

Twin satellites and ring bubbles from coalescing magnetically levitated air bubbles in water

N. Sampara ¹, G. Hunter-Brown ¹, K. A. Baldwin ², M. M. Scase ³, and R. J. A. Hill ^{1,*}

¹*School of Physics and Astronomy, [University of Nottingham](#), Nottingham NG7 2RD, United Kingdom*

²*School of Science and Technology, [Nottingham Trent University](#), Nottingham NG11 8NS, United Kingdom*

³*School of Mathematical Sciences, [University of Nottingham](#), Nottingham NG7 2RD, United Kingdom*



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We investigate the ejection of satellite bubbles during the coalescence of two air bubbles in water. Using magnetic levitation we suspend the bubbles, with diameters in the range 3–15 mm, in a stable, neutrally buoyant state, free from mechanical constraints. Our experiments reveal the simultaneous ejection of twin satellite bubbles from the poles of the bubble formed from the coalescence of equal-sized precursors. More generally, we find that the number and size of the ejected satellites are dependent on the initial size ratio of the precursor bubble pair. Numerical simulations, which agree well with our experiments, show that satellite production is governed by the combined effect of two distinct processes: the arrival of a capillary wave crest at a pole and the rapid retraction of that polar region. We propose a timing model that successfully predicts the range of size ratios for which single, double, or no satellites are formed. Our analysis also offers insight into the absence of satellite ejection in similarly sized liquid drop coalescence. Furthermore, we report the experimental observation of toroidal bubble ejection from precursor bubbles with approximately 2:1 diameter ratios, and we show that the formation of these toroidal bubbles is also governed by our timing model.

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I. INTRODUCTION

When two air bubbles in water coalesce, the reduction in surface area releases excess energy that is transformed into kinetic energy. This energy drives shape oscillations on the resultant, larger bubble, which are ultimately dissipated through viscous damping and the production of sound waves [1]. In some cases, these large-amplitude oscillations can pinch off a small satellite bubble [2]. Aside from fundamental interest, understanding the process of bubble coalescence is important for several technologies involving two-phase reactions. For example, the mass transfer rate in a CO₂ bubble column reactor, a nascent carbon capture technology, is dependent on the bubbles' size distribution, which is strongly influenced by coalescence [3] (see also Ref. [4] for other examples). Electrochemical systems are also significantly affected by bubble coalescence, which can enhance mass transfer through the stirring of the liquid around the coalescing bubbles [5], or inhibit it through the aggregation of large bubbles near electrodes, which is of pressing concern in new energy storage technologies [6].

*Contact author: richard.hill@nottingham.ac.uk

The ejection of satellite bubbles during coalescence was investigated by Ohnishi, Azuma, and Straub [2]. They compared results obtained from numerical modeling with those from experiments conducted in a drop tower, although the experiments did not produce quantitative data owing to limited control over the bubbles and the relatively short duration of the experiment. Later experiments by Tse *et al.* [7], conducted in a bubble column, identified the source of the ejected bubbles as an annular ripple that travels around the bubbles away from the point of coalescence. Further extensive experiments have since been performed under controlled conditions in which one or both of the bubbles was stationary but not free, either colliding one bubble with another attached to the end of a thin tube [8,9] or with both bubbles in contact with a surface [10] or needle [11].

Here, we study the coalescence of two air bubbles, with diameters in the range 3 to 15 mm, suspended freely and in neutral buoyancy using magnetic levitation. As we have recently demonstrated, this technique holds the bubbles in stable equilibrium without significantly deforming their shape [12]. This approach is free from mechanical constraints that can otherwise influence the dynamics, allowing us to realize a close approximation of the idealized problem: two initially stationary bubbles in hydrostatic equilibrium in the absence of gravity, completely surrounded by liquid, and contacting each other at a point.

Coalescence of two equal-sized bubbles may produce “twin” equal-sized satellite bubbles that are ejected simultaneously, one from each pole of the resultant bubble. Previous studies have suggested [8,10] that two satellite bubbles may be produced from a single coalescence, and indications of this have been noted in rising bubbles [7]. However, this is the first time, to our knowledge, that the simultaneous ejection of two satellites has been experimentally observed from two initially free and stationary precursor bubbles. In comparison, experiments have demonstrated that coalescence of drops does not produce the simultaneous, symmetric twin satellite droplets in the manner observed for bubbles here [13]. More generally, we find that varying the radius ratio of the initial bubble pair away from unity changes the number of satellites ejected and their radii relative to the precursor bubbles. Here we address the question of why this is the case and quantify and characterize the behavior. While previous work has noted this dependence [2,8], a complete model has remained elusive. Some have speculated that airflow through the growing neck connecting the bubbles may be responsible [8], while others have invoked energy-based arguments [2], but these do not adequately describe our observations. We find that the observed dependence can be explained in terms of the coincidence of two events, each arising from a distinct process with its own timescale: the arrival of a capillary wave crest at the poles of the resultant bubble and the onset of large-scale retraction of the polar regions.

II. EXPERIMENTAL METHOD

Stable magnetic levitation of diamagnetic materials including water and organic substances was first demonstrated by Beaugnon and Tournier using a vertical solenoid magnet [14]. Subsequently, this technique has been used to study free diamagnetic liquid drops, see, e.g., Refs. [15–19]; drops immersed in a paramagnetic liquid may also be trapped using permanent magnets [20]. In the vertical solenoid, stable levitation occurs at a point on the solenoid’s axis above its center [21,22]. Less well-known is the possibility of stable magnetic “levitation” of bubbles of gas in a paramagnetic liquid below the center of the solenoid, first demonstrated in bubbles of oxygen gas in liquid oxygen at cryogenic temperatures [23,24]. Recently we demonstrated the room temperature suspension of 6 mm-diameter air bubbles in water, with the addition of a small amount of a paramagnetic salt [12]. A significant advantage of this method of levitation is that the magnetic forces that trap a room temperature air bubble in place do not distort the bubble shape significantly [12].

We used a custom-built superconducting solenoid magnet (Cryogenic Ltd., London) to levitate air bubbles in a weakly paramagnetic aqueous solution of manganese chloride. This magnet has a 58-mm-diameter, room temperature vertical bore, into which we fitted a 1 liter cylindrical tank filled with the paramagnetic solution of concentration $0.110 \pm 0.002 \text{ mol l}^{-1}$ [25], with a volume magnetic susceptibility of $\chi = (1.15 \pm 0.04) \times 10^{-5}$ (SI units), calculated using Curie’s law [26].

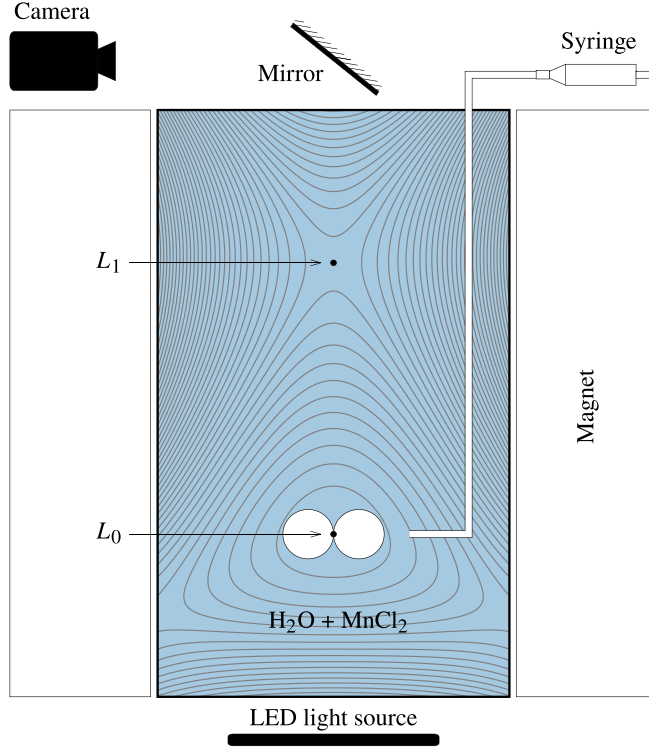


FIG. 1. Schematic of the experimental setup used to levitate two air bubbles and observe their coalescence (not to scale). The setup consists of a tank filled with a weakly paramagnetic aqueous solution of manganese chloride (shown in blue) inserted into the bore of the superconducting magnet. Grey contours indicate equipotential surfaces of the magnetogravitational potential E_{MG} of an air bubble in the liquid. Air bubbles reach stable equilibrium at the local minimum in this potential (denoted L_0). The system is lit from below using an LED light source and observed through a mirror. Images were recorded using a high-speed camera.

The density of this solution is $D = 1009.6 \pm 0.7 \text{ kg m}^{-3}$ [27,28]. The uncertainties in susceptibility and density arise from uncertainties in both the solution concentration and temperature, $21 \pm 3^\circ \text{C}$. The viscosity of the solution is approximately 1.04 times that of water, based on experimental data in Ref. [29]. Air bubbles were injected into the liquid using a 10 ml syringe connected to a stainless steel tube with internal diameter 1.0 mm.

The magnetogravitational potential E_{MG} describes the combined magnetic and gravitational energy of a bubble in the paramagnetic fluid, and depends on the differences in magnetic susceptibility and density between the bubble and the surrounding liquid, as well as the magnetic field strength and vertical position. For a full definition see Ref. [12]. The bubbles were created close to a point L_0 , where E_{MG} has a local minimum, corresponding to a point of stable equilibrium for an air bubble (see Fig. 1). At L_0 , the magnetic force balances buoyancy, and the liquid is also effectively weightless. For small displacements away from L_0 , the magnetogravitational force on a bubble, $-\nabla E_{MG}$, increases in magnitude approximately linearly with distance. Since this force also increases with the volume of the bubble [12], the bubble detaches from the end of the tube once it reaches a critical size, determined by the balance between surface tension and magnetogravitational force. We can thus control the size of the bubble released by varying the point at which it is formed relative to L_0 . (A second point L_1 exists where the magnetic force balances buoyancy, also shown in Fig. 1, but this location corresponds to an unstable equilibrium for air bubbles.)

