

Self-similar profiles of velocity moments in turbulent round jets: A symmetry approach

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(Received 10 April 2026; accepted 3 September 2026; published 5 October 2026)

The differential equations describing the self-similar profiles of velocity moments in turbulent round jets are investigated by the Lie-symmetry method. From this, solutions are constructed, which depend on free functions. Truncated Taylor expansion of the free functions yields expressions for the profiles. If the Taylor expansion is truncated after the leading term, the classical Gaussian profile for the mean velocity is recovered, constituting the first rigorous derivation of this well-known empirical result. Comparison to DNS data shows good agreement of the derived expressions with the actual profiles.

DOI: [10.1103/f6s1-3glr](https://doi.org/10.1103/f6s1-3glr)

I. INTRODUCTION

Turbulent round jets are among the most studied flow configurations. They develop when high-velocity fluid from a round nozzle flows into an infinite reservoir of fluid at rest. With increasing distance from the nozzle, the jet spreads, i.e., the velocity profile becomes flatter and wider; see Fig. 1. One of the most interesting properties of the turbulent round jet (and other free shear flows) is self-similarity. If the profiles and the radial coordinate are rescaled by an appropriate scale, usually a power of the downstream distance from the nozzle, the velocity profiles beyond a certain distance from the nozzle collapse. Research in the turbulence community has ever since focused on determining these scales. However, very little research has been conducted on the determination of the self-similar velocity and velocity moment profiles themselves. An approximation of the velocity profile was suggested by Pope [1] by assuming a constant turbulent viscosity over the full width of the jet. While this assumption is valid in the center of the jet, the turbulent viscosity decreases significantly at the edges of the jet [1]. A frequent empirical observation is that the velocity profile in the turbulent round jet closely follows a Gaussian curve [2]. This can be motivated theoretically by drawing an analogy between momentum transfer and heat diffusion [3]. Assuming a Gaussian curve for the mean axial velocity, Mh [4] derived the corresponding radial velocity profile and, through boundary layer approximation, a profile for the Reynolds shear stress. Nguyen *et al.* [5] proposed determining the profiles from repeated application of the invariant function method of Lie-symmetry theory. The present work aims to extend this proposed approach and eliminate some empirically based assumptions made in Ref. [5].

Specifically, we aim to compute highly accurate approximations for the velocity profiles and the profiles of the second-order velocity moments without a boundary layer approximation. Since we

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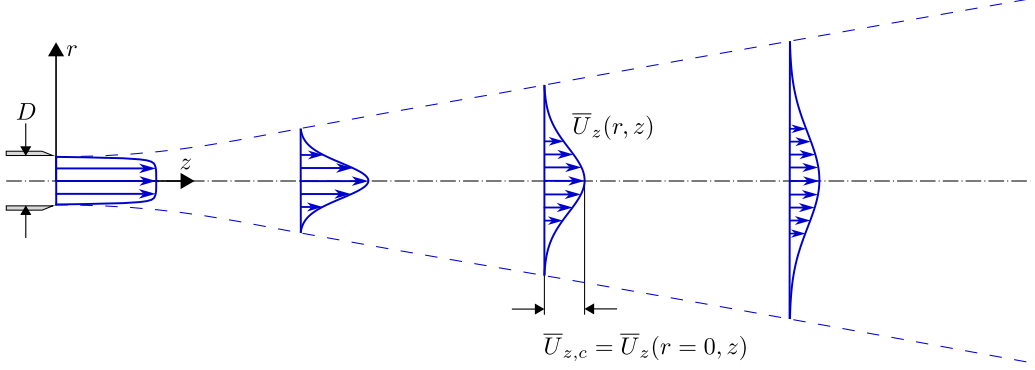


FIG. 1. Schematic view of a turbulent round jet. The flow exits from a round nozzle with diameter D . The shown profiles belong to the mean velocity $\overline{U}_z(r, z)$. The centerline velocity $\overline{U}_{z,c}$ characterizes the velocity decay with increasing distance from the nozzle.

cannot solve the closure problem, these profiles will still depend on parameters which have to be determined from data. In Sec. II we introduce the invariant function method, which is at the heart of the presented theory. In Sec. III, we explain how this theory applies to the self-similarity of the round jet. Section IV contains the results regarding the mean velocities, while Sec. V contains the results for the second-order velocity moments. Section VI concludes this work.

II. INVARIANT FUNCTION METHOD

The invariant function method by which we obtain approximations for the profiles of the statistical moments is based on Lie-symmetry theory, to which we give a short introduction in the following. Given a system of partial differential equations (PDE)

$$F(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(p)}) = 0, \quad (1)$$

a symmetry is a transformation of variables

$$\mathbf{x}^* = \boldsymbol{\phi}(\mathbf{x}, \mathbf{u}; a), \quad \mathbf{u}^* = \boldsymbol{\psi}(\mathbf{x}, \mathbf{u}; a), \quad (2)$$

which does not change the form of F , i.e.,

$$F(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(p)}) = 0 \iff F(\mathbf{x}^*, \mathbf{u}^*, \mathbf{u}^{*(1)}, \dots, \mathbf{u}^{*(p)}) = 0. \quad (3)$$

Here $\mathbf{x}$ are the independent variables, $\mathbf{u}$ are the dependent variables where the superscript encodes the order of a derivative of $\mathbf{u}$, and a is the group parameter of the transformation $(\boldsymbol{\phi}, \boldsymbol{\psi})$. Lie's first theorem states that the symmetries of a system of PDEs are fully described by their infinitesimal form

$$x_i^* = x_i + \xi_i(\mathbf{x}, \mathbf{u})a + O(a^2), \quad u_j^* = u_j + \eta_j(\mathbf{x}, \mathbf{u})a + O(a^2), \quad (4)$$

with infinitesimals

$$\xi_i(\mathbf{x}, \mathbf{u}) = \left. \frac{\partial \phi_i}{\partial a} \right|_{a=0}, \quad \eta_j(\mathbf{x}, \mathbf{u}) = \left. \frac{\partial \psi_j}{\partial a} \right|_{a=0}. \quad (5)$$

To prove the equivalence of (2) and (4), group properties are used, which is why the approach is often referred to as the Lie group method. The infinitesimal form of the transformation (4) is essentially a first-order Taylor expansion of (2) in a . The infinitesimals (5) define an infinitesimal generator

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta_k \frac{\partial}{\partial u_k} \quad (6)$$

needed subsequently. Accounting for the transformation of derivatives of order p of transformed quantities requires a prolonged infinitesimal generator, which is typically abbreviated by $X^{(p)}$, whereby detail may be taken from Bluman *et al.* [6], although subsequently not needed. The PDE system (1) of order p is then invariant under the transformation (2) if and only if the application of the prolonged infinitesimal generator of order p to the system yields zero, i.e.,

$$X^{(p)}\mathbf{F}|_{F=0} = 0. \quad (7)$$

Thus, condition (7) is fully equivalent to (3), since both verify that a certain transformation is a symmetry. The advantage of the infinitesimal form (7) is that it is a linear system of PDEs in itself that can be solved, which provides an algorithmic way of determining the infinitesimals and therefore the symmetries of a system of PDEs (1). For further details, we refer the reader to Bluman [7], Bluman *et al.* [6], and Olver [8].

