mathbb{C}$, $\mathbb{N}^{(vv)}$, and $\mathbb{N}^{(v\omega)}$ of particles with C_{2v} symmetry have the same form as for particles with D_{2h} symmetry (discussed above). But the broken reflection symmetry results in a nonzero Stokes translation-rotation coupling:

$$\mathbb{B} = \begin{bmatrix} 0 & B_{pq} & 0 \\ B_{qp} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (23)$$

as well as a nonzero $\mathbb{N}^{(v\omega)}$:

$$\begin{aligned} \mathbb{N}_{p::}^{(v\omega)} &= \begin{bmatrix} 0 & 0 & N_{ppn}^{(v\omega)} \\ 0 & 0 & 0 \\ N_{pnp}^{(v\omega)} & 0 & 0 \end{bmatrix}, & \mathbb{N}_{q::}^{(v\omega)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & N_{qqn}^{(v\omega)} \\ 0 & N_{qnq}^{(v\omega)} & 0 \end{bmatrix}, \\ \mathbb{N}_{n::}^{(v\omega)} &= \begin{bmatrix} N_{npp}^{(v\omega)} & 0 & 0 \\ 0 & N_{nqq}^{(v\omega)} & 0 \\ 0 & 0 & N_{nnn}^{(v\omega)} \end{bmatrix}. \end{aligned} \quad (24)$$

Now consider the corresponding expression for the fluid-inertia correction to the hydrodynamic torque. Since the general symmetry analysis of the unsteady contribution $\mathcal{U}$ is complicated, we only consider the steady limit, which requires $\omega = 0$, as discussed in the previous section. For the steady Re_p contributions to the torque we find:

$$T_p^{(1)} = -\frac{1}{32\pi} v B_{pq} [3A_{qq} - (A_{pp}\hat{v}_p^2 + A_{qq}\hat{v}_q^2 + A_{nn}\hat{v}_n^2)] v_q + 2N_{p(nq)}^{(vv)} v_n v_q, \quad (25a)$$

$$T_q^{(1)} = -\frac{1}{32\pi} v B_{qp} [3A_{pp} - (A_{pp}\hat{v}_p^2 + A_{qq}\hat{v}_q^2 + A_{nn}\hat{v}_n^2)] v_p + 2N_{q(np)}^{(vv)} v_n v_p, \quad (25b)$$

$$T_n^{(1)} = 2N_{n(qp)}^{(vv)} v_q v_p, \quad (25c)$$

in the basis $\mathbf{p}, \mathbf{q}, \mathbf{n}$ (Fig. 2). Comparing with Eq. (20) for the ellipsoid, we see that the form of the quadratic terms is unaffected by the reflection symmetry for $\omega = 0$. Broken reflection symmetry still changes the fluid-inertia torque, due to the nonzero translation-rotation coupling $\mathbb{B}$ in the Stokes limit, yielding a nonzero singular contribution from Eq. (17).

E. Pyramidal symmetry (C_{nv} with $n > 2$)

Particles with n -fold axisymmetry without fore-aft symmetry have C_{nv} symmetry, $n > 2$ [Fig. 2(e)]. Other examples invariant under C_{nv} are dumbbells with spheres of different radii [38,57] and straight fibers with continuous rotation symmetry but varying cross section [39]. For particles invariant under this symmetry, we find the following nonzero coefficients

$$A_{pp} = A_{qq} = A_{\perp}, \quad A_{nn} = A_{\parallel}, \quad C_{pp} = C_{qq} = C_{\perp}, \quad C_{nn} = C_{\parallel}, \quad B_{pq} = -B_{qp} = B, \quad (26a)$$

as well as

$$N_{ppn}^{(v\omega)} = N_{qpn}^{(v\omega)} = N_1^{(v\omega)}, \quad N_{pnp}^{(v\omega)} = N_{qnq}^{(v\omega)} = N_2^{(v\omega)}, \quad N_{npp}^{(v\omega)} = N_{nqq}^{(v\omega)} = N_3^{(v\omega)}, \quad N_{nnn}^{(v\omega)} = N_4^{(v\omega)}, \quad (26b)$$

$$N_{pqn}^{(vv)} = -N_{qp n}^{(vv)} = N_1^{(vv)}, \quad N_{pnq}^{(vv)} = -N_{npq}^{(vv)} = N_2^{(vv)}, \quad N_{npq}^{(vv)} = -N_{nqp}^{(vv)} = N_3^{(vv)}. \quad (26c)$$

The tensor $\mathbb{N}^{(\omega\omega)}$ has the same form as $\mathbb{N}^{(vv)}$. The steady Re_p contributions to the torque components are

$$T_p^{(1)} = \frac{-1}{32\pi} vB [A_\perp (3 - (\hat{v}_p^2 + \hat{v}_q^2)) - A_\parallel \hat{v}_n^2] v_q + (N_1^{(vv)} + N_2^{(vv)}) v_q v_n, \quad (27a)$$

$$T_q^{(1)} = \frac{1}{32\pi} vB [A_\perp (3 - (\hat{v}_p^2 + \hat{v}_q^2)) - A_\parallel \hat{v}_n^2] v_p - (N_1^{(vv)} + N_2^{(vv)}) v_p v_n, \quad (27b)$$

$$T_n^{(1)} = 0. \quad (27c)$$

Comparing with Eq. (22), we see that breaking of fore-aft symmetry introduces additional terms in the steady Re_p corrections to the torque, depending on the translation-rotation coupling B in the Stokes limit. The form of the quadratic terms in the steady torque does not change, as was mentioned for the fibers. The steady fluid-inertia torque (27) can be written as [30]

$$\mathbf{T}^{(1)} = v^2 g(\hat{v}_n) (\hat{v} \wedge \hat{n}), \quad (28)$$

with $g(\hat{v}_n) = -B[2A_\perp + (A_\perp - A_\parallel)\hat{v}_n^2]/(32\pi) + (N_1^{(vv)} + N_2^{(vv)})\hat{v}_n$. The general form of Eq. (28) is consistent with Eq. (2.11) in Roy *et al.* [39], who considered particles without fore-aft symmetry made out of nonslender rods with changing cross section or inhomogeneous mass distribution. However, Eq. (2.11) in Ref. [39] implies that g is proportional to $\hat{v}_n$. This is explained by the fact that Eq. (2.11) was derived in the slender-body approximation including terms up to $1/(\log \kappa)^2$, whereas the first two terms in the above expression for g are at least of order $1/(\log \kappa)^3$ (see Sec. V). Candelier and Mehlig [38] computed the torque on an asymmetric dumbbell settling in a quiescent fluid. In this case, g is a linear function of $\hat{v}_n$. We do not know why the coefficient of the quadratic term vanishes. It may be a consequence of the fact that the authors considered the special case of a slender and weakly asymmetric dumbbell, constraining the coefficients in Eq. (27) further.

F. Chiral octahedral symmetry $\mathcal{O}$

Kelvin's isotropic helicoid [53] is a particle which has *chiral octahedral symmetry* $\mathcal{O}$. That is, it has the same rotational symmetries as $\mathcal{O}_h$ but no reflection symmetries [Fig. 2(f)]. The rotation symmetries constrains $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ to be isotropic: $\mathbb{A} = A\mathbb{I}$, $\mathbb{B} = B\mathbb{I}$, $\mathbb{C} = C\mathbb{I}$. It thus has the same structure as a cube or sphere but with an isotropic translation-rotation coupling [13,53]. Here we conclude in addition that all $\mathbb{N}$ -tensors are proportional to ϵ . For the sphere we found $\mathbb{N}^{(v\omega)} = 0$, but here $\mathbb{N}^{(v\omega)} \propto \epsilon$ because there are no reflection symmetries, so $\det[\mathbb{O}] \equiv 1$. As a consequence, the steady translation-rotation coupling remains isotropic to order Re_p ,

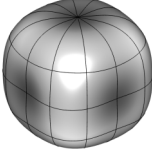
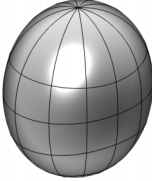
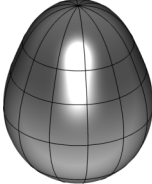
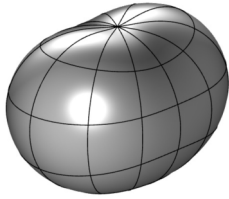
$$\mathbf{T}^{(1)} = -\frac{3v}{32\pi} B \left(\mathbb{I} - \frac{1}{3} \hat{v} \otimes \hat{v} \right) \cdot (A\mathbf{v} + B\boldsymbol{\omega}) + N^{(v\omega)} \mathbf{v} \wedge \boldsymbol{\omega}. \quad (29)$$

Note that $f^{(0)}$ is constant for an isotropic helicoid that translates with constant velocity $\mathbf{v}$ and rotates with constant angular velocity $\boldsymbol{\omega}$, regardless of how this vectors align. Therefore we quoted the steady result for nonzero frequency above. The last term on the right-hand side is the analog of the Rubinow-Keller lift force on a translating and rotating sphere.