For each experiment, two bubbles were injected into the liquid using the syringe (see Fig. 1). The first bubble came to rest at L_0 after release from the tube. The second bubble settled against the first but did not immediately coalesce. To control their relative orientation, we adjusted the solenoid current to modify the shape of the magnetogravitational equipotential surfaces (E_{MG}) [17]. In this study, the current was set to produce slightly oblate equipotentials close to L_0 , causing the two bubbles to settle horizontally on either side of L_0 . The liquid film separating the two bubbles ruptured typically 3–4 s after contact, initiating the coalescence. The process was recorded at up to 9000 frames per second using a high-speed Phantom camera through a window in the top of the tank. The coalescence of different pairs of bubbles was observed, each with radii of a few millimetres, and with varying size ratios.

The interfacial tension of the bubbles, γ , was determined from the angular frequency ω of small-amplitude oscillations of the fundamental ($l = 2$) shape mode of a single levitated bubble, using the relation derived by Lamb [30] for a bubble of radius R in a liquid of density D : $\omega^2 = (l + 1)(l - 1)(l + 2)\gamma/(DR^3)$. The Ohnesorge number, defined using the dynamic liquid viscosity μ as $Oh = \mu/\sqrt{D\gamma R}$, was between 2×10^{-3} and 4×10^{-3} in these experiments, indicating that viscous forces are small relative to inertial and surface tension forces.

III. NUMERICAL SIMULATIONS

Direct numerical simulations of the multiphase Navier-Stokes equations were carried out using the open source volume-of-fluid solver Basilisk [31,32]. The simulation was performed on an axisymmetric domain to reduce computational cost. A relatively large domain with side lengths of $18R$, where R is the radius of the smaller of the two simulated precursor bubbles, was used to minimize the influence of boundary effects on the flow field around the coalescing bubbles. An adaptive quad-octree mesh refinement scheme [31] was used to increase the mesh resolution at the bubble interface and also in areas of relatively large velocity, with a lower limit on the cell size of $0.0044R$. Such high local resolution was required to accurately model the flows at points where the interface has a small radius of curvature, particularly in the region of the neck between the two precursor bubbles and in the region of the neck connecting the resultant bubble to the satellite immediately prior to its ejection. Further details are given in Ref. [12].

Gravitational and magnetic forces are neglected in these simulations. This is justified by our previous analysis [12], which demonstrated that these forces are negligible compared to surface tension, resulting in static deformations of less than 0.05 % of the bubble radius. Consequently, these effects are negligible compared to the capillary and inertial forces governing the coalescence.

IV. EXPERIMENTAL RESULTS

Figure 2 shows the coalescence of two magnetically levitated equal-sized, 7.8-mm-diameter air bubbles (hereafter referred to as “precursor bubbles”) into a single 9.8 mm-diameter bubble (termed the “resultant bubble”); a high-speed video of the coalescence is available in the Supplemental Material [33]. Once started, coalescence proceeds first with the formation of a narrow neck between the two precursor bubbles, which expands rapidly in diameter ($t = 0.0$ to 4.0 ms in Fig. 2). As observed previously (e.g., Refs. [8,10,11]), the neck growth generates an axisymmetric capillary wave train that travels ahead of the growing neck region away from the point of contact. At later times, the resultant bubble takes the form of two spherical caps joined by a now-widened neck ($t = 8.0$ ms). The spherical caps shrink in area toward the two poles while maintaining approximately their initial radius of curvature. The ripples traveling over the spherical caps produce oscillations at the poles. When the final crest of these ripples reaches each pole, it forms a neck ($t = 26.0$ ms). This neck then pinches off, releasing two satellite bubbles, one from each pole ($t = 27.0$ ms). As the satellite bubbles are ejected, the poles of the resultant bubble retract quickly toward its center ($t = 27.0$ – 29.0 ms).

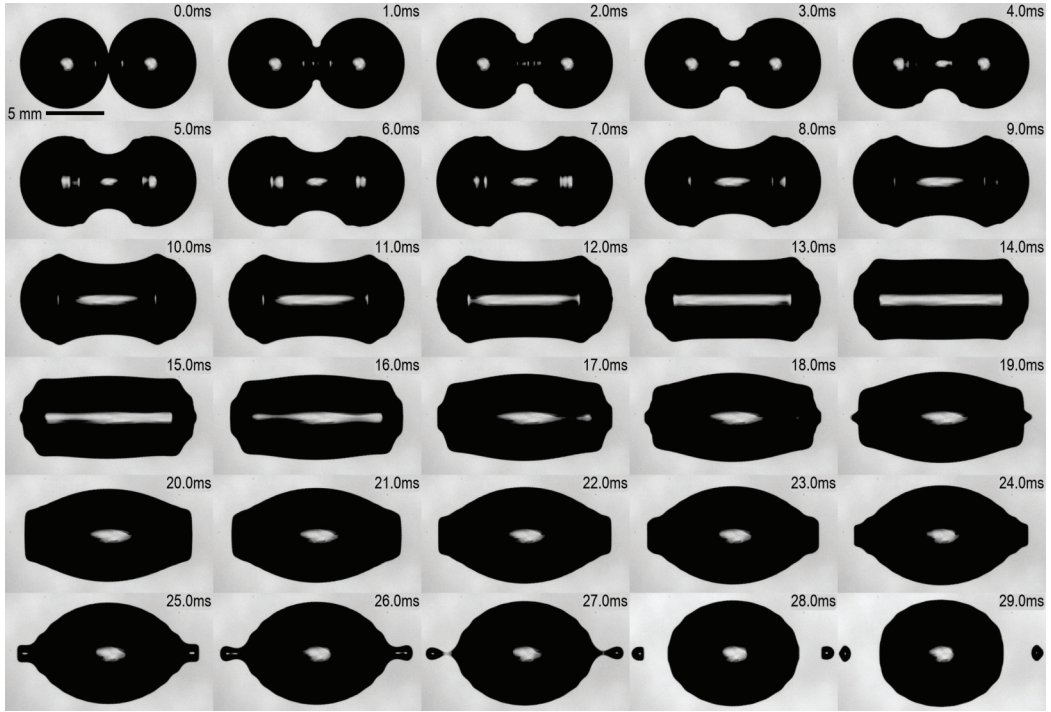


FIG. 2. Coalescence of two magnetically levitated, equal-sized air bubbles in water. (0.0ms) The two 7.8 mm precursor bubbles immediately prior to coalescence. (0.0–25.0 ms) The bubbles coalesce: capillary waves traveling away from the point of coalescence cause the two poles of the resultant bubble to oscillate. (25.0–29.0 ms) The final peak to arrive at each pole forms a protrusion which develops a neck that thins and breaks releasing a satellite bubble. At the same time, the polar region retracts sharply toward the bubble's center. See also Supplemental Material video [33].

Figure 3 demonstrates how the number and diameter of the ejected satellite bubbles varies with the diameter ratio of the precursor bubbles, from 1:1 (column a) to 1:2.11 (column e). The images in column a are from the same experiment as those shown in Fig. 2, and are included for comparison. The images in column b show the coalescence of two bubbles with diameters differing by a factor of 1.05. In this case, the neck growth and capillary wave propagation proceeded similarly to the coalescence of equal-sized bubbles. However, the satellite bubbles produced were unequal in size. The larger satellite bubble was ejected from the side corresponding to the smaller of the two precursor bubbles and was ejected first. For precursors with diameter ratios of 1:1.22 only one satellite was produced, from the pole corresponding to the smaller of the two precursor bubbles, Fig. 3(c). No satellite bubbles were observed from the coalescence of precursors with radii 1:1.63, Fig. 3(d), though an internal droplet is produced at 23.0 ms (see later discussion). We also observed other phenomena in coalescences of bubbles with larger diameter ratios. Figure 3(e) shows the ejection of a ring bubble from the coalescence of two bubbles with diameters in the ratio 1:2.11. A magnified view of the pole is shown in Fig. 4(a): after an isolated bubble has been ejected at approximately 8.5 ms, a ring bubble can be seen detaching from the pole between 10.00 and 10.25 ms. The top and bottom sections of the ring bubble (the sections intersecting the horizontal plane) appear as two black spots in these images. Both ring and satellite move away from the pole at approximately the same speed. Toroidal bubble ejection during bubble coalescence has been observed before, but under significantly different conditions: Zhang *et al.* [9] studied the coalescence of xenon bubbles in water with a much greater size difference—approximately a 1:30 diameter