The key to the present analysis is that the knowledge of the infinitesimals of the symmetries allows for the construction of invariant, self-similar solutions to the system of PDEs (1). To this end, we first consider an unknown invariant solution

$$\mathbf{u} = \mathbf{f}(\mathbf{x}). \quad (8)$$

Demanding the solution to be invariant under all symmetries by applying the infinitesimal generator $X^{(p)}$ leads to the invariant surface condition

$$X^{(p)}[\mathbf{u} - \mathbf{f}(\mathbf{x})]|_{\mathbf{u}=\mathbf{f}(\mathbf{x})} = 0. \quad (9)$$

The result is a hyperbolic system of PDEs, which implies a solution using the method of characteristics, resulting in the usual characteristic system

$$\frac{dx_1}{\xi_1(\mathbf{x}, \mathbf{u})} = \frac{dx_2}{\xi_2(\mathbf{x}, \mathbf{u})} = \dots = \frac{dx_i}{\xi_i(\mathbf{x}, \mathbf{u})} = \frac{du_1}{\eta_1(\mathbf{x}, \mathbf{u})} = \frac{du_2}{\eta_2(\mathbf{x}, \mathbf{u})} = \dots = \frac{du_k}{\eta_k(\mathbf{x}, \mathbf{u})}. \quad (10)$$

Integration of the characteristic system then leads to invariant solutions which, when scaling symmetries are involved, are referred to as self-similar solutions. Importantly, invariant solutions are not general solutions to the PDE system, but special solutions that allow for the reduction of the number of independent variables the PDE system depends on, the specifics of which we explain in the following. Integrating the infinitesimals ξ_i, ξ_k for $i \neq k$ of two independent variables x_i, x_k leads to a new independent variable $\tilde{x}(x_i, x_k)$ which is an invariant under (2). Integrating an infinitesimal η_j of a dependent variable u_j with an infinitesimal ξ_i of an independent variable x_i leads to a new dependent variable $\tilde{u}(x_i, u_j)$ which is an invariant function, i.e., a function which is itself invariant under the transformation (2). Crucially, inserting $u_j(x_i, \tilde{u})$ into the system of PDEs (1) leads to a dimensionally reduced system of equations which will only depend on $\tilde{x}$ and $\tilde{u}$ but not directly on x_i, x_k or u_j . These are called reduced equations, and they depend on one independent variable less than the original PDE system. Constructing invariant solutions to a system of PDEs,

$$\mathbf{F}(x_1, x_2, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(p)}) = 0, \quad (11)$$

with two independent variables x_1 and x_2 , which we later identify as r and z in the jet configuration, consequently results in a system of ordinary differential equations (ODE),

$$\tilde{\mathbf{F}}(\tilde{x}, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}^{(1)}, \dots, \tilde{\mathbf{u}}^{(p)}) = 0, \quad (12)$$

in which $\tilde{x}$ is the only independent variable. We see later that $\tilde{x}$ corresponds to the classical similarity coordinate $\eta = r/(z - z_0)$ in turbulent round jet flows. This reduction was done in Nguyen *et al.* [5] for the averaged Navier-Stokes equations governing turbulent round jets, i.e., Eq. (21) introduced in Sec. III.

The objective of the present work is to apply Lie-symmetry theory to the reduced ODEs once again to obtain the radial profiles of first and second velocity moments. In the following, we thus describe how the theory presented for PDEs so far can be applied to the reduced ODEs as well.

The reduced equations are first-order underdetermined ODEs of the form

$$\tilde{F}(\tilde{x}, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}^{(1)}) = 0. \quad (13)$$

Underdetermined means here that there are multiple dependent variables $\tilde{\mathbf{u}} = \tilde{u}_1, \dots, \tilde{u}_n$, while there is only one equation $\tilde{F}$, which stems from the closure problem of turbulence.

There are two important caveats coming with the ODEs studied in the following: (i) The symmetries of first-order ODEs cannot be computed algorithmically [9], so we need to rely on heuristics. (ii) The independent variable $\tilde{x}$ in the present ODEs is not involved in any of the symmetries, i.e., $\tilde{\xi}(\tilde{x}, \tilde{\mathbf{u}})$ is zero. Consequently, the characteristic system is as follows

$$\frac{d\tilde{x}}{0} = \frac{d\tilde{u}_1}{\tilde{\eta}_1(\tilde{x}, \tilde{\mathbf{u}})} = \frac{d\tilde{u}_2}{\tilde{\eta}_2(\tilde{x}, \tilde{\mathbf{u}})} = \dots = \frac{d\tilde{u}_k}{\tilde{\eta}_k(\tilde{x}, \tilde{\mathbf{u}})}, \quad (14)$$

which renders the classical approach of invariant solutions, i.e., reducing the number of independent variables, impossible. Therefore, we presently follow an unusual approach, as new invariant functions can only be computed by integrating infinitesimals $\tilde{\eta}_j, \tilde{\eta}_k$ of two dependent variables $\tilde{u}_j, \tilde{u}_k$, obtaining a relation of the form $\tilde{u}_k = \tilde{f}_k(\tilde{x}, \tilde{u}_j)$ where $\tilde{f}_k$ contains a free function. This is only helpful in determining the solution to a system of ODEs if $\tilde{f}_k$ is easier to determine than $\tilde{u}_k$. In the following, we always choose $j = 1$, thus obtaining functional dependencies of all other dependent variables on $\tilde{u}_1$. As the ODEs treated in the following are all linear, the invariant functions take the form

$$\tilde{u}_k = g_k(\tilde{x})\tilde{u}_1 + c_k\tilde{x}^{b_k}, \quad (15)$$

where g_k is a free function, the constant c_k is an unknown symmetry parameter, and the exponent b_k is known. Here, too, the problem of closure becomes apparent, i.e., the relationship between two unknown quantities is observed. Inserting (15) into (13) enables us to obtain ODEs

$$\frac{d\tilde{u}_1}{d\tilde{x}} = G(\tilde{x}, \tilde{u}_1, g_k(\tilde{x}), c_k), \quad (16)$$

which can be solved by separation of variables. Using (15), the solution for $\tilde{u}_1$,

$$\tilde{u}_1(\tilde{x}) = S(g_k(\tilde{x}), c_k), \quad (17)$$

yields solutions for all other unknown variables

$$\tilde{u}_l(\tilde{x}) = g_l(\tilde{x})S(g_k(\tilde{x}), c_k) + c_l\tilde{x}^{b_l}. \quad (18)$$