IV. NEARLY SPHERICAL PARTICLES

Table I summarizes the structure of the coupling tensors derived in the previous section for the symmetries $\mathcal{O}_h$, D_{2h} , $C_{\infty v}$, and C_{2v} . We tested these results for nearly spherical particles with

TABLE I. Summary of geometric constraints and contributions to the force and torque for different particle symmetries. The form of the tensor $\mathbb{C}$ is the same as $\mathbb{A}$ (with independent elements). The forms of $\mathbb{M}^{(vv)}$ and $\mathbb{M}^{(\omega\omega)}$ are the same as the form of $\mathbb{N}^{(v\omega)}$, and the forms of $\mathbb{M}^{(v\omega)}$ and $\mathbb{N}^{(\omega\omega)}$ are the same as that of $\mathbb{N}^{(vv)}$.

Symmetry	Constraints on $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{N}$	Examples (nearly spherical particles)
Full octahedral (O_h)	$\mathbb{A} = A\mathbb{I}$, $\mathbb{B} = \mathbb{0}$, $\mathbb{N}^{(v\omega)} = \mathbb{0}$, $\mathbb{N}^{(vv)} = N^{(vv)}\epsilon$ ($\epsilon \equiv$ Levi-Civita tensor).	 $r = 1 - \frac{1}{2}\varepsilon(\sin^4\vartheta\cos^4\varphi + \sin^4\vartheta\sin^4\varphi + \cos^4\vartheta)$ $(\varepsilon = 0.5)$
Prismatic (D_{2h})	$\mathbb{A} = \text{diag}[A_{pp}, A_{qq}, A_{nn}]$, $\mathbb{B} = \mathbb{0}$, $\mathbb{N}^{(v\omega)} = \mathbb{0}$, $\mathbb{N}^{(vv)} = N_{pqn}^{(vv)}\mathbf{p} \otimes \mathbf{q} \otimes \mathbf{n} + N_{pnq}^{(vv)}\mathbf{p} \otimes \mathbf{n} \otimes \mathbf{q}$ $+ N_{qpn}^{(vv)}\mathbf{q} \otimes \mathbf{p} \otimes \mathbf{n} + N_{qnp}^{(vv)}\mathbf{q} \otimes \mathbf{n} \otimes \mathbf{p}$ $+ N_{npq}^{(vv)}\mathbf{n} \otimes \mathbf{p} \otimes \mathbf{q} + N_{nqp}^{(vv)}\mathbf{n} \otimes \mathbf{q} \otimes \mathbf{p}$.	 $r = 1 - \varepsilon \sin^2\vartheta(\alpha - \alpha \sin^2\varphi + \sin^2\varphi)$ $(\varepsilon = 0.3, \alpha = 0.5)$
Pyramidal ($C_{\infty v}$)	$\mathbb{A} = \text{diag}[A_{\perp}, A_{\perp}, A_{\parallel}]$, $\mathbb{B} = B(\mathbf{p} \otimes \mathbf{q} - \mathbf{q} \otimes \mathbf{p})$, $\mathbb{N}^{(v\omega)} = N_1^{(v\omega)}(\mathbf{p} \otimes \mathbf{p} \otimes \mathbf{n} + \mathbf{q} \otimes \mathbf{q} \otimes \mathbf{n})$, $+ N_2^{(v\omega)}(\mathbf{p} \otimes \mathbf{n} \otimes \mathbf{p} + \mathbf{q} \otimes \mathbf{n} \otimes \mathbf{q})$, $+ N_3^{(v\omega)}(\mathbf{n} \otimes \mathbf{p} \otimes \mathbf{p} + \mathbf{n} \otimes \mathbf{q} \otimes \mathbf{q})$, $+ N_4^{(v\omega)}(\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n})$, $\mathbb{N}^{(vv)} = N_1^{(vv)}(\mathbf{p} \otimes \mathbf{q} \otimes \mathbf{n} - \mathbf{q} \otimes \mathbf{p} \otimes \mathbf{n})$, $+ N_2^{(vv)}(\mathbf{p} \otimes \mathbf{n} \otimes \mathbf{q} - \mathbf{q} \otimes \mathbf{n} \otimes \mathbf{p})$, $+ N_3^{(vv)}(\mathbf{n} \otimes \mathbf{p} \otimes \mathbf{q} - \mathbf{n} \otimes \mathbf{q} \otimes \mathbf{p})$.	 $r = 1 - \varepsilon(\sin^2\vartheta + \frac{\gamma}{2}\cos\vartheta)$ $(\varepsilon = 0.3, \gamma = 1.5)$
Orthorhombic pyramidal (C_{2v})	$\mathbb{A} = \text{diag}[A_{pp}, A_{qq}, A_{nn}]$, $\mathbb{B} = B_{pq}\mathbf{p} \otimes \mathbf{q} + B_{qp}\mathbf{q} \otimes \mathbf{p}$, $\mathbb{N}^{(v\omega)} = N_{ppn}^{(v\omega)}\mathbf{p} \otimes \mathbf{p} \otimes \mathbf{n} + N_{pnq}^{(v\omega)}\mathbf{p} \otimes \mathbf{n} \otimes \mathbf{p}$ $+ N_{qqn}^{(v\omega)}\mathbf{q} \otimes \mathbf{q} \otimes \mathbf{n} + N_{qnq}^{(v\omega)}\mathbf{q} \otimes \mathbf{n} \otimes \mathbf{q}$ $+ N_{npp}^{(v\omega)}\mathbf{n} \otimes \mathbf{p} \otimes \mathbf{p} + N_{npq}^{(v\omega)}\mathbf{n} \otimes \mathbf{q} \otimes \mathbf{p}$, $+ N_{nnn}^{(v\omega)}\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}$, $\mathbb{N}^{(vv)} = N_{pqn}^{(vv)}\mathbf{p} \otimes \mathbf{q} \otimes \mathbf{n} + N_{pnq}^{(vv)}\mathbf{p} \otimes \mathbf{n} \otimes \mathbf{q}$ $+ N_{qpn}^{(vv)}\mathbf{q} \otimes \mathbf{p} \otimes \mathbf{n} + N_{qnp}^{(vv)}\mathbf{q} \otimes \mathbf{n} \otimes \mathbf{p}$ $+ N_{npq}^{(vv)}\mathbf{n} \otimes \mathbf{p} \otimes \mathbf{q} + N_{nqp}^{(vv)}\mathbf{n} \otimes \mathbf{q} \otimes \mathbf{p}$.	 $r = 1 - \varepsilon(1 + \frac{\beta}{2}\cos\vartheta - \cos^2\varphi\sin^2\vartheta)$ $(\varepsilon = 0.3, \beta = 1.5)$

the shape symmetries listed in Fig. 2. To this end, we parameterized the particle boundary by the radius $r(\varphi, \vartheta)$. Here φ and ϑ are azimuthal and polar angles, that is, $x = r(\varphi, \vartheta) \sin \vartheta \cos \varphi$, $y = r(\varphi, \vartheta) \sin \vartheta \sin \varphi$, and $z = r(\varphi, \vartheta) \cos \vartheta$. Table I lists parameterizations for four particle shapes, namely O_h , D_{2h} , $C_{\infty v}$, and C_{2v} . Using the theory described in Sec. II, we computed hydrodynamic force and torque as a joint perturbation expansion in Re_p and ε . The method is described in Refs. [36,37]. The former is a singular expansion, the latter a regular one. This allows us to extract the elements of the tensors discussed in the previous section or at least of certain sums of their elements. Our results for the particles in Table I confirm the symmetry analysis in Sec. III

(Appendix B). We remark, however, that some coefficients vanish although they are allowed to be nonzero by particle-shape symmetries (Sec. V).

For example, for the special case $\alpha = 1$ in the second row of Table I, one obtains a parametrization of the hydrodynamic torque on a nearly spherical spheroid. In this case, the torque component T_n vanishes, confirming the results of the symmetry analysis, Eq. (21a). The case of a nearly spherical spheroid was also considered by Cox [35]. He parameterized the major axis as $1 + 2\epsilon$ and the minor axis as $1 - \epsilon$, in terms of a small parameter ϵ . Comparing with our parametrization ($c = 1$ and $a = b = 1 - \epsilon$), we see that $\epsilon = 3\epsilon$. Accounting for this factor of three, as well as a factor of 6π in the nondimensionalization of the torque in Ref. [35], one concludes that our results agree with those of Cox [35].

V. DISCUSSION

The results summarized in Sec. III show how particle-shape symmetries constrain the hydrodynamic torque, including fluid-inertia contributions to order Re_p , extending earlier work by Brenner and Cox on corresponding constraints on the hydrodynamic force [34]. They determined under which circumstances the contributions from the $\mathbb{M}$ tensor to the hydrodynamic force vanish by analyzing how reflections as well as $\frac{\pi}{2}$ and π rotations constrain $\mathbb{M}^{(vv)} : \mathbf{v} \otimes \mathbf{v}$. In a similar fashion, Cox [35] considered under which conditions the components of the vectors $\mathbb{N}^{(vv)} : \mathbf{v} \otimes \mathbf{v}$, $\mathbb{N}^{(v\omega)} : \mathbf{v} \otimes \boldsymbol{\omega}$ and $\mathbb{N}^{(\omega\omega)} : \boldsymbol{\omega} \otimes \boldsymbol{\omega}$ vanish.

