

Spatial structure of the Lamb vector in a separated three-dimensional shear flow around a wing

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The Lamb vector appears in the rotational form of the Navier-Stokes equations and is the source of vortical flow nonlinearity that results in the cascade of energy from large to small scales. We investigate the three-dimensional spatial structure of this vector in the near wake of a NACA0018 wing with square wingtip profile at 10° angle of attack and chord-based Reynolds number of 5000. At these conditions, a large recirculation zone forms in the wake and the flow transitions in the separating shear layer. We analyze the spatial footprint of the three components of the Lamb vector and its divergence (the latter for both the mean and the fluctuating velocity fields). By integrating the vertical component of the Lamb vector across the cross-stream direction (which is equal to the vorticity flux for two-dimensional flows) we obtain the local aerodynamic loading along the wing. The vorticity flux is equal to zero at the trailing edge, thereby confirming the Taylor-Sears condition away from the wing tip. We also apply Helmholtz decomposition to extract the potential and solenoidal parts of the Lamb vector. The former part is dominant in the attached and separated shear layers, while the latter is present downstream of the recirculation zone where the flow is turbulent. We decompose the Euclidean norms of the Lamb vector and its potential part into terms that are determined by the time-average field only, the fluctuating field only, as well as mixed terms. We find that the mixed terms play a dominant role and are the slowest to decay.

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I. INTRODUCTION

The Lamb vector L is defined as $L = \Omega \times U$, where Ω and U are the vorticity and velocity vectors of a flow field, respectively. L carries the convective nonlinearity of the Navier-Stokes equations, and the curl of L carries vortex stretching, the process behind the energy cascade from large to small scales. For this reason, L is considered the source of vortical flow nonlinearity [1].

In the field of aerodynamics, the Lamb vector is crucial for the development of a consistent and universal theory on lift from two-dimensional (2D) airfoils and three-dimensional (3D) wings. More specifically, the lift and lift-induced drag can be determined by the volume integral of L [1–4] when the right boundary of the integration domain is perpendicular to the approaching velocity, U_∞ . The application of this theory to viscous flows has led to the discovery of alternative forms of the Kutta-Joukowski condition. This condition is conventionally applied to potential flow theory

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and imposes a smooth velocity field around a sharp trailing edge to obtain the value of circulation [5]. An alternative form is that for steady, two-dimensional flows the integral of the vorticity flux in the streamwise direction should vanish in the wake along any straight boundary normal to U_∞ . This condition was first derived by Taylor [6], and it is necessary to make the volume integral of L independent of the location of the right boundary of the integration domain as it crosses the wake of the airfoil. This condition does not require sharp trailing edges and it is applicable for attached or separated, laminar or turbulent boundary layers. It was later used by Howarth [7] to determine theoretically the lift coefficient of a thin elliptic cylinder against the angle of attack. Sears extended this condition for unsteady flows [8,9] and it is now known as the Taylor-Sears condition, see also Refs. [10,11]. Its validity was verified computationally in Ref. [12] for the case of laminar flow over a flat plate at Reynolds number 200 and different angles of attack. Perhaps surprisingly, but to the best of our knowledge, there are no other computational verifications of this condition.

Apart from aerodynamics, the Lamb vector provides a very useful tool to probe the dynamical properties of fluid flows. The structure of the Lamb vector in the viscous sublayer has been studied in Refs. [13,14]. The divergence of the Lamb vector, $\nabla \cdot L$, can be used to systematically investigate coherent structures. Indeed, $\nabla \cdot L$ appears (with a minus sign) on the right-hand side of the Poisson equation for the Bernoulli function $H = \frac{P}{\rho} + \frac{1}{2}U^2$, where P is the static pressure and ρ the density of the fluid, see Eq. (11) of Sec. II of the paper. In [15] the authors derived the transport equation of $\nabla \cdot L$ and noted that it enables the description of the interaction between neighboring fluid parcels with high and low momentum, which results in a change of momentum over time. Other applications of $\nabla \cdot L$ involve the analysis of the dynamical mechanism of energy loss caused by cavitation in a hydrofoil [16], the effect of compressibility around a square cylinder with wavy axis [17], the transport of momentum between different regions in a transonic compressor cascade [18], and the noise characteristics of transonic deep buffeting around an airfoil [19], etc.

In [20] the Lamb vector was analyzed in the context of turbulence modeling. The authors found that the task of modeling the Reynolds stresses can be effectively addressed by focusing on modeling the mean turbulent Lamb vector, defined as $\langle \omega \times u \rangle$, where lowercase variables denote turbulent fluctuations and angular brackets the time-averaging operation. The transport equation of $\langle \omega \times u \rangle$ was derived, and interestingly, it was found that the mean turbulent Lamb vector is not advected by the mean velocity (it is thus a diffusive as opposed to advected quantity) and is determined by the nonlinear source term that contains double and triple correlations.

A small number of studies have focused on the Helmholtz decomposition of L into irrotational (potential) and solenoidal components in homogeneous isotropic turbulence (HIT). This decomposition can provide deep understanding of the turbulent dynamics because the vortex stretching process (responsible for the cascade of energy from large to small eddies) depends only on the solenoidal component (see Sec. II). For HIT, the relative magnitudes of the two components have been compared with respect to the reference case of Gaussian velocity and vorticity fields that satisfy only the kinematic condition of incompressibility. A reduction of the contribution of the solenoidal component in HIT is interpreted as a reduction of nonlinearity [21–23]. This suppression of nonlinearity is also known as “Eulerization.” There is a hypothesis that turbulent flows spend a significant amount of time in phase space near Euler solutions [24,25]. Recent direct numerical simulations (DNS) have, however, challenged this hypothesis. More specifically, Ref. [26] used a DNS database of HIT with Reynolds numbers (based on the Taylor microscale) in the range $Re_\lambda = 38 - 1500$ and determined the ratios between the potential and solenoidal parts of the Lamb vector. It was found that the ratio increases linearly with the logarithm of Re_λ , and for $Re_\lambda = 1300$ the two parts are almost equal. This indicates that the Navier-Stokes dynamics strengthens the solenoidal part at higher Reynolds numbers, contrary to its behavior at low Reynolds numbers, where the tendency is to suppress nonlinear advection. Furthermore, the authors also observed a substantial amount of dissipation in areas when nonlinearity was not present.

It clear from the above that the Lamb vector can provide deep insights into the aerodynamic characteristics of lifting bodies and the dynamics of turbulence. However, it has not received much

attention, especially in complex, highly nonhomogeneous, 3D turbulent flows. To fill this gap in the literature, in this paper we investigate the properties and the 3D spatial development of the Lamb vector and its divergence in a separated, transitional flow around a finite wing. We consider Lamb vectors due to mean as well as fluctuating fields. Moreover, this flow configuration offers an appropriate setting to examine the validity of the Taylor-Sears condition. We also investigate the Helmholtz decomposition of L , analyze the relative strengths of the potential and solenoidal parts, compare with HIT, compute the Euclidean norm of L , and quantify the contribution of the mean, fluctuating, and mixed terms.

The paper is organized as follows. In Sec. II we summarize the main properties of the Lamb vector, while the flow configuration is briefly described in Sec. III. Results are presented and discussed in Sec. IV, and we conclude in Sec. VI.

II. THE LAMB VECTOR AND ITS PROPERTIES

In this section we recall some basic properties of the Lamb vector. For incompressible flow, the Navier-Stokes and continuity equations take the form

$$\frac{\partial U}{\partial t} + U \cdot \nabla U = -\frac{1}{\rho} \nabla P + \nu \nabla^2 U, \quad (1a)$$

$$\nabla \cdot U = 0, \quad (1b)$$

where U is the velocity vector, P the static pressure, ρ the density of the fluid, and ν its kinematic viscosity. An alternative formulation of the convective acceleration (also known as the Lagrange acceleration formula [27]) is

$$U \cdot \nabla U = \underbrace{\Omega \times U}_{\text{Lamb vector, } L} + \underbrace{\nabla \frac{1}{2}(U \cdot U)}_{\text{kinetic energy}}, \quad (2)$$

where Ω is the vorticity and $L = \Omega \times U$ is the Lamb vector. Substituting (2) into (1a), the rotational form of the Navier-Stokes equations can be obtained [1]:

$$\frac{\partial U}{\partial t} + \Omega \times U = -\nabla \left(\frac{P}{\rho} + \frac{1}{2} U^2 \right) - \nu \nabla \times \Omega, \quad (3)$$

where we have applied the vector identity $\nabla \times \Omega = -\nabla^2 U$ (due to continuity) to write the diffusion term as a function of vorticity. The above form clearly shows that viscosity has no effect on the transport of momentum if vorticity is zero (irrotational flow).