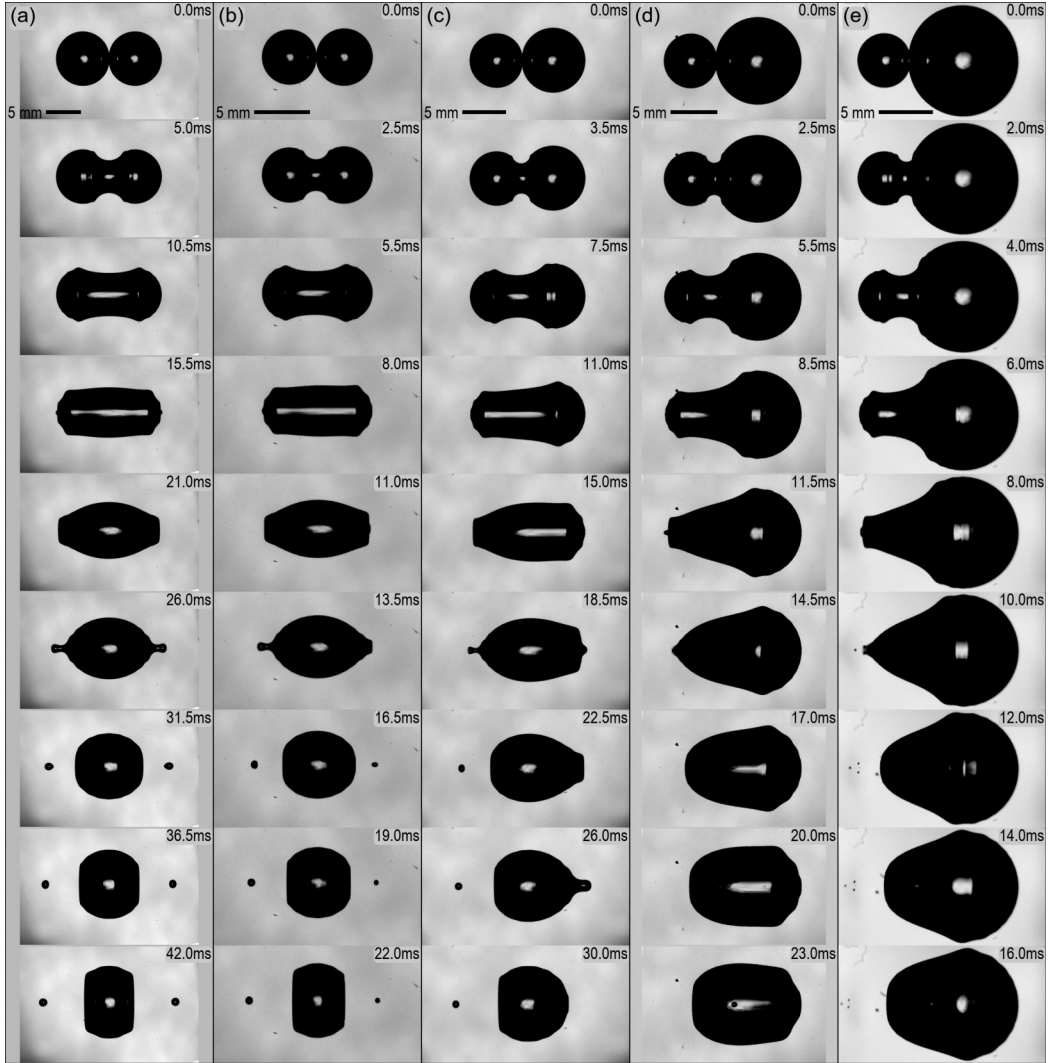


FIG. 3. The outcome of coalescence depends on the initial precursor bubble diameter ratio. (a) 1:1 ratio: double satellite ejection. (b) 1:1.05 ratio: double, unequal satellite ejection. (c) 1:1.22 ratio: single satellite ejection. (d) 1:1.63 ratio: no satellite ejection. (e) 1:2.11 ratio: bubble ring ejection.

ratio—and an inner-to-outer fluid density ratio of 3.3%, which was achieved by pressurizing the gas.

In both the 1:1.63 and 1:2.11 coalescences, we also observe the ejection of liquid *droplets* from the pole of the smaller precursor bubble, into the bubble's interior, in common with previous investigations [2,8]. The continuously changing refraction of the light through the oscillating bubble allows for some moments where this droplet can be seen to be traveling along the axis of the bubble toward the opposite pole, as seen in the series of images taken from the 1:1.63 coalescence shown in Fig. 4(b), but we cannot obtain an accurate measurement of the droplet's radius from these images. Usually, the droplet rebounds internally from the opposite pole. However, in the case shown in Fig. 4(b), the droplet penetrates the bubble wall. As it exits, it takes with it a thin layer of air, forming a short-lived (0.5 ms) “antibubble”—a liquid core within a gas shell—which subsequently bursts leaving a small satellite air bubble, Fig. 4(c).

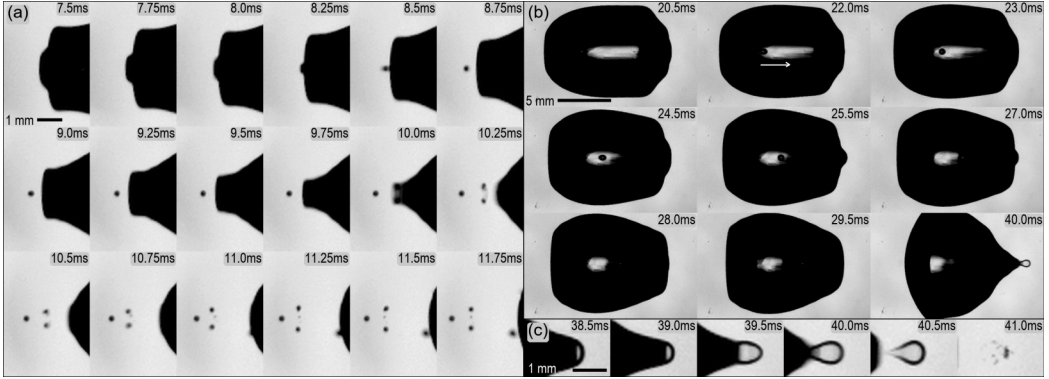


FIG. 4. Magnified views of ejection phenomena shown in Fig. 3. (a) The ejection of a ring bubble, which is preceded by the ejection of a smaller, secondary spherical satellite. (b) An internal liquid droplet, ejected from the left pole, travels across the bubble's interior. (c) Upon reaching the opposite pole, the droplet penetrates the bubble wall, emerging into the liquid as a short-lived (~ 0.5 ms) “antibubble”—a liquid droplet encapsulated by a thin gas shell.

Figure 5(a) compares the numerically simulated bubble shapes with experimental images for the coalescence of equal-sized bubbles, demonstrating good agreement in general, and reproducing the simultaneous pinch-off of the satellite bubbles from each pole. In these images, time is presented in nondimensional units, t/τ , where $\tau = \sqrt{DR^3/\gamma}$ is the capillary time of the precursor bubbles [30], of radius R . Superficially, there appear to be differences between the numerical and experimental

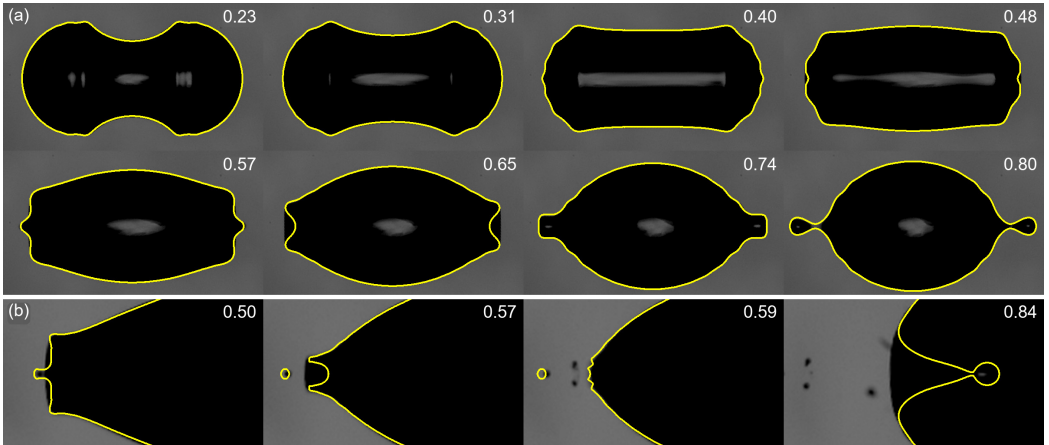


FIG. 5. Comparison of bubble shapes obtained from experiment (grey-scale images) and simulation (superimposed yellow contours). The figures top right in each frame indicate the nondimensional time t/τ , where τ is the capillary time of the precursor bubbles. (a) Coalescence of equal-sized bubbles. The simulation shows good agreement with the experiment, capturing the reentrant features at the poles (e.g., at $t/\tau = 0.48$ and 0.65) that are obscured from view by the foreground bubble surface in the experimental images. The experimental images are from the same experiments as those shown in Figs. 2 and 3(a). (b) Coalescence of precursor bubbles with a 1:2.11 diameter ratio, focusing on the left pole. The simulation reveals details obscured in the experiment: a small secondary satellite is ejected from a peak at the pole ($t/\tau = 0.50$); a toroidal bubble is formed from the rim of a bowl-like structure ($t/\tau = 0.57$); and an internal droplet is ejected from the tip of a reentrant liquid spike ($t/\tau = 0.84$). Here, τ is the capillary time of the smaller precursor. The images are from the same experiments as those shown in Figs. 3(e) and 4(a).