For general shapes, these contributions do not vanish. In this case, the question is How many parameters are needed to parameterize the dependence of the torque (and the force) upon particle velocity and angular velocity? Here we used symmetry analysis to answer this question. We determined how particle-shape symmetry constrains the nonzero elements of the coupling tensors. This determines the form of the torque and in particular how many parameters are needed in parametrizations. The general structure of the torque becomes more complex (involves more parameters to be determined) as more shape symmetries are broken. In particular, any given shape symmetry implies that the resistance tensors $\mathbb{A}$ and $\mathbb{C}$ commute with the generator of the corresponding symmetry transformation. Full octahedral symmetry O_h requires that $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ commute with all rotational generators. This implies that these tensors are proportional to the unit matrix (isotropy). Now consider how rotations constrain the rank-3 tensors: The symmetries in the point-group O_h require that all rank-3 tensors are proportional to the Levi-Civita tensor ϵ , the only antisymmetric rank-3 tensor. Reflection symmetries constrain the rank-3 tensors further, namely $\mathbb{N}^{(v\omega)}$ as well as $\mathbb{M}^{(vv)}$ and $\mathbb{M}^{(\omega\omega)}$ must vanish. Reflection symmetries also cause $\mathbb{B}$ to vanish. The last two statements apply to all particles in the top row of Fig. 2.

For nearly spherical particles with different shape symmetries (Sec. IV), we found some cases where the symmetry analysis from Sec. III allows a nonzero coupling, but the coupling nevertheless vanishes to a certain order in perturbation theory. For $C_{\infty v}$ symmetry for example, the translation-rotation coupling $\mathbb{B}$ vanishes to order ϵ^2 , although allowed by symmetry. In general, one expects that all coupling allowed by symmetry are nonzero unless there are additional symmetries not considered. The elements of $\mathbb{B}$ vanish to order ϵ^2 because the parametrization $r = 1 - \epsilon(\sin^2 \vartheta + \frac{\gamma}{2} \cos \vartheta)$ corresponds to a shifted spheroid. Expressing the torque with respect to the center of mass cancels any potential contributions to the torque up to order ϵ^2 . This happens also for C_{2v} symmetry. In the latter case, we also find that $N_{n(pq)}^{(vv)}$ and $N_{n(pq)}^{(\omega\omega)}$ evaluate to zero to order ϵ . We expect that higher orders in ϵ are nonzero.

Candelier *et al.* [25] used the slender-body approximation to calculate the steady hydrodynamic force and torque on curved atmospheric fibers with C_{2v} symmetry, settling in quiescent air. In slender-body theory [28], force and torque are expressed as

$$\mathbf{F} = \int d\mathbf{s} \mathbf{f}(s), \quad \mathbf{T} = \int d\mathbf{s} \mathbf{r}(s) \wedge \mathbf{f}(s), \quad (30)$$

replacing Eqs. (1). Here $\mathbf{f}(s)$ is the force on a small segment of a slender body. It is obtained by asymptotic matching, as an expansion in $1/\log \kappa$, where κ is the (large) aspect ratio of the fiber. The results of Khayat and Cox [28] are obtained including the order $1/(\log \kappa)^2$ and assuming that the particle Reynolds number with respect to the (small) radius a of the fiber is small (the particle Reynolds number with respect to the fiber length can be of order unity). It has been shown that the predictions of Khayat and Cox [28] are accurate for aspect ratios greater than 30 and for small Reynolds numbers based on the fiber radius, i.e., $v_0 a/\nu \leq 0.1$. To order $\text{Re}_p = v_0 a/\nu$, the resulting hydrodynamic force and torque contain at most third powers of $\hat{\mathbf{v}}$, just as Eqs. (13) and (14). Let us compare the slender-body results for the torque on curved fibers with C_{2v} symmetry (Eq. (3) in Candelier *et al.* [25]) with the present symmetry analysis, Eq. (25). The two expressions are consistent. The differences arise from a difference in the orientation of the basis. Here the reflection symmetry $\mathbf{n} \rightarrow -\mathbf{n}$ is broken, while in Ref. [25], the symmetry $\mathbf{p} \rightarrow -\mathbf{p}$ is broken. However, the slender-body approximation has fewer terms than allowed by the symmetry analysis, similarly to the nearly spherical particles discussed above, although different terms are zero in that case. The allowed terms that vanish in the slender-body approximation are proportional to $B_{qn}A_{nn}$, while allowed terms proportional to $B_{nq}A_{qq}$ appear. This can be understood as follows. Computing the Stokes resistance tensors for planar fibers with constant radius of curvature using $\mathbf{f}(s)$ from Ref. [28], one concludes that A_{qq} and A_{nn} are of order $1/\log \kappa$ and so is B_{nq} [58]. The coefficient B_{qn} , by contrast, is proportional to $1/(\log \kappa)^2$. This means that the corresponding contribution to the torque is of order $1/(\log \kappa)^3$, and thus not contained in the slender-body theory of Khayat and Cox [28], to the order quoted there. The above dependencies on κ are confirmed by numerical results for B_{nq}/B_{qn} for a bead model obtained by modeling the fibers as chains of hydrodynamically interacting, identical beads (Table S1 in the Supplemental Material of Ref. [25]): The numerical results are consistent with $B_{nq}/B_{qn} \sim \log \kappa$. The accuracy is approximately 2% when comparing $\kappa = 20$ with $\kappa = 40$.

This shortcoming of slender-body theory stems from approximations for the inner solution, namely that hydrodynamic interactions between different segments of the fiber are neglected (one has to go to higher orders in $1/(\log \kappa)$ to account for these interactions [59]). It is instructive to compare these conclusions with results for the bead model [25]. In that case, $\mathbb{B} = 0$ without hydrodynamic interactions in the center of mass of the chain. This is no longer true if the beads have different sizes or in the slender-body approximation where the fiber segments have different orientations.

In slender-body theory, higher orders in $\text{Re}_p = av_0/\nu$ result in higher powers of $\hat{\mathbf{v}}$ in the expressions for hydrodynamic force and torque. To obtain reasonable agreement between theory and experiment, Candelier *et al.* [25] found it necessary to include higher orders in Re_p , not contained in Eq. (25) in Sec. III (although for spheroids, Bhowmick *et al.* [24] found that the first-order approximation in Re_p works well if the prefactor is adjusted by an Re_p dependence). It is likely that higher-order singular contributions to force and torque are entirely determined by the Stokes tensors $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$. If this is the case, then any other symmetry-breaking effects at larger particle Reynolds numbers must come from the inner solution.

At small but finite Re_p , fluid inertia torques tend to align highly symmetric particles with the slip velocity. This is important in the atmospheric sciences, where the alignment of symmetric ice crystals (D_{nh} symmetry with $n > 2$) settling in the quiescent or weakly turbulent atmosphere [16–20,24]. Calculations of torque and force for nearly spherical spheroids and slender fibers with circular cross sections [28] show that such particles—with D_{nh} symmetry with $n > 2$ —align so that they settle in quiescent air with their broad sides down. The fixed points of angular dynamics are dictated by rotational and fore-aft symmetry. Their stability is determined by the sign of the coefficient $N_1^{(vv)} + N_2^{(vv)}$ in Eq. (22) which depends on the aspect ratio of the spheroid. The coefficient changes sign at aspect ratio $\lambda = 1$ (sphere). Dabade *et al.* [29] calculated how $N_1^{(vv)} + N_2^{(vv)}$ depends on the particle aspect ratio, although their result differs from that of Cox [35] in the nearly spherical limit.

Andersen *et al.* [60] used an empirical torque model to describe fluttering and tumbling of platelets settling in a quiescent fluid at Re_p of order 10^3 . For planar motion, they parameterize the out-of-plane component of the torque as $T_3 = c_1\omega + c_2\omega^2$ [their Eq. (3.9)]. The first term is consistent with Stokes torque (5). However, our symmetry analysis (Sec. III) does not produce the suggested ω^2 correction. Instead, we find $(N_1^{(\omega\omega)} + N_2^{(\omega\omega)})\omega^2\sin 2\theta$ for a spheroid to order Re_p . In the spherical limit, the quadratic correction to the torque vanishes to this order in Re_p , as observed by Rubinow and Keller [56]. Analysis of *ab initio* data for force and torque on a thin disk (aspect ratio $\lambda = 0.2$) in unsteady motion in a quiescent fluid at $\text{Re}_p \sim 2$ shows that there is a small ω^2 contribution to the torque consistent with $(N_1^{(\omega\omega)} + N_2^{(\omega\omega)})\omega^2\sin 2\theta$. Any possible contribution of the form $c_2\omega^2$ is small. This is consistent with the direct numerical simulations of Pierson *et al.* [61] who considered the torque on a cylinder rotating perpendicular to its symmetry axis (with $\theta = 0$). They observe an ω^2 contribution, but only at large Reynolds numbers, not attainable from a first-order expansion in Re_p .

