

Extending the Duchon-Robert framework for anomalous dissipation to compressible fluid flows

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Anomalous dissipation, the persistence of a finite mean kinetic energy dissipation rate as the Reynolds number tends to infinity, occurs in flows with sufficiently spatially rough velocity fields. Compressible turbulence adds further anomalous dissipation mechanisms, which we investigate in this work. To this end, the Duchon-Robert framework (DR) for anomalous dissipation is extended from the incompressible to the compressible Navier-Stokes flow case. We obtain three integral dissipation terms, two anomalous and a viscous, which arise from the pressure-dilatation and density variations, differently from the incompressible case. Subsequently, fully compressible one-dimensional flows with traveling and mutually crossing shock waves are analyzed in detail. In such flows, DR reveals a local maximum of anomalous dissipation at the shock front. Furthermore, DR is compared with a coarse-grained cascade theory of compressible turbulence due to Aluie (AL) and the relevant dissipation flux terms of both frameworks are identified and compared with each other. The comparison shows that each contribution related to the compressibility effects in DR has an analog in AL. Finally, a piecewise linear shock-type velocity profile, which approximates the crossing of two shock waves from the simulations, is used for an analytical analysis of the anomalous dissipation terms of DR to investigate the dependence of the terms on the local Hölder exponent. Our work is a first step towards a comparison of coherent flow structures in higher-dimensional compressible turbulent flows and related anomalous dissipation.

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I. INTRODUCTION

In a fully developed three-dimensional compressible Navier-Stokes flow $\mathbf{u}(\mathbf{x}, t)$, the kinetic energy of fluid motion is dissipated at smaller scales in multiple ways [1,2]. A fraction is directly converted into heat by the finite molecular dynamic viscosity μ of the fluid in the viscous subrange, the latter of which starts at a typical mean viscous scale, such as the Kolmogorov length [3]. However, unlike in incompressible turbulence, density and pressure fluctuations introduce additional mechanisms for energy dissipation. These genuine effects due to compressibility can change the kinetic energy transfer in the inertial subrange, especially in regimes with Mach numbers $\text{Ma} \gtrsim 1$, when the characteristic velocity U of the problem at hand is larger than the speed of sound a_s . As a result, the dissipation of kinetic energy in compressible flows is regulated not only by dynamic viscosity μ , but also by complex interactions of shock waves and significant gradients of pressure, density, and velocity [4].

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The concept of *anomalous dissipation* or *dissipative anomaly*, loosely speaking, a dissipation without molecular viscosity, originates from the incompressible turbulence literature [3,5]. The mean rate of kinetic dissipation has been found to be finite even as the viscosity μ goes to zero. Onsager’s conjecture [6], made in 1949, states that in the limit of zero viscosity, weak solutions to the Euler equation, namely, those with “sufficiently rough” velocity fields, should have a nonzero energy loss rate even when viscosity is zero. Empirical evidence for the existence of a dissipative anomaly has been collected from experimental and numerical simulation data in Refs. [7–9]; see, also, [10]. These are known as “dissipative Euler solutions” [11]. In other words, the dissipative anomaly is the statement that if the spatial irregularity or roughness of the velocity field is strong enough in the inertial subrange, the energy continues to cascade downwards to smaller scales and be dissipated when $\mu \rightarrow 0$. A characteristic of spatial roughness is the Hölder exponent h , which is determined through the velocity increments over a spatial distance $\xi = |\xi|$, i.e., $|\mathbf{u}(\mathbf{x} + \xi) - \mathbf{u}(\mathbf{x})| \sim |\xi|^h$. If $h < 1/3$, the flow is rough enough to maintain the existence of anomalous dissipation, which was proven by Isett for the Euler case [12]. Conversely, for a spatially smooth field (as in the viscous subrange), we have a Hölder exponent of $h = 1$.

In a recent work [13], we analyzed precursors of anomalous dissipation by means of a framework due to Duchon and Robert [14], termed the Duchon-Robert framework (DR) in the following, for three-dimensional incompressible Navier-Stokes turbulence. This framework starts with a local kinetic energy balance that is obtained by combining the velocity field $\mathbf{u}$ and the corresponding velocity field $\mathbf{u}^\varepsilon$, which is regularized over a filter or regularization scale ε . In the limit of $\varepsilon \rightarrow 0$, an additional anomalous dissipation term $D(\mathbf{u})$ in the balance follows. Our intention in [13] was to connect characteristic coherent features of incompressible turbulence, such as stretched vortices obtained in kinematic models [15–18] and direct numerical simulations [19–22], to anomalous dissipation.

The present work focuses on two major points. First, we extend the DR framework for the kinetic energy balance to compressible turbulence. This extension results in two additional terms next to $D(\mathbf{u})$ that will contribute to anomalous dissipation. We will subsequently compare these terms with an alternative formulation based on density-weighted coarse graining, known as Favre averaging [23], which was introduced by Aluie [24] and subsequently discussed in detail in Refs. [25,26]. We will term this framework the Aluie framework (AL) in the following. Second, we compare the anomalous dissipation terms that arise in both frameworks for two simple one-dimensional compressible flows: (i) a fully compressible case with an explicit balance equation for the internal energy, and (ii) a configuration where the energy balance is reduced to a polytropic equation of state. We analyze the physical processes that are responsible for this kind of dissipation in compressible flows. Therefore, here we study one-dimensional problems from gas dynamics. Particularly, we highlight the birth of shock waves and describe how they provide an effective way to transfer energy through scales. Indeed, theoretical analysis by Eyink and Drivas [25] has found an explicit “pressure-work” channel of anomalous dissipation, by which work that is done by the compression of fluid elements is necessarily transformed into internal energy. We relate this process to the real dynamics of shocks: incipient shock waves become steeper with increasing local Mach number, significantly increasing anomalous dissipation. To this end, successively coarse-grained analysis and analytical modeling are assisted by direct numerical simulations (DNS) of nonlinear problems, which cause the shock discontinuities. By performing a spatial filtering operation using a Gaussian-type kernel [27], we analyze energy fluxes at various scales and calculate anomalous dissipation directly. Finally, we discuss anomalous dissipation in a simple analytical model, which approximates the velocity profile during the crossing of shocks as piecewise linear, to obtain closed expressions of the anomalous dissipation terms in dependence on the local Hölder regularity.

The modeling helps us to identify the local flow characteristics that act as precursors to anomalous dissipation, such as the emergence of steep gradient areas or colliding waves indicating the emergence of a shock. The detection of such precursors in the velocity, density, and pressure fields helps us to attribute the emergence of anomalous dissipation to the evident real-time flows that are

observed. We also calculate the anomalous dissipation terms through DR and AL to qualitatively validate the similarity of these methods.

The outline of the manuscript is as follows. In Sec. II, we construct the Duchon-Robert framework for fully compressible Navier-Stokes flows by deriving the local energy balance. Technical details of this derivation are found in Appendix A. Section III introduces the Aluie framework in brief and formulates the corresponding energy budget. In Sec. IV, we compare DR and AL, identifying their similar structure. Section V presents our numerical implementation of fully compressible one-dimensional flow, the solution of which is analyzed using both frameworks. Details of the numerics are in Appendix B. We quantify the anomalous kinetic energy dissipation in Sec. V A. In Sec. V B, we compare fully compressible flow with flow under adiabatic conditions. Section VI provides the analytical investigation of anomalous dissipation in a one-dimensional configuration of crossing shock waves. We finish the work with a summary and an outlook in Sec. VII

II. FRAMEWORK BY DUCHON AND ROBERT [14]

In the study of turbulent flows, DR provides one possibility to analyze energy dissipation. Originally developed for incompressible flows [14], this approach is now extended to compressible turbulence to account for density variations as well as preshocks and shocks. Here, we adapt this framework to the compressible Navier-Stokes (NS) equations, aiming to derive a local kinetic energy balance that will include anomalous dissipation terms arising from nonlinear interactions between velocity, density, and pressure fields. We begin with the compressible three-dimensional NS equations written in terms of the density-weighted velocity field,

$$\mathbf{w}(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} \mathbf{u}(\mathbf{x}, t), \quad (1)$$

where $\rho(\mathbf{x}, t)$ is the mass density field and $\mathbf{u} = (u_x, u_y, u_z)$ is the velocity field. For the rest of the work, we switch to index notation for vector and tensor fields and apply the Einstein summation rule. Indices $i, j, k = 1, 2, 3$, or x, y, z for the three-dimensional case. This choice of density-weighted velocity was made because the square of it is equal to the local kinetic energy, $E(x_j, t) = w_i^2(x_j, t)/2$ [28]. In index form, the momentum balance or Navier-Stokes (NS) equation is then given by

$$\begin{aligned} \partial_t w_i + \partial_j (u_j w_i) - \frac{w_i \theta}{2} &= \frac{1}{\sqrt{\rho}} \partial_j \tau_{ij} - \frac{1}{\sqrt{\rho}} \partial_i p, \\ \theta &= \partial_j u_j, \\ \tau_{ij} &= \mu \left(\partial_j u_i + \partial_i u_j - \frac{2}{3} \delta_{ij} \partial_k u_k \right). \end{aligned} \quad (2)$$

Here, p is the pressure field, τ_{ij} is the viscous stress tensor, and θ is the velocity divergence. The dynamic viscosity $\mu = \rho \nu$ with the kinematic viscosity ν . Furthermore, the second dynamic or bulk viscosity is set to zero in the strain-stress relation above. The variables u_j and w_j denote the j th components of the velocity fields $\mathbf{u}$ and $\mathbf{w}$, respectively, and ∂_j is the partial derivative with respect to the j th spatial coordinate x_j . It is also important to note that in analogy with the incompressible DR framework, we derive a local kinetic energy balance directly from the momentum equation and do not explicitly use the internal energy equation.

The detailed derivation of the compressible DR framework is provided in Appendix A. In short, the NS equations (2) are regularized at a scale ε to give corresponding equations for w_i^ε ; the regularized NS equations are then dotted with w_i and the original NS equations with w_i^ε . We obtain the regularized velocity field by convolution with a $\varphi^\varepsilon \in C^\infty(V)$,

$$w_i^\varepsilon = \varphi^\varepsilon * w_i = \int_V \varphi^\varepsilon(\boldsymbol{\xi}) w_i(\mathbf{x} + \boldsymbol{\xi}) d\boldsymbol{\xi}. \quad (3)$$

Both equations are summed with respect to all spatial directions i to obtain a balance for $w_i w_i^\varepsilon$. The limit $\varepsilon \rightarrow 0$ is taken, which leads to the following final form of the kinetic energy balance:

$$\partial_t E + \partial_j(u_j E) = u_i \partial_j \tau_{ij} - u_j \partial_j p - \lim_{\varepsilon \rightarrow 0} (D_{wuu} + D_{w\chi p} - D_{w\chi\tau} + \mathcal{D}^\varepsilon). \quad (4)$$

We denote $\chi = 1/\sqrt{\rho}$ and $\lambda_i = \partial_i \chi$. Spatial increments are denoted, for example, as $\delta w_i = w_i(\mathbf{x} + \boldsymbol{\xi}) - w_i(\mathbf{x})$ for the field w_i . The dissipation terms D on the right-hand side of (4) are defined as follows:

$$D_{wuu}(\varepsilon, \mathbf{x}) = \frac{1}{4} \int \partial_j \varphi^\varepsilon(\boldsymbol{\xi}) \delta w_i \delta w_i \delta u_j d^3 \boldsymbol{\xi}, \quad (5a)$$

$$D_{w\chi p}(\varepsilon, \mathbf{x}) = \frac{1}{2} \int \partial_i \varphi^\varepsilon(\boldsymbol{\xi}) \delta w_i \delta \chi \delta p d^3 \boldsymbol{\xi}, \quad (5b)$$

$$D_{w\chi\tau}(\varepsilon, \mathbf{x}) = \frac{1}{2} \int \partial_j \varphi^\varepsilon(\boldsymbol{\xi}) \delta w_i \delta \chi \delta \tau_{ij} d^3 \boldsymbol{\xi}, \quad (5c)$$

and $\mathcal{D}^\varepsilon$ collect defects that couple fields with their gradients. These terms need not vanish as $\varepsilon \rightarrow 0$. In particular, three contributions appear to exist for $\mathcal{D}^\varepsilon$,

$$\begin{aligned} \mathcal{D}^\varepsilon &= \mathcal{D}_\tau^\varepsilon + \mathcal{D}_\theta^\varepsilon + \mathcal{D}_\lambda^\varepsilon = \partial_j [(\chi \tau_{ij} w_i)^\varepsilon - \chi (\tau_{ij} w_i)^\varepsilon] + [(w_i \tau_{ij})^\varepsilon \lambda_j - w_i (\tau_{ij} \lambda_j)^\varepsilon] \\ &\quad + \frac{1}{2} [w_i (w_i \theta)^\varepsilon - (w_i w_i)^\varepsilon \theta] + [w_i (p \lambda_i)^\varepsilon - (w_i p)^\varepsilon \lambda_i]. \end{aligned} \quad (6)$$