We define the Bernoulli function H as [1,15,28]

$$H = \frac{P}{\rho} + \frac{1}{2} U^2, \quad (4)$$

and substituting in (3) we get

$$\frac{\partial U}{\partial t} + \Omega \times U = -\nabla H - \nu \nabla \times \Omega. \quad (5)$$

The Bernoulli function H denotes the total energy of a fluid particle (per unit mass) in an incompressible flow. For a steady and inviscid flow Eq. (5) becomes

$$L = \Omega \times U = -\nabla H. \quad (6)$$

Taking the dot product with the unit vector tangent to a streamline $s = U/|U|$ we find $(\Omega \times U) \cdot s = -(\nabla H) \cdot s \Rightarrow 0 = -(\nabla H) \cdot s = -\frac{\partial H}{\partial s}$, because L and s are perpendicular. This means that under these conditions H is constant along a streamline, but it can vary from one streamline to another; this is the well-known Bernoulli equation. In order for H to be constant throughout the domain,

the Lamb vector should vanish; this would be the case for an irrotational flow (where $\Omega = 0$) or Beltrami flow (where Ω is parallel to U). We will return to these special cases later in this section.

Taking the curl of Eq. (3), we obtain the transport equation for vorticity:

$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times U) = \nu \nabla^2 \Omega. \quad (7)$$

This equation is more convenient than the original set (1) due to the absence of pressure. The curl of the Lamb vector $\nabla \times L = \nabla \times (\Omega \times U)$ can be decomposed as

$$\nabla \times (\Omega \times U) = \underbrace{U \cdot \nabla \Omega}_{\text{convective term}} - \underbrace{\Omega \cdot \nabla U}_{\text{vortex stretching}}. \quad (8)$$

The vortex stretching term $\Omega \cdot \nabla U$ is responsible for the cascade of energy from large to small scales [29,30] and the generation of small-scale fluctuations. Equation (8) shows that the curl of the Lamb vector is intrinsically linked with this process.

The following relation can be obtained using vector algebra [1]:

$$|\Omega \times U|^2 + |U \cdot \Omega|^2 = |U|^2 |\Omega|^2, \quad (9)$$

where $U \cdot \Omega$ is known as helicity. For a two-dimensional flow, helicity is 0 (because U and Ω are perpendicular); thus the cosine of the angle between Ω and U ,

$$\cos(U, \Omega) = \frac{U \cdot \Omega}{|U| |\Omega|}, \quad (10)$$

is a measure of the three-dimensionality of the flow.

Taking the divergence of (3), we obtain

$$\nabla \cdot L = -\nabla^2 H. \quad (11)$$

This can be considered as a Poisson equation from which H (and therefore pressure) can be obtained from the divergence of the Lamb vector that acts as source term. Hamman *et al.* [15] derived the transport equation of $\nabla \cdot L$ and showed that it accurately describes the dynamical mechanisms by which adjacent high- and low-momentum fluid parcels interact to effect a time rate of change of momentum and generate forces such as drag.

Using Helmholtz decomposition, L can be written as

$$L = \nabla \alpha + \nabla \times \beta, \quad (12)$$

where $\nabla \alpha$ is the potential component and $\nabla \times \beta$ the solenoidal component. For bounded domains this splitting is not unique and is invariant under a gauge transformation, i.e., $\nabla \alpha \rightarrow \nabla \alpha + \nabla \Psi$ and $\nabla \times \beta \rightarrow \nabla \times \beta - \nabla \Psi$, where $\nabla^2 \Psi = 0$, see Ref. [20]. This arbitrariness arises because boundary conditions exist for the unsplit vector L but not for each component. However, for unbounded domains with uniform conditions at infinity or homogeneous flows with periodic boundary conditions, the splitting is unique. It can easily be shown that for HIT (homogeneous isotropic turbulence) the Fourier transform of the potential part is parallel to the wave number vector while that of the solenoidal part is perpendicular to it. Using the convolution property of the Fourier transform, one can then easily show that $\langle \nabla \alpha \cdot \nabla \times \beta \rangle_{\text{HIT}} = 0$ [21,26], where $\langle \cdot \rangle_{\text{HIT}}$ denotes the spatial-averaging operator.

Substituting (12) into (5) we get

$$\frac{\partial U}{\partial t} + \nabla \times \beta = -\nabla(H + \alpha) - \nu \nabla \times \Omega, \quad (13)$$

where the potential part was absorbed into the gradient term. This means that the vortical nonlinearity of the governing equations is captured only by the solenoidal part. The importance of this part can be also seen by substituting (12) into (8). Rearranging, we find

$$\Omega \cdot \nabla U = U \cdot \nabla \Omega + \nabla^2 \beta, \quad (14)$$

where, without loss of generality, we have assumed $\nabla \cdot \beta = 0$, thus $\nabla \times (\Omega \times U) = -\nabla^2 \beta$. This means that only the solenoidal component of the Lamb vector contributes to vortex stretching.

Using the fact that $U \cdot \nabla U = \nabla \cdot (UU)$ and solving for L from Eq. (2) we find $L = \Omega \times U = \nabla \cdot (UU - \frac{U \cdot U}{2} I)$, where I is the identity matrix, i.e., L can be written as the divergence of a tensor. This implies that for HIT, $\langle L \rangle_{\text{HIT}} = 0$ and only the statistics of $\langle L \cdot L \rangle_{\text{HIT}}$ are considered. Since $\langle \nabla \alpha \cdot \nabla \times \beta \rangle_{\text{HIT}} = 0$, we find

$$\langle L \cdot L \rangle_{\text{HIT}} = \langle \nabla \alpha \cdot \nabla \alpha \rangle_{\text{HIT}} + \langle \nabla \times \beta \cdot \nabla \times \beta \rangle_{\text{HIT}}. \quad (15)$$

The contribution of the two terms to $\langle L \cdot L \rangle_{\text{HIT}}$ has been examined by Tsinober [21]. Assuming that U and Ω are isotropic random and statistically independent solenoidal vector fields (i.e., $\Omega \neq \nabla \times U$), it can be shown analytically that the contributions of the potential and solenoidal parts are almost equal. If, on the other hand, U is an isotropic Gaussian solenoidal random vector field (so that its fourth moments can be expressed in terms of its second moments) and $\Omega = \nabla \times U$, then the potential part is larger than the solenoidal part, $\langle (\nabla \alpha)^2 \rangle_{\text{HIT}} \approx 0.7 \langle L^2 \rangle_{\text{HIT}}$ and $\langle (\nabla \times \beta)^2 \rangle_{\text{HIT}} \approx 0.3 \langle L^2 \rangle_{\text{HIT}}$, or $\mathcal{R} = \langle (\nabla \times \beta)^2 \rangle_{\text{HIT}} / \langle (\nabla \alpha)^2 \rangle_{\text{HIT}} \approx 0.43$. Therefore the kinematic constraint $\Omega = \nabla \times U$ results in reduction of nonlinearity. While at low Re_λ the dynamics of governing equations contribute to further reduction in nonlinearity [31], recent direct numerical simulations performed by [26] have shown that $\mathcal{R}$ increases logarithmically with Re_λ (Reynolds number based on the Taylor microscale) as $\mathcal{R} = 0.18 \ln(\text{Re}_\lambda) - 0.38$; thus nonlinearity increases as Re_λ grows. In Sec. V we compute the ratio $\langle (\nabla \alpha)^2 \rangle / \langle L^2 \rangle$ for a three-dimensional, separated flow around a finite wing.

Taking the time average of (5) we get

$$\langle \Omega \times U \rangle = -\nabla \langle H \rangle - \nu \nabla \times \langle \Omega \rangle, \quad (16)$$

where $\langle \cdot \rangle$ denotes the time-averaging operator. This is a natural Helmholtz decomposition of $\langle L \rangle$. In the present paper, we take

$$\nabla \alpha = -\nabla H = -\nabla \left(\frac{P}{\rho} + \frac{1}{2} U^2 \right), \quad (17)$$

as in previous literature; see Eq. 8(b) of Ref. [20]. Additionally, taking the time average of (17) gives $\nabla \langle \alpha \rangle = -\nabla \langle H \rangle$, which is consistent with the natural decomposition (16). In Sec. V we employ (17) to derive an analytic expression of the Euclidean norm $\langle \nabla \alpha \cdot \nabla \alpha \rangle$ in terms of mean velocity, pressure, and correlations of their fluctuations.