For particles with prismatic symmetry [Fig. 2(a)], the symmetry analysis predicts that all three torque components are nonzero in general. For specific orientations the torque may vanish. For example, if $\boldsymbol{\omega} = 0$, and $\hat{\mathbf{v}} = -\hat{\mathbf{e}}_3$, then we have $\hat{\mathbf{v}}_p = \sin \theta \cos \psi$, $\hat{\mathbf{v}}_q = -\sin \theta \sin \psi$, and $\hat{\mathbf{v}}_n = -\cos \theta$. We see that there are six orientations where the torque vanishes, each corresponding to one side of the particle in Fig. 2(a) pointing down. These are potential steady states of the angular dynamics at small Re_p .

The laboratory experiments of Bhowmick *et al.* [43] show that ellipsoidal particles settling in quiescent air approach a steady orientation with their broadest side down. Equation (20) explain this observation for nearly spherical ellipsoids, as the linear-stability analysis described in Appendix C shows. More generally, the symmetry analysis of Sec. III explains that particles with broken reflection symmetries ($C_{\infty v}$ and C_{2v} , for example) align differently compared to more symmetric particles discussed in the previous paragraph, such as spheroids and ellipsoids. The reason is that both Stokes and fluid-inertia torque are nonzero and can balance for certain orientations, provided that Re_p is not too large [38–40,57]. Reference [25] illustrates how this gives rise to a bifurcation that changes the stability of existing fixed points of the angular dynamics and introduces new fixed points corresponding to inclined settling (with nonzero tilt angle). When particles align at a nonzero tilt angle as they settle in quiescent air, they move horizontally. This can affect the transport of such particles in the atmosphere, but standard models for the transport of atmospheric particles assume symmetrical shapes (spheroids) [41].

The previous paragraph discussed the steady-state problem, the orientation of a steadily settling particle. Now consider how the particle orientation changes as the steady state is approached. As explained in Sec. II, the transient is affected by history forces and torques arising from inertial fluid acceleration (the term $\partial_t \mathbf{w}$ in the unsteady Stokes equation). In this limit, the history force is well understood for a spherical particle settling in a quiescent fluid. The history force arises from the outer solution and decays slowly, as $t^{-1/2}$, after a sudden change of the particle dynamics [62]. Fluid-inertia corrections at small Re_p modify the power-law decay in a nonuniversal fashion [47], but the decay remains slow at small Reynolds numbers. The history torque, by contrast, decays rapidly for a sphere. Its kernel is essentially a Dirac delta function [63]. Since the torque is determined by the inner solution, it is expected that small- Re_p corrections do not change this fact. Equations (9) and (13) show that the situation is quite different for particles with nonzero translation-rotation coupling $\mathbb{B}$, because $\mathbb{B}$ results in a slow decay of the history torque as well. This history torque is expected to affect the transient dynamics of asymmetric particles settling at nonzero Reynolds numbers [25]. For spheroids ($\mathbb{B} = 0$) the relaxation of the orientational dynamics is well described [24], but as yet there is no reliable theory for the relaxation of particles with nonzero translation-rotation coupling outside the Stokes limit [25]. The main complication are the above-mentioned slowly decaying history torques.

Note that we computed the torque with respect to the center of mass of the particle and assumed a homogeneous mass-density distribution. An inhomogeneous distribution of the particle-mass

density can shift the center of mass, breaking further shape symmetries. Adjusting its center of mass allows gliders settling in a quiescent fluid to navigate to land on a certain spot [64].

Our symmetry analysis is based on the theory for hydrodynamic forces and torques described in Sec. II, which is valid at nonzero but small Re_p . How small the Reynolds number must be for the theory to work qualitatively or quantitatively depends on particle shape. For a spherical particle, the left of Eqs. (17) simplifies to the Oseen force (Appendix B), and it is well known that this works up to $\text{Re}_p \sim 0.5$ [65]. The Rubinow-Keller lift force for a spherical particle [Eq. (B2)] is in good qualitative agreement with experiments up to $\text{Re}_p \sim 5$ [66]. For fibers, Fintzi *et al.* [67] demonstrated that it is necessary to include the full functional form of the singular contribution at moderate Reynolds numbers ($\text{Re}_p \geq 0.5$), especially for large aspect ratios, even though this range of Reynolds numbers probably lies beyond the validity of the present asymptotic analysis.

Bhowmick *et al.* [24] used empirical parametrizations of the fluid-inertia torque for spheroidal particles based on Eq. (22), with Re_p -dependent coefficients. This works well up to $\text{Re}_p \sim 20$, although the functional dependence of the coefficients has to be determined by comparison with particle-resolving simulations. In a similar way, the simulation results of Jiang *et al.* [30] and the experiments of Cabrera *et al.* [33] indicate that the functional form (22) is correct for fibers and spheroids; see also Refs. [3,67,68]. As mentioned above, slender-body theory [28] allows us to include higher-order corrections in Re_p , albeit only to order $1/(\log \kappa)^2$. The *ab initio* simulations of Fintzi *et al.* [67] (see also Ref. Kharrouba *et al.* [69]) indicate that these refined predictions fail quantitatively for $\kappa = 30$ and $\text{Re}_p \geq 0.5$ (with Reynolds number based on the radius of the fiber), because the large-aspect-ratio limit required by the theory of Khayat and Cox [28] was most likely not attained in the simulations. Candelier *et al.* [25], on the other hand, found qualitative agreement between the full slender-body theory for the torque on curved atmospheric fibers and experiments. Roy *et al.* [39] compared theory and experiment for the settling of straight slender fibers with broken fore-aft symmetry. They found good agreement between measurements and slender-body theory (including high-order Re_p corrections). These results indicate that the results presented here help to constrain fluid-inertia torques on particles of general shape. In the past, some authors have attempted to constrain the functional form of the fluid-inertia force by comparing with the potential-flow limit. The results of Candelier *et al.* [63] indicate, however, that this approach may be fallacious, because the expressions for the hydrodynamic force on a sphere in a linear flow are qualitatively different in the two limits. Whether it may help to constrain fluid-inertia torques remains an open question. In summary, only direct numerical simulations and dedicated experiments can serve as the final benchmark to test the range of validity of the theory present here.

We determined how shape symmetry constrains the hydrodynamical torque on an isotropic helicoid, an isotropic particle with nonzero translation-rotation coupling [53]. Our analysis shows that the coupling remains isotropic to order Re_p , just as in the potential-flow limit analyzed by Kelvin. We also saw that symmetry allows for a nonzero contribution $\mathbb{N}^{(v\omega)} : \mathbf{v} \otimes \boldsymbol{\omega} \propto \mathbf{v} \wedge \boldsymbol{\omega}$ to the fluid-inertia torque, analogously to the Rubinow-Keller lift force on a rotating sphere translating in a fluid at rest. In the Stokes limit, helical ribbons exhibit an infinite sequence of bifurcations as the translation-rotation coupling tensor passes through axisymmetry [15]. The theory described here can be used to determine fluid-inertia torques (and forces) on helical ribbons. This allows us to determine how fluid-inertia corrections change the phase portraits of the angular dynamics.

The results described up to this point ignore the unsteady term of the disturbance equation in the inner solution, as one must in the stated limit $1 \gg \text{Re}_p \gg \text{Re}_p \text{Sl}$. To conclude the discussion, we briefly consider the opposite limit $1 \gg \text{Re}_p \text{Sl} \gg \text{Re}_p$ (details in Appendix D). In this limit, there are additional contributions to the regular parts of hydrodynamic force and torque,

$$\mathbf{F}_u^{(2)} = \text{Re}_p \text{Sl} (\mathbb{P} \cdot \dot{\mathbf{v}} + \mathbb{Q} \cdot \dot{\boldsymbol{\omega}}), \quad (31a)$$

$$\mathbf{T}_u^{(2)} = \text{Re}_p \text{Sl} (\mathbb{Q}^T \cdot \dot{\mathbf{v}} + \mathbb{R} \cdot \dot{\boldsymbol{\omega}}). \quad (31b)$$

Here dots denote time derivatives, and $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{R}$ are second-rank tensors that transform as Eqs. (6). Therefore they obey the same constraints as $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$. Equations (31) were first derived by Gavze [70] who obtained a general solution for the unsteady Stokes equation for a particle of arbitrary shape. We note that the unsteady solution also contains history terms. In Appendix D we show that these history terms agree with those derived in Sec. II to order $\sqrt{\text{Re}_p \text{Sl}}$. Collis *et al.* [37] solved the unsteady Stokes equation for nearly spherical particles with several different shapes in the frequency domain. One must obtain the kernels derived in Appendix D by integration over frequency.

VI. CONCLUSIONS

In this paper we used symmetry analysis to constrain the hydrodynamic torque on particles moving in a quiescent fluid at small but nonzero particle Reynolds number. For highly symmetrical particles, such as slender fibers with circular cross section and spheroids, fluid inertia torques were well understood [27–29] and known to describe how highly symmetric particles align as they settle in quiescent or weakly turbulent fluids [24].