These terms (5) and (6) represent further terms of energy dissipation in the compressible DR. The first two terms will be related to anomalous dissipation. Term (5a) is due to velocity-velocity coupling (D_{wuu}), term (5b) is due to pressure-density-velocity interactions ($D_{w\chi p}$), and term (5c) is due to viscous stress-density-velocity coupling. Term $\mathcal{D}^\varepsilon$ determines conversion or transport defects that appear due to unresolved covariance with θ , λ , and τ_{ij} . For finite viscosity, fields are smooth below the Kolmogorov length $\eta_K = (\nu^3/\bar{\varepsilon})^{1/4}$ with the mean kinetic energy dissipation rate $\bar{\varepsilon}$. For $\varepsilon < \eta_K$, one has $\delta f \sim |\boldsymbol{\xi}|$ so that each of the three increment integrals scales like

$$\int \partial \varphi^\varepsilon (\delta f)^3 d^3 \boldsymbol{\xi} \sim \varepsilon^2 \rightarrow 0.$$

The same scale law is valid for the defects $\mathcal{D}^\varepsilon$. All these terms are expected not to vanish for sufficient roughness of the velocity, pressure, and density fields in the inertial subrange for $\varepsilon \rightarrow 0$ for the Euler case $\mu \rightarrow 0$ (or $\text{Re} \rightarrow \infty$). In dimensionless form with Reynolds number Re and Mach number Ma , the local kinetic energy balance becomes

$$\partial_t E + \partial_j(u_j E) = -\frac{1}{\gamma \text{Ma}^2} u_j \partial_j p + \frac{1}{\text{Re}} u_i \partial_j \tau_{ij} - \lim_{\varepsilon \rightarrow 0} \left[D_{wuu} + \frac{1}{\gamma \text{Ma}^2} D_{w\chi p} - \frac{1}{\text{Re}} D_{w\chi\tau} + \mathcal{D}^\varepsilon \right]. \quad (7)$$

The dimensionless parameters are given by

$$\gamma = \frac{C_p}{C_v}, \quad \text{Ma} = \frac{U_{\text{rms}}}{a_s}, \quad \text{and} \quad \text{Re} = \frac{\rho_0 U_{\text{rms}} L_0}{\mu}. \quad (8)$$

Here, γ is the heat capacity ratio (also known as adiabatic index), C_v the specific-heat capacity at constant volume, C_p the specific-heat capacity at constant pressure. L_0 represents an outer scale of turbulence, e.g., the integral length scale [29], and $U_{\text{rms}} = \sqrt{\langle u_i^2 \rangle_{V,t}}$ is the root-mean-square velocity, which measures the strength of velocity fluctuations. a_s is the speed of sound and ρ_0 is a reference density, e.g., the mean density $\rho_0 = \langle \rho \rangle_{V,t}$. For constant density $\rho(\mathbf{x}, t) = \rho_0$, the terms $D_{w\chi p}$, $D_{w\chi\tau}$, and $\mathcal{D}^\varepsilon$ vanish since the density increment is zero, $\delta \chi = 0$. Moreover, the viscous work term simplifies to $u_i \partial_j \tau_{ij} = \partial_j^2 (u_i^2/2) - (\partial_j u_i \partial_j u_i)$. Thus, the local kinetic energy balance (4) simplifies to the incompressible kinetic energy balance with a single dissipation term [13, 14]. We

recover the incompressible DR case,

$$\partial_t E + \partial_j(u_j E) = -u_j \partial_j p + \frac{1}{\text{Re}} [\Delta E - (\partial_j u_i)^2] - \lim_{\varepsilon \rightarrow 0} D_{www}, \quad (9)$$

where D_{www} corresponds to the anomalous dissipation derived by Duchon and Robert [14], termed $D(\mathbf{u})$ in the introduction. This confirms the generality of the framework.

III. FRAMEWORK BY ALUIE [24]

In the study of turbulent flows, different frameworks have been developed to capture the physical mechanisms responsible for energy transfer and dissipation. The DR approach emphasizes energy dissipation arising from a lack of smoothness in weak solutions of the Euler and Navier-Stokes equations, while the framework developed by Aluie [24], which we are going to outline in the following, is developed for the analysis of kinetic energy transfer across scales in compressible turbulence by using coarse graining or Favre filtering [23]. Starting point are again the continuity and compressible Navier-Stokes equations in standard form,

$$\begin{aligned} \partial_t \rho + \partial_j(\rho u_j) &= 0, \\ \partial_t(\rho u_i) + \partial_j(\rho u_j u_i) &= -\partial_i p + \partial_j \tau_{ij} + \rho f_i, \end{aligned} \quad (10)$$

where the viscous stress tensor is given by (2). All flow variables are decomposed into large-scale (filtered) and small-scale components by applying a spatial filter at scale ε . Commuting the filter operation (3) with space derivatives in the momentum and continuity equations yields

$$\begin{aligned} \partial_t \bar{\rho} + \partial_j \bar{\rho} \bar{u}_j &= 0, \\ \partial_t \bar{\rho} \bar{u}_i + \partial_j \bar{\rho} \bar{u}_j \bar{u}_i &= -\partial_i \bar{p} + \partial_j \bar{\tau}_{ij} + \bar{\rho} \bar{f}_i, \end{aligned} \quad (11)$$

where the overbar denotes spatial filtering, $\bar{a} = \varphi^\varepsilon * a = a^\varepsilon$. Even though this procedure is again a kind of regularization, we introduce a separate notation to distinguish this routine from DR. The scale decomposition employed in the large-scale momentum balance (11) is equivalent to traditional filtering in large-eddy simulations (see, for example, [30]). For any field $a(x, t)$ for which a density-weighted average is meaningful (such as velocity components or Reynolds stresses), a Favre filter leads to

$$\tilde{a}(\varepsilon, \mathbf{x}) = \frac{\bar{\rho} \bar{a}}{\bar{\rho}}. \quad (12)$$

Since viscous effects are dominant at small scales, their contribution to the kinetic energy balance decreases as the coarse-graining scale moves into the inertial range. Note that for Favre filtering, viscous effects at large scales are negligible, as shown in Ref. [31]. Now, using density-weighted quantities defined by (12), we can rewrite momentum balance in the following form:

$$\bar{\rho} \partial_t \tilde{u}_i + \bar{\rho} \tilde{u}_j \partial_j \tilde{u}_i = -\partial_j [\bar{\rho} \tilde{\sigma}(u_i, u_j)] - \partial_i \bar{p} + \partial_j \bar{\tau}_{ij} + \bar{\rho} \tilde{f}_i, \quad (13)$$

with

$$\bar{\rho} \tilde{\sigma}(u_i, u_j) = \bar{\rho} (\tilde{u}_j \tilde{u}_i - \tilde{u}_j \tilde{u}_i). \quad (14)$$

By multiplying (13) by density-weighted velocity $\tilde{u}_i$ and substituting the continuity equation, which is presented in the form

$$\bar{\rho} \partial_t \tilde{u}_i^2 = \partial_t (\bar{\rho} \tilde{u}_i^2) + \partial_j (\bar{\rho} \tilde{u}_i^2 \tilde{u}_j) - 2 \bar{\rho} \tilde{u}_i \tilde{u}_j \partial_j \tilde{u}_i,$$

into the momentum equation, one gets the following local kinetic energy balance [24]:

$$\begin{aligned}
 \partial_t \frac{\bar{\rho} \tilde{u}_i^2}{2} + \partial_j J_j &= -\Pi - \Lambda + \bar{p} \partial_i \tilde{u}_i - \bar{\tau}_{ij} \partial_j \tilde{u}_i + \bar{\rho} \tilde{f}_i \tilde{u}_i, \\
 J_j(\varepsilon, \mathbf{x}) &= \left(\bar{\rho} \tilde{u}_j \frac{\tilde{u}_i^2}{2} + \bar{p} \tilde{u}_j + \bar{\rho} \tilde{u}_i \tilde{\sigma}(u_i, u_j) - \tilde{u}_i \bar{\tau}_{ij} \right), \\
 \Pi(\varepsilon, \mathbf{x}) &= -\bar{\rho} \tilde{\sigma}(u_i, u_j) \partial_j \tilde{u}_i, \\
 \Lambda(\varepsilon, \mathbf{x}) &= \frac{1}{\bar{\rho}} \bar{\sigma}(\rho, u_j) \partial_j \bar{p}, \\
 \bar{\sigma}(\rho, u_j)(\varepsilon, \mathbf{x}) &= (\bar{\rho} \tilde{u}_j - \bar{p} \tilde{u}_j), \\
 \bar{\rho} \tilde{\sigma}(u_i, u_j)(\varepsilon, \mathbf{x}) &= \bar{\rho} (\tilde{u}_j \tilde{u}_i - \tilde{u}_i \tilde{u}_j).
 \end{aligned} \tag{15}$$

Here, $\bar{\rho}$, $\bar{p}$, and $\tilde{\mathbf{u}}$ are the Favre-averaged density, pressure, and velocity fields coarse grained at a scale ε , respectively. Furthermore, J_j is the spatial flux of the large-scale kinetic energy, $\bar{\tau}_{ij} \partial_j \tilde{u}_i$ the viscous dissipation rate, and $\bar{\rho} \tilde{f}_i \tilde{u}_i$ the energy injection rate at large scales due to external (volume) forcing $\mathbf{f}$.

Furthermore, the term $\bar{p} \partial_i \tilde{u}_i$ is the pressure dilatation at the resolved scale. It represents energy exchange between resolved kinetic energy and internal energy. This term does not contribute to energy transfer between scales. It is given by the product of the large-scale pressure and the large-scale dilatation. In incompressible Navier-Stokes flow, $\bar{p} \partial_i \tilde{u}_i = 0$. However, in the compressible case, it can become significant.

The term $\bar{\tau}_{ij} \partial_j \tilde{u}_i$ is the viscous dissipation, the rate at which molecular viscosity directly dissipates kinetic energy coming from scales larger than ε into heat. For high-Reynolds-number flows, $\bar{\tau}_{ij} \partial_j \tilde{u}_i$ is essentially zero for all but the very finest scales; it can be shown rigorously [24] that viscous dissipation vanishes in the limit of decreasing ε well above the dissipation scale of the flow.

The two remaining terms $\Pi(\varepsilon, \mathbf{x})$ and $\Lambda(\varepsilon, \mathbf{x})$ on the right-hand side of Eq. (15) are the subgrid-scale kinetic energy fluxes—they act as sinks of kinetic energy in the large-scale kinetic energy balance and thus as a source in the complementary small-scale balance. The term $\Pi(\varepsilon, \mathbf{x})$ is the *deformation work*, analogous to the turbulent energy flux in incompressible flow. It arises from large-scale velocity gradients that work against the subgrid stress. In this term, $\partial_j \tilde{u}_i$ is the resolved strain-rate tensor (on scales larger than ε) and $\tilde{\sigma}(u_i, u_j)$ is the turbulent stress. Physically, $\Pi(\varepsilon, \mathbf{x})$ represents the part of kinetic energy transferred downscale by the action of the large-scale strain and turbulent stress. A positive Π means that large scales lose kinetic energy, which is passed down to smaller ones. This term arises from nonlinear interactions.

The quantity $\Lambda(\varepsilon, \mathbf{x})$ is termed *baropycnal work*, a scale-to-scale kinetic energy transfer mechanism that exists only in compressible flows where the density is scale dependent. Here $\bar{\sigma}(\rho, u_j) = \bar{\rho} \tilde{u}_j - \bar{p} \tilde{u}_j$ is the subgrid mass flux. The baropycnal work term $\Lambda(\varepsilon, \mathbf{x})$ represents the work done by the large-scale pressure-gradient force on the small-scale mass flux. In other words, it transfers energy across scales due to variations in pressure and density, thus a “baropycnal” work. Physically, Λ captures effects similar to baroclinicity in the energy budget, as discussed in [26]. Clearly, this term vanishes in the incompressible limit and becomes significant when density gradients interact with pressure gradients. Together, Π and Λ constitute the total energy flux at scale ε . They are the only two processes through which kinetic energy can be transferred directly from small to large scales across ε (and vice versa) in a variable-density flow.

In summary, the coarse-grained kinetic energy balance in AL separates the effects of interscale energy transfer from the rest. The deformation work Π and baropycnal work Λ are the two channels of kinetic energy cascade across scales, analogous to the turbulent energy flux in incompressible turbulence, with Λ being a unique contribution due to the variable density in the flow, that has no analog in incompressible turbulence. Meanwhile, the pressure-dilatation term appears as a distinct term in the large-scale energy equation, representing the conversion between kinetic and internal

TABLE I. Comparison of local kinetic energy balance terms related to anomalous and viscous dissipation between the frameworks DR (left column) and AL (right column).