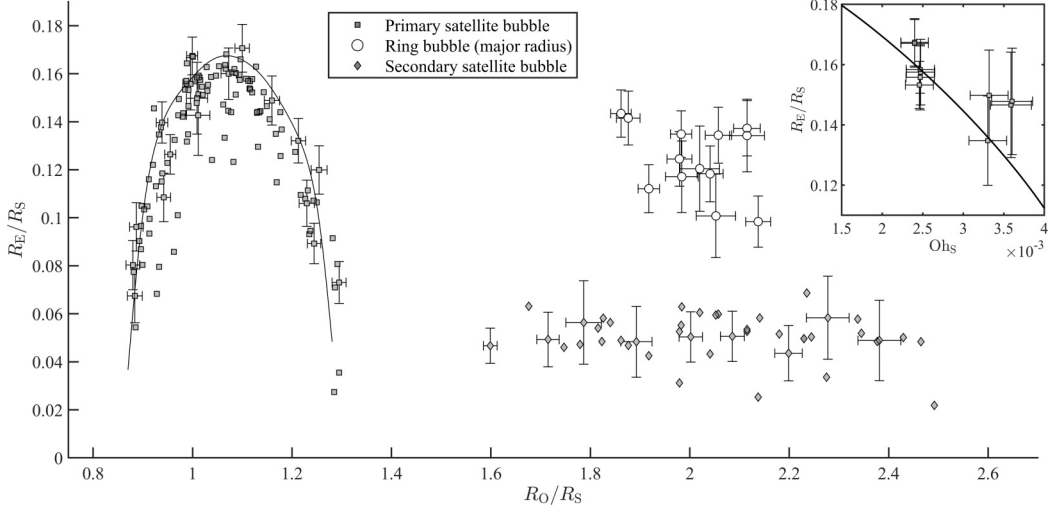


FIG. 6. Normalized satellite bubble radius (R_E/R_S) as a function of the precursor bubble radius ratio (R_O/R_S). Experimental data includes primary satellite bubbles ejected in the range $0.85 < R_O/R_S < 1.30$ ($\square$), smaller secondary satellite bubbles ejected in the range $1.60 < R_O/R_S < 2.50$ ($\diamond$) and ring bubbles ejected in the range $1.85 < R_O/R_S < 2.15$ ($\circ$). For clarity, error bars are shown for a representative subset of the data; the uncertainties for the remaining points are of a similar magnitude. Inset: The dependence of the satellite size on the Ohnesorge number of the source precursor bubble, for a 1:1 coalescence. Solid lines shows the results from numerical simulation.

results around the two poles in some of the images, for example in the fourth and sixth images ($t/\tau = 0.48$ and $t/\tau = 0.65$, respectively). This occurs because the camera briefly loses sight of the poles as they are hidden from view behind the surrounding surface of the bubble. Figure 5(b) compares the numerically simulated bubble shapes with experimental images for the 1:2.11 coalescence. Here, the simulation reveals the origin of the bubble ring. The region of the surface surrounding the pole of the bubble is obscured from view in the video images at the moment that the bubble ring is produced; simulation reveals that the pole forms a sharply peaked bowl-like structure at this moment (second image $t/\tau = 0.57$; τ here is the capillary time of the smaller precursor). Although this simulation is not able to reproduce the pinch-off, it is clear from comparing the experimental images with the simulation results that the bubble ring is produced from the rim of this bowl (second and third images $t/\tau = 0.57, 0.59$). The structure from which the preceding small bubble is produced is similarly partially hidden from view in experiment (first image $t/\tau = 0.50$). Simulation shows that the bubble is formed from the pinch-off of a small peak that forms at the pole, within a shallow bowl. The internal droplet is ejected as the pole retracts following the ejection of the bubble ring. Simulation shows that as the pole retracts in this case, a narrow spike of liquid forms, growing rapidly inward toward the center of the bubble, hidden from view of the camera (fourth image $t/\tau = 0.84$). The tip of this cone subsequently pinches off to leave a free droplet within the bubble.

The various bubble ejection phenomena we observe occur within specific ranges of the precursor bubble radius ratio. We define R_S as the radius of the precursor bubble that is the source of the ejected satellite (i.e., the left precursor for leftward ejected satellites and the right precursor for rightward ejected satellites, as viewed by the camera) and R_O as the radius of the opposite precursor bubble. Figure 6 shows the results from all experiments, plotting the normalized radius of the ejected bubble, R_E/R_S , against the precursor radius ratio, R_O/R_S . The plot includes data from satellites ejected both to the left and right and shows data for the primary satellites [examples of which are shown in Figs. 3(a)–3(c)], and the smaller secondary satellites and bubble rings [e.g., Fig. 4(a)]. In the case of the spherical satellite bubbles, R_E was measured after the satellite's shape oscillations had decayed

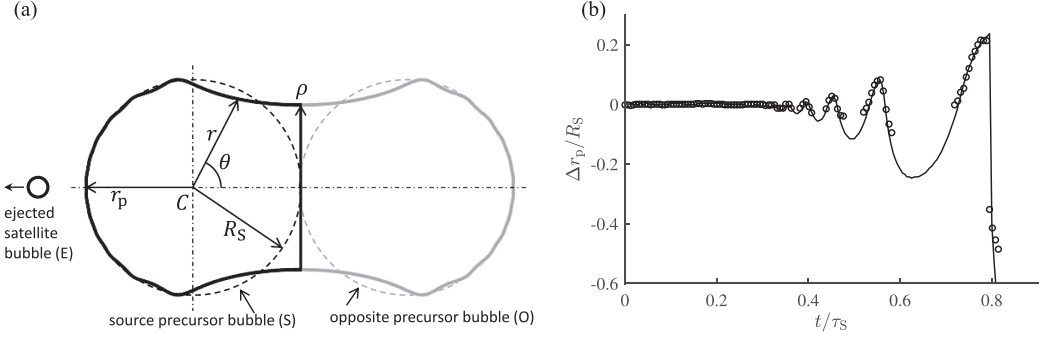


FIG. 7. (a) Schematic defining the coordinate system and geometric parameters used in the analysis, including the source polar radius, r_p , and the neck radius, ρ , which is measured at the original contact point between the two precursor bubbles. The thick dark line indicates the portion of the resultant bubble used in the analysis of the spherical surface harmonics. (b) Oscillations of the source pole in a 1:1 coalescence. The plot shows the normalized displacement of the source pole, $\Delta r_p/R_S = r_p/R_S - 1$, versus time normalized by the capillary time of the source precursor bubble, τ_S . Experimental data (markers) are compared with results from the numerical simulation (solid line).

away, approximately 3 ms after its ejection; for the bubble rings, R_E is the major radius of the torus, measured immediately after ejection. The error bars reflect the uncertainty in measuring the bubble radii from the video images, which is primarily limited by image resolution.

The primary satellite bubbles are ejected in the range $0.87 < R_O/R_S < 1.30$, with their scaled radius peaking at $R_O/R_S \approx 1.07$. Our numerical simulations (solid line), calculated for a source bubble with an Ohnesorge number $Oh_S = \mu/\sqrt{D\gamma R_S} = 2.5 \times 10^{-3}$, agree well with these experimental results. The inset shows how the satellite size depends on the Ohnesorge number for a 1:1 coalescence. The simulations predict a critical value of $Oh_S = 5.85 \times 10^{-3}$, above which no primary satellite bubbles are ejected, which is close to the value of 6.5×10^{-3} reported previously [13]. The variation in the size of the source precursor bubble in experiment leads to the Ohnesorge number ranging between $Oh_S = 2.0 \times 10^{-3}$ and 4.0×10^{-3} , contributing to the scatter of the experimental data.

Outside the range for primary satellite ejection, other outcomes are observed. No bubbles are ejected for $R_O/R_S < 0.87$ or in the interval $1.30 < R_O/R_S < 1.60$. The smaller secondary satellite bubbles, with scaled radius $R_E/R_S \approx 0.05$, appear in the range $1.60 < R_O/R_S < 2.50$. Within that range, bubble rings are ejected when $1.85 < R_O/R_S < 2.15$. Finally, the internally ejected droplets are observed in cases where primary satellites are absent; these are not plotted as their radii cannot be measured accurately.

For the remainder of this article we concentrate mainly on the primary satellite bubbles produced from the coalescence of bubble pairs in the range $0.87 \lesssim R_O/R_S \lesssim 1.30$. In terms of the radii R_L and R_R of the left-hand and right-hand precursor bubbles, respectively, referring now to their position in the camera's view, we observe ejection from the left pole of the resultant bubble in the range $0.87 < R_R/R_L < 1.30$ and ejection from the right pole in the range $1/1.30 < R_R/R_L < 1/0.87$ (i.e., $0.77 < R_R/R_L < 1.15$). In the overlapping interval $0.87 < R_R/R_L < 1.15$, two satellite bubbles are ejected, one from the left and one from the right pole.

V. OSCILLATIONS OF THE POLE

We now focus on the process of bubble ejection from one end of the resultant bubble, which we label the “source pole”. We first analyze the capillary waves arriving at the pole, which experiment and simulation show are the origin of the satellites. We define r_p as the distance between the source pole and the center, C , of the corresponding precursor bubble, as illustrated in Fig. 7(a).

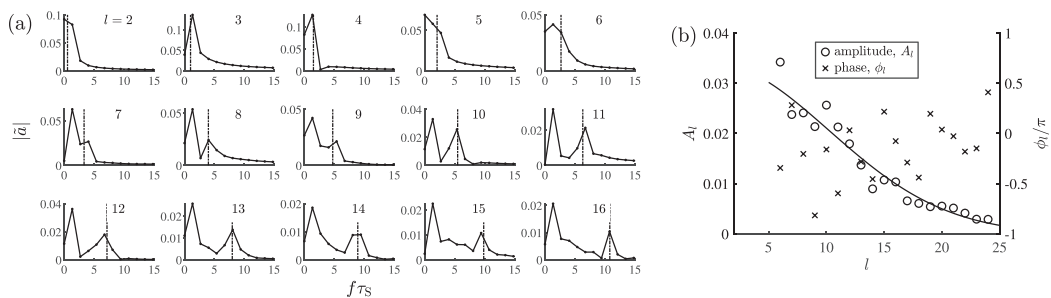


FIG. 8. Spectral analysis of the shape oscillations of the source bubble, obtained from simulation, during the period from the onset of coalescence to the ejection of a satellite bubble. The data shown are for a 1:1 coalescence, but similar spectra are obtained for other ratios. (a) Amplitude spectra, $|\tilde{a}_l(f)|$, for spherical harmonic modes $l = 2$ to $l = 16$. For modes $l \geq 7$, a distinct peak is visible close to the frequency of the source precursor bubble's normal shape modes f_l , given by Eq. (2) (vertical dashed lines). (b) The amplitude A_l (left axis, $\circ$) and phase constants ϕ_l (right axis, $\times$) of each mode, extracted from the spectra. The A_l are well-approximated by a Gaussian (solid curve) with a half-width of $\Delta l \approx 12$. The phase constants are distributed around $\phi = 0$; the scatter is due to noise in the spectra.