However, the solutions will be parametrized by the unknown functions g_k . We then approximate g_k by a truncated Taylor expansion in $\tilde{x}$,

$$g_k(\tilde{x}) \approx \sum_{i=0}^{n_k} \alpha_{i,k} \tilde{x}^i, \quad (19)$$

to obtain approximations for the dependent variables $\tilde{u}_k$

$$\tilde{u}_l(\tilde{x}) = \left(\sum_{j=0}^{n_l} \alpha_{j,l} \tilde{x}^j \right) S(\alpha_{i,k}, c_k) + c_l \tilde{x}^{b_l}. \quad (20)$$

As previously mentioned, the methodology we present does not, of course, allow the full closure of underdetermined ODEs since the approximations (20) still contain the unknown parameters $\alpha_{i,k}$ and c_k . However, in the physical problem we consider, knowing the form of the solution is valuable in itself, for example for the calibration of turbulence models, even if there are parameters which must be fitted to data.

III. SELF-SIMILARITY AND REDUCED EQUATIONS

The equations describing the turbulent round jet are the momentum balances in the axial and radial directions,

$$\begin{aligned} \frac{\partial \overline{U_z^2}}{\partial z} + \frac{1}{r} \frac{\partial r}{\partial r} \overline{U_z U_r} + \frac{\partial \overline{P}}{\partial z} &= 0 \\ \frac{\partial \overline{U_z U_r}}{\partial z} + \frac{1}{r} \frac{\partial r}{\partial r} \overline{U_r^2} + \frac{\partial \overline{P}}{\partial r} - \frac{\overline{U_\varphi^2}}{r} &= 0, \end{aligned} \quad (21)$$

as well as the continuity equation

$$\frac{\partial \overline{U_z}}{\partial z} + \frac{1}{r} \frac{\partial r}{\partial r} \overline{U_r} = 0, \quad (22)$$

both in cylindrical coordinates. Here, and in the following, we have eliminated the viscous terms, as for large-scale quantities such as the mean velocity, they are negligibly small in the large Reynolds number limit. The averaged momentum equations (21) are equivalent to the Reynolds-averaged Navier-Stokes equations. However, to simplify the symmetry analysis, they lack the Reynolds decomposition. Furthermore, it is important to note that we do not apply any boundary layer approximation. By applying the invariant function method, Nguyen *et al.* [5] found that the PDEs (21) and (22) can be reduced to ODEs only depending on the invariant

$$\eta = \frac{r}{z - z_0} \quad (23)$$

by inserting the scaling laws

$$\overline{U_z^n U_r^m U_\varphi^k}(r, z) = \frac{\alpha_z e^{c_z(n+m+k)}}{(z - z_0)^{(n+m+k)}} \widetilde{\overline{U_z^n U_r^m U_\varphi^k}}(\eta), \quad \overline{P}(r, z) = \frac{\alpha_z e^{2c_z}}{(z - z_0)^2} \widetilde{\overline{P}}(\eta) + a_P. \quad (24)$$

Here, $z_0 = 0$ is the virtual origin and $\alpha_z = 0.941$, $c_z = 2.06$ and $a_P = 0$ are empirically determined parameters. This essentially leads to the same result as assuming that the jet is self-similar, as (24) takes a very similar form to the usual self-similar ansatz functions [1]. Inserting the scaling laws (24) into the continuity equation (22) and the momentum balances (21), the resulting reduced continuity equation reads

$$\frac{d\eta \widetilde{\overline{U_r}}(\eta)}{d\eta} - \eta \frac{d\eta \widetilde{\overline{U_z}}(\eta)}{d\eta} = 0, \quad (25)$$

while the momentum balances become

$$\frac{d}{d\eta} (\eta \widetilde{\overline{U_z U_r}}(\eta) - \eta^2 [\widetilde{\overline{U_z^2}}(\eta) + \widetilde{\overline{P}}(\eta)]) = 0 \quad (26)$$

and

$$\frac{d}{d\eta} (\eta [\widetilde{\overline{U_r^2}}(\eta) + \widetilde{\overline{P}}(\eta)] - \eta^2 \widetilde{\overline{U_z U_r}}(\eta)) = \widetilde{\overline{P}}(\eta) + \widetilde{\overline{U_\varphi^2}}(\eta). \quad (27)$$

Integrating (26) yields

$$\eta \widetilde{\overline{U_z U_r}}(\eta) - \eta^2 [\widetilde{\overline{U_z^2}}(\eta) + \widetilde{\overline{P}}(\eta)] = K, \quad (28)$$

which can be evaluated at $\eta = 0$, yielding $K = 0$. This result can be used to eliminate the pressure in (27), yielding

$$\frac{d}{d\eta} (\eta [\widetilde{\overline{U_r^2}}(\eta) - \widetilde{\overline{U_z^2}}(\eta)] + (1 - \eta^2) \widetilde{\overline{U_z U_r}}(\eta)) = \widetilde{\overline{U_\varphi^2}}(\eta) - \widetilde{\overline{U_z}}(\eta) + \frac{1}{\eta} \widetilde{\overline{U_z U_r}}(\eta). \quad (29)$$

The ODEs (25) and (29) are underdetermined and thus formally admit infinitely many solutions, or in other words, additional constraints are needed to construct a concrete solution. A viable path here is to apply the invariant function method once again, as described in Sec. II, to get approximate solutions for the mean velocity profiles $\widetilde{U}_z$ and $\widetilde{U}_r$, and for the second velocity moment profiles $\widetilde{U}_z^2$, $\widetilde{U}_z\widetilde{U}_r$, $\widetilde{U}_r^2$, and $\widetilde{U}_\varphi^2$.

The ODEs (25) and (29) possess six symmetries. There are two scaling symmetries

$$\overline{T}_{Ss,1} : \widetilde{U}_z^* = e^{a_{Ss1}} \widetilde{U}_z, \quad \widetilde{U}_r^* = e^{a_{Ss1}} \widetilde{U}_r \quad (30)$$

and

$$\overline{T}_{Ss,2} : \widetilde{U}_z^{*2} = e^{a_{Ss2}} \widetilde{U}_z^2, \quad \widetilde{U}_z\widetilde{U}_r^* = e^{a_{Ss2}} \widetilde{U}_z\widetilde{U}_r, \quad \widetilde{U}_r^{*2} = e^{a_{Ss2}} \widetilde{U}_r^2, \quad \widetilde{U}_\varphi^{*2} = e^{a_{Ss2}} \widetilde{U}_\varphi^2, \quad (31)$$

and four η -dependent “translation-type” symmetries

$$\begin{aligned} \overline{T}_{z,1} : \widetilde{U}_z^* &= \widetilde{U}_z + \frac{a_{z1}}{\eta}, \\ \overline{T}_{r,1} : \widetilde{U}_r^* &= \widetilde{U}_r + \frac{a_{r1}}{\eta}, \\ \overline{T}_{z,2} : \widetilde{U}_z^{*2} &= \widetilde{U}_z^2 + a_{z2}, \\ \overline{T}_{r,2} : \widetilde{U}_r^{*2} &= \widetilde{U}_r^2 + \frac{a_{r2}}{\eta}. \end{aligned} \quad (32)$$