	Duchon-Robert framework (DR)	Aluie framework (AL)
Deformation work	$-D_{wuu}(\varepsilon, \mathbf{x}) + \mathcal{D}_\theta^\varepsilon(\varepsilon, \mathbf{x})$	$-\Pi(\varepsilon, \mathbf{x})$
Baropycnal work	$-D_{w\chi\rho}(\varepsilon, \mathbf{x}) + \mathcal{D}_\lambda^\varepsilon(\varepsilon, \mathbf{x})$	$-\Lambda(\varepsilon, \mathbf{x})$
Viscous work	$D_{w\chi\tau}(\varepsilon, \mathbf{x}) + \mathcal{D}_\tau^\varepsilon(\varepsilon, \mathbf{x}) + u_i \partial_j \tau_{ij}(\mathbf{x})$	$-\bar{\tau}_{ij} \partial_j \tilde{u}_i(\varepsilon, \mathbf{x}) + \partial_j [\tilde{u}_i \bar{\tau}_{ij}](\varepsilon, \mathbf{x})$

energy at the resolved scales. We are going to compare DR and AL in the next section. Thus, we provide the dimensionless form of the local kinetic energy balance at the end of this section, which is given by

$$\begin{aligned} \partial_t \frac{\bar{\rho} \tilde{u}_i^2}{2} + \partial_j J_j = & -\Pi - \frac{1}{\gamma \text{Ma}^2} \Lambda + \frac{1}{\gamma \text{Ma}^2} \bar{\rho} \partial_i \bar{u}_i - \frac{1}{\text{Re}} \bar{\tau}_{ij} \partial_j \tilde{u}_i + \frac{1}{F^2} \bar{\rho} \tilde{f}_i \tilde{u}_i, \\ J_j(\varepsilon, \mathbf{x}) = & \left(\bar{\rho} \tilde{u}_j \frac{\tilde{u}_i^2}{2} + \frac{1}{\gamma \text{Ma}^2} \bar{\rho} \bar{u}_j + \bar{\rho} \tilde{u}_i \tilde{\sigma}(u_i, u_j) - \frac{1}{\text{Re}} \tilde{u}_i \bar{\tau}_{ij} \right). \end{aligned} \quad (16)$$

Here, Mach number Ma and Reynolds number Re are defined in the same way as in DR. An additional dimensionless parameter, $F = U_{\text{rms}}/\sqrt{|f|L_0}$, measures the ratio of inertial to external forcing in turbulent flow.

IV. DIRECT COMPARISON OF BOTH FRAMEWORKS

By comparing the local kinetic energy balance derived from frameworks DR (4) and AL (15), we can identify analogous terms with respect to anomalous dissipation. They are compactly summarized in Table I to initiate the comparison. Both approaches rely on spatial filtering with the same test function, a compact-support convolution kernel $\varphi^\varepsilon \in C^\infty(V)$, and capture the same physical mechanisms of kinetic energy transfer in compressible turbulence. However, they differ slightly in how these mechanisms are expressed and derived. In the following, we show in detail the correspondence between the main terms of both frameworks and discuss their physical meaning.

(1) *Deformation work (velocity coupling)*. The anomalous term D_{wuu} in DR stands for the inertial energy flux from large scales to small scales in the coarse-graining approach. Aluie represented the inertial transfer by the deformation work Π . In the coarse-grained framework, the deformation work Π is defined to be Galilean invariant and to represent the flux of kinetic energy across scales. The corresponding contribution to the spatial transport is expressed through a divergence term $\partial_j [\bar{\rho} \tilde{u}_i \tilde{\sigma}(u_i, u_j)]$, which can be rewritten as $\tilde{u}_i \tilde{\sigma}(u_i, u_j) \partial_j \bar{\rho} + \bar{\rho} \tilde{u}_i \partial_j \tilde{\sigma}(u_i, u_j)$. This flux can be rewritten in a form that expresses the effect of density gradients more explicitly,

$$\partial_j \tilde{\sigma}(u_i, u_j) = (2\tilde{u}_i \tilde{u}_j - \tilde{u}_j \tilde{u}_i) \frac{\partial_j \bar{\rho}}{\bar{\rho}} + \frac{\partial_j \bar{\rho} u_j \tilde{u}_i}{\bar{\rho}} - \tilde{u}_i \frac{\partial_j \bar{\rho} u_j}{\bar{\rho}} - \tilde{u}_j \frac{\partial_j \bar{\rho} u_i}{\bar{\rho}}.$$

It can be seen that the inertial energy transfer in a variable-density flow has two contributions; one is Π , associated with the strain-rate work, analogous to the incompressible case. The other part is solely related to compressibility effects, and in the absence of density variations, the divergence of the flux of stresses vanishes. In DR, the interscale transfer is captured by D_{wuu} ; the additional $\mathcal{D}_\theta^\varepsilon$ determines the dilatational defect. It captures unresolved covariance between local kinetic energy $E = w_i^2/2$ and dilatation θ . The term D_{wuu} represents the nonlinear transfer of kinetic energy that remains finite in the limit $\varepsilon \rightarrow 0$ (taken after $\mu \rightarrow 0$). Physically, D_{wuu} and Π describe the same process, namely, the kinetic energy that is transferred from large eddies to smaller ones. When the density is constant, this term reduces to the usual subgrid-scale stress. When density varies, the extra flux contribution accounts for (additional) energy fluxes that appear due to compressibility.

In summary, the quantity D_{www} corresponds to the deformation work Π and the energy flux related to the density gradient is part of the resolved transport term J_j in AL and partly corresponds to the DR commutator $\mathcal{D}_\theta^\varepsilon$. These represent interscale kinetic energy transfer.

(2) *Baropycnal work (pressure-density coupling)*. The term $D_{w\chi p}$ in DR corresponds directly to the baropycnal work term, which is denoted by Λ in AL. Both $D_{w\chi p}$ and Λ arise from fluctuations in pressure and density. In physical terms, this is the work done by pressure forces in a variable-density flow that transfers kinetic energy across scales. In DR, the baropycnal transfer is captured by $D_{w\chi p}$; the additional $\mathcal{D}_\lambda^\varepsilon$ is a baropycnal conversion defect across the density gradient. In AL, only the baropycnal work Λ carries interscale transfer. In AL, Λ appears as a separate contribution to the energy balance, representing how large-scale pressure gradients acting on small-scale density fluctuations. In DR, $D_{w\chi p}$ is defined by an integral of the product of three increments, δw , $\delta \chi$, δp ; see Eq. (5b). The term explicitly couples variations in the density field ($\chi = 1/\sqrt{\rho}$) with pressure and velocity and should not vanish in the limit of $\varepsilon \rightarrow 0$ (taken after $\mu \rightarrow 0$) if the fields are sufficiently rough. For example, this can be the case near shock fronts or strong compressions, indicating anomalous dissipation due to compressible effects. Importantly, the baropycnal term has no counterpart in incompressible turbulence, showing how compressibility introduces a new source of cascading energy. In particular, for constant density, both $D_{w\chi p}$ and the defect $\mathcal{D}_\lambda^\varepsilon$ vanish, matching AL, $\Lambda = 0$.

(3) *Viscous work (stress-density coupling)*. Viscous effects appear differently in both frameworks, but they are equivalent. In DR, the viscous dissipation in the kinetic energy equation comes from three terms: the regular viscous work $u_i \partial_j \tau_{ij}$ on the right side of Eq. (4); the additional term $D_{w\chi\tau}$, which involves density and velocity fluctuations with the viscous stress [see Eq. (5c)]; and the viscous defect $\mathcal{D}_\tau^\varepsilon$, arising from the noncommutation of filtering with products involving χ and τ_{ij} . The divergence part of $\mathcal{D}_\tau^\varepsilon$ is a transport defect; it redistributes kinetic energy in space and its average in the periodic domain is zero. The other part of $\mathcal{D}_\tau^\varepsilon$ is a production defect caused by the coupling of viscous stress with the density gradient. It can act as a local sink of kinetic energy, converting it into internal energy. The sum $u_i \partial_j \tau_{ij} + D_{w\chi\tau} + \mathcal{D}_\tau^\varepsilon$ represents the total rate at which kinetic energy is converted into internal energy by viscosity. In AL, one finds comparable terms $-\bar{\tau}_{ij} \partial_j \tilde{u}_i(\varepsilon, \mathbf{x}) + \partial_j [\tilde{u}_i \bar{\tau}_{ij}](\varepsilon, \mathbf{x})$ in the filtered energy balance, which is the viscous dissipation at scale ε , the work of the viscous stress $\bar{\tau}_{ij}$ against the filtered velocity $\tilde{u}_i$, and the divergence of viscous spatial transport. Thus, $\bar{\tau}_{ij} \partial_j \tilde{u}_i$ in AL corresponds to $\tau_{ij} \partial_j u_i$ plus the anomalous subgrid dissipation $D_{w\chi\tau}$. Both frameworks agree in the point that viscous forces are a sink of kinetic energy in the turbulent cascade. However, DR explicitly displays how much of this dissipation can be attributed to small-scale stress-velocity correlations with the term $D_{w\chi\tau}$ when the solution is not smooth.

Beyond identifying these corresponding terms, it is useful to show the structural differences between both frameworks. First, both frameworks employ an explicit filtering operation using a kernel φ^ε to separate scales. In DR, the convolution smoothens the fields and one considers the limit of the filter scale of $\varepsilon \rightarrow 0$ to isolate singular contributions. In AL, the filter scale is kept finite to analyze energy fluxes at that scale. Only in the limit of very large Reynolds numbers Re with an increasingly extended inertial range, one might consider $\varepsilon \rightarrow 0$ which is taken after $\mu \rightarrow 0$.

Despite this difference, the filtering function can be chosen identically in both approaches, e.g., as a Gaussian function or any smooth compact-support mollifier normalized to 1, as done in [13] for the incompressible case; see, also, the review by Dubrulle [5]. In fact, mathematically, one can express filtered derivatives in the same way for both frameworks. For instance, an n th-order spatial derivative of a filtered field can be written as an integral over the field increments,

$$\partial_j^n \bar{a} = (-1)^n \int \partial_j^n \varphi^\varepsilon(\xi) a(\mathbf{r} + \xi) d^3 \xi = (-1)^n \int \partial_j^n \varphi^\varepsilon(\xi) [a(\mathbf{r} + \xi) - a(\mathbf{r})] d^3 \xi, \quad n \in \mathbb{N}. \quad (17)$$

This identity is exactly the same as the relation used to derive both of the anomalous terms in DR and AL; it shows that both frameworks are based on the same filtering formalism. Notably, for the DR, we use a density-weighted velocity $w_i = \sqrt{\rho} u_i$ to simplify the kinetic energy density, whereas AL

uses Favre filtering for velocity, $\tilde{u}_i = \overline{\rho u_i} / \bar{\rho}$. These are mathematically related choices to handle the variable density. The latter approach is driven by perspective which is used in large-eddy simulations. Despite the different notation, the study of compressibility effects through filtering is thus consistent between the two methods.

The Duchon-Robert framework yields a local kinetic energy balance that is exact in the distributional sense. It states that locally, there can be a loss of kinetic energy through the anomalous dissipation terms D_{wuu} , $D_{w\chi p}$, $D_{w\chi\tau}$, and $\mathcal{D}^\varepsilon$ if the fields are not smooth. These terms are essentially zero for smooth solutions and become nonzero only when the solution develops sufficiently rough fields. Specifically, the terms D_{wuu} and $D_{w\chi p}$ represent anomalous dissipation and $D_{w\chi\tau}$ is an additional contribution to the viscous dissipation arising from compressibility effects.

Both frameworks point to three key physical processes in compressible turbulence: (1) the “classical” anomalous dissipation term is captured by D_{wuu} in DR and by Π in AL, (2) the second dissipation term, baropycnal work, is governed by pressure-density fluctuations and captured by $D_{w\chi p}$ and Λ , and (3) the last term, viscous dissipation, is captured by $D_{w\chi\tau}$ and the filtered viscous term $\bar{\tau}_{ij}\partial_j\tilde{u}_i$, and represents kinetic energy losses to internal energy.

Both approaches show a consistent physical picture. Large-scale kinetic energy is passed down to smaller scales by rough velocity, pressure, and density fields, and, finally, viscosity at the smallest scales converts the kinetic energy into thermal energy.

Note again, in order to obtain a finite limit of the dissipation, we have to take two limits successively, i.e., the coarse-graining scale limit to zero $\varepsilon \rightarrow 0$ following the inviscid limit $\nu \rightarrow 0$ (or $\mu \rightarrow 0$). In AL, the kinetic energy, built from the Favre-filtered velocity $\tilde{u}_i = \overline{\rho u_i} / \bar{\rho}$, satisfies the “inviscid criterion,” which was first introduced in [32]. In Ref. [31], it was shown that the density-weighted velocity $w_i = \sqrt{\rho}u_i$ does not satisfy this criterion. This implies that this scale decomposition does not localize viscous contributions to the smallest scales and they can remain significant at larger scales in compressible flows, here as the term $D_{w\chi\tau}$ which is given by Eq. (5c).

In conclusion, the Duchon-Robert and Aluie frameworks for compressible turbulence are in close agreement on the nature of energy transfer and dissipation. They apply the same filtering methodology, and equivalent physical terms can be identified in their balances: deformation work with compressible effects, baropycnal work, and viscous dissipation. The primary difference comes from their specific formulation and starting point. Despite this difference, for an incompressible flow case, both frameworks reduce to the classic incompressible energy cascade result. In the following, we are going to analyze these terms from both frameworks for a simple one-dimensional flow.