A. Special cases

When the streamlines are parallel to the vorticity lines, i.e., for Beltrami flows, $L = \Omega \times U = 0$ and $|\cos(U, \Omega)| = 1$. In these flows, the fluid motion has a clearly defined helical shape, with the axis of rotation along the direction of motion. Under these conditions, the vorticity equation (7) becomes linear,

$$\frac{\partial \Omega}{\partial t} = \nu \nabla^2 \Omega, \quad (18)$$

and accurate vortex flow solutions can be obtained analytically. The momentum equation (3) becomes

$$\frac{\partial U}{\partial t} = -\nabla H + \nu \nabla^2 U, \quad (19)$$

and H can be found from the solution of the Laplace equation $\nabla^2 H = 0$, see (11), with appropriate boundary conditions. The velocity field can then be found easily from (19). The absence of vortical nonlinearity when $L = 0$ makes the solution of the flow problem particularly easy (similar to a potential flow problem).

TABLE I. Summary of different flow types.

Flow type	$\Omega \times U$	$\Omega \cdot U$	Main characteristic
Irrotational	0	0	$\Omega = 0$
2D planar	Finite	0	Absence of vortex stretching
Beltrami	0	Finite	Ω parallel to U , $L = 0$
Generalized Beltrami	Finite	Finite	$\nabla \times (\Omega \times U) = 0$

In a generalized Beltrami flow, the strict condition $L = \Omega \times U = 0$ is replaced by the weaker condition,

$$\nabla \times (\Omega \times U) = 0, \quad (20)$$

and thus $L = \Omega \times U = -\nabla \zeta$ is a solution [1]. This more general condition also allows the linearization of the Helmholtz vorticity equation (7). This flow setting belongs to one of three major flow types for which exact solutions of Navier-Stokes equations can be obtained, the other two being parallel flows and similarity solutions [32]. Note that for an inviscid and steady flow (which is governed by the Euler equations), the momentum equation (5) becomes $\Omega \times U = -\nabla H$; this is the generalized Beltrami flow condition.

In 2D or plane flows, vortex stretching is absent (because there is no out-of-plane velocity component); thus $\Omega \cdot \nabla U = 0$ and helicity $U \cdot \Omega = 0$, and from (9) the Lamb vector $L = \Omega \times U$ is maximized. The vorticity equation (7) can be simplified to

$$\frac{\partial \Omega}{\partial t} + U \cdot \nabla \Omega = \nu \nabla^2 \Omega. \quad (21)$$

We summarize the different special cases in Table I.

III. FLOW CONFIGURATION AND COMPUTATIONAL DETAILS

We consider the flow around a NACA 0018 wing at 10° angle of attack. The wing has a square wingtip profile and aspect ratio equal to 2. The geometry is the same as in Ref. [33], but the Reynolds number (based on the chord C and the free stream velocity U_∞) is $Re_c = 5000$ (while in the aforementioned reference it is 10^4). In order to reduce the computational cost, only half of the wing is simulated. More details about the code and validation for $Re_c = 10^4$ can be found in Ref. [33]. The simulation was performed with approximately 80 million cells, which resulted in a very fine resolution; the grid size was at worst 1.2 times the Kolmogorov length scale; see [34] for details.

In the following, x_1, x_2, x_3 (interchangeably used with x, y, z) represent the streamwise, cross-stream, and spanwise directions, respectively. The wing tip is located at $z/C = -1$ and the symmetry plane at $z/C = 0$, see [33]. The velocity components in the three directions (normalized with U_∞) are denoted as U_1, U_2, U_3 , respectively. Time averages are designated with angle brackets and fluctuations with lowercase variables. For example, $\langle U_i \rangle, u_i$ are the mean and fluctuating velocities, respectively, in the i th direction x_i , and $U_i = \langle U_i \rangle + u_i$. The Reynolds decomposition is applied also to all other variables, for example, $P = \langle P \rangle + p$ for static pressure.

IV. RESULTS AND DISCUSSION

A. The structure of the Lamb vector of the mean velocity field ($x - y$ plane)

The Lamb vector has a strong 3D spatial structure. We first focus on the $x - y$ plane $z/C = -0.2$, which is close to the symmetry plane, and in the next section we consider different $y - z$ planes at different x locations.

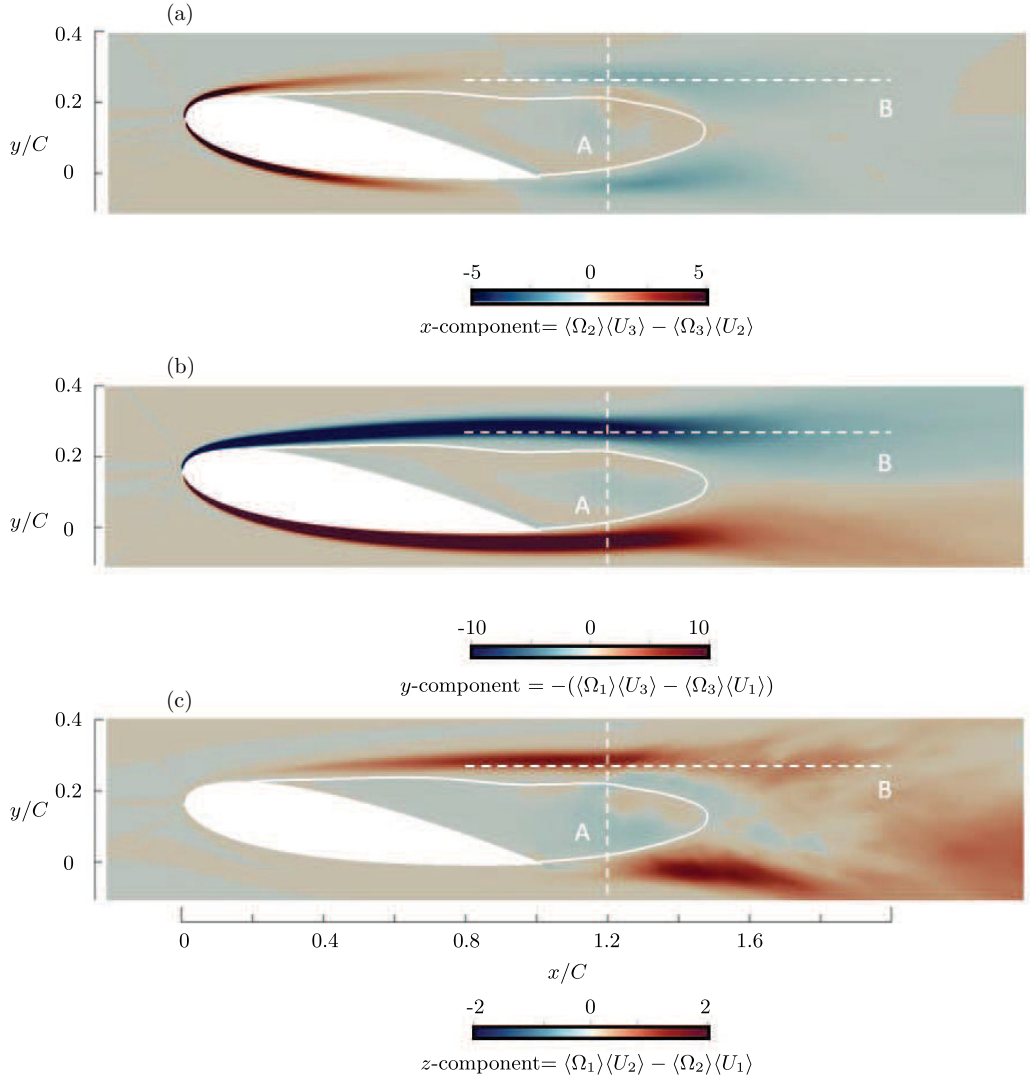


FIG. 1. Contours of the Lamb vector components of the mean field $\langle \Omega \rangle \times \langle U \rangle$, (a) x component, (b) y component, and (c) z component, in the $y - z$ plane at $z/C = -0.2$. The solid white line marks the boundary of the recirculation zone, where $\langle U_1 \rangle = 0$. Line A (placed at $x/C = 1.2$) goes through the recirculating zone and B (placed at $y/C = 0.28$) marks the separated shear layer.