Note that the variables that are unaffected by the respective group parameters are omitted for brevity. From these symmetries (30)–(32), the characteristic system is

$$\frac{d\widetilde{U}_z}{a_{Ss1}\widetilde{U}_z + \frac{a_{z1}}{\eta}} = \frac{d\widetilde{U}_r}{a_{Ss1}\widetilde{U}_r + \frac{a_{r1}}{\eta}} = \frac{d\widetilde{U}_z^2}{a_{Ss2}\widetilde{U}_z^2 + a_{z2}} = \frac{d\widetilde{U}_r^2}{a_{Ss2}\widetilde{U}_r^2 + \frac{a_{r2}}{\eta}} = \frac{d\widetilde{U}_z\widetilde{U}_r}{a_{Ss2}\widetilde{U}_z\widetilde{U}_r} = \frac{d\widetilde{U}_\varphi^2}{a_{Ss2}\widetilde{U}_\varphi^2}. \quad (33)$$

IV. MEAN VELOCITIES

We now use the methodology introduced in Sec. II to determine the form that the mean velocity profiles $\widetilde{U}_z$ and $\widetilde{U}_r$ take in the turbulent round jet. We find a functional dependency of $\widetilde{U}_r$ on $\widetilde{U}_z$ of the type (15). We insert this functional dependency into the reduced continuity equation (25) to obtain a solvable ODE like (13), the solution of which is parametrized by an unknown free function; see (18). By approximating this free function by a truncated Taylor series like (19), we obtain a very precise approximation of the profiles of $\widetilde{U}_z$ and $\widetilde{U}_r$, which is of the type (20).

Integrating the characteristic system (33) for the mean velocities $\widetilde{U}_z$ and $\widetilde{U}_r$ yields

$$\widetilde{U}_r(\eta) = C_1(\eta) \left[\widetilde{U}_z(\eta) + \frac{\tilde{a}_{z1}}{\eta} \right] - \frac{\tilde{a}_{r1}}{\eta}, \quad (34)$$

with $\tilde{a}_{z1} = a_{z1}/a_{Ss1}$ and $\tilde{a}_{r1} = a_{r1}/a_{Ss1}$. As η is an invariant, the constant of integration $C_1(\eta)$ is a free function of η . We rewrite the invariant function (34) as

$$\eta \widetilde{U}_r(\eta) = \left[\eta + \frac{1}{G'_1(\eta)} \right] [\eta \widetilde{U}_z(\eta) + \tilde{a}_{z1}] - \tilde{a}_{r1}, \quad (35)$$

where we define

$$G'_1(\eta) = \frac{1}{C_1(\eta) - \eta}. \quad (36)$$

To obtain physical results, the free function G_1 has to be chosen in this specific way, as outlined in the Appendix. Inserting (35) into (25) yields

$$\frac{d\eta\tilde{U}_z(\eta)}{d\eta} = \left(\frac{G_1''(\eta)}{G_1'(\eta)} - G_1'(\eta) \right) (\eta\tilde{U}_z(\eta) + \tilde{a}_{z1}). \quad (37)$$

The solution of (37) by separation of variables is

$$\begin{aligned} \tilde{U}_z(\eta) &= A_1 \frac{G_1'(\eta)}{\eta} e^{-G_1(\eta)} - \frac{\tilde{a}_{z1}}{\eta}, \\ \tilde{U}_r(\eta) &= A_1 \left(G_1'(\eta) + \frac{1}{\eta} \right) e^{-G_1(\eta)} - \frac{\tilde{a}_{r1}}{\eta}. \end{aligned} \quad (38)$$

The solution (38) is not yet useful, as the function G_1 is still unknown. However, due to the axisymmetric geometry of the problem, G_1 must be an even function of η [1], and can therefore be written as a Taylor series

$$G_1(\eta) = \gamma_1 \eta^2 + \sum_{k=2}^{\infty} \lambda_{1,2k} \eta^{2k}. \quad (39)$$

We choose this form, as the coefficient of η^0 can be morphed into the constant A_1 , while the special naming of the quadratic coefficient γ_1 helps us to stay consistent with the work of Nguyen *et al.* [5]. In the following, we use that $G_1(0) = 0$ and implicitly assume that $\lim_{\eta \rightarrow \infty} G_1(\eta) = \infty$. Inserting (39) into (38) and evaluating at $\eta = 0$ yields

$$A_1 = \frac{1}{2\gamma_1}, \quad \tilde{a}_{z1} = 0, \quad (40)$$

since the boundary condition $\tilde{U}_z(\eta = 0) = 1$ must hold. The boundary condition $\lim_{\eta \rightarrow 0} \tilde{U}_r(\eta) = 0$ gives

$$\tilde{a}_{r1} = A_1 = \frac{1}{2\gamma_1}. \quad (41)$$

Truncating the series (39) after a certain number of terms and then empirically determining its coefficients leads to highly accurate approximations of the mean velocity profiles.

The reason why this is true might lie in the fact that the velocity profiles tend to zero for large η . Approximations of the velocity profile are therefore needed, especially for small η , which is exactly where a truncated Taylor series can be expected to be a good approximation of the actual function. There is of course a trade-off between accuracy and the number of fitted parameters, as with an increasing number of parameters, the assumed form of the profiles becomes less meaningful; see the popular story of “von Neumann’s elephant”. If the Taylor series (39) is truncated after the first term, the resulting profile for the axial velocity is

$$\tilde{U}_z(\eta) = e^{-\gamma_1 \eta^2}, \quad (42)$$

which is the Gaussian curve that has been observed empirically in numerous studies [2]. It is remarkable that this profile directly results as a first-order approximation from the symmetries of the moment equations. So far, there has been no theoretical derivation of this profile that goes beyond loose analogies to heat transfer. The previous work by Nguyen *et al.* [5] also only motivates and does not thoroughly derive this profile from Lie-symmetry theory, using a spurious scaling symmetry in η and some empirical assumptions in the process. The mean radial velocity profile resulting from the axial velocity following a Gaussian curve is

$$\tilde{U}_r(\eta) = \eta e^{-\gamma_1 \eta^2} - \frac{1}{2\gamma_1 \eta} (1 - e^{-\gamma_1 \eta^2}), \quad (43)$$

TABLE I. Best-fit coefficients for the Gaussian profile (42), the profile derived by Pope (47), and the profile derived in the present work (45). The fits were determined on the data of Nguyen and Oberlack [10] (Re = 3500). The parameter for Pope's profile was determined from (48).