V. ONE-DIMENSIONAL COMPRESSIBLE FLOW EXAMPLES AND ANOMALOUS DISSIPATION ANALYSIS

To calculate and compare the anomalous dissipation terms of both frameworks, we study two one-dimensional (1D) flow configurations. First, we look at a fully compressible 1D flow as in a simple gas dynamics process; second, we look at a 1D flow with a polytropic gas law to show the difference in the evolution of flow parameters and dissipation between both cases. We consider 1D fully compressible Navier-Stokes equations for an ideal gas with spatial coordinate x , time t , and velocity field $u(x, t)$. The balances of mass, momentum, and energy together with the equation of state in physical dimensions are given by

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \rho \partial_t u + \rho u \partial_x u &= -\partial_x p + \frac{4}{3}\mu \partial_{xx} u, \\ \rho C_p d_t T - \beta T d_t p &= \mu \Phi_{\text{diss}} + \lambda \partial_{xx} T + q_v, \\ p &= \rho R T, \end{aligned} \tag{18}$$

with $d_t = \partial_t + u \partial_x$. Here, T is the temperature field, R is the ideal gas constant, β is the volumetric expansion coefficient, and q_v is a volumetric heat source. All material parameters, dynamic viscosity

μ , thermal conductivity λ , and specific heat at constant pressure, C_p , are assumed to be constant with respect to x and t . Furthermore, $\Phi_{\text{diss}} = \frac{4}{3}(\partial_x u)^2$, $q_v = 0$, and $\beta \approx 1/T$. By eliminating pressure using the equation of state, substituting the density from the continuity equation into the equation of energy, and using the relation $R = C_p - C_v$ for gas constant, the equations simplify to

$$\begin{aligned}\partial_t \rho + \partial_x(\rho u) &= 0, \\ \rho \partial_t u + \rho u \partial_x u &= -R \partial_x(\rho T) + \frac{4\mu}{3} \partial_{xx} u, \\ \rho C_v(\partial_t T + u \partial_x T) + \rho R T \partial_x u &= \frac{4\mu}{3} (\partial_x u)^2 + \lambda \partial_{xx} T.\end{aligned}\tag{19}$$

The dimensionless form with Mach and Reynolds numbers [cf. definitions (8)] is given by

$$\begin{aligned}\partial_t \rho + \partial_x(\rho u) &= 0, \\ \rho \partial_t u + \rho u \partial_x u &= -\frac{1}{\gamma \text{Ma}^2} \partial_x(\rho T) + \frac{4}{3\text{Re}} \partial_{xx} u, \\ \partial_t T + u \partial_x T + (\gamma - 1) T \partial_x u &= \frac{4\text{Ma}^2}{3\text{Re}} \gamma(\gamma - 1) \frac{(\partial_x u)^2}{\rho} + \frac{\gamma}{\text{PrRe}} \frac{\partial_{xx} T}{\rho}.\end{aligned}\tag{20}$$

Here, $\text{Pr} = \mu C_p / \lambda$ is the Prandtl number. We take a periodic velocity profile with constant density and temperature distributions as the initial conditions,

$$\begin{aligned}u(x, 0) &= \sin(2\pi x) \\ \rho(x, 0) &= 1 \\ T(x, 0) &= 1.\end{aligned}\tag{21}$$

Any pressure or density changes at $t > 0$ will be generated by the motion of the gas. In our numerical approach for solving these equations numerically, we employ a fourth-order Runge-Kutta method for time integration. Spatial derivatives are computed using an eighth-order compact finite-difference scheme, which ensures precise resolution of steep gradients and shock features. The domain is the periodic interval $x \in [0, 1]$ and is discretized by 10^4 uniform mesh cells, which results in a grid spacing of $dx = 10^{-4}$, while the time integration is performed over 3×10^5 steps from $t = 0$ to 1.5 time units (given by L_0/U_{rms}), resulting in a time step of $dt = 5 \times 10^{-6}$. Periodic boundary conditions are applied for computation at the ends of the one-dimensional interval. This high-order scheme is chosen to accurately capture the steep gradients and complex features of compressible flow, enhancing both the stability and precision of the simulation. For more details on the numerical scheme, we refer the reader to Appendix B.

Figure 1 shows how the initial velocity distribution and the other fields evolve, in particular how shock waves form, cross each other, and thus interact. The simple initial velocity profile consists of a single broad perturbation, which leads to the formation of two symmetric shock waves. As the simulation begins, the fluid in the perturbed region accelerates and compresses the gas ahead of it, resulting in the formation of shock waves. We observe in Fig. 1 that after a short time $t \approx 0.34$, sharp discontinuities in u and ρ develop. They are indicated as thin red and blue lines in the panels for $\partial_x u$ and $\partial_x \rho$, indicating the presence of shock fronts. The two generated shock waves propagate in opposite directions. The formed shock waves pass by each other every time interval $\Delta t \approx 0.34$ due to periodic boundary conditions. At this moment, the two waves merge, which leads to the appearance of a new shock that carries the combined compression and strength of both initial waves, resulting in a velocity jump and almost negligible density and pressure gradients. After this short moment of merging, the two shock waves instantly split, returning to the previous amplitudes of velocity, density, and pressure and moving in opposite directions until the next time they merge. Over time, due to nonvanishing viscosity, the strength of the shock waves decreases until the shock waves completely disappear.

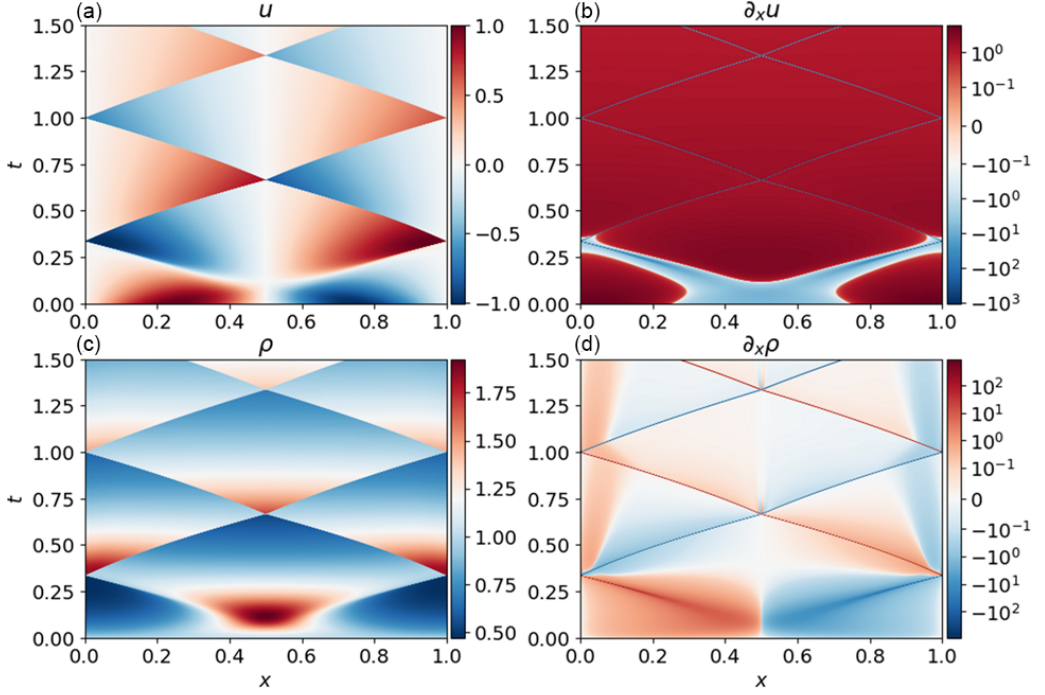


FIG. 1. Evolution of velocity and density fields as well as their derivatives in a 1D compressible flow. Each panel is a space-time diagram (x on the horizontal axis, t on the vertical axis) for a specific variable. (a) Velocity $u(x, t)$, (b) velocity derivative $\partial_x u(x, t)$, (c) density $\rho(x, t)$, and, finally, (d) density derivative $\partial_x \rho(x, t)$. The shocks meet at $t \approx 0.34$ at $x = 0$ (and $x = 1$ due to periodic boundary conditions) and at $t \approx 0.67$ at $x = 0.5$, and so on. Here, $\text{Re} = 5000$, $\gamma = 1.4$, and $\text{Ma} = 0.7$.

A more detailed spatial profile of the fields at the shock interaction instant is presented in Fig. 2. While the velocity develops steep gradients, the density and pressure gradients are much weaker. However, a different type of nonsmooth behavior can develop during a merging event of shock fronts, namely, a cusp in the density and pressure fields. For example, a discontinuity $\delta\rho(x \approx 0, \xi) \sim |\xi|^h$ is seen in the orange curve of density (and pressure). During the shock crossing event, the minimum value of the Hölder exponent is $h_{\min} \approx 0.15$. It means that the baropycnal work [see Eq. (5b)] involves interactions between different types of singularities, sharp shocks with nearly discontinuous velocity gradients, and local regions where density and pressure have Hölder-type distributions.

We also studied whether this structure changes or disappears for different Re numbers. With further calculations of $\text{Re} = 2 \times 10^4$ and $\text{Re} = 5 \times 10^4$, it is found that the shocks forming and the Hölder-type distribution are stable. The time of the first shock interaction is again $t \approx 0.34$. However, all gradients in the shock fronts become steeper. It leads to smaller values of the Hölder exponent at the moment of shock interaction. For $\text{Re} = 2 \cdot 10^4$, $h_{\min} \approx 0.12$, for $\text{Re} = 5 \cdot 10^4$, $h_{\min} \approx 0.11$.

A. Fully compressible flow case

Using the 1D flow configuration described above, we now apply the kinetic energy balance (4) and calculate the anomalous dissipation terms (5). With the same dimensionless parameters as in Eq. (20), the local kinetic energy balance equation for $E = \rho u^2/2$ in dimensionless form is given

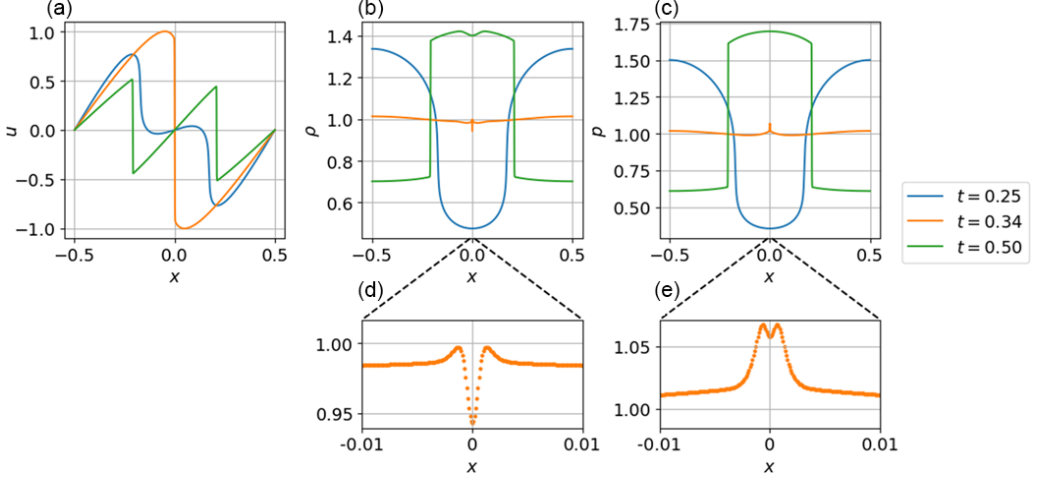


FIG. 2. Spatial profiles of (a) velocity, (b) density, and (c) pressure at the time instant of shock wave crossing. The bottom row contains magnifications of the vicinity of the merging region to highlight the cusplike profile for (e) density and (f) pressure. Parameters are the same as in Fig. 1. Note that the x coordinate is shifted from $[0,1]$ to $[-0.5, 0.5]$ to clarify the shock-merging event at $x = 0$, in the middle of the interval.

by

$$\partial_t E + \partial_x(uE) = u \left(\frac{4}{3\text{Re}} \partial_{xx} u - \frac{1}{\gamma M^2} \partial_x p \right) - \lim_{\varepsilon \rightarrow 0} \left(D_{wuu} + \frac{1}{\gamma M^2} D_{w\chi p} - \frac{1}{\text{Re}} D_{w\chi\tau} \right). \quad (22)$$

Individual dissipation contributions are now given by

$$\begin{aligned} D_{wuu}(\varepsilon, x) &= \frac{1}{4\varepsilon^2} \int \psi_1 \delta w^2 \delta u d\xi, \\ D_{w\chi p}(\varepsilon, x) &= \frac{1}{2\varepsilon^2} \int \psi_1 \delta w \delta \chi \delta p d\xi, \\ D_{w\chi\tau}(\varepsilon, x) &= \frac{1}{2\varepsilon^2} \int \psi_1 \delta w \delta \chi \delta \tau_{11} d\xi, \end{aligned} \quad (23)$$

with $w(x, t) = \sqrt{\rho} u$ and $\tau_{11}(x, t) = (4/3) \partial_x u$. Here, ε again defines the filter on a coarse-grained scale. In the one-dimensional case, we choose the Gaussian-type test function. To calculate the dissipation term in (22), we have to use the first-order Gaussian which is given by

$$\psi_1(\xi) = \partial_x [\exp(-\xi^2/2)] = -\xi \exp(-\xi^2/2). \quad (24)$$

With this choice, the dissipation terms (22) are the one-dimensional cases of the DR integrals in Eqs. (5a)–(5c) with the Gaussian kernel. This kernel provides a smooth filter to measure kinetic energy transfer across scale ε .