The present study was motivated by recent interest in the angular dynamics of asymmetrical atmospheric particles [25, 38–40, 57], such as curved fibers, asymmetric ice crystals, and ash particles. Here we show how breaking of shape symmetry allows new terms to appear in the hydrodynamic torque (and force) that can significantly change how particles align as they settle. The analysis shows that the bent disks of Miara *et al.* [14], and planar curved atmospheric fibers with constant radius of curvature [25], obey the same equations of motion, albeit with different parameters. The form of the equation of motion determines the possible dynamical behaviors by constraining the bifurcations that may occur as the parameters are varied. For example, in the Stokes limit the transition from alignment at small Re_p (curved fiber) to oscillations (bent disk) occurs when the product $B_{pq}B_{qp}$ of the nonzero translation-rotation coupling coefficients in Eq. (23) changes sign [25]. The results summarized here allow to determine how the bifurcations change for larger ribbons as fluid-inertia corrections become important.

Regarding the hydrodynamic torque, our results generalize the theory of Ref. [35] to include unsteady effects, leading to slowly decaying history torques for asymmetrical particles with translation-rotation coupling (for symmetrical particles, history torques decay rapidly). Such unsteady effects influence the transient angular dynamics. In order to understand the transient angular dynamics for asymmetrical, curved atmospheric fibers [24], we expect history torques to play a role. Similar conclusions are expected for bent disks [14], which have the same point-group shape symmetry as planar fibers with constant radius of curvature.

Even in the absence of history torques, one expects complex settling trajectories even for symmetrical particles settling in quiescent air. For example, our theory shows that the transient dynamics of ellipsoids is governed by three couplings [the six nonzero elements of the rank-3 tensor $\mathbb{N}^{(vv)}$, which take different values depending on the aspect ratios of the ellipsoidal particle]. This—together with the fact that all three components of the hydrodynamical torque are nonzero—explains the rich variety of settling behaviors observed in the experiment [43], but a detailed computation of the transient dynamics remains a question for the future. Regarding the steady limit, our calculations show that nearly spherical ellipsoids settle with their broadest side down, as observed in the experiment [43]. To confirm that this conclusion is valid for ellipsoidal particles with arbitrary aspect ratios, one needs to compute the three symmetrized coefficients in Eq. (20) for the torque by direct numerical simulation.

The results discussed here were obtained in the limit of small Reynolds numbers, $\text{Re}_p \ll 1$. An important question is how the particle dynamics changes as the Reynolds number increases. For spheroids in shear flow, bifurcations that occur at larger shear Reynolds numbers Re_s have been mapped out, and it turns out that the small-Re theory remains qualitatively valid for Re_s up to order unity. For spheroids settling in a quiescent fluid, one can obtain quantitative agreement with experiments using empirical torque models similar to those discussed here [24]. For particles with broken shape symmetry [25], the expressions derived here yield qualitative but not quantitative

agreement with the experiments. How the accuracy of small-Re approximations breaks down for highly asymmetric particles as the Reynolds number increases remains an open question. It may be instructive to compare the results of the present analysis with the potential-flow limit, the limit considered by Kelvin in his analysis of the isotropic helicoid [53]. Corresponding expressions for hydrodynamic force and torque can be obtained for particles of arbitrary shape using either the Lagrangian formalism [71] or by direct integration of the pressure on the body [72]. However, the results of Candelier *et al.* [63] indicate that even for a spherical particle, hydrodynamic force and torque can be quite different in the two limits. As a consequence it may be challenging to find convincing interpolation formulas between the two limits.

Last, symmetry analysis as described here can be used to constrain hydrodynamical forces and torques on motile micro-organisms. The standard model assumes that the organisms are spherical, but this is an idealization. In the future we will analyze how the interaction of microswimmers with the surrounding flow changes when shape symmetries are broken. To this end, one needs to generalize the present analysis to account for strain and vorticity of the undisturbed flow. For special cases this is known [8,11,45,46] but not in general.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: FLUID-INERTIA KERNELS

We use asymptotic matching [48,49] to find a solution to the disturbance equation (2) assuming that $1 \gg \text{Re}_p \gg \text{Re}_p \text{Sl}$. The calculation is very similar to the one in Ref. [50] except for two differences. First, here we consider a passive particle which simplifies the expression for $\mathbf{f}_0$. Second, here we consider nonspherical particles and allow force and particle velocity to point in different directions. The outer solution in Ref. [50] was obtained for a spherical particle, where $\mathbf{v}$ and $\mathbf{f}_0$ are collinear. This allows certain simplifications of the outer solution (and the corresponding kernels) that are not admissible in the present case. The general result for the kernels in Eq. (9) is as follows:

$$[\mathbb{K}^{(1)}(t, \tau)]_{ij} = -\frac{1}{8\pi} \left(\frac{\exp(-A(t, \tau)^2)}{\sqrt{\pi}A(t, \tau)^2} + \frac{\text{erf}(A(t, \tau))(A(t, \tau)^2 - \frac{1}{2})}{A(t, \tau)^3} \right) \frac{\delta_{ij}}{\sqrt{t - \tau}} \\ + \frac{1}{8\pi} \left(\frac{3 \exp(-A(t, \tau)^2)}{2\sqrt{\pi}A(t, \tau)^2} + \frac{\text{erf}(A(t, \tau))(A(t, \tau)^2 - \frac{3}{2})}{2A(t, \tau)^3} \right) \frac{\delta_{ij} - \hat{a}_i(t, \tau)\hat{a}_j(t, \tau)}{\sqrt{t - \tau}}, \quad (\text{A1})$$

and

$$[\mathbb{K}^{(2)}(t, \tau)]_{ij} = \frac{1}{8\pi} \frac{1}{4A(t, \tau)} \left(\frac{3 \exp(-A(t, \tau)^2)}{\sqrt{\pi}A(t, \tau)^2} + \frac{\text{erf}(A(t, \tau))(A(t, \tau)^2 - \frac{3}{2})}{A(t, \tau)^3} \right) \\ \times \frac{v_i(\tau)\hat{a}_j(t, \tau) + \hat{a}_i(t, \tau)v_j(\tau) - 4[\mathbf{v}(\tau) \cdot \hat{\mathbf{a}}(t, \tau)]\hat{a}_i(t, \tau)\hat{a}_j(t, \tau)}{t - \tau}$$

$$\begin{aligned}
 & -\frac{1}{8\pi} \frac{1}{4A(t, \tau)} \left(\frac{(-4A(t, \tau)^2 - 3) \exp(-A(t, \tau)^2)}{\sqrt{\pi} A(t, \tau)^2} + \frac{\operatorname{erf}(A(t, \tau))(A(t, \tau)^2 + \frac{3}{2})}{A(t, \tau)^3} \right) \\
 & \times [\mathbf{v}(\tau) \cdot \hat{\mathbf{a}}(t, \tau)] \frac{\delta_{ij} - \hat{a}_i(t, \tau) \hat{a}_j(t, \tau)}{t - \tau}.
 \end{aligned} \tag{A2}$$

In these expressions, $\mathbf{f}^{(0)}$ is the force exerted by the particle upon the fluid in the Stokes limit, $A(t, \tau)$ is the norm of the pseudodisplacement vector $\mathbf{a}(t, \tau)$ initially introduced by Lovalenti and Brady [47]

$$\mathbf{a}(t, \tau) = \frac{1}{2} \sqrt{\frac{\operatorname{Re}_p}{\operatorname{Sl}}} \frac{1}{\sqrt{t - \tau}} \int_{\tau}^t dt' \mathbf{v}(t'), \tag{A3}$$

and $\hat{\mathbf{a}} = \mathbf{a}/A(t, \tau)$. In the steady limit, Eq. (A3) implies $\mathbf{a} = \frac{1}{2} \sqrt{\frac{\operatorname{Re}_p}{\operatorname{Sl}}} \sqrt{t - \tau} \mathbf{v}$ and therefore $\hat{\mathbf{a}} = \hat{\mathbf{v}}$. Using these results, we see that Eq. (9) simplifies to Eq. (16) in the steady limit. These kernels are equivalent to those obtained by Lovalenti and Brady [47] [but the details differ because the decomposition into two integrals, one involving $\mathbf{f}^{(0)}$ and the other involving $d\mathbf{f}^{(0)}/d\tau$ is not unique]. This follows because in the outer solution, particle shape enters only as the amplitude of a source term in the disturbance equation. Therefore, the outer solution for a particle of general shape must be a convolution of this shaped-dependent amplitude with the universal kernel of Lovalenti and Brady [47].

APPENDIX B: COUPLING TENSORS FOR NEARLY SPHERICAL PARTICLES

In this Appendix, we quote results for the elements of the coupling tensors obtained for the nearly spherical particles listed in Table I. The results were obtained as follows. To each order in the asymptotic-matching calculation one needs to solve a Stokes problem, as described in Ref. [50]. These Stokes problems are solved for nearly spherical particles by expanding in the small nonsphericity parameter ε as described in the Appendix of Ref. [36].