Gaussian	Pope		Present work	
γ_1	β	γ_1	$\lambda_{1,4}$	$\lambda_{1,6}$
94.7	56.6	88.6	-266	19 000

which was first found by Mih [4]. We find here that truncating (39) after the third term, i.e.,

$$G_1(\eta) = \gamma_1 \eta^2 + \lambda_{1,4} \eta^4 + \lambda_{1,6} \eta^6, \quad (44)$$

provides very good results while keeping the number of parameters low. The resulting profiles are

$$\tilde{U}_z(\eta) = \left(1 + 2 \frac{\lambda_{1,4}}{\gamma_1} \eta^2 + 3 \frac{\lambda_{1,6}}{\gamma_1} \eta^4\right) e^{-\gamma_1 \eta^2 - \lambda_{1,4} \eta^4 - \lambda_{1,6} \eta^6} \quad (45)$$

and

$$\tilde{U}_r(\eta) = \left(\eta + 2 \frac{\lambda_{1,4}}{\gamma_1} \eta^3 + 3 \frac{\lambda_{1,6}}{\gamma_1} \eta^5\right) e^{-\gamma_1 \eta^2 - \lambda_{1,4} \eta^4 - \lambda_{1,6} \eta^6} - \frac{1}{2\gamma_1 \eta} (1 - e^{-\gamma_1 \eta^2 - \lambda_{1,4} \eta^4 - \lambda_{1,6} \eta^6}). \quad (46)$$

Another common profile

$$\tilde{U}_z(\eta) = \frac{1}{(1 + \beta \eta^2)^2} \quad (47)$$

was derived by Pope [1] under the assumption of constant turbulent viscosity. Here, the parameter β is related to γ_1 in (42) as

$$\beta = \frac{\sqrt{2} - 1}{\ln(2)} \gamma_1, \quad (48)$$

by equating the spreading rate $S = r_{1/2}(z)/(z - z_0)$ associated with both profiles. The half radius $r_{1/2}$ is implicitly defined as $\tilde{U}_z(r_{1/2}, z) = \frac{1}{2} \tilde{U}_z(r = 0, z)$. The corresponding radial velocity profile is

$$\tilde{U}_r(\eta) = \frac{\eta - \beta \eta^3}{2(1 + \beta \eta^2)^2}. \quad (49)$$

Fitting the coefficients of the classic Gaussian profile (42) and the profile (45) derived here on the direct numerical simulation (DNS) data of a turbulent round jet at Re = 3500 by Nguyen and Oberlack [10] results in the values reported in Table I, where the coefficient β associated with the profile (47) was determined via (48). Note that, to yield physical results, $\lim_{\eta \rightarrow \infty} G_1(\eta) = \infty$ must hold; thus, the leading coefficient $\lambda_{1,6}$ is forced to be positive. When fitting, only the axial velocity profile $\tilde{U}_z$ is used to determine the parameters; the profile of $\tilde{U}_r$ serves as validation. The visualization of the velocity profiles in Fig. 2 also reveals that the approximation (45) is visually indistinguishable from the data and fits significantly better than the often reported Gaussian profile (42). The profile (47) derived by Pope deviates from the data at the edges of the jet. This follows from the fact that the assumption of constant turbulent viscosity becomes invalid at the edge of the jet [1]. The radial velocity data do not collapse well on the outer edge of the jet, yet the profile (46) derived in this work and the profile (43) derived from the Gaussian show reasonable agreement with the data. For $\eta < 0.2$, the profile (46) matches better than (43), while for $\eta > 0.2$ the data is too spread out to judge which one fits better. Pope's profile (49) only fits well close to the center and shows qualitatively wrong behavior for $\eta > 0.1$.

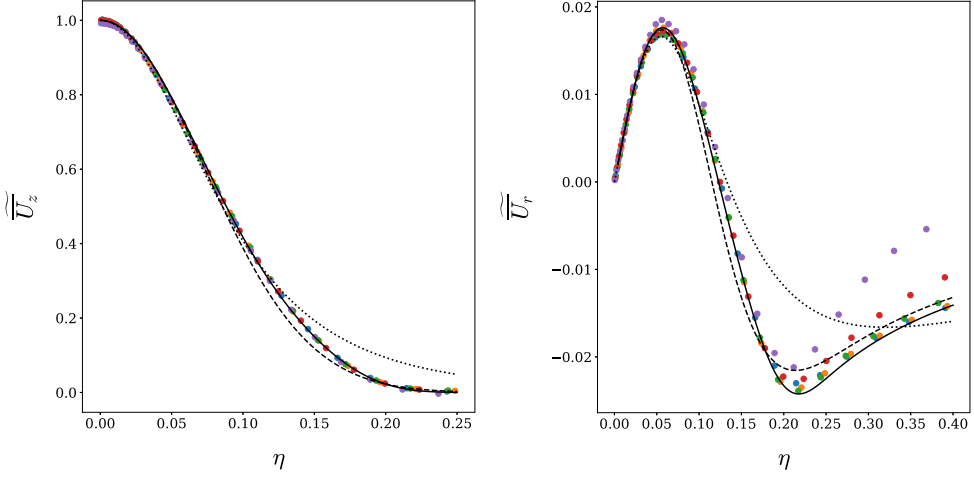


FIG. 2. Left: Comparison of the approximation (45) for $\widetilde{U_z}$ derived in this work (solid), the well-known Gaussian profile (42) (dashed), and the profile (47) derived by Pope [1] (dotted) to the DNS data from Nguyen and Oberlack [10] (Re = 3500, colored dots, blue: $z = 25D$, red: $z = 35D$, orange: $z = 45D$, purple: $z = 55D$, green: $z = 65D$). Right: Comparison of the corresponding profiles of $\widetilde{U_r}$, (46), (43), and (49). $\widetilde{U_z}$ is used for fitting profiles, $\widetilde{U_r}$ for validation.