Figure 7(b) shows the oscillations in r_p as a function of the time t scaled by the capillary time of the source precursor bubble, $\tau_S = \sqrt{DR_S^3/\gamma}$. There is good agreement between the results obtained from experiment and from numerical simulation, excepting gaps in the experimental data that appear as the view of the pole from the camera is periodically obscured by the surrounding surface of the bubble. The last of the wave crests to arrive at the pole leads to the pinch-off of a satellite bubble at $t \approx 0.8\tau_S$. These data were obtained for the coalescence of equal-sized precursors (shown in Fig. 2). Similar data are obtained for coalescences with other size ratios, except that the normalized pinch-off time, t/τ_S , varies with size ratio.

To understand the nature of the waves arriving at the source pole, we now analyze the evolving shape of the simulated resultant bubble, decomposing it into a series of spherical surface harmonics and performing a frequency analysis on their time-dependent amplitudes.

The shape of the simulated bubble was decomposed in polar coordinates (r, θ) as a series of zonal spherical surface harmonics about the original center C of the *source precursor* bubble, as indicated in Fig. 7(a),

$$r(\theta, t) = \sum_{l=0} r_l(\theta, t) = R_S + \sum_{l=1} a_l(t) P_l(\cos \theta), \quad (1)$$

where the functions P_l are Legendre polynomials and the coefficients $a_l(t)$ are determined using the standard orthogonality relations. Since the coalescence is axisymmetric there are no contributions from the tesseral or sectoral harmonics. For the representation of the shape in polar coordinates $r(\theta)$ to remain single-valued at early times, and to avoid including the oscillations of the capillary waves at the opposite pole, we have truncated the resultant bubble shape in this analysis by a planar cut through the point of initial contact between the two precursor bubbles, retaining the portion of the surface corresponding to the source precursor, as shown in Fig. 7(a). Note that this shape is generally not symmetric about the equatorial plane of the source precursor ($\theta = \pi/2$), so that the sum involves both even and odd Legendre polynomials.

Figure 8(a) shows the amplitude spectra $|\tilde{a}_l(f)|$ for the surface harmonics of this truncated shape ($l = 2-16$), determined from the Fourier transform of the coefficients $a_l(t)$. These spectra were obtained for the case of equal-sized precursor bubbles; similar spectra are observed for coalescences with $R_O/R_S \neq 1$. Distinct peaks are visible in the spectra of harmonics $l \geq 7$ at frequencies close to

those of the normal shape modes of a spherical bubble of radius R_S , as given by Lamb [30],

$$f_l = \frac{\sqrt{(l+1)(l-1)(l+2)}}{2\pi\tau_S}. \quad (2)$$

Additionally, the spectra also show a peak at approximately the same relatively low frequency, $f \approx 1.3\tau_S^{-1}$, for all surface harmonics l , which is related to the growth of the neck. For harmonics $l < 7$, this low-frequency feature tends to obscure the normal mode peaks.

Proceeding on the basis that the capillary wave oscillations arriving at the source pole are composed of the source bubble's normal modes, we develop a simple model that assumes sinusoidal oscillation of the harmonic coefficients $a_l(t)$ at the normal mode angular frequencies $\omega_l = 2\pi f_l$ [Eq. (2)]:

$$a_l(t) = A_l \cos(-\omega_l t + \phi_l). \quad (3)$$

The displacement of the source pole, $r_p(t) \equiv r(\theta = \pi, t)$, is given generally by the sum $r_p(t) = R_S + \sum_{l=1} r_{p,l}(t)$, where $r_{p,l}(t) = a_l(t)P_l(\cos \pi)$ are the individual contributions from each harmonic. We restrict our model to the shape oscillations $l \geq 2$, excluding the $l = 1$ mode which corresponds to translation of the center of mass rather than a deformation. Substituting Eq. (3) into the expression for the individual harmonic contributions, and evaluating $P_l(\cos \pi)$, gives

$$\begin{aligned} r_{p,l}(t) &= a_l(t)P_l(\cos \pi) \\ &= A_l \cos(-\omega_l t + \phi_l) \times (-1)^l \\ &= A_l \cos(l\pi - \omega_l t + \phi_l). \end{aligned} \quad (4)$$

Crests and troughs are expected in $r_{p,l}$ when the phase $l\pi - \omega_l t + \phi_l$ equals $2\pi n$ and $2\pi n + \pi$, respectively, with n an integer. We define a phase index N to label the extrema, where $N = 2n$ corresponds to crests and $N = 2n + 1$ to troughs. The normalized time t_N/τ_S at which a crest or trough with index N occurs is given by

$$\frac{t_N(l)}{\tau_S} = \frac{\pi(l - N) + \phi_l}{\sqrt{(l+1)(l-1)(l+2)}}. \quad (5)$$

Maxima and minima in the time rate of change $\dot{r}_{p,l} = dr_{p,l}(t)/dt$ correspond to phase shifts of $\pi/2$, and are therefore expected at half-integer indices $N = 2n + \frac{1}{2}$ and $N = 2n - \frac{1}{2}$, respectively. The amplitude spectra in Fig. 8(a) show that the coalescence excites a broad range of surface harmonics. The amplitudes of each mode, A_l , obtained from the height of the peaks in the spectra, are well-approximated by a Gaussian profile with a half-width of $\Delta l \approx 12$, as shown in Fig. 8(b). The phase constants ϕ_l , obtained from the Fourier transform of $a_l(t)$, are scattered around $\phi = 0$, also shown in Fig. 8(b). Taken together, the wide range of shape modes excited and their apparent phase coherence imply a narrow, localized deformation of the source bubble's surface at $\theta = 0$ (the location of the neck) at $t = 0$. Since the amplitudes remain significant over the range of modes that contribute to the polar oscillations ($l \lesssim 20$) and the phase constants are distributed around zero, we simplify the model by setting a constant amplitude $A_l = A$ and phase constant $\phi_l = 0$ for all modes. This corresponds to an initial condition in which all shape modes begin with maximum positive values at $t = 0$. Physically, this choice describes a source bubble shape that is spherical everywhere ($r = R_S$) except for a sharp deformation at $\theta = 0$.

To validate this model, we compare it with the time evolution of the surface harmonics in the simulation. Figure 9(a) shows the velocity of the harmonics at the pole, $\dot{r}_{p,l}(t)$, for three different ratios R_O/R_S . Plotting the rate of change increases the visibility of the capillary wave oscillations, while deemphasizing the more slowly varying changes in the position of the pole. Bright yellow and dark blue colors indicate maxima and minima in $\dot{r}_{p,l}(t)$, respectively. Note that the spherical harmonic decomposition is only valid for times before the surface profile becomes multi-valued, so data are not presented beyond this limit (e.g., for $t/\tau_S > 0.80$ in the case $R_O/R_S = 1.00$). The times

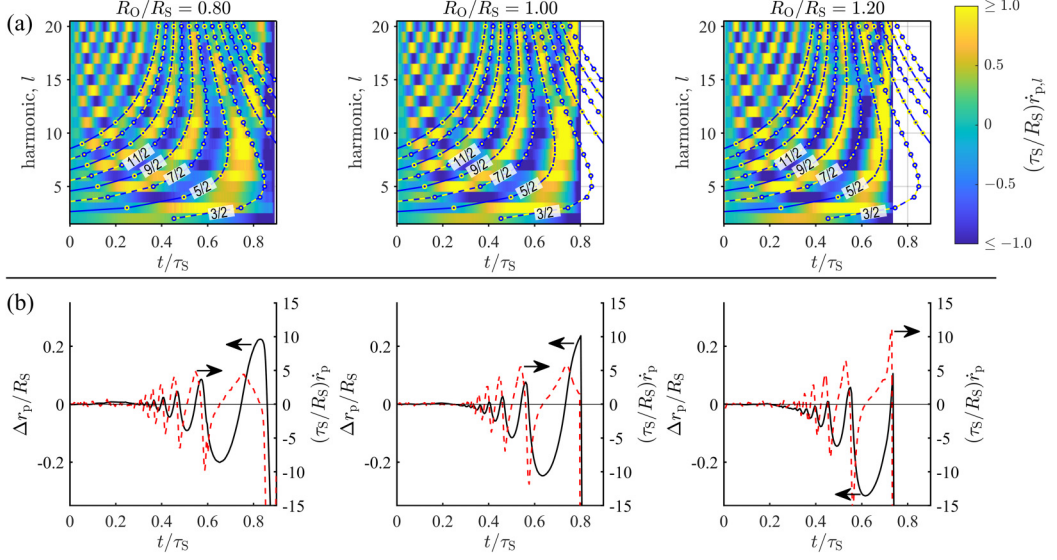


FIG. 9. Analysis of the simulated pole oscillations. (a) Time evolution of the pole displacement velocity contributed by each spherical surface harmonic ($\dot{r}_{p,l} = dr_{p,l}/dt$) for three different precursor radius ratios (R_O/R_S). Bright yellow and dark blue indicate positive and negative velocity peaks, respectively. The overlaid curves show the predicted timing of these peaks from our linear wave model (Sec. V). Labels on selected curves show the value of the phase index N defined in the text. (b) The total pole displacement ($\Delta r_p/R_S = r_p/R_S - 1$, solid line) and total pole velocity ($\dot{r}_p = dr_p/dt$, dashed line), which result from the superposition of the surface harmonics.

of maxima and minima in $\dot{r}_{p,l}$ predicted by our model [Eq. (5)] are overlaid on Fig. 9(a) as marker points joined by solid (maxima) and broken (minima) curves. Reasonable agreement is observed with the rates of change of the simulated surface harmonics as can be seen by the alignment of the marker points with the bright yellow and dark blue bands of color. However, at later times, the maxima and minima in the simulation tend to arrive slightly later than predicted by this model. This may be due to a slowing of the oscillation frequency with increasing amplitude as the waves converge on the pole, a nonlinear effect described by Refs. [12,34]. Nevertheless, Eq. (5) reproduces the key features of these data.