Figure 1 shows contours of the three components of the Lamb vector of the time-average flow, $\langle \Omega \rangle \times \langle U \rangle$, in the $x - y$ plane. Due to the large angle of attack, a recirculation zone forms on the suction side and its boundary is demarcated by a solid white line. Two dashed lines are also shown in the figure; the vertical line A crosses the recirculation zone, while line B is placed at the height of the separated shear layer along which the flow transitions. High values of the x component of the Lamb vector, see panel (a), are found around the leading edge, the attached flow in the pressure side, and to a lesser extent to the area bordering the recirculation zone. There is a sign change after the flow passes the trailing edge in the pressure side. This change of sign is attributed to $\langle U_2 \rangle$, which initially takes negative values but turns positive slightly downstream of the trailing edge (while $\langle \Omega_3 \rangle$ retains its sign).

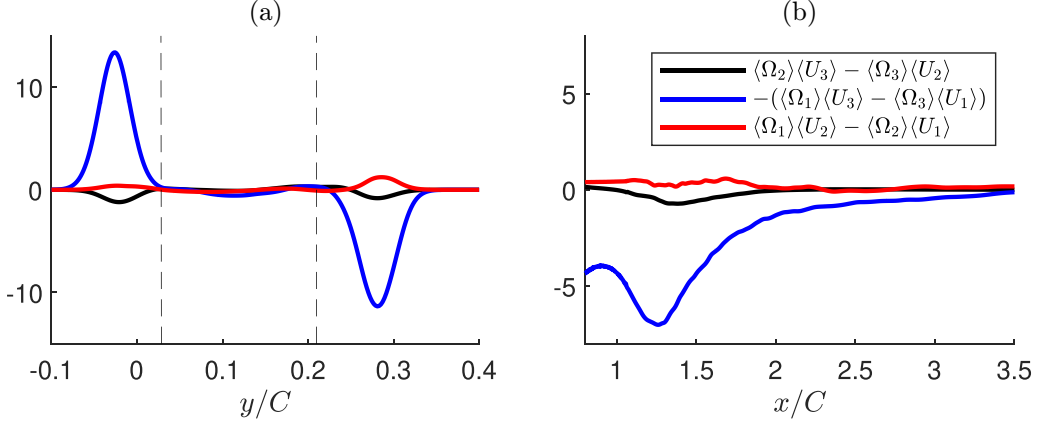


FIG. 2. Variation of the three components of $\langle \Omega \rangle \times \langle U \rangle$ (a) along line A and (b) along line B.

The y component of the Lamb vector is shown in panel (b). This is by far the strongest of the three components, as can be seen also from Fig. 2 that displays the values along lines A and B. Since $|\langle \Omega_3 \rangle \langle U_1 \rangle| \gg |\langle \Omega_1 \rangle \langle U_3 \rangle|$, the specific shape and size of the y component is determined by the profiles of spanwise vorticity $\langle \Omega_3 \rangle$ and average axial velocity $\langle U_1 \rangle$. This means that the sign of this component is determined by the sign of $\langle \Omega_3 \rangle$; this observation explains the opposite signs in the suction and pressure side of the wing, respectively; see also Fig. 2(a). Along line B, the y component has a minimum negative value at $x/C \approx 1.3$ and progressively approaches zero as the spanwise vorticity decays; see Fig. 2(b). Recall that the volume integral of the y component yields the vortical force in this direction (lift), while the integral of the x component yields the lift-induced drag; see Ref. [1]. We elaborate on the distribution of the integral along the streamwise direction in Sec. IV C below.

The third component along the z direction, shown in Fig. 1(c), contains the streamwise and cross-stream vorticities $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ and arises due to the three-dimensionality of the flow field (mean flow inhomogeneity in the spanwise direction stemming from the wing tip effect, see more details in Ref. [33]). It is the weakest of the three components.

B. The structure of the Lamb vector of the mean velocity field ($y - z$ planes)

We now examine the three components of the Lamb vector of the mean flow $\langle \Omega \rangle \times \langle U \rangle$ at three $y - z$ planes placed at streamwise locations $x/C = 0.02, 0.9, 1.2$. The results are shown in Fig. 3. The solid white line marks the boundary of the recirculation region where $\langle U_1 \rangle = 0$.

The x component, $\langle \Omega_2 \rangle \langle U_3 \rangle - \langle \Omega_3 \rangle \langle U_2 \rangle$, is shown in panel (a). Very close to the leading edge, at $x/C = 0.02$, the flow is still attached and 2D; thus the Lamb vector has a uniform distribution on the wall along the spanwise direction. At $x/C = 0.9$ and $x/C = 1.2$, the flow has separated and strong spanwise inhomogeneity is observed close to the wing tip, located at $z/C = -1$.

The y component, $-(\langle \Omega_1 \rangle \langle U_3 \rangle - \langle \Omega_3 \rangle \langle U_1 \rangle)$, is shown in panel (b). At all planes, it has opposite signs in the pressure and suction sides. After the flow separates, it has strong footprint in the shear layer outside the recirculating zone. Notice also the complex structure close to the tip due to the presence of the wing tip vortex. At $x/C = 1.2$, the main shape persists but diffuses as the wake evolves downstream.

The z component, $\langle \Omega_1 \rangle \langle U_2 \rangle - \langle \Omega_2 \rangle \langle U_1 \rangle$, is shown in panel (c). This component is due to the flow three-dimensionality. As can be seen, it has a strong presence within highly concentrated regions around the wing tip.

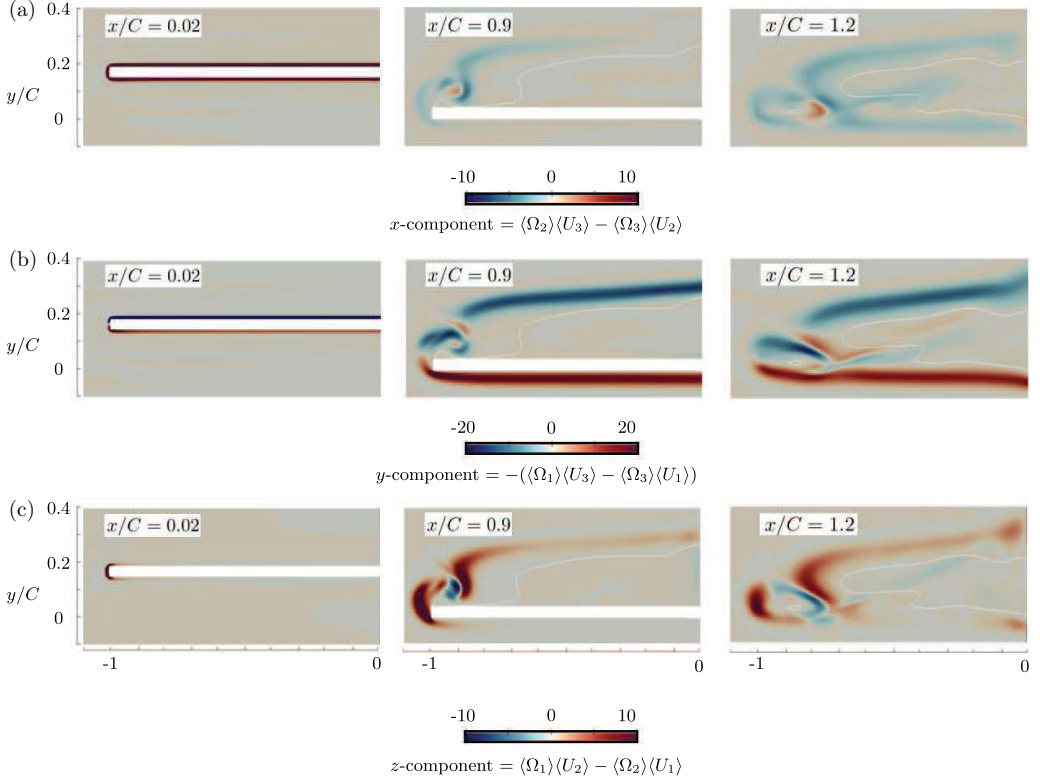


FIG. 3. Contours of the three components of the Lamb vector of the mean flow field $\langle \Omega \rangle \times \langle U \rangle$, (a) x component, (b) y component, and (c) z component, in the $y - z$ plane at $x/c = 0.02, 0.9, 1.2$, respectively. The solid white line marks the boundary of the recirculation zone, where $\langle U_1 \rangle = 0$.