V. SECOND VELOCITY MOMENTS

We now repeat the procedure done for the mean velocities to find the profiles of the second-order velocity moments $\widetilde{U_z^2}$, $\widetilde{U_z U_r}$, $\widetilde{U_r^2}$ and $\widetilde{U_\varphi^2}$. To turn (29) into a solvable ODE, we again integrate the characteristic system (33), yielding

$$\begin{aligned}\widetilde{U_z U_r}(\eta) &= C_{zr2}(\eta)(\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}), & \widetilde{U_\varphi^2}(\eta) &= C_{\varphi\varphi2}(\eta)(\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}), \\ \widetilde{U_r^2}(\eta) &= C_{rr2}(\eta)(\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}) - \frac{\tilde{a}_{r2}}{\eta},\end{aligned}\quad (50)$$

with unknown free functions C_{zr2} , $C_{\varphi\varphi2}$, and C_{rr2} . Inserting into (29) leads to

$$\begin{aligned}\frac{d}{d\eta} \left([\eta(C_{rr2}(\eta) - 1) + (1 - \eta^2)C_{zr2}(\eta)] [\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}] \right) \\ = \left[\frac{1}{\eta} C_{zr2}(\eta) + C_{\varphi\varphi2}(\eta) - 1 \right] [\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}].\end{aligned}\quad (51)$$

By redefining the free functions to

$$\begin{aligned}H_2(\eta) &= C_{rr2}(\eta) - C_{\varphi\varphi2}(\eta) - \eta C_{zr2}(\eta), \\ F_2(\eta) &= -\frac{1}{\eta} C_{zr2}(\eta) + 1,\end{aligned}\quad (52)$$

and

$$G'_2(\eta) = \frac{H_2(\eta)}{\eta C_{rr2}(\eta) - \eta^3 - (\eta - \eta^3)F_2(\eta)},\quad (53)$$

equation (51) becomes

$$\frac{d}{d\eta} \left(\frac{H_2(\eta)}{G'_2(\eta)} [\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}] \right) = \left(-G'_2(\eta) + \frac{1}{\eta} \right) \left(\frac{H_2(\eta)}{G'_2(\eta)} [\widetilde{U_z^2}(\eta) + \tilde{a}_{z2}] \right). \quad (54)$$

Analogously to the first-order moments, the definition of the free function G_2 in (53) follows the procedure laid out in the Appendix to yield physical results. Solving (54) yields

$$\widetilde{U_z^2}(\eta) = A_{2,2} \frac{\eta G'_2(\eta)}{H_2(\eta)} e^{-G_2(\eta)} - \tilde{a}_{z2}. \quad (55)$$

Using (50) and (52), the expressions

$$\begin{aligned} \widetilde{U_z U_r}(\eta) &= A_{2,2} \eta^2 [1 - F_2(\eta)] \frac{G'_2(\eta)}{H_2(\eta)} e^{-G_2(\eta)}, \\ \widetilde{U_\phi^2}(\eta) &= A_{2,2} F_2(\eta) \frac{\eta G'_2(\eta)}{H_2(\eta)} e^{-G_2(\eta)} + A_{2,2} (1 - \eta G'_2(\eta)) e^{-G_2(\eta)}, \\ \widetilde{U_r^2}(\eta) &= A_{2,2} C_{rr2}(\eta) \frac{\eta G'_2(\eta)}{H_2(\eta)} e^{-G_2(\eta)} - \frac{\tilde{a}_{r2}}{\eta} \end{aligned} \quad (56)$$

follow. However, we want to eliminate the free function C_{rr2} by expressing it in terms of G_2 , H_2 , and F_2 . By rearranging (53), we obtain

$$C_{rr2}(\eta) = F_2(\eta) + \eta^2 [1 - F_2(\eta)] + \frac{H_2(\eta)}{G'_2(\eta)} \frac{1}{\eta}. \quad (57)$$

Thus, the adequate expression for $\widetilde{U_r^2}$ becomes

$$\widetilde{U_r^2}(\eta) = A_{2,2} \{F_2(\eta) + \eta^2 [1 - F_2(\eta)]\} \frac{\eta G'_2(\eta)}{H_2(\eta)} e^{-G_2(\eta)} - \frac{\tilde{a}_{r2}}{\eta} + A_{2,2} e^{-G_2(\eta)}. \quad (58)$$

Assuming again that $\widetilde{U_z^2}$ is an even function of η , G_2 and H_2 can be developed as an even Taylor series

$$\begin{aligned} G_2(\eta) &= \gamma_2 \eta^2 + \sum_{k=2}^{\infty} \lambda_{2,2k} \eta^{2k}, \\ H_2(\eta) &= \sum_{k=0}^{\infty} \sigma_{2k} \eta^{2k}. \end{aligned} \quad (59)$$

Additionally, $\widetilde{U_\phi^2}$ and $\widetilde{U_r^2}$ should be even and $\widetilde{U_z U_r}$ should be odd. The logical choice then is to develop F_2 as an even series

$$F_2(\eta) = \sum_{k=0}^{\infty} \mu_{2k} \eta^{2k} \quad (60)$$

as well. When inserting (59) and (60) into (55), the boundary conditions $\widetilde{U_z^2}(\eta=0) = 1$ and $\lim_{\eta \rightarrow \infty} \widetilde{U_z^2}(\eta) = 0$ enforce

$$\sigma_0 = 0, \quad A_{2,2} = \frac{\sigma_2}{2\gamma_2}, \quad \tilde{a}_{z2} = 0, \quad (61)$$

while

$$\tilde{a}_{r2} = 0 \quad (62)$$

TABLE II. Best-fit coefficients for the second-order velocity-moment profiles (64) derived in the present work. The fits were determined on the data of Nguyen and Oberlack [10] (Re = 3500).

G_2	H_2		F_2	
γ_2	σ_2	σ_4/σ_2	μ_0	μ_2
85.2	0.883	108	0.0311	15.8

was inferred from the fact that $\widetilde{U_r^2}$ must be bounded in the limit $\eta \rightarrow 0$. We find that, in order to obtain good results, G_2 can be truncated after the first term, while H_2 and F_2 retain a second term; thus,

$$\begin{aligned} G_2(\eta) &= \gamma_2 \eta^2, \\ H_2(\eta) &= \sigma_2 \eta^2 + \sigma_4 \eta^4, \\ F_2(\eta) &= \mu_0 + \mu_2 \eta^2. \end{aligned} \quad (63)$$

The profiles of the second velocity moments then follow to be

$$\begin{aligned} \widetilde{U_z^2}(\eta) &= \frac{1}{1 + \frac{\sigma_4}{\sigma_2} \eta^2} e^{-\gamma_2 \eta^2}, \\ \widetilde{U_z U_r}(\eta) &= \frac{\eta(1 - \mu_0 - \mu_2 \eta^2)}{1 + \frac{\sigma_4}{\sigma_2} \eta^2} e^{-\gamma_2 \eta^2}, \\ \widetilde{U_\varphi^2}(\eta) &= \frac{\mu_0 + \mu_2 \eta^2}{1 + \frac{\sigma_4}{\sigma_2} \eta^2} e^{-\gamma_2 \eta^2} + \sigma_2 \left(\frac{1}{2\gamma_2} - \eta^2 \right) e^{-\gamma_2 \eta^2}, \\ \widetilde{U_r^2}(\eta) &= \frac{\mu_0 + (1 - \mu_0 + \mu_2) \eta^2 - \mu_2 \eta^4}{1 + \frac{\sigma_4}{\sigma_2} \eta^2} e^{-\gamma_2 \eta^2} + \frac{\sigma_2}{2\gamma_2} e^{-\gamma_2 \eta^2}. \end{aligned} \quad (64)$$