1. Analysis in the Duchon-Robert framework

With the given parameters, the anomalous dissipation terms can be calculated for the initial condition (21). For the following, we fix the comparison study to a Reynolds number $\text{Re} = 5000$, an adiabatic index $\gamma = 1.4$, and a Mach number $\text{Ma} = 0.7$ (strictly speaking, there exist no shocks for $\text{Ma} \lesssim 1$; however, we will stick to this notion for the work). The behavior of shocks is similar starting from time $t = 0.5$, as it becomes periodic with decaying amplitudes. For the dissipation analysis, we will therefore focus only on the time interval $t \leq 0.5$. In Fig. 3, the evolution of the dissipation terms $D_{wuu}(\varepsilon, x)$ and $-D_{w\chi p}(\varepsilon, x)$ is shown at four different time instants. They have

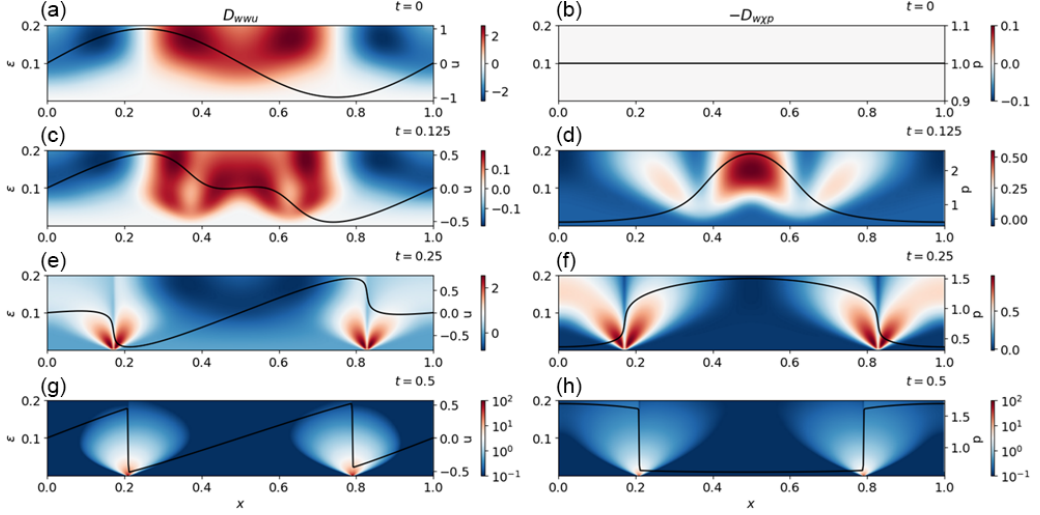


FIG. 3. Temporal evolution of the anomalous dissipation terms $D_{wwu}(\varepsilon, x)$ (left column) and $-D_{w\chi p}(\varepsilon, x)$ (right column) in DR. Snapshots are shown at different times to illustrate how these terms change and where they get enhanced in the flow. (a), (b) $t = 0$, (c), (d) $t = 0.125$, (e), (f) $t = 0.25$, and (g), (h) $t = 0.5$ time units. The smallest value of ε , which is taken in the analysis in all panels, is $\varepsilon = 10^{-4}$. This is the same order as the grid spacing. The black lines in the left and right columns show the velocity and pressure distributions, respectively.

similar behavior when two shocks appear, but differ when these shocks interact. This interaction occurs periodically at times $t \approx 0.34$ and 0.67 at $x = 0$ and 0.5 , respectively, and continues alternating between both points. See, again, Fig. 1. At the interaction points, the velocity gradient remains large, while variations in density, pressure, and temperature are almost absent, as can be seen in Fig. 4, where we plot the dissipation terms. Both figures display symmetric contours, but at strongly different amplitudes. The dissipation in Fig. 4(b) is almost absent, indicating weak coupling between velocity, density, and pressure. This corresponds to $t \approx 0.34$, when two shocks interact, and the density and pressure increments are nearly zero. In contrast, Fig. 4(a) shows strong dissipation, highlighting the dominant nonlinear coupling of velocity.

In order to simplify the analysis and subsequent comparisons with AL, it is useful to consider spatially averaged anomalous dissipation terms, $\langle D \rangle_x$. They are displayed in Fig. 5 for the DR case. Both figures show the spatially averaged dissipation, with significantly higher magnitudes in Fig.

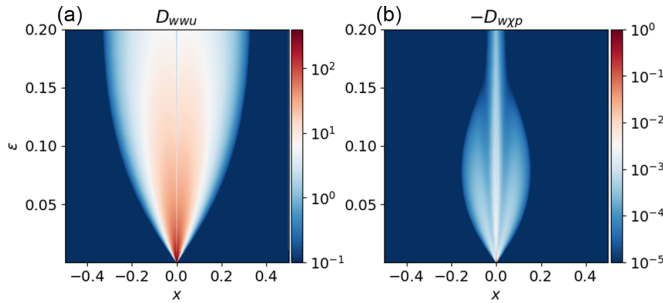


FIG. 4. Dissipation near the shock-interaction region. Here, the time is $t \approx 0.34$ when both shocks meet at $x = 0$. Due to periodic boundary conditions, we continue the x axis to the left with negative values. The interaction strongly influences the spatial and temporal distribution of both terms. (a) $D_{wwu}(\varepsilon, x)$, (b) $-D_{w\chi p}(\varepsilon, x)$.

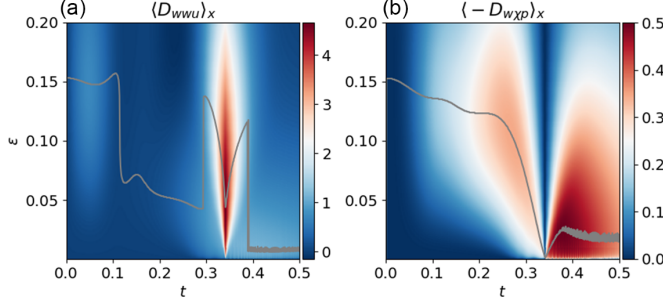


FIG. 5. Spatially averaged anomalous dissipation terms vs time t and filter scale ε . (a) $\langle D_{www} \rangle_x$ and (b) $\langle -D_{w\chi p} \rangle_x$. The gray line in both contour plots stands for $\max_{\varepsilon} \langle D \rangle_x$ at time t .

5(a) than in Fig. 5(b). The dissipation peaks around $t \approx 0.34$, where shock interactions occur. While the left panel displays a concentrated and intense peak at this time instant, the right panel shows an almost negligible value at this point and generally has a broader distribution. The gray lines in both panels stand for scale ε at which the maximum dissipation occurs. For example, in Fig. 5(a), this is the case for $\varepsilon \approx 0.16$ at $t = 0$.

The scaling behavior of D_{www} with ε in the viscous range for $\varepsilon \ll \eta_K$ is shown in Fig. 6 for three different Reynolds numbers (see legend). For regular velocity fields, the theory predicts that the anomalous dissipation vanishes as $D \sim \varepsilon^2$ when the coarse-graining scale $\varepsilon \rightarrow 0$. The data are taken at the specific time instant $t \approx 0.34$ and at $x \approx 0$, which corresponds to the shock-interaction region. For higher Reynolds number, and thus lower viscosity, Kolmogorov length η_K decreases, which leads to an extension of the inertial range, that is indicated by vertical dashed lines. In the viscous range, i.e., $\varepsilon \ll \eta_K$, the dissipation term indeed vanishes approximately with the predicted rate, $D_{www} \sim \varepsilon^\zeta$ and $\zeta \simeq 1.7$. Considering $\varepsilon = 0$ makes everything vanish identically. Note that the grid resolution was the same for all three runs.

2. Comparison with the Aluie framework

The newly obtained terms can be directly compared to AL. We therefore analyze the deformation work $\Pi(\varepsilon, x)$ and the baropycnal work $\Lambda(\varepsilon, x)$. In Fig. 7, their evolution is shown with respect to time. Before the shocks interact, AL yields sharper peaks, making it easier to locate the regions of interest. However, once the shocks begin to interact, the results become qualitatively similar to those obtained from DR. In both approaches, Π and D_{www} remain mostly positive, indicating that energy

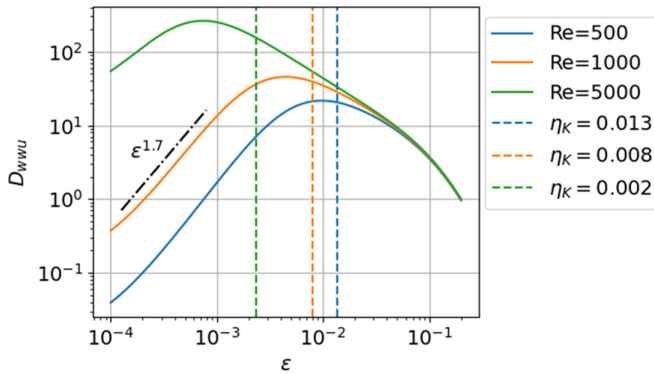


FIG. 6. Anomalous dissipation at the shock-interaction instant at $x = 0$ for different Reynolds numbers. Vertical dashed lines show the corresponding Kolmogorov length scales η_K .

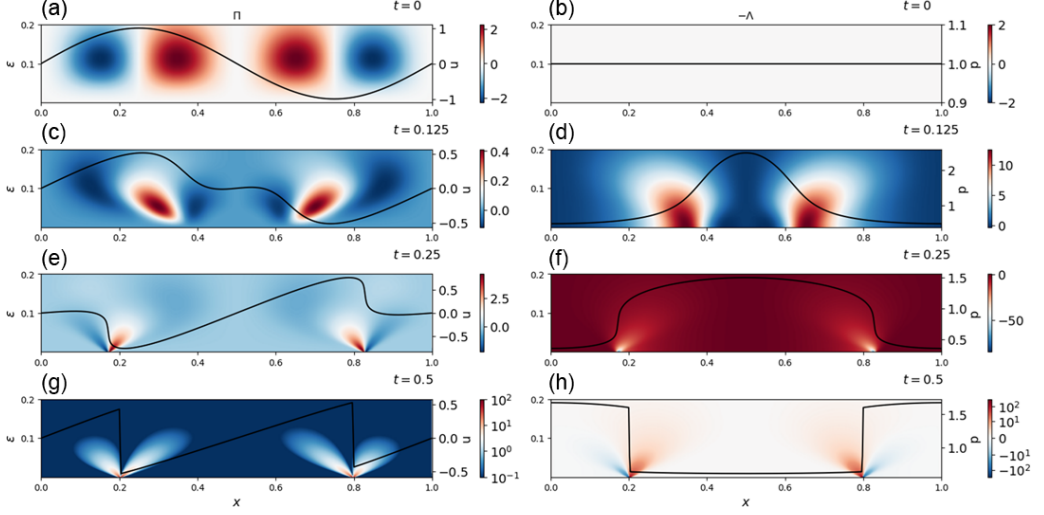


FIG. 7. Temporal evolution of the deformation work $\Pi(\varepsilon, x)$ (left column) and the baropycnal work $\Lambda(\varepsilon, x)$ (right column) in AL. Snapshots are shown at different times to illustrate how these terms change and where they concentrate in the flow. (a), (b) $t = 0$, (c), (d) $t = 0.125$, (e), (f) $t = 0.25$, and (g), (h) $t = 0.5$. The figure has to be compared to Fig. 3. The smallest value of ε , which is taken in the analysis in all panels, is $\varepsilon = 10^{-4}$.

is dissipated at a specific coarse-grained scale ε due to strong velocity fluctuations. Meanwhile, baropycnal work Λ generally reaches higher magnitudes than the deformation work Π , except in the shock-interaction zone where they become comparable, as seen in Fig. 8. Here, the density and pressure fields are relatively smooth, causing baropycnal work to be much smaller than deformation work.

Similarly to the DR case, it is also interesting to analyze the spatially averaged terms $\langle \Pi \rangle_x$ and $\langle \Lambda \rangle_x$. The result is displayed in Fig. 9. Both panels exhibit different behaviors with respect to their DR counterparts. The left panel of the figure shows the deformation work, which has nearly the same behavior as the averaged dissipation term $\langle D_{wuu} \rangle_x$ in Fig. 5(a), however with smaller magnitudes. In contrast, the contours in Fig. 9(b) differ significantly in comparison to $\langle -D_{w\chi p} \rangle_x$. The mean baropycnal work oscillates in time between positive and negative values, depending on the location of the shocks, and has, in general, bigger magnitudes than the deformation work. We observe notable changes near $t \approx 0.34$ that coincide with shock interactions. This time instant is indicated by the

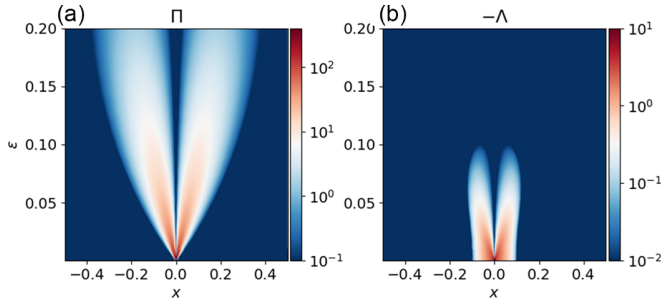


FIG. 8. (a) Deformation $\Pi(\varepsilon, x)$ and (b) baropycnal work $\Lambda(\varepsilon, x)$ near the shock-interaction region. The time is now $t \approx 0.34$ when both shocks meet at $x = 0$. Due to periodic boundary conditions, we continue the x axis to the left with negative values.

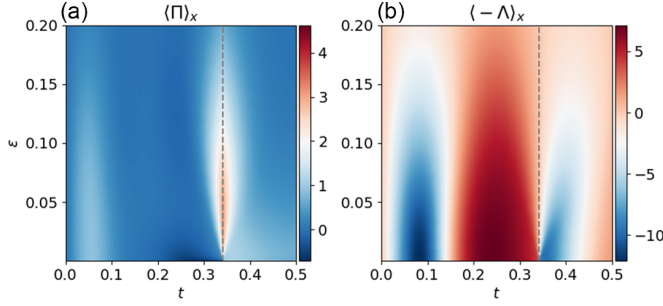


FIG. 9. Spatially averaged work terms vs time t and filter scale ε . (a) Deformation work $\langle \Pi \rangle_x$ and (b) baropycnal work $\langle -\Lambda \rangle_x$. The gray dashed line in both contour plots stands for the shock-interaction instant of $t \approx 0.34$.

gray dashed line in both figures. In contrast to deformation work, baropycnal work Λ changes sign at exactly this instant.