1. O_h symmetry

The results for O_h symmetry are well known [10]. The tensors $\mathbb{A}$ and $\mathbb{C}$ are isotropic:

$$A_{pp} = A_{qq} = A_{nn} = 6\pi - \frac{9}{5}\pi\varepsilon + \frac{116}{525}\pi\varepsilon^2, \quad \text{and} \quad C_{pp} = C_{qq} = C_{nn} = 8\pi - \frac{36}{5}\pi\varepsilon + \frac{1678}{525}\pi\varepsilon^2. \tag{B1a}$$

The tensor $\mathbb{B}$ vanishes. For $\varepsilon = 0$, Eq. (17) yields the Oseen drag on a spherical particle,

$$\mathbf{F} = 6\pi \left(1 - \frac{3}{8}\operatorname{Re}_p\right) \mathbf{v}. \tag{B1b}$$

The following rank-3 tensors are nonzero: $\mathbb{N}^{(v\omega)} = M^{(v\omega)}\boldsymbol{\epsilon}$, $\mathbb{N}^{(vv)} = N^{(vv)}\boldsymbol{\epsilon}$, and $\mathbb{N}^{(\omega\omega)} = N^{(\omega\omega)}\boldsymbol{\epsilon}$. The tensors $\mathbb{N}^{(vv)}$ and $\mathbb{N}^{(\omega\omega)}$ do not contribute because they are doubly contracted with a symmetrical tensor. For $M^{(v\omega)}$ we find to order ε ,

$$M^{(v\omega)} = -\left(\pi - \frac{9}{10}\pi\varepsilon\right). \tag{B1c}$$

Equation (B1c) yields the Keller-Rubinow lift force for a nearly spherical particle with O_h symmetry. In dimensional form, we obtain

$$\mathbf{F} = -\varrho_f \nu a v_0 \operatorname{Re}_p \pi \left(1 - \frac{9}{10}\varepsilon\right) \mathbf{v} \wedge \boldsymbol{\omega} / (v_0 \omega_0) = \varrho_f a^3 \pi \left(1 - \frac{9}{10}\varepsilon\right) \boldsymbol{\omega} \wedge \mathbf{v}. \tag{B2}$$

For $\varepsilon = 0$, this is Eq. (71) in Ref. [56].

2. D_{2h} symmetry

For D_{2h} symmetry, we find the following nonzero coefficients:

$$\begin{aligned}
 A_{pp} &= 6\pi + \left(-\frac{12}{5}\pi - \frac{6}{5}\alpha\pi\right)\varepsilon, \\
 A_{qq} &= 6\pi + \left(-\frac{6}{5}\pi - \frac{12}{5}\alpha\pi\right)\varepsilon, \\
 A_{nn} &= 6\pi + \left(-\frac{12}{5}\alpha\pi - \frac{12}{5}\pi\right)\varepsilon, \\
 C_{pp} &= 8\pi + \left(-\frac{48}{5}\pi - \frac{24}{5}\alpha\pi\right)\varepsilon, \\
 C_{qq} &= 8\pi + \left(-\frac{24}{5}\pi - \frac{48}{5}\alpha\pi\right)\varepsilon, \\
 C_{nn} &= 8\pi + \left(-\frac{48}{5}\pi - \frac{48}{5}\alpha\pi\right)\varepsilon, \\
 2N_{p(nq)}^{(\omega\omega)} &= \frac{1}{15}\pi\varepsilon, \quad 2N_{q(np)}^{(\omega\omega)} = -\frac{1}{15}\pi\alpha\varepsilon, \quad 2N_{n(pq)}^{(\omega\omega)} = \frac{1}{15}\pi(\alpha - 1)\varepsilon, \\
 2N_{p(nq)}^{(vv)} &= \frac{29}{20}\pi\varepsilon, \quad 2N_{q(np)}^{(vv)} = -\frac{29}{20}\pi\alpha\varepsilon, \quad 2N_{n(pq)}^{(vv)} = \frac{29}{20}\pi(\alpha - 1)\varepsilon, \\
 M_{pqn}^{(v\omega)} &= -\pi + \left(\frac{2}{5}\alpha\pi + \frac{37}{20}\pi\right)\varepsilon, \quad M_{pnq}^{(v\omega)} = \pi - \left(\frac{2}{5}\alpha\pi + \frac{3}{4}\pi\right)\varepsilon, \\
 M_{qnp}^{(v\omega)} &= \pi - \left(\frac{2}{5}\alpha\pi + \frac{3}{4}\pi\right)\varepsilon, \quad M_{qpn}^{(v\omega)} = \pi - \left(\frac{2}{5}\pi + \frac{37}{20}\alpha\pi\right)\varepsilon, \\
 M_{nqp}^{(v\omega)} &= \pi - \left(\frac{3}{4}\alpha\pi + \frac{37}{20}\pi\right)\varepsilon, \quad M_{npq}^{(v\omega)} = -\pi + \left(\frac{3}{4}\pi + \frac{37}{20}\alpha\pi\right)\varepsilon,
 \end{aligned} \tag{B3}$$

confirming the symmetry analysis in Sec. III [Eq. (19)]. For $\alpha = 1$ the particle shape is spheroidal ($C_{\infty h}$). As explained in the main text, this constrains more coefficients to vanish [Eq. (22)].

3. $C_{\infty v}$ symmetry

For a particle with broken fore-aft symmetry (third row in Table I), we find the following nonzero coupling coefficients:

$$\begin{aligned}
 A_{\perp} &= 6\pi - \frac{18}{5}\pi\varepsilon + \left(\frac{198}{175}\pi + \frac{9}{20}\gamma^2\pi\right)\varepsilon^2, \\
 A_{\parallel} &= 6\pi - \frac{24}{5}\pi\varepsilon + \left(\frac{192}{175}\pi + \frac{3}{5}\gamma^2\pi\right)\varepsilon^2, \\
 C_{\perp} &= 8\pi - \frac{72}{5}\pi\varepsilon + \left(\frac{2736}{175}\pi + \frac{9}{5}\gamma^2\pi\right)\varepsilon^2, \\
 C_{\parallel} &= 8\pi - \frac{96}{5}\pi\varepsilon + \left(\frac{3264}{175}\pi + \frac{12}{5}\gamma^2\pi\right)\varepsilon^2, \\
 N_1^{(vv)} + N_2^{(vv)} &= \frac{29}{20}\pi\varepsilon \quad \text{and} \quad N_1^{(\omega\omega)} + N_2^{(\omega\omega)} = \frac{\pi}{15}, \\
 N_1^{(v\omega)} &= \pi\gamma\varepsilon, \quad \text{and} \quad N_3^{(v\omega)} = -\pi\gamma\varepsilon, \\
 M_1^{(\omega\omega)} + M_2^{(\omega\omega)} &= -\frac{1}{6}\pi\gamma\varepsilon, \quad \text{and} \quad M_3^{(\omega\omega)} = \frac{1}{6}\pi\gamma\varepsilon, \\
 M_1^{(v\omega)} &= \pi - \frac{23}{20}\pi\varepsilon, \quad M_2^{(v\omega)} = -\pi + \frac{9}{4}\pi\varepsilon, \quad \text{and} \quad M_3^{(v\omega)} = -\pi + \frac{13}{5}\pi\varepsilon.
 \end{aligned} \tag{B4}$$

The tensors $\mathbb{B}$, $\mathbb{N}^{(vv)}$, and $\mathbb{N}^{(\omega\omega)}$ were computed with respect to the center of mass $\mathbf{x}_{c.m.} = -\frac{\gamma}{2}\varepsilon\hat{\mathbf{e}}_3$. The coupling coefficients $M_4^{(vv)}$ and $M_4^{(\omega\omega)}$ vanish to order ε although they are allowed to be nonzero by symmetry. The tensor $\mathbb{B}$ vanishes to order ε^2 , although allowed to be nonzero by the symmetry analysis. We quote the elements of $\mathbb{A}$ and $\mathbb{C}$ to second order in ε to show their γ dependence which occurs only at this order.

4. C_{2v} symmetry

For a nearly spherical particle with C_{2v} symmetry (fourth row in Table I), we find the following nonzero coupling coefficients:

$$\begin{aligned}
 A_{pp} &= 6\pi - \frac{24\pi}{5}\varepsilon + \left(\frac{192\pi}{175} + \frac{9\pi}{20}\beta^2\right)\varepsilon^2, \\
 A_{qq} &= 6\pi - \frac{18\pi}{5}\varepsilon + \left(\frac{198\pi}{175} + \frac{9\pi}{20}\beta^2\right)\varepsilon^2, \\
 A_{nn} &= 6\pi - \frac{18\pi}{5}\varepsilon + \left(\frac{198\pi}{175} + \frac{3\pi}{5}\beta^2\right)\varepsilon^2, \\
 C_{pp} &= 8\pi - \frac{96}{5}\pi\varepsilon + \left(\frac{3264\pi}{175} + \frac{9\pi}{5}\beta^2\right)\varepsilon^2, \\
 C_{qq} &= 8\pi - \frac{72\pi}{5}\varepsilon + \left(\frac{2736\pi}{175} + \frac{9\pi}{5}\beta^2\right)\varepsilon^2, \\
 C_{nn} &= 8\pi - \frac{72}{5}\pi\varepsilon + \left(\frac{2736\pi}{175} + \frac{12\pi}{5}\beta^2\right)\varepsilon^2, \\
 2N_{q(pn)}^{(vv)} &= \frac{29}{20}\pi\varepsilon, \quad 2N_{n(pq)}^{(vv)} = -\frac{29}{20}\pi\varepsilon, \\
 2N_{q(pn)}^{(\omega\omega)} &= \frac{1}{15}\pi\varepsilon, \quad 2N_{n(pq)}^{(\omega\omega)} = -\frac{1}{15}\pi\varepsilon, \\
 N_{pn}^{(v\omega)} &= N_{qnq}^{(v\omega)} = \beta\pi\varepsilon, \quad N_{np}^{v\omega} = N_{nqq}^{v\omega} = -\beta\pi\varepsilon, \\
 2M_{p(pn)}^{(\omega\omega)} &= 2M_{q(np)}^{(\omega\omega)} = -\frac{1}{6}\pi\beta\varepsilon, \quad 2M_{n(pp)}^{(\omega\omega)} = 2M_{n(qq)}^{(\omega\omega)} = \frac{1}{6}\pi\beta\varepsilon, \\
 M_{pqn}^{(v\omega)} &= -M_{pnq}^{(v\omega)} = -\pi + \frac{13}{5}\pi\varepsilon, \\
 M_{qpn}^{(v\omega)} &= -M_{npq}^{(v\omega)} = \pi - \frac{23}{20}\pi\varepsilon, \quad M_{qnp}^{(v\omega)} = -M_{nqp}^{(v\omega)} = -\pi + \frac{9}{4}\pi\varepsilon.
 \end{aligned} \tag{B5}$$