To obtain physical results, the denominator and therefore σ_4/σ_2 must be positive; otherwise, all profiles would have a pole. An interesting observation is that

$$\lim_{\eta \rightarrow 0} \widetilde{U_r^2}(\eta) = \lim_{\eta \rightarrow 0} \widetilde{U_\varphi^2}(\eta) = \mu_0 + \frac{\sigma_2}{2\gamma_2}, \quad (65)$$

which must be true because at small radii, the radial and azimuthal components of the flow become indistinguishable [1]. That this result follows directly from the present theory without being explicitly forced serves as a further validation of the presented approach. Fitting the coefficients of the axial velocity profiles in (64) on the DNS data leads to the values seen in Table II. For the fitting, only the data of $\widetilde{U_z^2}$, $\widetilde{U_z U_r}$, and $\widetilde{U_\varphi^2}$ were used; thus, $\widetilde{U_r^2}$ serves as validation here. Since the coefficient σ_2 is relatively small, it was more stable to use σ_4/σ_2 in fitting, instead of σ_4 . It is also interesting to note that γ_1 and γ_2 are quite similar in their numerical value. The visualization of the velocity profiles in Fig. 3 also reveals that the approximations fit the true profiles well. However, for the validation case $\widetilde{U_r^2}$, this is only true for $\eta > 0.15$. This is likely due to the fact that for $\eta < 0.15$, the terms $-\eta \widetilde{U_z^2}$ and $(1 - \eta^2) \widetilde{U_z U_r}$ on the left-hand side of (29) are significantly larger than $\eta \widetilde{U_r^2}$. Therefore, Eq. (29) is already quite accurately fulfilled for $\eta < 0.15$ if the profiles of $\widetilde{U_z^2}$ and $\widetilde{U_z U_r}$ are accurate, which is the case, as seen in Fig. 3. This leaves little information in (29) to determine $\widetilde{U_r^2}$ from, leading to the observed deviation.

Figure 3 also compares the profile for $\widetilde{U_z U_r}$ to the one derived by Mih [4] using the boundary layer approximation. Up until the peak at $\eta \approx 0.06$, Mih's profile shows good agreement with the

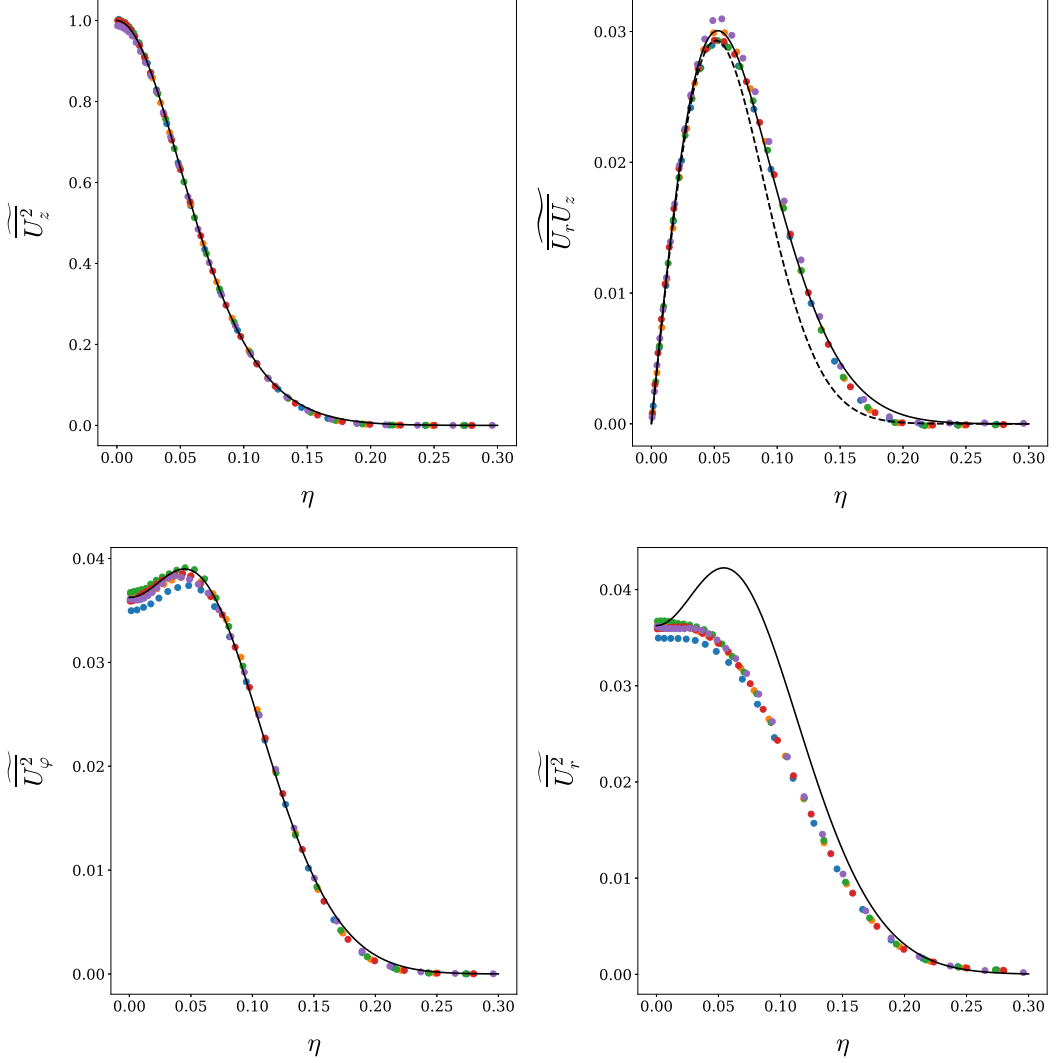


FIG. 3. Comparison of the approximations (64) derived in this work (solid) and the profile for $\widetilde{U_z U_r}$ derived by Mih [4] (dashed) to the DNS data from Nguyen and Oberlack [10] ($\text{Re} = 3500$, colored dots, blue: $z = 25D$, red: $z = 35D$, orange: $z = 45D$, purple: $z = 55D$, green: $z = 65D$). $\widetilde{U_z^2}$, $\widetilde{U_z U_r}$, and $\widetilde{U_\varphi^2}$ are used for fitting, $\widetilde{U_r^2}$ for validation.

data. However, for $\eta > 0.06$, it deviates from the data while maintaining a qualitatively correct trend. The profile presented in the present work captures the profile better, only slightly deviating at $\eta \approx 0.18$. This provides evidence that employing the boundary layer approximation yields qualitatively correct, but quantitatively inaccurate results.