B. Polytropic flow case

After exploring the fully compressible one-dimensional Navier-Stokes case, i.e., the case with a full internal energy balance, we now consider a simplified flow case with a polytropic gas law which is given by

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \rho \partial_t u + \rho u \partial_x u &= -\partial_x p + \frac{4}{3} \mu \partial_{xx} u, \\ \frac{p}{p_0} &= \left(\frac{\rho}{\rho_0} \right)^\gamma. \end{aligned} \quad (25)$$

The former energy equation is now closed by a polytropic equation of state, which is often also referred to as an adiabatic gas law when the polytropic index γ equals the heat capacity ratio. We will stick with the notion of “polytropic” for the following. The dimensionless form of these equations, with the pressure substituted from the equation of state, can be written as

$$\begin{aligned} \partial_t \rho &= -\partial_x(\rho u), \\ \partial_t u &= -u \partial_x u - \frac{\gamma}{\text{Ma}^2} \rho^{\gamma-2} \partial_x \rho + \frac{4}{3 \rho \text{Re}} \partial_{xx} u. \end{aligned} \quad (26)$$

Figure 10 illustrates the difference between the present flow and the former fully compressible case with respect to time. A value of zero means that both solutions predict the same value at that point, while nonzero colors indicate where they deviate. The results show that the differences between the polytropic and fully compressible case are very small in most of the space-time domain; they are concentrated primarily around the shock waves. This also implies that in smooth flow regions, heat flux does not play a significant role. It can be seen that the difference at the shock locations is characterized by the same amplitudes; furthermore, the overall evolution remains the same. The only difference is the velocity of the shock-wave propagation. At later times, a gradual widening of the colored bands is detected, which shows the distance between the shock waves in the polytropic and fully compressible cases. As a result, the comparison in Fig. 10 indicates that using the present flow configuration has only a minor effect on the overall flow dynamics. The primary impact of heat fluxes is to slightly smooth and weaken the shocks. We have varied the Prandtl number between $\text{Pr} = 0.07$ and $\text{Pr} = 7.0$ and could not detect a significant impact of the variation of this parameter.

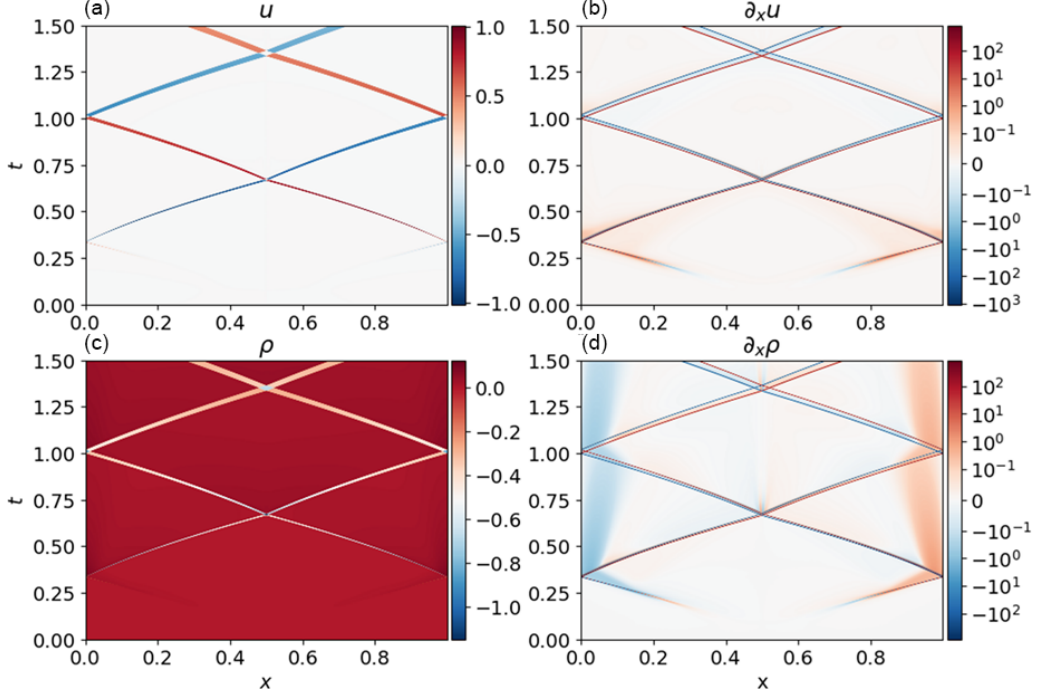


FIG. 10. Difference between the evolution of the polytropic and fully compressible flow cases at $Pr = 0.7$ for the simple initial condition (21). Each panel is a space-time diagram (x on the horizontal axis, t on the vertical one) for a specific variable. (a) Velocity u , (b) velocity gradient $\partial_x u$, (c) density ρ , and (d) density gradient $\partial_x \rho$.

It is also important to understand the influence of the polytropic flow conditions on the anomalous dissipation terms in DR. For this reason, we take the mean dissipation values corresponding to the gray lines in both panels of Fig. 5, and replot them together with the corresponding ones for the polytropic case in Fig. 11. Blue and orange circles correspond to the case of the polytropic flow, while green and red squares to the flow with $Pr = 0.7$. For larger coarse-grained scales

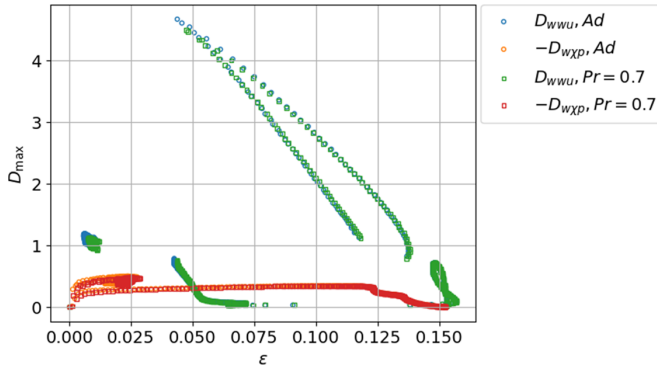


FIG. 11. Maxima of the DR mean dissipation terms for the polytropic (Ad) and the fully compressible ($Pr = 0.7$) flow cases, denoted $D_{\max}$, vs coarse-grained scale ε . Points denote the maxima at different time instants in the course of the evolution for the different coarse-grained scales.

$\varepsilon > 0.5$, there is almost no difference between the results (not shown). However, at smaller scales, the polytropic flow shows slightly higher dissipation at the same scale as the fully compressible flow. In the polytropic case, gradients cannot be smoothed by heat fluxes. As a result, the shocks are sharper and more localized, with steeper gradients in velocity and density. This results in greater roughness at small scales. The anomalous dissipation is sensitive to steep gradients and velocity increments $\sim \delta u^3$. Thus, we can conclude that the influence of adiabatic conditions in the one-dimensional case is nearly negligible, which means that it is possible to use these simplified conditions to analyze dissipation in the present one-dimensional flow example.

VI. DISSIPATION TERMS IN ANALYTICAL SHOCK CONFIGURATION

Neither of the reviewed cases can fully capture what happens to the dissipation terms in the limit where the coarse-graining scale approaches zero, $\lim_{\varepsilon \rightarrow 0} [D_{wuu}(\varepsilon, x) + D_{w\chi p}(\varepsilon, x) + D_{w\chi\tau}(\varepsilon, x)]$. To better understand the limiting behavior, we consider an idealized shocklike velocity profile that approximates real flow conditions, as illustrated in Fig. 1. It is important to note that the following shock profile is not an exact solution to the compressible Navier-Stokes equations. This analytical profile captures, however, a discontinuity in the velocity field, approximating two crossing shocks, as observed in our numerical simulations. The velocity $u(x)$ is defined piecewise linear over a periodic domain $x \in [-0.5, 0.5]$ as follows:

$$u = \begin{cases} A(2x + 1), & -0.5 \leq x \leq -0.5 + d \\ 2Ax, & -0.5 + d < x < 0.5 - d \\ A(2x - 1), & 0.5 - d \leq x \leq 0.5. \end{cases} \quad (27)$$

Here, A represents the amplitude of the shock wave, determining its strength, and d denotes the distance of the discontinuity from the periodic boundary. Now we can examine specific time instants, particularly when two shocks interact, such as at $t \approx 0.34$ in the numerical examples, or for $d = 0.5$ in this case. In the latter case, the middle segment of (27) vanishes and the profile simplifies to a single centered shock:

$$u = \begin{cases} A(2x + 1), & x < 0 \\ A(2x - 1), & x > 0, \end{cases} \quad (28)$$

which jumps from $u(0^-) = A$ to $u(0^+) = -A$ at $x = 0$. This individual shock solution can be also obtained from the inviscid limit of the Burgers equation, as discussed in Ref. [5]. At this moment, pressure and density variations are negligible, as it was shown before, so the dissipation is dominated by the velocity increment term $D_{wuu}(\varepsilon, x) \approx \rho_c D_{uuu}(\varepsilon, x)$ as $\rho \approx \rho_c$. Thus, we discuss D_{uuu} only for the following. The velocity increment $\delta u(x, \xi) = u(x + \xi) - u(x)$ for the piecewise linear shock profile becomes

$$\delta u = -u + \begin{cases} A(2(x + \xi) + 1), & \xi < -x \\ A(2(x + \xi) - 1), & \xi > -x. \end{cases} \quad (29)$$

The anomalous dissipation term can then be evaluated analytically as a convolution of the velocity increments with a Gaussian filter [see, again, Eq. (24)],

$$D_{uuu}(\varepsilon, x) = -\frac{1}{4\varepsilon^3} \int_{-\infty}^{+\infty} \xi \exp\left(-\frac{\xi^2}{2\varepsilon^2}\right) (\delta u)^3 d\xi. \quad (30)$$

At the exact location of the shock $x = 0$, the result simplifies to

$$D_{uuu}(\varepsilon, 0) = \frac{A^3}{2} \left(\frac{1}{\varepsilon} - 3\sqrt{2\pi} + 24\varepsilon - 12\sqrt{2\pi}\varepsilon^2 \right). \quad (31)$$

For small $\varepsilon \ll 1$, the first term inside the brackets dominates. This implies that an infinitely steep shock, a discontinuous velocity jump, produces a dissipation term that infinitely grows as $\varepsilon \rightarrow 0$,

$$\lim_{\varepsilon \rightarrow 0} D_{uuu}(\varepsilon, 0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} = \infty. \quad (32)$$

In other words, a shock with an infinitely sharp gradient shows an anomalous dissipation that tends to infinity for vanishing filter scale.

Away from the shock at $x \neq 0$ in the smooth regions, the dissipation results in the following respective functions to the left $x < 0$ and to the right $x > 0$ of the shock:

$$\frac{D_{uuu}}{A^3} = \frac{2 + 12\varepsilon^2}{\varepsilon} \exp\left(-\frac{x^2}{2\varepsilon^2}\right) - 6\sqrt{2\pi}\varepsilon^2 + 6\frac{x^2 \pm x}{\varepsilon} \exp\left(-\frac{x^2}{2\varepsilon^2}\right) - 3\sqrt{2\pi}\left[1 \pm \operatorname{erf}\left(\frac{x}{\sqrt{2}\varepsilon}\right)\right]. \quad (33)$$

One can verify that as $\varepsilon \rightarrow 0$, dissipation tends to zero for any fixed $x \neq 0$. Physically, the anomalous dissipation vanishes in regions where the flow is smooth (as it is expected). Meanwhile, at the shock position, it is extremely large in an infinitely thin region around $x = 0$. If we compute the mean dissipation over one full periodic region, the divergent contribution at the shock has infinitely small spatial width, yielding a finite average. Integrating $D(x)$ over the whole domain and taking the limit $\varepsilon \rightarrow 0$ gives an average dissipation of

$$\langle D_{uuu} \rangle_x = \frac{A^3}{2}, \quad (34)$$

which is finite and independent of coordinate x and filter scale ε . This simple example confirms that if local dissipation at the ideal shock is singular, the mean energy dissipation rate remains finite.