The oscillations of the pole result from the superposition of these surface harmonics. The polar radius r_p and its rate of change with time, $\dot{r}_p = dr_p/dt$, are shown in the three plots in Fig. 9(b). Maxima and minima in $\dot{r}_p$ occur as the pole passes through its equilibrium point. Comparing the oscillations in $\dot{r}_p$ in Fig. 9(b) with the oscillations in the surface harmonics $\dot{r}_{p,l}$ in Fig. 9(a) we see that maxima and minima in $\dot{r}_p$ are the result of constructive interference, which occurs when a group of different harmonics reach their peak velocity at the same time, at the pole. As an example, the maximum in $\dot{r}_p$ at $t/\tau_S \approx 0.53$ – 0.54 (and corresponding zero in displacement $\Delta r_p(t)$) visible in all plots in Fig. 9(b), occurs when the $\dot{r}_{p,l}$ of surface harmonics of degree $l \approx 13$ – 17 reach maxima at approximately the same time, as can be seen from the alignment of the yellow bands of color in Fig. 9(a) for this group of surface harmonics. The corresponding prediction of the model is found where $dt_N/dl = 0$, with t_N given by Eq. (5) (treating l as a continuous variable), yielding the following cubic relation

$$l^3 - 3\tilde{N}l^2 + (1 - 4\tilde{N})l + \tilde{N} + 4 = 0, \quad (6)$$

where $\tilde{N} = N - \phi_l/\pi$. The value of l for which $dt_N/dl = 0$ is the $l \geq 2$ root of the above equation. Taking $\phi_l = 0$ again, for curve $N = 9/2$, we obtain $l = 15$ (rounding to the nearest integer) and

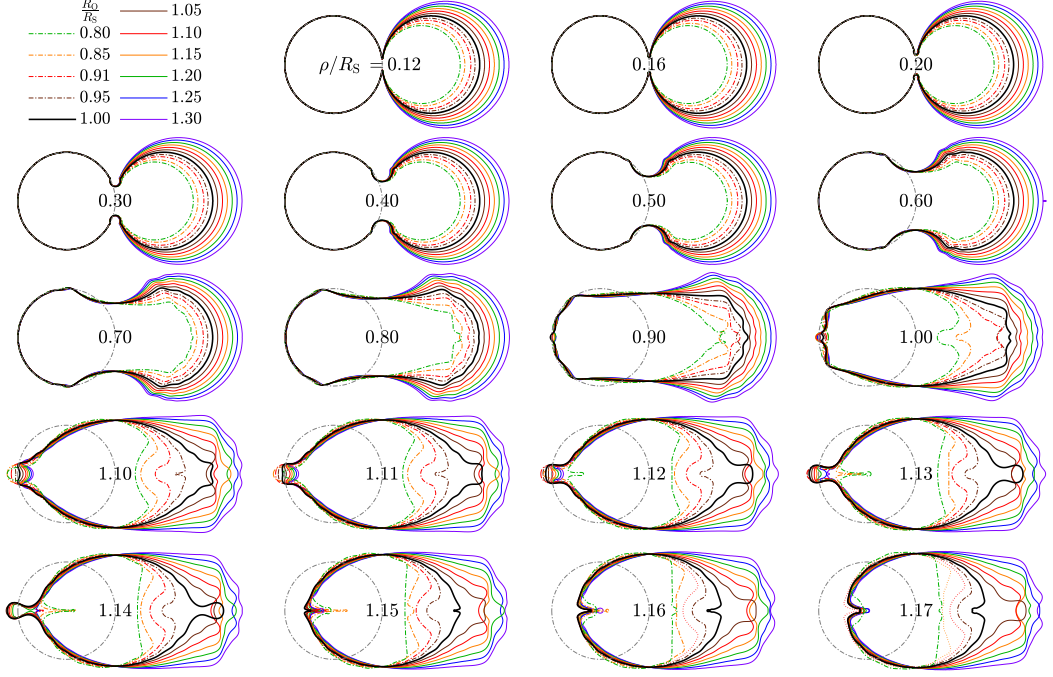


FIG. 10. Simulated evolution of the bubble shape as a function of the growing neck radius, ρ . Each panel shows bubble profiles at a specific neck radius, scaled by the source bubble radius, R_S . The different colored contours correspond to different precursor bubble size ratios, R_O/R_S , in the range 0.80 to 1.30. The source precursor bubble (dot-dash circle) is shown with the same size in all cases. Ejected satellites are not shown. For all size ratios shown the rapid retraction of the pole begins when the neck reaches a value of $\rho \approx 1.15R_S$.

corresponding time $t_{9/2}(l) = 0.535\tau_S$ in good agreement with the time of the observed maximum in $\dot{r}_p$. The adjacent minima in $\dot{r}_p$ at $t \approx 0.47\tau_S$ and $0.58\tau_S$, occur due to constructive interference between a group of surface harmonics including $l = 17$ and adjacent degree, and the group including $l = 12$ and adjacent degree, respectively. The times of the minima obtained from the model, $t/\tau_S = 0.489$ and 0.597 (curves $N = 11/2$ and $7/2$) are in good agreement. The final maximum and minimum in $\dot{r}_p$ (and corresponding zeros in Δr_p), occur at $t/\tau_S \approx 0.74$ and 0.874 , respectively (the latter only observable in the $R_O/R_S = 0.8$ coalescence). The model ($N = 5/2$ and $N = 3/2$) gives $t/\tau_S = 0.689$ and 0.849 , a little earlier than observed.

VI. RETRACTION OF THE POLE

In addition to the relatively rapid capillary oscillations, the pole also undergoes a slower, larger-scale bulk motion. As seen in Fig. 9(b), the pole's quasi-equilibrium position retracts toward the bubble's center over time. This retraction is more pronounced for larger size ratios R_O/R_S [compare the case $R_O/R_S = 1.20$ with 0.80 in Fig. 9(b)]. The sudden decreases in $\dot{r}_p$ at times t/τ_S between 0.7 and 0.9, seen in Figs. 7(b) and 9(b), mark the times when the retraction of the source pole suddenly accelerates. Previous work on bubbles and drops has noted that the pole retraction is necessary to thin the neck for pinch-off [9,35], but the sudden acceleration of this retraction has not been fully explored. We now focus on the timing of this rapid retraction and its relationship to satellite bubble ejection. Figure 10 compares the simulated shapes of the resultant bubble for R_O/R_S in the range 0.8 to 1.3. The plots are scaled such that the source precursor bubble is shown with the same size for all R_O/R_S . The panels show the bubble's evolution chronologically, with each panel corresponding to a specific value of the nondimensional neck radius ρ/R_S . The simulations reproduce the experimental

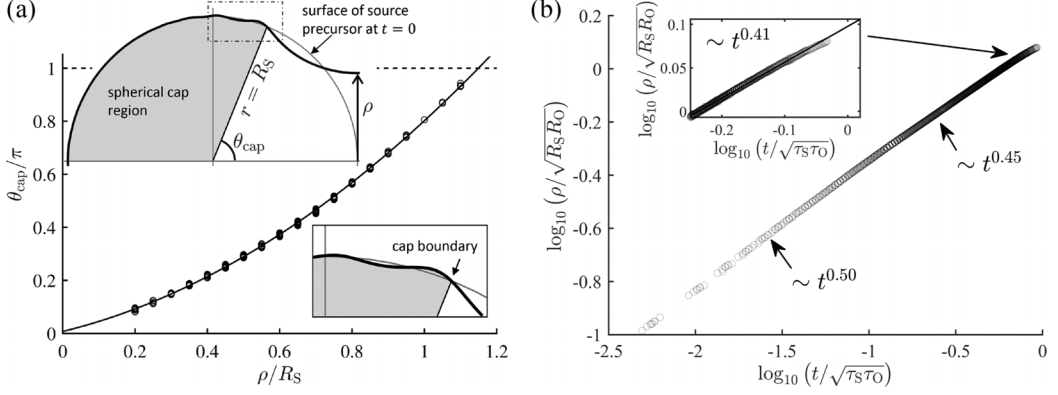


FIG. 11. Determining the time-evolution of the shrinking spherical cap in simulation, for precursor bubble radius ratios in the range 0.80 to 1.30. The boundary of the cap is defined as the angular position $\theta = \theta_{\text{cap}}$ where $r(\theta)$ crosses $r = R_S$ for the second time with increasing θ (illustrated in the top-left inset of panel a, with a magnified view of the second crossing point shown in the bottom-right inset). (a) The angular position of the boundary, θ_{cap} , collapses onto a single curve when plotted as a function of the normalized neck radius ρ/R_S . (b) Neck radius ρ , scaled by the geometric mean of the precursor bubble radii, $\sqrt{R_S R_O}$ versus time t , scaled by the geometric mean of the capillary times, $\sqrt{\tau_S \tau_O}$. This scaling collapses data from different size ratios onto a single power-law curve. The inset shows the later stage of growth when satellite ejection occurs.

observations, showing that as the neck between the two precursor bubbles expands, the resultant bubble takes the form of two spherical caps over which capillary waves travel away from the growing neck region. It is apparent that the fractional area of the spherical cap of the source bubble decreases with increasing ρ/R_S in the same proportion for all R_O/R_S . By the time $\rho = 0.7R_S$, the cap has shrunk to a hemisphere for all size ratios R_O/R_S shown and continues to reduce in area with increasing ρ .