C. Lift and the Lamb vector

It is well known [1] that the volume integral of the cross-stream component of the Lamb vector (in the present case the y component) in a domain with the right boundary normal to U_∞ yields the lift force, i.e.,

$$L = -\rho \int_V (\boldsymbol{\Omega} \times \mathbf{U}) \cdot \hat{\mathbf{e}}_L dx dy dz = \rho \int_V (\langle \Omega_1 \rangle \langle U_3 \rangle - \langle \Omega_3 \rangle \langle U_1 \rangle) dx dy dz, \quad (22)$$

where is $\hat{\mathbf{e}}_L$ the unit vector in the lift direction. If we define

$$l(x, z) = - \int_{y_1}^{y_2} (\boldsymbol{\Omega} \times \mathbf{U}) \cdot \hat{\mathbf{e}}_L dy = \rho \int_{y_1}^{y_2} (\langle \Omega_1 \rangle \langle U_3 \rangle - \langle \Omega_3 \rangle \langle U_1 \rangle) dy, \quad (23)$$

then

$$L = \int_Z \int_X l(x, z) dx dz, \quad (24)$$

and thus $l(x, z)$ can be considered as local aerodynamic loading. In a two-dimensional flow, $\langle U_3 \rangle = 0$ and $\langle \Omega_1 \rangle = 0$, so only the second term remains in (23) and the integral represents the vorticity flux in the streamwise direction. To compute $l(x, z)$ we chose $y_1 = -0.1$ and $y_2 = 0.4$ so the separated flow is captured entirely, see Fig. 1(b).

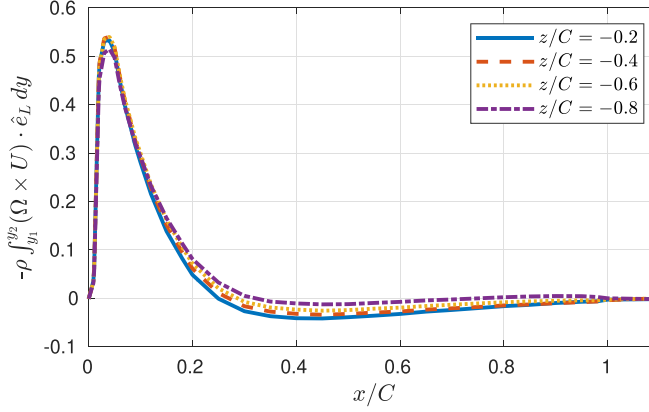


FIG. 4. Local aerodynamic loading $l(x, z) = -\rho \int_{y_1}^{y_2} (\Omega \times U) \cdot \hat{e}_L dy$ along the chord at various spanwise positions.

Figure 4 plots the loading $l(x, z)$ along x at four spanwise locations $z/C = -0.2, -0.4, -0.6$, and -0.8 . There is a linear increase from the leading edge to the peak (at $x/C = 0.04$) followed by decay as the flow separates and evolves downstream. For all spanwise positions the profiles nearly collapse close to the leading edge, suggesting that the flow is 2D. Further downstream, as the three-dimensionality takes effect, the profiles start to (slightly) deviate.

Note that although high values of the y component of the Lamb vector are observed in the separated region, see Fig. 3(b), their contribution to the integrated aerodynamic load cancels out and the value is zero at the trailing edge. This indicates that the Taylor-Sears condition [6,8,9] is satisfied at $x/C = 1$. This condition requires that the integrated vorticity flux vanishes in the wake and, strictly speaking, is valid for a two-dimensional flow only. Here we show that away from the wing tip, until $z/C = -0.8$ where the flow is (approximately) 2D as shown in Fig. 3, this condition is also valid.

D. Divergence of the Lamb vector

We now consider the divergence of the Lamb vector and employ the vector identity [15]

$$\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle) = \langle U \rangle \cdot \nabla \times \langle \Omega \rangle - \langle \Omega \rangle \cdot \langle \Omega \rangle, \quad (25)$$

where $\langle U \rangle \cdot \nabla \times \langle \Omega \rangle$ is known as the mean flexion product and $\nabla \times \langle \Omega \rangle$ is the mean flexion vector [27]. The flexion product can also be written as [15]

$$\langle U \rangle \cdot \nabla \times \langle \Omega \rangle = \langle S \rangle : \langle S \rangle + \frac{1}{2} [\langle \Omega \rangle \cdot \langle \Omega \rangle - \nabla^2 (\langle U \rangle \cdot \langle U \rangle)], \quad (26)$$

leading to the alternative expression for the divergence,

$$\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle) = \langle S \rangle : \langle S \rangle - \frac{1}{2} \langle \Omega \rangle \cdot \langle \Omega \rangle - \frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle). \quad (27)$$

Figure 5 shows the contours of the divergence of the Lamb vector and of the second and third terms on the right-hand side of (27). The divergence $\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle)$, shown in panel (a), is active in the areas of high shear (attached boundary layer and separated shear layer), where it is split into two regions of positive and negative sign adjacent to each other. The enstrophy of the mean flow $\frac{1}{2} \langle \Omega \rangle \cdot \langle \Omega \rangle$, shown in panel (b), is clearly present in the shear layers surrounding the recirculating bubble. The isoline $\langle S \rangle : \langle S \rangle = 250$ is superimposed (colored light blue) and is present around the stagnation point at the leading edge and the shear layers. The term $\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle)$, which represents the Laplacian of the kinetic energy, is shown in panel (c). It has similar shape to the divergence, but

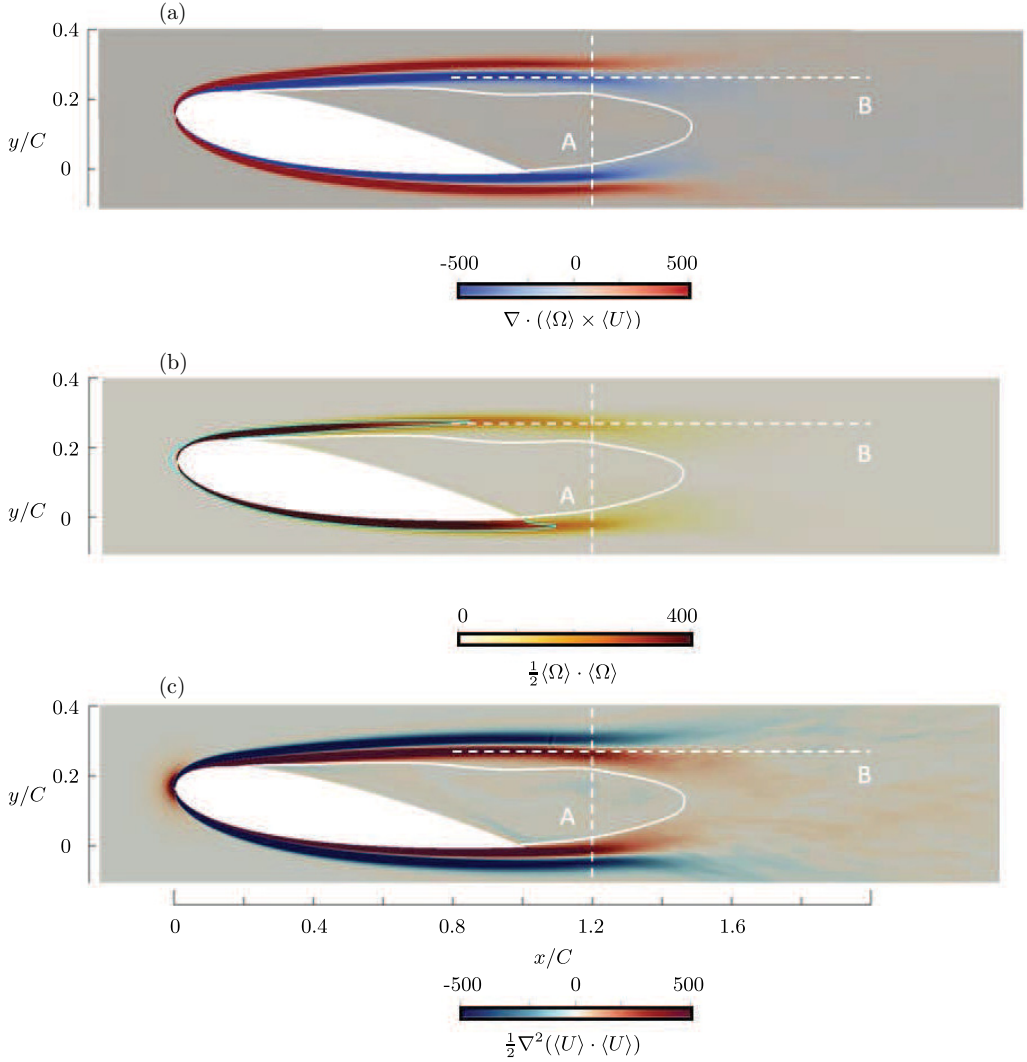


FIG. 5. Contours of (a) $\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle)$, (b) $\frac{1}{2} \langle \Omega \rangle \cdot \langle \Omega \rangle$, and (c) $\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle)$ in the $x-y$ plane at $z/C = -0.2$. The light blue isoline in (b) represents $\langle S \rangle : \langle S \rangle = 250$.

a key distinction lies in the fact that the stagnation point area also exhibits significant values (due to flow deceleration).