The other way to show finite energy dissipation is to use a different type of singularity for the velocity profile. Therefore, we consider a profile of the form

$$u(x) = A|x|^h, \quad (35)$$

where h is the Hölder exponent, characterizing the steepness of the velocity singularity. While this profile does not correspond to the shock-wave configuration discussed earlier, it serves a different purpose. First, in compressible flow, density and pressure fields show Hölder-type distribution near shocks. Thus, this profile offers a simplified way to show the influence of such singularities on kinetic energy dissipation. Second, the Hölder exponent is the threshold for the anomalous dissipation. It is known that finite anomalous dissipation occurs only when $h \leq 1/3$. This makes the Hölder-type velocity profile a natural test case for analyzing anomalous dissipation. At the point of singularity $x = 0$, its increment becomes

$$\delta u(x = 0, \xi) = A|\xi|^h. \quad (36)$$

As this form is symmetric, we cannot use the first-order Gaussian kernel, as this results to a trivial zero value. We switch to the second-order Gaussian test function. It is known as the “Mexican hat” function and is given by

$$\psi_2(\xi) = (1 - \xi^2) \exp(-\xi^2/2). \quad (37)$$

Substituting the velocity increment and test function into the dissipation integral results in

$$D_{uuu}(\varepsilon, 0) = \frac{A^3}{2\varepsilon^2} \int_0^\infty \left(1 - \frac{\xi^2}{\varepsilon^2}\right) \xi^{3h} \exp\left(-\frac{\xi^2}{2\varepsilon^2}\right) d\xi, \quad (38)$$

which leads to

$$D(\varepsilon, h) = \frac{3}{2} A^3 \frac{h}{(\sqrt{2}\varepsilon)^{1-3h}} \Gamma\left(\frac{1+3h}{2}\right), \quad (39)$$

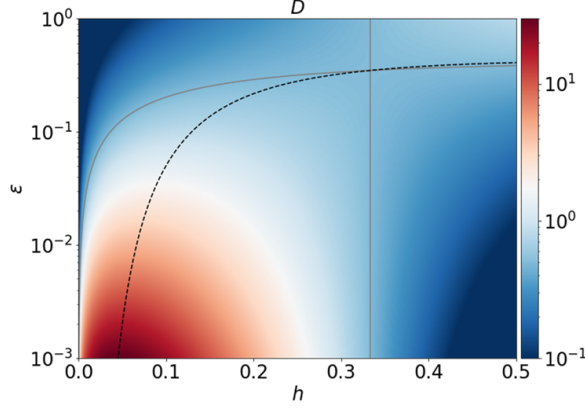


FIG. 12. Dissipation for Hölder-type profile of the velocity field $u(x)$. The gray line shows the constant dissipation $D = 1/2$ corresponding to the Hölder exponent $h = 1/3$. The black dashed line determines the maximum dissipation given by (40).

with the Gamma function $\Gamma(y)$ and $D_{uuu}(\varepsilon, x = 0; h) = D(\varepsilon, h)$. The resulting function $D(\varepsilon, h)$ is shown as a heat map in Fig. 12. The integral is finite in the limit $\varepsilon \rightarrow 0$ only when the Hölder exponent $h = 1/3$, leading to $D = A^3/2$; a Hölder exponent $h = 1/3$ of the velocity leads to a finite anomalous dissipation locally. For $h < 1/3$, we again receive local singularity. The smaller the h ($h > 0$), the stronger the divergence. The roughness parameter h for the strongest singularity can be determined through the maximum dissipation $d_h D(\varepsilon, h) = 0$. This results in

$$\varepsilon = \frac{1}{\sqrt{2}} \exp \left[-\frac{1}{3h_{\max}} - \frac{1}{2} \Gamma_p \left(\frac{1 + 3h_{\max}}{2} \right) \right], \quad (40)$$

where $\Gamma_p(y)$ is a Polygamma function. For the smoother velocity case of $h > 1/3$, the dissipation vanishes in the limit $\varepsilon \rightarrow 0$. If the velocity field is smoother than $|x|^{1/3}$ at $x = 0$, then

$$\lim_{\varepsilon \rightarrow 0} D(\varepsilon, h) = 0, \quad (41)$$

which implies that no anomalous energy dissipation occurs at the point of singularity. These findings are consistent with Onsager's theoretical criterion for energy conservation in ideal flows [6]. Onsager conjectured that a Hölder exponent of $h = 1/3$ is the critical threshold separating dissipative and nondissipative anomalous behavior in turbulent flows. However, if local dissipation is constant or even singular, an average over the full domain results in 0, as while the width decreases with $\sim \varepsilon$, the height increases with $\sim \varepsilon^{3h-1}$ for $h < 1/3$. It means that for any $h \in [0, 1)$, the average dissipation is $\langle D \rangle_x = 0$.

It is also possible to analyze integral (5a), $D_{www} \approx \rho_c D_{uuu}$, in a different way. Assuming that the velocity field near the origin $x \approx 0$ is Hölder-continuous and smooth away from $x \approx 0$, we split the integral into a small ball $B_R = \{x : |x| < R\}$ around the origin, where the velocity is just Hölder, and the rest are where it is smooth. Near the origin, Hölder regularity gives $\delta u \sim |x|^h \sim \varepsilon^h$, also $\partial \varphi^\varepsilon \sim \varepsilon^{-d-1}$. Integrating over ξ contributes a volume $\sim \varepsilon^d$. This results in the following bounds:

$$\int_{|x| < R} D_{uuu} dx \lesssim \varepsilon^{3h-1} R^d. \quad (42)$$

In the smooth zone, ∇u cannot blow up faster than R^{h-1} . For the increments, we obtain $\delta u \sim |\xi| |\nabla u| \lesssim \varepsilon R^{h-1}$. As a result, we obtain bounds for the smooth zone,

$$\int_{|x| > R} D_{uuu} dx \lesssim \varepsilon^2 R^{3(h-1)}. \quad (43)$$

Balancing these contributions results in

$$R = \varepsilon^\alpha \quad \text{with} \quad \alpha = \frac{3(1-h)}{3(1-h)+d}, \quad (44)$$

which leads to the following bounds of the deformation work:

$$\int_V D_{uuu} dx \lesssim \varepsilon^{3h-1+\alpha d}. \quad (45)$$

For the one-dimensional case ($d = 1$), we receive that D_{uuu} becomes a constant for $h = (2 - \sqrt{3})/3 \approx 0.089$. For $d \geq 2$ and $h \geq 0$, dissipation vanishes with the rate $3h - 1 + \alpha d$.

In the first part of this section, we found that during shock merging, coupling of different types of singularities can happen. This influences baropycnal work, which we want to discuss finally. We assume the following fields at this moment near $x \approx 0$:

$$w = \sqrt{\rho}u = \begin{cases} A(2x+1), & x < 0 \\ A(2x-1), & x > 0, \end{cases} \quad (46)$$

$$\chi = \frac{1}{\sqrt{\rho}} = a_\chi + b_\chi |x|^{h_1}, \quad (47)$$

$$p = a_p + b_p |x|^{h_2}. \quad (48)$$

Their increments at point $x = 0$ are given by

$$\delta w = \begin{cases} A(2\xi+1), & \xi < 0 \\ A(2\xi-1), & \xi > 0, \end{cases} \quad (49)$$

$$\delta \chi = b_\chi |\xi|^{h_1}, \quad (50)$$

$$\delta p = b_p |\xi|^{h_2}. \quad (51)$$

For this discussion, we again use the first-order test function, which results in

$$D_{w\chi p}(\varepsilon, 0, h_1, h_2) = -\frac{b_\chi b_p}{2\varepsilon^3} \int_{-\infty}^{+\infty} \xi \exp\left(-\frac{\xi^2}{2\varepsilon^2}\right) |\xi|^{h_1+h_2} \delta w \, d\xi. \quad (52)$$

This leads to

$$D_{w\chi p}(\varepsilon, 0, h_1, h_2) = Ab_\chi b_p \frac{1}{(\sqrt{2}\varepsilon)^{1-h_1-h_2}} \left[\sqrt{2}\Gamma\left(\frac{2+h_1+h_2}{2}\right) - 4\varepsilon\Gamma\left(\frac{3+h_1+h_2}{2}\right) \right]. \quad (53)$$

In the case of the same roughness of the pressure p and field χ , $h_1 = h_2 = h$, the dissipation term remains finite for fields with Hölder exponent $h = 1/2$, which is shown in a heat map in Fig. 13.

For the baropycnal work, we can again split the region into just Hölder for pressure and density and shock for velocity around the origin in the small ball B_R and the outer smooth region. The only change to the previous case is that the velocity gradient in the smooth region is bounded by $1/R$ and $\delta u \sim 1$ for $x \in B_R$,

$$\int_{|x|<R} D_{w\chi p} dx + \int_{|x|>R} D_{w\chi p} dx \lesssim \varepsilon^{2h-1} R^d + R^{2h-3} \varepsilon^2. \quad (54)$$

After balancing the contributions, baropycnal work is bounded from above by

$$\int_V D_{w\chi p} dx \lesssim \varepsilon^{2h-1+\beta d}, \quad \beta = \frac{2h-3}{2h-3-d}. \quad (55)$$

For the one-dimensional case, $D_{w\chi p}$ is constant for $h = (2 - \sqrt{3})/2 \approx 0.134$. For $d \geq 2$ and $h \geq 0$, this dissipation term vanishes with the rate $2h - 1 + \beta d$.

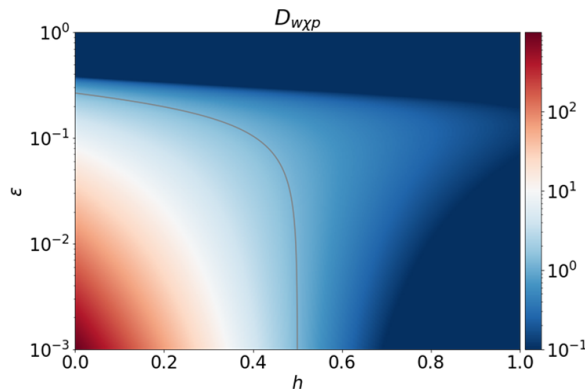


FIG. 13. Baropycnal dissipation term with the shock-type distribution of the velocity field and Hölder-type distribution of the pressure and density field. The gray line shows the constant dissipation $D_{w\chi p} = \sqrt{\pi/2}$ corresponding to the Hölder exponent $h = 1/2$.

VII. SUMMARY AND OUTLOOK

The purpose of this study was to identify mechanisms of anomalous dissipation for fully compressible turbulent flows. To this end, we extended the Duchon-Robert framework (DR) [14], which is originally formulated for incompressible flows, to the case of a fully compressible flow. We then considered one-dimensional fully compressible flow examples from gas dynamics, where the dissipation mechanism is essentially determined by the generation of shock waves and their crossings due to periodic boundary conditions.

The anomalous dissipation contributions, represented as different terms $D(\varepsilon, x)$, were determined within DR and compared with the framework that was suggested by Aluie (AL) in Ref. [24]. We could relate dissipation terms of DR and AL to each other, as summarized in Table I. The kinetic energy dissipation appears in the form of three additional integral terms in the local kinetic energy balance. While two terms, D_{wuu} and $D_{w\chi p}$, of DR can be directly identified as anomalous dissipation terms, the third one, $D_{w\chi\tau}$, contains a viscous stress contribution and thus does not represent an anomalous dissipation term.

The limit $\varepsilon \rightarrow 0$ of the coarse-grained or filter scale, following after the viscosity limit $\mu \rightarrow 0$, has to be taken to determine its magnitude. The dependency of $D(\varepsilon, x)$ on the finite filter scale ε and on the position x away from a shock position was investigated in two examples numerically, using a high-order compact finite-difference scheme. To this end, we considered two one-dimensional examples: a first case with an internal energy balance (termed the fully compressible case) and a second case for which the internal energy balance is closed by a polytropic gas law for the state variables, $p \sim \rho^\gamma$. The analysis showed that the results of the two models exhibit qualitatively (as well as quantitatively, to a good approximation) similar behavior for the anomalous dissipation terms. DR and AL terms for anomalous dissipation could be related to each other; similarities and differences, which arise from both frameworks, were discussed. Both frameworks started from different perspectives, as we have detailed in Sec. IV.

Moreover, we approximated the shock profiles in a simple analytical model for the discontinuity to describe the behavior of the anomalous dissipation terms within the inertial subrange. The model profile was guided by the numerical simulation results. For the simple piecewise profile, our discussion showed that the anomalous dissipation increases as the filtering scale $\varepsilon \rightarrow 0$. For the more singular shock structures, dissipation grows at the finer scales, while the mean kinetic energy dissipation rate remains finite. Considering a Hölder continuous profile, it was shown that the dissipation remains finite even locally for $h = 1/3$.

Possible extensions of the present study include (1) the enlargement of the consideration to two- and three-dimensional compressible flows, where shock surfaces and their interactions can result in higher orders of anomalous dissipation; and (2) the extension of the approach to fully compressible three-dimensional homogeneous isotropic turbulence, where additional dynamical mechanisms, such as the stretching of compressible vortices, generation due to baroclinic vorticity production, and shock-vortex interactions contribute to the energy cascade as well as dissipation. These studies are in progress and will be reported elsewhere.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [33].

APPENDIX A: DERIVATION OF THE COMPRESSIBLE FRAMEWORK

To apply the DR to compressible flow, the original NS equations (2) for w_i have to be rewritten in an alternative form that isolates terms involving density-pressure and density-velocity coupling,

$$\begin{aligned} \partial_t w_i + \partial_j(u_j w_i) - \frac{w_i \theta}{2} &= -\partial_i(\chi p) + \partial_j(\chi \tau_{ij}) + p \lambda_i - \tau_{ij} \lambda_j, \\ \chi &= \frac{1}{\sqrt{\rho}}, \\ \lambda_i &= \partial_i \left(\frac{1}{\sqrt{\rho}} \right). \end{aligned} \quad (\text{A1})$$

This reformulation facilitates the identification of terms contributing to energy flux and dissipation when convolution with a test function is performed, as detailed next.