We define the boundary of the spherical cap region as the angular position $\theta = \theta_{\text{cap}}$ where the surface $r(\theta)$ crosses $r = R_S$ for the second time with increasing θ , from $r(\theta) < R_S$ to $r(\theta) > R_S$, as illustrated in the insets of Fig. 11(a). Figure 11(a) plots the angle θ_{cap} against the neck radius ρ for the various precursor size ratios, obtained from simulation. The plot demonstrates that all data collapse onto a single curve when plotted against the normalized neck radius ρ/R_S . We define ρ_{ret} as the neck radius at which the spherical cap boundary reaches the pole ($\theta_{\text{cap}} = \pi$). Since the surface $r(\theta)$ becomes multi-valued near the pole, as the neck forms at the base of the satellite bubble, this prevents us from measuring directly the value of θ_{cap} in the final moments before pinch-off. However, extrapolating the trend in Fig. 11(a) to the pole ($\theta_{\text{cap}} = \pi$) gives a value of $\rho_{\text{ret}} = (1.15 \pm 0.01)R_S$ for all size ratios shown. Figure 10 shows that the polar region begins to rapidly retract toward the center of the bubble when the neck radius reaches ρ_{ret} , including in cases where no satellite bubble is ejected from the source pole (e.g., for $R_O/R_S = 0.80$).

We now consider the time dependence of the neck radius, which, given the data collapse shown in Fig. 11(a), determines the time dependence of the spherical cap boundary and the timing of the retraction of the pole. The growth of the neck is influenced by the sizes of both precursor bubbles. Figure 11(b) shows a log-log plot of simulated neck radius ρ , scaled by the geometric mean of the precursor bubble radii, $\sqrt{R_S R_O}$, versus time t scaled by the geometric mean of the precursor bubbles' capillary times $\sqrt{\tau_S \tau_O}$ for ten ratios R_O/R_S between 1.00 and 1.30. Note that, by symmetry, data for $R_S/R_O < 1.0$ are identical to those for $R_O/R_S > 1.0$. The plot demonstrates that the neck grows as

$$\frac{\rho}{\sqrt{R_S R_O}} = a \left(\frac{t}{\sqrt{\tau_S \tau_O}} \right)^b, \quad (7)$$

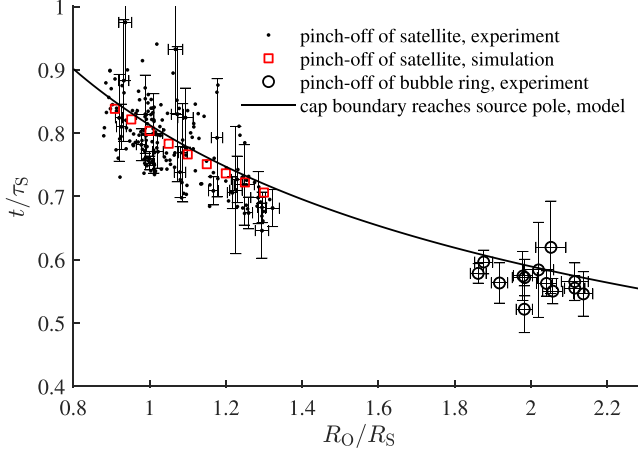


FIG. 12. Effect of the precursor bubble radius ratio on the pinch-off time of the primary satellite bubbles. Experimental ($\cdot$) and simulated ($\square$) pinch-off times are plotted against the precursor radius ratio, R_O/R_S . For clarity, error bars are shown for a representative subset of the experimental data; the uncertainties for the remaining points are of a similar magnitude. Pinch-off times for bubble rings ($\circ$) are also included. The solid curve shows the predicted time of rapid pole retraction based on our model [Eq. (8)].

with the data collapsing onto a single curve with these scalings for time and neck radius. This data collapse can be motivated by symmetry arguments. If we assume a power-law relationship, $\rho \propto t^b$, dimensional analysis implies that $\rho/R_S = f(R_O/R_S)(t/\tau_S)^b$, where f is an unknown function of the precursor radius ratio. By symmetry, exchanging the two precursor bubbles must yield the same physics, leading to the scaling relationship Eq. (7) for ratios R_O/R_S near unity (see Appendix).

At early times $t \lesssim 0.05\sqrt{\tau_S\tau_O}$ the power is $b = 0.50$, in agreement with previous findings from experiments and numerical simulations of bubble coalescence [9–11, 36–38] and also drops [39–41]. At later times, b falls to $b = 0.45$, and remains constant over a range $0.10 \lesssim t/\sqrt{\tau_S\tau_O} \lesssim 0.53$. A further drop in b occurs as ρ approaches $\sqrt{R_S R_O}$. Over the range of times during which the retraction of the pole is observed ($0.60 \lesssim t/\sqrt{\tau_S\tau_O} \lesssim 1.0$), $b = 0.41 \pm 0.01$ and $a = 1.25 \pm 0.01$. We define the retraction time of the source pole, t_{ret} , as the time at which the neck radius reaches ρ_{ret} (i.e., when $\rho(t_{\text{ret}}) = \rho_{\text{ret}}$). Rearranging Eq. (7), we obtain the retraction time on the timescale of the capillary wave oscillations τ_S ,

$$\frac{t_{\text{ret}}}{\tau_S} \approx \left(\frac{1}{a} \frac{\rho_{\text{ret}}}{R_S} \right)^{\frac{1}{b}} \left(\frac{R_O}{R_S} \right)^{\frac{3}{4} - \frac{1}{2b}}. \quad (8)$$

Figure 12 shows the curve t_{ret}/τ_S versus R_O/R_S obtained from Eq. (8), substituting the values of a and b found above. Marker points show times of separation of satellite bubbles from the source pole, obtained from both experiment and simulation. The vertical error bars on the experimental data primarily reflect the uncertainty in identifying the exact frame of satellite pinch-off, which is determined by the camera's temporal resolution. Both simulation and experimental data points fall on or close to the times given by the retraction curve, demonstrating that the ejection of a satellite bubble is triggered by the retraction of the pole.

VII. DISCUSSION

Our model explains the dependence of satellite ejection on the size ratio of the precursors in terms of two distinct processes, each with its own timescale: (i) the arrival of capillary wave crests

at the pole, governed by the source bubble's capillary time, τ_S [Eq. (5)]; and (ii) the rapid retraction of the polar region, the timing of which is sensitive to the size ratio [Eq. (8)].

Equations (5) and (6) predict that as the final crest of the capillary wave arrives at the pole, the pole's displacement increases from zero at $t_1 = 0.689\tau_S$ to a maximum and then back to zero again at $t_2 = 0.849\tau_S$, independent of the precursor radius ratio R_O/R_S . Equation (8) predicts that the time that the polar region begins to rapidly retract toward the center of the bubble coincides with t_1 for precursors with $R_O/R_S = 1.4 \pm 0.1$, and with t_2 for precursors with $R_O/R_S = 0.91 \pm 0.06$. Between these two limits, $0.91 < R_O/R_S < 1.4$, the rapid retraction of the pole occurs as the final peak passes through it. Increasing or decreasing R_O/R_S delays or advances the timing of the polar retraction, respectively, relative to the time of the arrival of the wavecrests at the pole, varying the displacement of the pole at the time that the satellite bubble is ejected, and so varying the size of the satellite bubble ejected. These limits on R_O/R_S obtained from the model are in reasonably good agreement with the range over which satellite bubble ejection is observed in experiment and simulation, $0.87 \lesssim R_O/R_S \lesssim 1.30$ (Fig. 6). If we now apply the model to the left and right poles, respectively, as viewed by the camera in experiment, then we predict satellite ejection from the left when $0.91 < R_R/R_L < 1.4$ and from the right when $0.91 < R_L/R_R < 1.4$. In the overlap between these two intervals, $0.91 < R_R/R_L < 1/0.91$ (i.e., $0.91 < R_R/R_L < 1.10$), the model predicts that both poles eject a satellite bubble, in good agreement with the observed range, $0.87 < R_R/R_L < 1.15$. The small discrepancy between the range of R_O/R_S predicted by the model to produce satellite bubbles, and that observed is due to the decrease in the oscillation frequency of the surface harmonics, mentioned in Sec. V, that is not accounted for in this linear model.

In early investigations combining microgravity experiments with simulations, Ohnishi *et al.* [2] studied the production of satellite bubbles from coalescence of similarly sized bubbles. They proposed criteria to observe satellite ejection based on the surface energy available to form a satellite. However, while this energy model predicts a lower critical value of R_O/R_S for ejection, it fails to predict the upper cut-off that we observe; this suggests that satisfying the energy threshold, while necessary, is an insufficient condition. Our results are consistent with the experiments of Tse *et al.* [7] and later Zhang and Thoroddsen [8], who demonstrated that satellite ejection results from the focusing of capillary waves at the pole, but did not reach a conclusion regarding the dependence of satellite ejection on the precursor bubble ratio. Our investigation shows that this dependence arises from an additional condition on satellite bubble ejection, that bulk retraction of the pole is required to complete the pinch-off of the neck. A similar mechanism was noted by Zhang *et al.* [9] in the context of bubbles and drops with large diameter ratios $\approx 1:30$ (and also by Blanchette and Bigioni in drops merging with a liquid pool [35]); they observed that the motion of the source bubble cap toward the system center was critical for thinning the neck to induce scission. Based on the rate of drainage of gas from the smaller to the larger bubble, driven by the capillary pressure difference between the two precursor bubbles, their model explains the dependence of satellite size on the inner-to-outer fluid density ratio. However, this model cannot be applied to satellite production in the $R_O/R_S \sim 1$ regime, as observed here and previously [7,8], where the capillary pressures of the precursors are effectively equal. In this case, the retraction of the cap of the source bubble is governed not by drainage into the opposite bubble but instead by the expansion of the neck region. Consequently, satellite ejection depends on the timing of the capillary wave crest arrival at the pole relative to the timing of the neck growth, as captured by the model presented here.