To gain deeper insight, the three components of $\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle)$ are plotted along lines A and B in Fig. 6. Interestingly, $\langle S \rangle : \langle S \rangle$ and $\frac{1}{2} \langle \Omega \rangle \cdot \langle \Omega \rangle$ show a nearly identical distribution. Based on this finding, we derive the following approximation:

$$\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle) \approx -\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle). \quad (28)$$

Along line A, outside the recirculation zone, $-\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle)$ has two peaks of equal size but opposite sign; this is explained by the curvature of the velocity profile. Above the recirculation zone, for $y/C > 0.3$, $-\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle) \geq 0$ (i.e., it becomes superharmonic). Between the boundary of the recirculation and $y/C \approx 0.3$, $-\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle)$ switches rapidly to subharmonic, i.e., $-\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle) \leq 0$. Inside the recirculation bubble, $-\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle)$ reaches a quasiarmonic

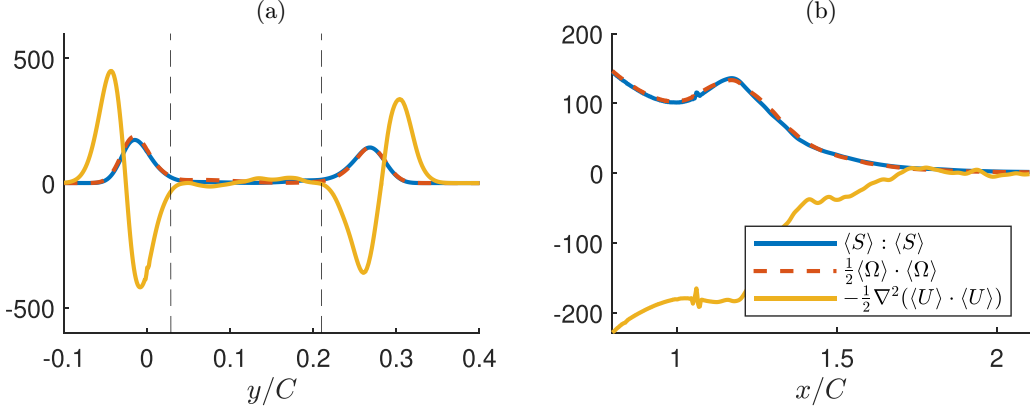


FIG. 6. The three components of the divergence $\nabla \cdot (\langle \Omega \rangle \times \langle U \rangle)$ along (a) line A and (b) line B. The vertical dashed lines in (a) denote the top and bottom boundaries of the recirculation zone.

state, i.e., approximately equal to zero. Below the recirculation zone the pattern is very similar; it is a mirror image of the pattern above. Along line B, large values of enstrophy $\frac{1}{2} \langle \Omega \rangle \cdot \langle \Omega \rangle$ are present and $-\frac{1}{2} \nabla^2 (\langle U \rangle \cdot \langle U \rangle)$ remains negative. Downstream of $x/C \approx 1.5$, all terms decay and eventually become negligible.

E. The Lamb vector of the fluctuating velocity field

We now turn our attention to the mean Lamb vector of the fluctuating field $\langle \omega \times u \rangle$. Using the fluctuating velocity field, Eq. (27) becomes

$$\nabla \cdot (\omega \times u) = s : s - \frac{1}{2} \omega \cdot \omega - \frac{1}{2} \nabla^2 (u \cdot u). \quad (29)$$

The time average and instantaneous values of the terms of (29) are plotted in the left and right columns of Fig. 7, respectively. The results in the left column show the Laplacian of the kinetic energy is negligible and based on this observation, the following approximation can be made:

$$\nabla \cdot \langle \omega \times u \rangle \approx \langle s : s \rangle - \frac{1}{2} \langle \omega \cdot \omega \rangle. \quad (30)$$

Interestingly, this is the opposite of the relation (28), in the sense that strain-squared and enstrophy canceled each other out for the mean field, while their difference approximates well the divergence of the fluctuating Lamb vector. The quantity $\langle s : s \rangle - \frac{1}{2} \langle \omega \cdot \omega \rangle$ can be considered as an indicator of small-scale inhomogeneity (it is zero for homogeneous turbulence, see [35]). Comparison of Figs. 5(a) and 7(a) indicates that $\nabla \cdot \langle \omega \times u \rangle \ll \nabla \cdot (\langle \Omega \rangle \times \langle U \rangle)$. This is not a general property; it depends on the inhomogeneity of the mean flow field and the Reynolds number.

In the right column, isolines with $Q = 2$ (defined as $Q = \frac{1}{2} (\Omega \cdot \Omega - S : S)$) are superimposed to visualize eddies of different sizes. Structures *E1* and *E2* are formed as a result of the Kelvin-Helmholtz instability that develops on the separating shear layer in the early stages of the transition process. *E3* is another large eddy stemming from the suction side. *E4* is a fine-scale eddy with high vorticity concentration, while *E5* represents a group of eddies when vortex breakdown has completed. As can be seen in panel (b), $\nabla \cdot (\omega \times u)$ changes sign across the Q isoline that demarcates the boundaries of the structures *E1* and *E2*. In panel (d), there is excess enstrophy $\frac{1}{2} \omega \cdot \omega$ inside the eddies while excess strain is found outside the Q isolines, as expected by the definition. The presence of a recirculating bubble prevents structures such as *E1* and *E2* from mixing prematurely with the eddies from the pressure side. Note also that the time-average values are at least two orders of magnitude smaller than the instantaneous. For HIT all the variables shown in the left column

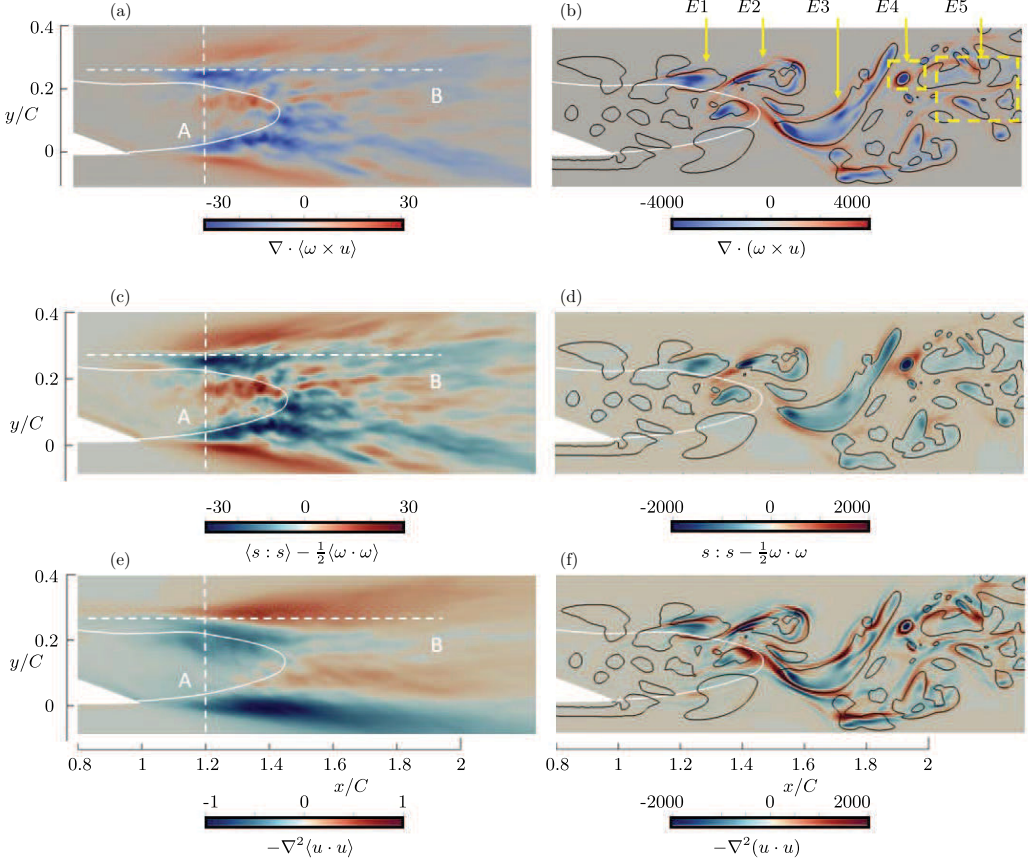


FIG. 7. Contours of $\nabla \cdot \langle \omega \times u \rangle$ and its components in the $x - y$ plane (left column); instantaneous snapshots of the same quantities (right column). Vortical structures are visualized with solid black lines with $Q = 2$ (right column).

are equal to 0 due to flow homogeneity. The (small) nonzero values in the our case are due to the small-scale inhomogeneity.