The Duchon-Robert framework employs spatial filtering to analyze energy transfer across scales. Let $\varphi \in C^\infty(V)$ be an infinitely differentiable function with compact support in the flow volume V (which will be a 3-torus for simplicity) and normalized so that $\int_V \varphi dV = 1$. We then define the rescaled function for the spatial coordinates $\xi = (\xi_x, \xi_y, \xi_z)$ as

$$\varphi^\varepsilon(\xi) = \frac{1}{\varepsilon^3} \varphi\left(\frac{\xi}{\varepsilon}\right), \quad (\text{A2})$$

where $\varepsilon > 0$ serves as the filtering or coarse-grained scale. This procedure corresponds to a convolution as given by Eq. (3) in the main text. By denoting the convolutions $w_i^\varepsilon = \varphi^\varepsilon * w_i$, $(u_j w_i)^\varepsilon = \varphi^\varepsilon * (u_j w_i)$, $(w_i \theta)^\varepsilon = \varphi^\varepsilon * (w_i \theta)$ as well as $(\chi p)^\varepsilon = \varphi^\varepsilon * (\chi p)$, $(\chi \tau_{ij})^\varepsilon = \varphi^\varepsilon * (\chi \tau_{ij})$, $(p \lambda_i)^\varepsilon = \varphi^\varepsilon * (p \lambda_i)$, and $(\tau_{ij} \lambda_j)^\varepsilon = \varphi^\varepsilon * (\tau_{ij} \lambda_j)$, we can rewrite Eqs. (A1) in regularized form,

$$\partial_t w_i^\varepsilon + \partial_j(u_j w_i)^\varepsilon - \frac{(w_i \theta)^\varepsilon}{2} = -\partial_i(\chi p)^\varepsilon + \partial_j(\chi \tau_{ij})^\varepsilon + (p \lambda_i)^\varepsilon - (\tau_{ij} \lambda_j)^\varepsilon, \quad (\text{A3})$$

where superscript ε denotes convolution with φ^ε . This step regularizes the fields while preserving the structure of the NS equations, enabling a systematic analysis of scale interactions for scales smaller and larger than the filter scale ε . To derive the kinetic energy balance equation that combines w_i and w_i^ε , we multiply the original equation (A1) by w_i^ε and the regularized equation (A3) by w_i , then sum

both to obtain

$$\begin{aligned}
 & \partial_i(w_i w_i^\varepsilon) + \partial_j(u_j w_i w_i^\varepsilon + w_j^\varepsilon \chi p - w_i^\varepsilon \chi \tau_{ij}) \\
 &= -w_i \partial_j(u_j w_i)^\varepsilon + u_j w_i \partial_j w_i^\varepsilon + w_i \frac{(w_i \theta)^\varepsilon}{2} + \frac{w_i^\varepsilon w_i \theta}{2} \\
 &\quad - w_i \partial_i(\chi p)^\varepsilon + \chi p \partial_i w_i^\varepsilon + w_i (p \lambda_i)^\varepsilon + w_i^\varepsilon p \lambda_i \\
 &\quad + w_i \partial_j(\chi \tau_{ij})^\varepsilon - \chi \tau_{ij} \partial_j w_i^\varepsilon - w_i (\tau_{ij} \lambda_j)^\varepsilon - w_i^\varepsilon \tau_{ij} \lambda_j.
 \end{aligned} \tag{A4}$$

Furthermore, we can summarize some terms to

$$\begin{aligned}
 & \int_{\mathbb{R}^3} \partial_j \varphi^\varepsilon(\xi) \delta w_i \delta w_i \delta u_j d^3 \xi = -\partial_j(u_j w_i w_i)^\varepsilon + u_j \partial_j(w_i w_i)^\varepsilon - w_i w_i \partial_j u_j^\varepsilon \\
 &\quad + 2w_i \partial_j(u_j w_i)^\varepsilon - 2w_i u_j \partial_j w_i^\varepsilon, \\
 & \int_{\mathbb{R}^3} \partial_i \varphi^\varepsilon(\xi) \delta w_i \delta \chi \delta p d^3 \xi = -\partial_i(p w_i \chi)^\varepsilon + \chi \partial_i(p w_i)^\varepsilon + p \partial_i(w_i \chi)^\varepsilon - w_i \chi \partial_i p^\varepsilon - w_i p \partial_i \chi^\varepsilon \\
 &\quad - p \chi \partial_i w_i^\varepsilon + w_i \partial_i(p \chi)^\varepsilon, \\
 & \int_{\mathbb{R}^3} \partial_j \varphi^\varepsilon(\xi) \delta w_i \delta \chi \delta \tau_{ij} d^3 \xi = -\partial_j(\tau_{ij} w_i \chi)^\varepsilon + \chi \partial_j(\tau_{ij} w_i)^\varepsilon + \tau_{ij} \partial_j(w_i \chi)^\varepsilon - w_i \chi \partial_j \tau_{ij}^\varepsilon \\
 &\quad - w_i \tau_{ij} \partial_j \chi^\varepsilon - \tau_{ij} \chi \partial_j w_i^\varepsilon + w_i \partial_j(\tau_{ij} \chi)^\varepsilon,
 \end{aligned} \tag{A5}$$

with $\delta f(\xi) = f(\mathbf{x} + \xi) - f(\mathbf{x})$ the function increments over a distance vector ξ , and $f = (w, u, \chi, p, \tau_{ij})$. We will frequently use the shorthand notation of δf to ease the longer mathematical expressions. Substituting the last two terms from (A5) into (A4) leads to

$$\begin{aligned}
 & \partial_i(w_i w_i^\varepsilon) + \partial_j \left[u_j w_i w_i^\varepsilon + w_j^\varepsilon \chi p - w_i^\varepsilon \chi \tau_{ij} + \frac{(u_j w_i w_i)^\varepsilon}{2} + (p w_j \chi)^\varepsilon - (\tau_{ij} w_i \chi)^\varepsilon \right] \\
 &= -\frac{1}{2} \int \partial_j \varphi^\varepsilon(\xi) \delta w_i \delta w_i \delta u_j d^3 \xi - \int \partial_i \varphi^\varepsilon(\xi) \delta w_i \delta \chi \delta p d^3 \xi + \int \partial_j \varphi^\varepsilon(\xi) \delta w_i \delta \chi \delta \tau_{ij} d^3 \xi \\
 &\quad + \frac{1}{2} [u_j \partial_j(w_i w_i)^\varepsilon - w_i w_i \partial_j u_j^\varepsilon + w_i (w_i \theta)^\varepsilon + w_i^\varepsilon w_i \theta] \\
 &\quad + \chi \partial_i(p w_i)^\varepsilon + p \partial_i(w_i \chi)^\varepsilon - w_i \chi \partial_i p^\varepsilon - w_i p \partial_i \chi^\varepsilon + w_i (p \lambda_i)^\varepsilon + w_i^\varepsilon p \lambda_i \\
 &\quad - \chi \partial_j(\tau_{ij} w_i)^\varepsilon - \tau_{ij} \partial_j(w_i \chi)^\varepsilon + w_i \chi \partial_j \tau_{ij}^\varepsilon + w_i \tau_{ij} \partial_j \chi^\varepsilon - w_i (\tau_{ij} \lambda_j)^\varepsilon - w_i^\varepsilon \tau_{ij} \lambda_j.
 \end{aligned} \tag{A6}$$

With further transformations, the expression (A6) results in

$$\begin{aligned}
 & \partial_i(w_i w_i^\varepsilon) + \partial_j \left[u_j w_i w_i^\varepsilon + \chi (w_j^\varepsilon p + w_j p^\varepsilon) - \chi (w_i^\varepsilon \tau_{ij} + w_i \tau_{ij}^\varepsilon) \right] \\
 &\quad + \frac{1}{2} [(u_j w_i w_i)^\varepsilon - u_j (w_i w_i)^\varepsilon] + (p w_j \chi)^\varepsilon - \chi (p w_j)^\varepsilon - (\tau_{ij} w_i \chi)^\varepsilon + \chi (\tau_{ij} w_i)^\varepsilon \\
 &= -\frac{1}{2} \int \partial_j \varphi^\varepsilon(\xi) \delta w_i \delta w_i \delta u_j d^3 \xi - \int \partial_i \varphi^\varepsilon(\xi) \delta w_i \delta \chi \delta p d^3 \xi + \int \partial_j \varphi^\varepsilon(\xi) \delta w_i \delta \chi \delta \tau_{ij} d^3 \xi \\
 &\quad + \frac{1}{2} [-(w_i w_i)^\varepsilon \theta + w_i (w_i \theta)^\varepsilon + w_i (w_i^\varepsilon \theta - w_i \partial_j u_j^\varepsilon)] \\
 &\quad - (p w_i)^\varepsilon \lambda_i + w_i (p \lambda_i)^\varepsilon + p (w_i^\varepsilon \lambda_i - w_i \partial_i \chi^\varepsilon) + p \partial_i(w_i \chi)^\varepsilon + p^\varepsilon \partial_i w_i \chi \\
 &\quad + (\tau_{ij} w_i)^\varepsilon \lambda_j - w_i (\tau_{ij} \lambda_j)^\varepsilon - \tau_{ij} (w_i^\varepsilon \lambda_j - w_i \partial_j \chi^\varepsilon) - \tau_{ij} \partial_j(w_i \chi)^\varepsilon - \tau_{ij}^\varepsilon \partial_j w_i \chi.
 \end{aligned} \tag{A7}$$

In the limit of $\varepsilon \rightarrow 0$, many terms in (A7) compensate each other. The term $(w_i w_i^\varepsilon)/2$ converges, for example, to the kinetic energy E . In this limit, we obtain the local kinetic energy balance equation,

$$\partial_t E + \partial_j (u_j E) = u_i \partial_j \tau_{ij} - u_j \partial_j p - \lim_{\varepsilon \rightarrow 0} (D_{w w u} + D_{w \chi p} - D_{w \chi \tau} + \mathcal{D}^\varepsilon). \quad (\text{A8})$$

This is Eq. (4) from Sec. II of the main text.

APPENDIX B: DETAILS OF THE NUMERICAL SCHEME

For time integration, we use the classic explicit fourth-order Runge-Kutta (RK4) method. This scheme evaluates the right-hand side four times, k_1, k_2, k_3, k_4 , to achieve overall fourth-order accuracy in time,

$$\begin{aligned} u_i^{n+1} &= u_i^n + \frac{\Delta t}{6} (k_{1u} + 2k_{2u} + 2k_{3u} + k_{4u}), & \rho_i^{n+1} &= \rho_i^n + \frac{\Delta t}{6} (k_{1\rho} + 2k_{2\rho} + 2k_{3\rho} + k_{4\rho}), \\ k_{1u} &= f_u(t^n, u^n, \rho^n), & k_{1\rho} &= f_\rho(t^n, u^n, \rho^n), \\ k_{2u} &= f_u\left(t^n + \frac{\Delta t}{2}, u^n + \frac{k_{1u}}{2}, \rho^n + \frac{k_{1\rho}}{2}\right), & k_{2\rho} &= f_\rho\left(t^n + \frac{\Delta t}{2}, u^n + \frac{k_{1u}}{2}, \rho^n + \frac{k_{1\rho}}{2}\right), \\ k_{3u} &= f_u\left(t^n + \frac{\Delta t}{2}, u^n + \frac{k_{2u}}{2}, \rho^n + \frac{k_{2\rho}}{2}\right), & k_{3\rho} &= f_\rho\left(t^n + \frac{\Delta t}{2}, u^n + \frac{k_{2u}}{2}, \rho^n + \frac{k_{2\rho}}{2}\right), \\ k_{4u} &= f_u(t^n + \Delta t, u^n + k_{3u}, \rho^n + k_{3\rho}), & k_{4\rho} &= f_\rho(t^n + \Delta t, u^n + k_{3u}, \rho^n + k_{3\rho}). \end{aligned} \quad (\text{B1})$$

Detailed descriptions of RK methods can be found in textbooks on numerical differential equations, such as in Ref. [34].

For spatial derivatives, we apply a high-order compact finite-difference scheme. Compact schemes couple neighboring points to compute derivatives. In our case, an eighth-order compact scheme is used [35], which means a tridiagonal system has to be solved to find the derivative values at each point,

$$\begin{aligned} \beta_c C_{i+2} + \alpha_c C_{i+1} + C_i + \alpha_c C_{i-1} + \beta_c C_{i-2} &= c_c \frac{u_{i+3} - u_{i-3}}{6\Delta} + b_c \frac{u_{i+2} - u_{i-2}}{4\Delta} + a_c \frac{u_{i+1} - u_{i-1}}{2\Delta}, \\ \beta_v V_{i+2} + \alpha_v V_{i+1} + V_i + \alpha_v V_{i-1} + \beta_v V_{i-2} &= c_v \frac{u_{i+3} - 2u_i + u_{i-3}}{9\Delta^2} + b_v \frac{u_{i+2} - 2u_i + u_{i-2}}{4\Delta^2} \\ &\quad + a_v \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta^2}. \end{aligned} \quad (\text{B2})$$

Here, C and V refer to the first- and second-order derivatives, respectively. Constants in the scheme are defined as

$$\begin{aligned} a_c &= \frac{25}{16}, & b_c &= \frac{1}{5}, & c_c &= -\frac{1}{80}, & \alpha_c &= \frac{3}{8}, & \beta_c &= 0, \\ a_v &= \frac{147}{152}, & b_v &= \frac{51}{95}, & c_v &= -\frac{23}{760}, & \alpha_v &= \frac{9}{38}, & \beta_v &= 0. \end{aligned}$$

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