Although not the focus of the present study, we can also apply this analysis to the coalescence of similarly sized liquid drops to gain insight into why they do not eject twin satellite drops in the manner observed for bubbles. Figure 13 compares the simulated pole oscillations for equal-sized bubbles, Fig. 13(a), with those of equal-sized drops, Fig. 13(b). The data shown in Fig. 13(b) are from simulations by Menchaca-Rocha *et al.* [40] ($Oh \approx 4 \times 10^{-4}$), which are in good agreement with their experimental results and also with simulations by Deka *et al.* [42]. Vertical lines indicate the predicted times of crests (even N , solid lines) and troughs (odd N , dashed lines) in the polar displacement, according to our linear model. In the case of bubbles, these times are given by Eqs. (5)

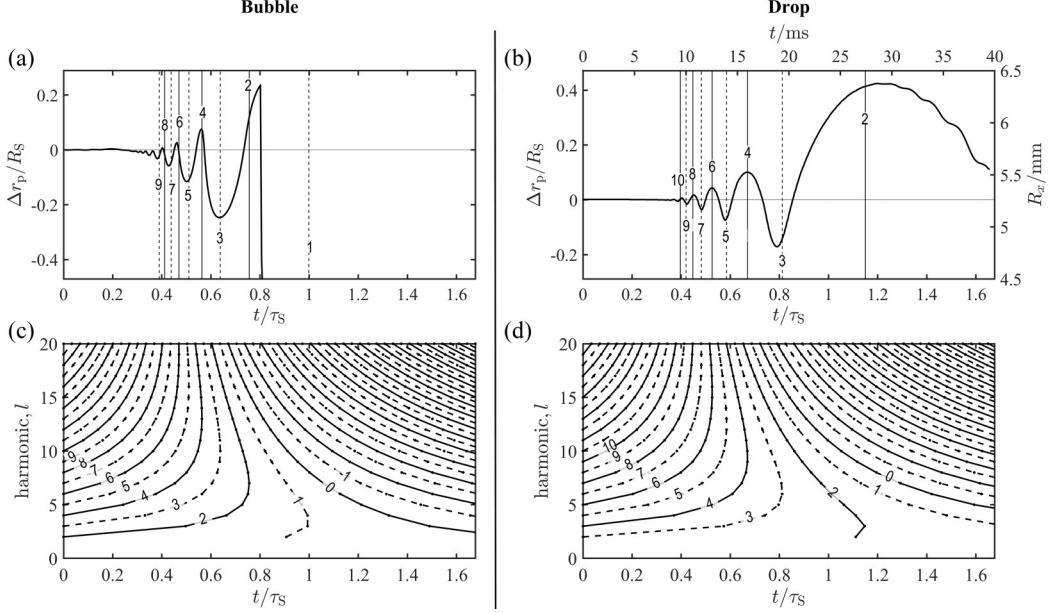


FIG. 13. Comparison of simulated polar oscillations in the coalescence of equal-sized bubbles and drops. (a), (b) The normalized pole displacement ($\Delta r_p/R_s = r_p/R_s - 1$). The data in panel (a) are the same as those in Fig. 9(b) ($R_O/R_S = 1.00$). The data in panel (b) are from simulations of coalescing 1.0 g mercury drops by Menchaca-Rocha *et al.* [40]; here, the top and right axes show the original absolute time and axial radius R_x (distance from the contact point to the pole, equal to $2R_S$ at $t = 0$), respectively. For direct comparison with the bubble data, the time normalized by the precursor drops' capillary time τ_S is indicated on the bottom axis. The relative pole displacement is indicated on the left axis as $\Delta r_p/R_s = R_x/R_S - 2$, where R_S is the radius of the precursor drops. (c), (d) The times $t_N(l)$ of crests (solid lines) and troughs (dashed lines) for each harmonic l at the poles, predicted by our linear wave model [Eqs. (5) and (9)]. Vertical lines in panels (a) and (b) correspond to the stationary points $dt_N(l)/dl = 0$ [obtained from Eqs. (6) and (10)], where constructive interference produces crests and troughs at the poles. Labels show the value of the phase index N defined in the text.

and (6), with $\phi_l = 0$. In the case of drops, the equivalent expressions are

$$\frac{t_N(l)}{\tau_S} = \frac{\pi(l - N) + \phi_l}{\sqrt{l(l-1)(l+2)}} \quad (9)$$

and

$$l^3 - 3\tilde{N}l^2 + (2 - 2\tilde{N})l + 2\tilde{N} = 0 \quad (10)$$

(with $\tilde{N} = N - \phi_l/\pi$), obtained as in Sec. V, but assuming the normal mode frequencies of a spherical drop, $\omega_l = \sqrt{l(l-1)(l+2)}/\tau_S$ [30]. Good agreement between the predicted and observed timing of the oscillation peaks and troughs is obtained with the simulation data of Menchaca-Rocha *et al.* [40], if we take $\phi_l = \pi$ for all modes, as shown in Fig. 13(b). This phase constant corresponds to an initial condition where all modes begin with maximum negative value, rather than maximum positive value in the case of bubbles. This difference leads to a markedly different shape in the final peak arriving at the poles: in bubbles, the final peak is the result of constructive interference of harmonics $l \approx 6-8$, as shown in Fig. 13(c), producing a relatively narrow protrusion of the interface. In drops, the final peak arises from lower-degree, longer-wavelength harmonics $l \approx 2-4$, as shown in Fig. 13(d), producing a broader and more slowly varying peak. We suggest, therefore,

that the lack of satellite ejection in 1:1 drop coalescence is due to this final peak being too broad to form the narrow neck required for pinch-off.

Finally, we remark on the timing of the ejection of bubble *rings* from the coalescence of bubble pairs with radii ratios of approximately 1:2, as described in Sec. IV. The time of their ejection falls on the same curve given by Eq. (8), shown in Fig. 12. That is to say that ring bubble ejection is triggered by the same rapid retraction of the pole that causes the primary satellites to be ejected. In this case, however, the retraction coincides with the arrival at the pole of a *trough* that precedes the final peak, forming the bowl-shaped feature shown in Fig. 5(b) from which the ring is pinched off. The “secondary” droplet ejection that occurs shortly beforehand occurs due to the pinch-off of the penultimate wavecrest to arrive at the pole.

VIII. SUMMARY AND CONCLUSION

We have investigated the coalescence of magnetically levitated air bubbles in water, observing for the first time the simultaneous ejection of equal-sized ‘twin’ satellite bubbles from equal-sized precursor bubbles. We found that for precursor bubbles with diameter ratios between 1:1 and 1:1.3, the coalescence yields two, one, or no satellites, depending on the initial size ratio. We propose a timing model that explains this dependence, showing that a satellite is ejected when a capillary wave crest arrives at a pole as the pole begins a rapid retraction toward the center of the bubble. This model successfully predicts the range of size ratios for which single, double, or no satellites are formed. The same model is also consistent with our observation of the ejection of ring bubbles from a collapsing bowl-like structure at the pole. Furthermore, our analysis of the capillary waves suggests a reason for the absence of satellite ejection in the coalescence of similarly sized drops.

These findings are enabled by magnetic levitation, which holds the bubbles in stable, neutral buoyancy without deforming their shape. We suggest this technique is highly suitable for future experimental studies of bubble dynamics in idealized, unconstrained conditions.

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APPENDIX

The physical quantities that govern the growth of the neck ρ , are t , R_1 , R_2 , density D , and interfacial tension γ , where R_1 and R_2 are the radii of the two precursor bubbles. The relationship between these five quantities can be written in terms of three dimensionless groups, $\pi_I = g(\pi_{II}, \pi_{III})$, where g is an unknown function. Choosing to nondimensionalize the neck radius by dividing by R_1 gives the first group, $\pi_I = \rho/R_1$. Nondimensionalizing the time t by the capillary time formed from R_1 , D and γ ($T_1 = \sqrt{DR_1^3/\gamma}$) gives the second group, $\pi_{II} = t/T_1$. The influence on the neck growth of the radius of the second bubble, R_2 , can be expressed as a dimensionless ratio of the two characteristic lengths, giving the third group $\pi_{III} = R_2/R_1$. Hence,

$$\frac{\rho}{R_1} = g\left(\frac{t}{T_1}, \frac{R_2}{R_1}\right). \quad (\text{A1})$$

We now assume a power-law relationship, modeling the system over a time interval where its behavior can be approximated by the form $\rho \propto t^b$ with a constant exponent b . The function g must therefore take the form

$$\frac{\rho}{R_1} = f\left(\frac{R_2}{R_1}\right)\left(\frac{t}{T_1}\right)^b, \quad (\text{A2})$$

where f is an unknown function of R_2/R_1 . By symmetry we must also have

$$\frac{\rho}{R_2} = f\left(\frac{R_1}{R_2}\right)\left(\frac{t}{T_2}\right)^b. \quad (\text{A3})$$

Equating the two expressions for ρ [Eqs. (A2) and (A3)] gives

$$f\left(\frac{R_2}{R_1}\right)\left(\frac{t}{T_1}\right)^b R_1 = f\left(\frac{R_1}{R_2}\right)\left(\frac{t}{T_2}\right)^b R_2. \quad (\text{A4})$$

Writing $x = R_2/R_1$ and $\Gamma = 1 - \frac{3}{2}b$, and noting that $T_1 \propto R_1^{3/2}$ and $T_2 \propto R_2^{3/2}$, yields the following functional equation:

$$f(x) = x^\Gamma f(x^{-1}). \quad (\text{A5})$$

The solution has the form $f(x) = x^{\Gamma/2} h(\ln x)$ where h is any even function. Expanding h about $x = 1$, we have $h(\ln x) = a + a_2(\ln x)^2 + O[(\ln x)^4]$. Multiplying Eqs. (A2) and (A3), and taking the square root, we obtain the following to first order in $\ln x$,

$$\frac{\rho}{\sqrt{R_1 R_2}} = h(\ln x) \left(\frac{t}{\sqrt{T_1 T_2}} \right)^b \approx a \left(\frac{t}{\sqrt{T_1 T_2}} \right)^b, \quad (\text{A6})$$

where a is a dimensionless constant.

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