V. EUCLIDEAN NORMS OF THE LAMB VECTOR AND ITS POTENTIAL COMPONENT

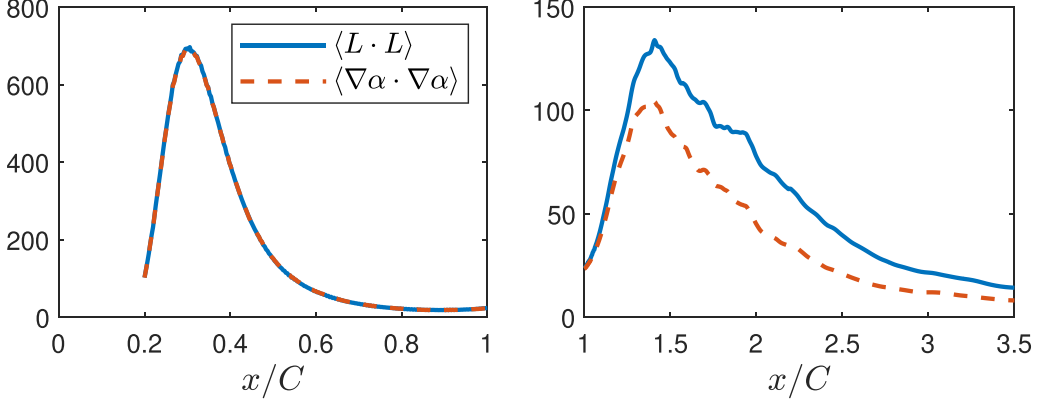
Using (12) we can express the square of the Euclidean norm of the Lamb vector $\langle L \cdot L \rangle$ as

$$\langle L \cdot L \rangle = \langle \nabla \alpha \cdot \nabla \alpha \rangle + 2 \langle \nabla \alpha \cdot \nabla \times \beta \rangle + \langle (\nabla \times \beta) \cdot (\nabla \times \beta) \rangle. \quad (31)$$

By employing Reynolds decomposition we can write $L = (\langle \Omega \rangle + \omega) \times (\langle U \rangle + u)$, and expanding we get (see Appendix for details)

$$\begin{aligned} \langle L \cdot L \rangle = & \langle (\langle \Omega \rangle \times \langle U \rangle)^2 \rangle + \langle ((\langle \Omega \rangle \times u)^2) \rangle + \langle (\omega \times \langle U \rangle)^2 \rangle + \langle (\omega \times u)^2 \rangle \\ & + 2 \langle (\langle \Omega \rangle \times \langle U \rangle) \cdot \langle \omega \times u \rangle \rangle + 2 \langle ((\langle \Omega \rangle \times u) \cdot (\omega \times u)) \rangle \\ & + 2 \langle (\omega \times \langle U \rangle) \cdot (\omega \times u) \rangle + 2 \langle (\omega \times \langle U \rangle) \cdot ((\langle \Omega \rangle \times u)) \rangle. \end{aligned} \quad (32)$$

As can be seen, the norm consists of terms that involve time-average quantities only, fluctuating quantities only, or mixed terms that comprise both. Similarly for the potential component we have

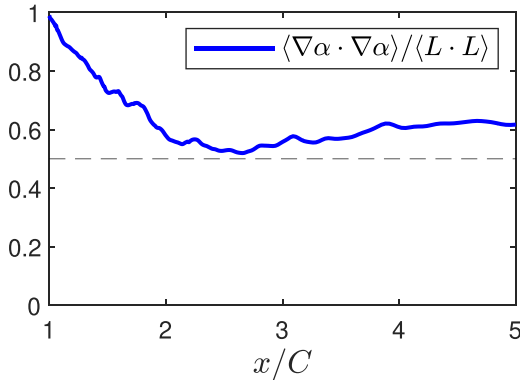

 FIG. 8. Streamwise evolution of $\langle L \cdot L \rangle$ and $\langle \nabla \alpha \cdot \nabla \alpha \rangle$ along line B.

(again, see Appendix for details)

$$\begin{aligned}
 \langle \nabla \alpha \cdot \nabla \alpha \rangle = & \frac{1}{\rho^2} \nabla \langle P \rangle \cdot \nabla \langle P \rangle + \frac{1}{\rho^2} \langle \nabla p \cdot \nabla p \rangle + \frac{1}{\rho} \nabla \langle P \rangle \cdot \nabla \langle U \rangle^2 + \frac{1}{\rho} \nabla \langle P \rangle \cdot \nabla \langle u^2 \rangle \\
 & + \frac{1}{\rho} \langle \nabla p \cdot \nabla (u \cdot \langle U \rangle) \rangle + 2 \frac{1}{\rho} \langle \nabla p \cdot \nabla u^2 \rangle + \frac{1}{4} \nabla \langle U \rangle^2 \cdot \nabla \langle U \rangle^2 + \frac{1}{2} \nabla \langle U \rangle^2 \cdot \nabla \langle u^2 \rangle \\
 & + \langle \nabla (\langle U \rangle \cdot u) \cdot \nabla u^2 \rangle + \langle (\nabla \langle U \rangle \cdot u)^2 \rangle + \frac{1}{4} \langle \nabla u^2 \cdot \nabla u^2 \rangle.
 \end{aligned} \tag{33}$$

Figure 8 shows the distribution of $\langle L \cdot L \rangle$ and $\langle \nabla \alpha \cdot \nabla \alpha \rangle$ along line B. It can be seen that for $x/C < 1$, $\langle L \cdot L \rangle = \langle \nabla \alpha \cdot \nabla \alpha \rangle$ and their values peak at $x/C \approx 0.3$ as the line intersects the separated shear layer. For $x/C > 1$, $\langle L \cdot L \rangle$ starts to diverge from $\langle \nabla \alpha \cdot \nabla \alpha \rangle$, but both peak at $x/C \approx 1.5$.

Figure 9 shows the ratio of $\langle \nabla \alpha \cdot \nabla \alpha \rangle / \langle L \cdot L \rangle$ as transition occurs. At $x/C = 1$ the ratio is close to 1 (consistent with Fig. 8) but is then reduced until $x/C \approx 2.6$, reaching a minimum value of ≈ 0.52 , and further downstream it stabilizes at ≈ 0.60 . The ratio $\mathcal{R} = \langle (\nabla \times \beta)^2 \rangle / \langle (\nabla \alpha)^2 \rangle \approx 0.33$ is thus smaller than the value of 0.43 from purely kinematic constraints (see Sec. II). This is probably due to the relatively small Re_λ , which reaches a maximum value of approximately 80 in the examined flow, see Ref. [34]. Note, however, that direct comparison with HIT is not appropriate, because in the definition of L we have used both the mean and the fluctuating fields. A more appropriate


 FIG. 9. Streamwise evolution of the ratio $\langle \nabla \alpha \cdot \nabla \alpha \rangle / \langle L \cdot L \rangle$ along line B. The horizontal dashed line is drawn at 0.5.