The elements of $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{N}$ were computed with respect to the center of mass $\mathbf{x}_{c.m.} = -\frac{\beta}{2}\varepsilon\hat{\mathbf{e}}_1$. The tensor $\mathbb{B}$ vanishes to order ε^2 , and the combinations $N_{n(qp)}^{(vv)}$ and $N_{n(qp)}^{(\omega\omega)}$ vanish to order ε , although it is allowed to be nonzero by the symmetry analysis.

APPENDIX C: STABILITY OF A SETTLING ELLIPSOID

We consider the sedimentation of an ellipsoidal particle in a quiescent fluid. Such particles have D_{2h} symmetry. The force and torque are given by

$$\mathbf{F} = -\mathbb{A} \cdot \mathbf{v} + \text{Re}_p \mathbf{F}^{(1)} - G\hat{\mathbf{e}}_3, \tag{C1a}$$

$$\mathbf{T} = -\mathbb{C} \cdot \boldsymbol{\omega} + \text{Re}_p \mathbf{T}^{(1)}. \tag{C1b}$$

Here $\mathbf{F}^{(1)} = \mathbf{F}_{\text{reg}}^{(1)} + \mathbf{F}_{\text{sing}}^{(1)}$, where $\mathbf{F}_{\text{reg}}^{(1)}$ takes the form in Eq. (14) with $M^{(vv)} = M^{(\omega\omega)} = 0$ and $M^{(v\omega)}$ being of the same form as $N^{(vv)}$ due to the symmetry constraints posed by Eqs. (18), and $\mathbf{F}_{\text{sing}}^{(1)}$ is given in Eq. (17). Moreover, $\mathbf{T}^{(1)} = \mathbf{T}_{\text{reg}}^{(1)}$ is given in Eq. (20) [$\mathbf{T}_{\text{sing}}^{(1)} = 0$ since $\mathbb{B} = 0$ for D_{2h} symmetry]. Finally, $-G\hat{\mathbf{e}}_3$ is the buoyancy force, with $G = (\rho_p - \rho_f)V_p g / (\mu v_0 a)$, where V_p is the particle volume and ρ_p and ρ_f are the densities of the particle and fluid, respectively. We express

the dynamics in the coordinate system $\mathbf{p}, \mathbf{q}, \mathbf{n}$ that rotates with the particle. In this system, $\mathbb{A}$ and $\mathbb{C}$ are diagonal for D_{2h} . We obtain

$$\dot{v}_p = -A_{pp}v_p + F_p^{(1)} + G \cos \psi \sin \theta + \omega_n v_q - \omega_q v_n, \quad (\text{C2a})$$

$$\dot{v}_q = -A_{qq}v_q + F_q^{(1)} - G \sin \psi \sin \theta + \omega_p v_n - \omega_n v_p, \quad (\text{C2b})$$

$$\dot{v}_n = -A_{nn}v_n + F_n^{(1)} - G \cos \theta + \omega_q v_p - \omega_p v_q, \quad (\text{C2c})$$

$$\dot{\theta} = \omega_q \cos \psi + \omega_p \sin \psi, \quad \dot{\psi} = \omega_n + \cot \theta [\omega_p \cos \psi - \omega_q \sin \psi], \quad (\text{C2d})$$

$$\dot{\omega}_p = \frac{1}{I_{pp}} [-C_{pp}\omega_p + \text{Re}_p T_p^{(1)} + \omega_n \omega_q (I_{qq} - I_{nn})], \quad (\text{C2e})$$

$$\dot{\omega}_q = \frac{1}{I_{qq}} [-C_{qq}\omega_q + \text{Re}_p T_q^{(1)} + \omega_p \omega_n (I_{nn} - I_{pp})], \quad (\text{C2f})$$

$$\dot{\omega}_n = \frac{1}{I_{nn}} [-C_{nn}\omega_n + \text{Re}_p T_n^{(1)} + \omega_q \omega_p (I_{pp} - I_{qq})]. \quad (\text{C2g})$$

Here $I_{pp}, \dots$ are the elements of the inertia tensor of the ellipsoidal particle. The $\omega_i v_j$ and $\omega_i \omega_j$ terms result from the time derivative of the particle orientation. We consider the dynamics in the subsystem $\mathbf{V} = [v_p, v_q, v_n, \theta, \psi, \omega_p, \omega_q, \omega_n]^T$, not including $\mathbf{x}$ and ϕ , since these coordinates do not affect $\mathbf{V}$.

We search for a rotation-free steady state, $\boldsymbol{\omega} = 0$. By solving $\dot{\mathbf{V}} = 0$ under this condition, the system (C2) has six fixed points corresponding to the particle settling with either $\mathbf{p}, \mathbf{q}, \mathbf{n}$ aligned with or against gravity. The settling speed is obtained as a solution to a second-order polynomial. For small Reynolds numbers, it takes the form $v_g^* = G/A^{(g)}$, where $A^{(g)}$ is the component of the resistance tensor in the direction that aligns or antialigns with gravity.

The 8×8 stability matrix $\mathbb{J}$ has elements $J_{ij} = \partial_{V_j} \dot{V}_i$. Evaluating $\mathbb{J}$ at either of the fixed points and suitably permuting the rows and columns, we find that it is block-diagonal with two 1-blocks and two 3-blocks. The eigenvalues of the 1-blocks are the negative resistance elements $\lambda_1 = -A^{(g)}$ and $\lambda_2 = -C^{(g)}/I^{(g)}$. The eigenvalues of the 3-blocks each satisfy a third-order polynomial being equal to zero, giving the stability exponents λ_{3-5} and λ_{6-8} , respectively. For particles settling with $\mathbf{n}$ pointing downwards, these become (neglecting contributions of order Re_p^2),

$$\begin{aligned} 0 &= \lambda^3 + P_2 \lambda^2 + P_1 \lambda + P_0 \quad \text{and} \quad 0 = \lambda^3 + Q_2 \lambda^2 + Q_1 \lambda + Q_0 \quad \text{with} \\ P_0 &= -\frac{2N_{p(qn)}^{(vv)}}{I_{pp}} \frac{G^2 \text{Re}_p}{A_{nn}}, \quad Q_0 = \frac{2N_{q(np)}^{(vv)}}{I_{qq}} \frac{G^2 \text{Re}_p}{A_{nn}}, \\ P_1 &= \frac{A_{qq} C_{pp}}{I_{pp}} \left(1 - \frac{G \text{Re}_p}{32\pi} \frac{A_{nn} - 3A_{qq}}{A_{nn}} \right) - 2N_{p(qn)}^{(vv)} \frac{G^2 \text{Re}_p}{A_{nn}^2 I_{pp}}, \\ Q_1 &= \frac{A_{pp} C_{qq}}{I_{qq}} \left(1 - \frac{G \text{Re}_p}{32\pi} \frac{A_{nn} - 3A_{pp}}{A_{nn}} \right) + 2N_{q(np)}^{(vv)} \frac{G^2 \text{Re}_p}{A_{nn}^2 I_{qq}}, \\ P_2 &= A_{qq} + \frac{C_{pp}}{I_{pp}} - \frac{G \text{Re}_p}{32\pi} \frac{A_{qq}}{A_{nn}} (A_{nn} - 3A_{qq}), \quad Q_2 = A_{pp} + \frac{C_{qq}}{I_{qq}} - \frac{G \text{Re}_p}{32\pi} \frac{A_{pp}}{A_{nn}} (A_{nn} - 3A_{pp}). \end{aligned} \quad (\text{C3})$$

A necessary condition for these polynomials to only have negative solutions is that all polynomial coefficients are positive, $P_i > 0$ and $Q_i > 0$, which requires

$$N_{p(qn)}^{(vv)} < 0 \quad \text{and} \quad N_{q(np)}^{(vv)} > 0. \quad (\text{C4})$$