VI. CONCLUSION

In the present work, we use Lie-symmetry theory to determine approximations for the self-similar profiles of the mean velocities and the second velocity moments in the turbulent round jet. Critically, we do not employ the boundary layer approximation. Since we cannot solve the closure problem of turbulence, these approximations still contain a number of parameters which must be fitted

empirically. To this end, we use recently published DNS data. Although the derivation employs minimal assumptions, the resulting approximate profiles show very good agreement with the data. We show that the mean velocity profiles we derive represent the data better than the well-known Gaussian profile for the axial velocity or profiles based on the assumption of constant eddy viscosity. Furthermore, the Gaussian profile, for which so far no theoretical derivation existed, follows as the first-order approximation of the profile directly from the symmetries of the underlying equations. The profiles we derive for the second velocity moments also fit the data well. The profile for the mixed radial/axial moment therein turns out to be more accurate than the profile derived from the boundary layer approximation, supporting our approach of omitting this approximation. Beyond that, the theory contains the known result that radial and azimuthal velocity moments must be equal on the centerline, without assuming this beforehand.

ACKNOWLEDGMENTS

The authors acknowledge being funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Project No. 577383178 (OB 96/59–1). The authors thank Johannes Conrad for the fruitful discussions.

DATA AVAILABILITY

The data that support the findings of this article are openly available [10].

APPENDIX: SYSTEMATIC CHOICE OF FREE FUNCTIONS

The definitions of the free functions G_1 and G_2 , i.e., Eqs. (36) and (53), might at first glance seem arbitrary. However, there is a well-defined logic behind the choice of these functions, which we explain in the following Appendix.

Both differential equations that we solve in this work, specifically

$$\frac{d}{d\eta}(\eta\tilde{U}_z + \tilde{a}_{z1}) = -\frac{C'_1(\eta)}{C_1(\eta) - \eta}(\eta\tilde{U}_z + \tilde{a}_{z1}), \quad (\text{A1})$$

which is Eq. (37) without the substitution (36), and

$$\begin{aligned} \frac{d}{d\eta}(\{\eta[C_{rr2}(\eta) - 1] + (1 - \eta^2)C_{zr2}(\eta)\}[\tilde{U}_z^2(\eta) + \tilde{a}_{z2}]) \\ = \left[\frac{1}{\eta}C_{zr2}(\eta) + C_{\varphi\varphi2}(\eta) - 1 \right][\tilde{U}_z^2(\eta) + \tilde{a}_{z2}], \end{aligned} \quad (\text{A2})$$

which is Eq. (51), are of the form

$$\frac{d}{d\eta}[L(\eta)U(\eta)] = R(\eta)U(\eta). \quad (\text{A3})$$

Here, U is the function to be determined and L and R are arbitrary functions. If $L(\eta) \neq 0 \forall \eta$, the differential equation can be rewritten as

$$\frac{d}{d\eta}[L(\eta)U(\eta)] = \frac{R(\eta)}{L(\eta)}L(\eta)U(\eta), \quad (\text{A4})$$

implying the solution

$$U(\eta) = \frac{C}{L(\eta)} e^{\int \frac{R(\eta)}{L(\eta)} d\eta} \quad (\text{A5})$$

by separation of variables. The fraction $\frac{R(\eta)}{L(\eta)}$ can be decomposed as

$$\frac{R(\eta)}{L(\eta)} = -G'(\eta) + \frac{P'_+(\eta)}{P_+(\eta)} - \frac{P'_-(\eta)}{P_-(\eta)}, \quad (\text{A6})$$

where G is an unknown function and P_+ and P_- can be determined from the definition of L and R . The solution (A5) can then be rewritten as

$$U(\eta) = \frac{CP_+(\eta)}{L(\eta)P_-(\eta)} e^{-G(\eta)}. \quad (\text{A7})$$

For the ODE of the first-order moment (A1) specifically, the definitions

$$\begin{aligned} U(\eta) &= \eta \widetilde{U}_z + \tilde{a}_{z1}, \\ L(\eta) &= 1, \\ R(\eta) &= -\frac{C'_1(\eta)}{C_1(\eta) - \eta} \end{aligned} \quad (\text{A8})$$

hold. We therefore obtain that

$$\frac{R(\eta)}{L(\eta)} = -\frac{C'_1(\eta)}{C_1(\eta) - \eta} = -\frac{1}{C_1(\eta) - \eta} - \frac{C'_1(\eta) - 1}{C_1(\eta) - \eta}. \quad (\text{A9})$$

Thus, it follows that

$$\begin{aligned} G'(\eta) &= \frac{1}{C_1(\eta) - \eta}, \\ P_+(\eta) &= 1, \\ P_-(\eta) &= C_1(\eta) - \eta. \end{aligned} \quad (\text{A10})$$

This is the definition of the free function behind the substitution (36). We observe that omitting the decomposition (A6) leads to unphysical results. It can also be readily checked that the solution implied by (A7),

$$\eta \widetilde{U}_z + \tilde{a}_{z1} = \frac{C}{C_1(\eta) - \eta} e^{-G(\eta)}, \quad (\text{A11})$$

is equivalent to the solution derived in Sec. IV, which is Eq. (38).

For the second-order moment ODE (A2), we find

$$\begin{aligned} U(\eta) &= \widetilde{U}_z^2 + \tilde{a}_{z2}, \\ L(\eta) &= \eta[C_{rr2}(\eta) - 1] + (1 - \eta^2)C_{zr2}(\eta), \\ R(\eta) &= \frac{1}{\eta}C_{zr2}(\eta) + C_{\varphi\varphi2}(\eta) - 1. \end{aligned} \quad (\text{A12})$$

The decomposition (A6) then yields

$$\frac{R(\eta)}{L(\eta)} = \frac{C_{\varphi\varphi2}(\eta) - C_{rr2}(\eta) + \eta C_{zr2}(\eta)}{\eta[C_{rr2}(\eta) - 1] + (1 - \eta^2)C_{zr2}(\eta)} + \frac{1}{\eta}, \quad (\text{A13})$$

and consequently the definitions

$$\begin{aligned} G'(\eta) &= \frac{C_{rr2}(\eta) - C_{\varphi\varphi2}(\eta) - \eta C_{zr2}(\eta)}{\eta[C_{rr2}(\eta) - 1] + (1 - \eta^2)C_{zr2}(\eta)}, \\ P_+(\eta) &= \eta, \\ P_-(\eta) &= 1. \end{aligned} \quad (\text{A14})$$

Using the simplifying definitions (52), the definition of the free function G_2 in Sec. V, i.e., (53) follows. Again, the ensuing solution of type (A7),

$$\frac{\widetilde{U_z^2}}{\widetilde{U_z^2} + \widetilde{a}_{z2}} = \frac{C\eta}{\eta[C_{rr2}(\eta) - 1] + (1 - \eta^2)C_{zr2}(\eta)} e^{-G(\eta)} \quad (\text{A15})$$

agrees with the solution (55) derived in Sec. V, if the free functions are defined as in (52).

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