A sufficient condition (Routh-Hurwitz criterion) is that additionally the product of the coefficients of λ and λ^2 is larger than the constant coefficient:

$$P_1 P_2 > P_0 \quad \text{and} \quad Q_1 Q_2 > Q_0. \quad (\text{C5})$$

Alignment with $\mathbf{n}$ pointing against gravity (instead of along) gives the same stability exponents. The stability exponents for the remaining four orientations are obtained by permutations of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{n}$. The conditions corresponding to Eq. (C4) when settling along or against $\mathbf{p}$ and $\mathbf{q}$ are

$$N_{n(pq)}^{(vv)} < 0 \quad \text{and} \quad N_{p(qn)}^{(vv)} > 0 \quad \text{for } \mathbf{q} \parallel \mathbf{e}_3, \quad (\text{C6a})$$

$$N_{q(np)}^{(vv)} < 0 \quad \text{and} \quad N_{n(pq)}^{(vv)} > 0 \quad \text{for } \mathbf{p} \parallel \mathbf{e}_3. \quad (\text{C6b})$$

The three conditions (C4) and (C6) are mutually exclusive. This means that at most two orientations can be stable (with one of the principal axes collinear with $\mathbf{e}_3$). In the overdamped limit (large components of the resistance tensor), Eqs. (C5) are satisfied and the ellipsoid obtain a stable orientation provided that at least two of the components $N_{n(pq)}^{(vv)}$, $N_{q(np)}^{(vv)}$, and $N_{n(pq)}^{(vv)}$ have opposite signs. This orientation may become unstable for less inertial particles, for which Eqs. (C5) are no longer satisfied.

For nearly spherical ellipsoids, the shape parameters are given in Eq. (B2). Substituting the results into Eqs. (C4)–(C6) shows that, as long as $|\epsilon|$ is not too large, the particle aligns $\mathbf{n}$ with gravity if $\epsilon < 0$ and $\alpha > 0$ and $\alpha > 0$, aligns $\mathbf{q}$ if $\epsilon > 0$ and $\alpha < 1$ and aligns $\mathbf{p}$ if $\epsilon > 0$ and $\alpha > 1$ or if $\epsilon < 0$ and $\alpha < 0$. All these cases correspond to the particle aligning the broadest cross-section perpendicular to gravity.

APPENDIX D: ASYMPTOTIC MATCHING FOR LARGE STROUHAL NUMBERS

Here we show how to obtain Eqs. (31). When the Strouhal number is large, in the limit $1 \gg \text{Re}_p \text{Sl} \gg \text{Re}_p$, the equations for the outer flow may be simplified by neglecting the convective terms. Force and torque on the particle can still be obtained using matched-asymptotic expansions, but extended to the second order, following Candelier *et al.* [63], but setting the undisturbed flow to 0. In short, the idea is to seek the disturbance flow in the form

$$\begin{aligned} \mathbf{w}_{\text{in}} &= \mathbf{w}_{\text{in}}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} \mathbf{w}_{\text{in}}^{(1)} + \text{Re}_p \text{Sl} \mathbf{w}_{\text{in}}^{(2)} + \dots \quad \text{and} \\ p_{\text{in}} &= p_{\text{in}}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} p_{\text{in}}^{(1)} + \text{Re}_p \text{Sl} p_{\text{in}}^{(2)} + \dots \end{aligned} \quad (\text{D1})$$

To second order, the outer flow equations require two source terms, $\mathbf{f}^{(0)}$ and $\mathbf{f}^{(1)}$,

$$\text{Re}_p \text{Sl} \frac{\partial \mathbf{w}}{\partial t} = -\nabla p + \Delta \mathbf{w} + (\mathbf{f}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} \mathbf{f}^{(1)}) \delta(\mathbf{r}). \quad (\text{D2})$$

The solution of this outer equation takes the form

$$\begin{aligned} \mathbf{w}_{\text{out}} &= \frac{1}{8\pi} \left(\frac{\mathbb{I}}{r} + \frac{\mathbf{r} \otimes \mathbf{r}}{r^3} \right) \cdot (\mathbf{f}^{(0)} + \sqrt{\text{Re}_p \text{Sl}} \mathbf{f}^{(1)}) + \sqrt{\text{Re}_p \text{Sl}} \mathcal{U}_1 \\ &+ \text{Re}_p \text{Sl} \left(\frac{3\mathbf{r}}{32\pi} \left(\mathbb{I} - \frac{1}{3} \frac{\mathbf{r} \otimes \mathbf{r}}{r^2} \right) \cdot \frac{d\mathbf{f}^{(0)}}{dt} + \mathcal{U}_2 \right), \end{aligned} \quad (\text{D3})$$

where

$$\mathcal{U}_1 = - \int_0^t \frac{d\tau}{6\pi} \frac{1}{\sqrt{\pi(t-\tau)}} \frac{d\mathbf{f}^{(0)}}{d\tau} \quad \text{and} \quad \mathcal{U}_2 = - \int_0^t \frac{d\tau}{6\pi} \frac{1}{\sqrt{\pi(t-\tau)}} \frac{d\mathbf{f}^{(1)}}{d\tau}. \quad (\text{D4})$$

The matching procedure leads us to $\mathbf{f}^{(1)} = -\mathbb{A} \cdot \mathcal{U}_1$. The corresponding singular contributions to force and torque read:

$$\begin{aligned} \mathbf{F}_{\text{sing}}^{(1)} &= \mathbb{A} \cdot \mathcal{U}_1, \quad \mathbf{F}_{\text{sing}}^{(2)} = \mathbb{A} \cdot \mathcal{U}_2, \\ \mathbf{T}_{\text{sing}}^{(1)} &= \mathbb{B} \cdot \mathcal{U}_1, \quad \mathbf{T}_{\text{sing}}^{(2)} = \mathbb{B} \cdot \mathcal{U}_2. \end{aligned}$$

In the limit $Sl \gg 1$, the two kernels $\mathbb{K}^{(1)}$ and $\mathbb{K}^{(2)}$, defined in Eqs. (A1) and (A2), simplify as

$$\mathbb{K}^{(1)} \longrightarrow \frac{1}{6\pi\sqrt{t-\tau}}, \quad \mathbb{K}^{(2)} \longrightarrow 0, \quad (\text{D5})$$

the same as the kernel $\mathcal{U}_1$ in Eq. (D4). The derivation underlying Eqs. (A1) and (A2) assumed $Sl \ll 1$. Equation (D5) shows that the result remains uniformly valid at order $\sqrt{\text{Re}_p Sl}$. To order $\text{Re}_p Sl$ there are additional contributions to the memory kernel in the form of a double convolution. For an isotropic particle where $\mathbb{A} \propto \mathbb{I}$, the double convolution simplifies to an instantaneous contribution connected to the added mass [63].

Now consider the regular contribution. The regular expansion for the disturbance velocity in the inner region is linearly related to the particle velocity $\mathbf{v}$ and its angular velocity $\boldsymbol{\omega}$,

$$\mathbf{w}_{\text{in}}^{(0)} = \mathbb{W}_v(\mathbf{X}) \cdot \mathbf{v} + \mathbb{W}_\omega(\mathbf{X}) \cdot \boldsymbol{\omega}, \quad (\text{D6})$$

where X_i are coordinates in the particle-fixed coordinate system. Equation (2), however, is written in a coordinate system that translates with particle (but does not rotate with it). Evaluating the time derivative of $\mathbf{w}_{\text{in}}^{(0)}$ in the translating coordinate system yields an expression that depends linearly on $\dot{\mathbf{v}}$ and $\dot{\boldsymbol{\omega}}$, and there are also second-order terms proportional to $\boldsymbol{\omega} \otimes \mathbf{v}$ and $\boldsymbol{\omega} \otimes \boldsymbol{\omega}$. Hydrodynamic force and torque must have the same tensorial structure. There are quadratic terms just like Eq. (14). But there are also unsteady contributions to force and torque of the form

$$\mathbf{F}_{\text{u}}^{(2)} = \text{Re}_p Sl (\mathbb{P} \cdot \dot{\mathbf{v}} + \mathbb{Q} \cdot \dot{\boldsymbol{\omega}}), \quad (\text{D7a})$$

$$\mathbf{T}_{\text{u}}^{(2)} = \text{Re}_p Sl (\mathbb{Q}^T \cdot \dot{\mathbf{v}} + \mathbb{R} \cdot \dot{\boldsymbol{\omega}}), \quad (\text{D7b})$$

distinct from the terms discussed in the main text. Here $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ are second-order tensors, constrained by symmetry in the same way as the second-order Stokes resistance tensors. The fact that $\mathbb{Q}^T$ appears in the expression for the torque follows from the reciprocal theorem. The terms (D7) that look like added-mass contributions. They were first obtained by Gavze [70], who derived a general solution of the unsteady Stokes equation for a particle of arbitrary shape. More recently, Collis *et al.* [37] considered the same limit as Gavze [70]. They solved the unsteady Stokes equation for several nearly spherical particles to obtain hydrodynamic force and torque in the unsteady limit at large Strouhal numbers. No history terms appear in their expressions, because they work in the frequency domain. Integration over frequency must give these terms, in the same way as for a sphere oscillating in a fluid at rest [62].

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