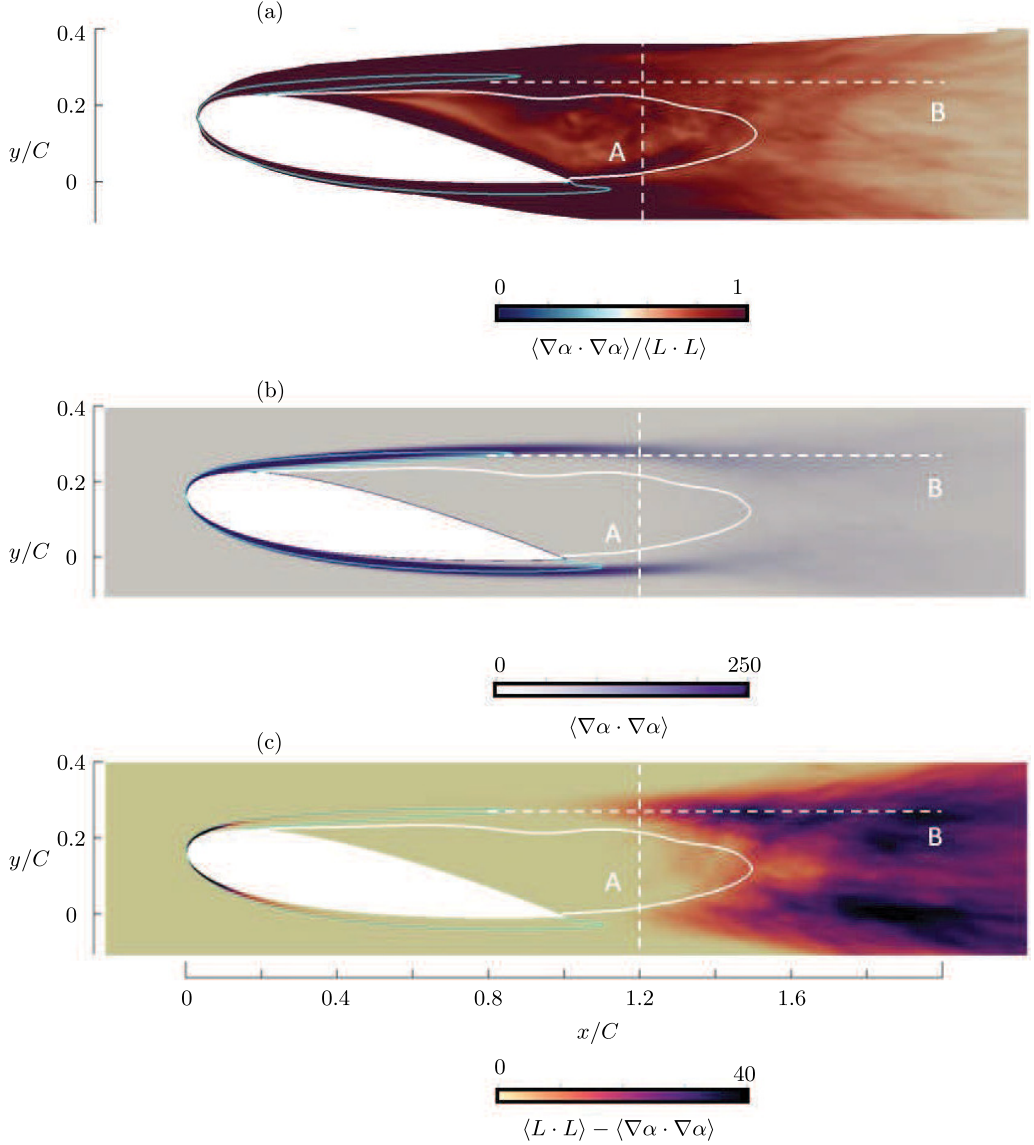
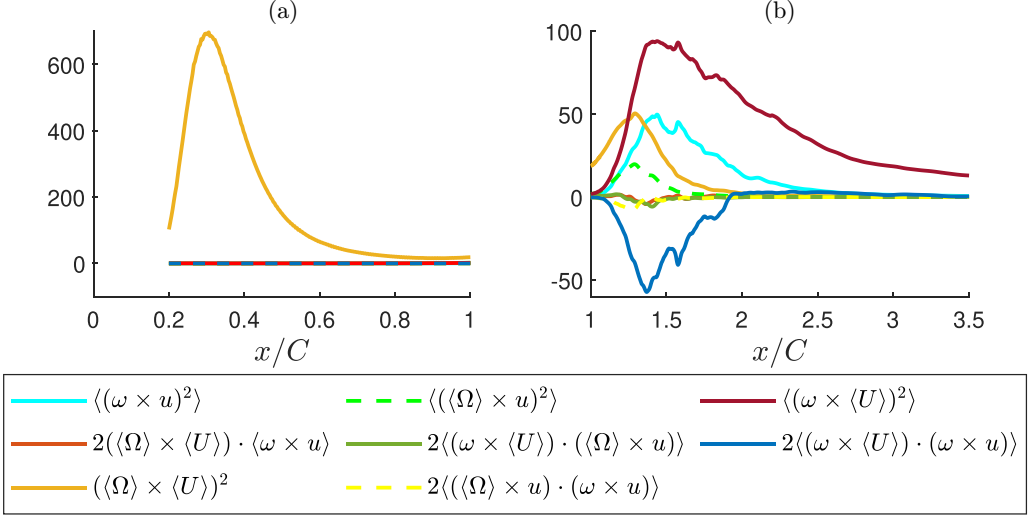


FIG. 10. Contours of (a) $\langle \nabla \alpha \cdot \nabla \alpha \rangle / \langle L \cdot L \rangle$, (b) $\langle \nabla \alpha \cdot \nabla \alpha \rangle$, and (c) $\langle L \cdot L \rangle - \langle \nabla \alpha \cdot \nabla \alpha \rangle$ in the $x-y$ plane at $z/C = -0.2$. The light blue isoline represents $\frac{1}{2} \langle \Omega \cdot \Omega \rangle = 250$.

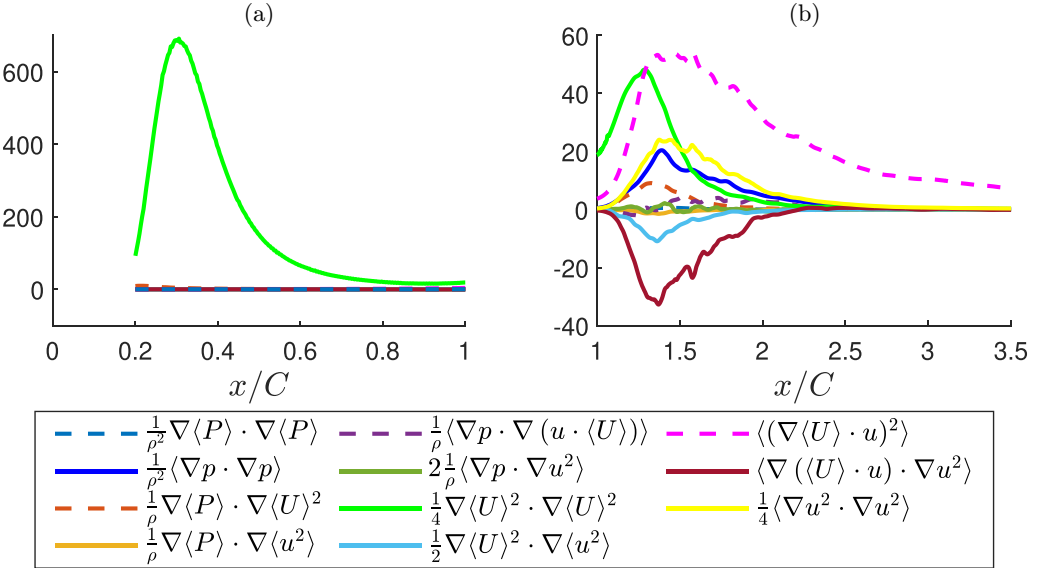
comparison would have been to apply the Helmholtz decomposition directly to the fluctuating vector $\omega \times u$.

To get further insight, contours of $\langle \nabla \alpha \cdot \nabla \alpha \rangle / \langle L \cdot L \rangle$ are shown in Fig. 10(a) in the plane $x-y$. The light blue isoline representing $\frac{1}{2} \langle \Omega \cdot \Omega \rangle = 250$ marks the two laminar shear layers (one detaching from the leading edge and the other attached in the pressure side); inside these layers the ratio is close to 1. Downstream of the recirculation zone, in the wake region, the ratio is reduced in agreement with Fig. 9. Panel (b) shows contours of the potential term $\langle \nabla \alpha \cdot \nabla \alpha \rangle$. High values are found in the shear layers, but inside the recirculation zone it has negligible values. Downstream $x/C \approx 1.2$, the contours become more diffuse because of mixing caused by the developing turbulence. Panel (c) shows contours of the difference $\langle L \cdot L \rangle - \langle \nabla \alpha \cdot \nabla \alpha \rangle$. A layer


 FIG. 11. Streamwise evolution of the components of $\langle L \cdot L \rangle$ along line B.

with significant values forms in the vicinity of the leading edge where the flow is attached; this layer is thinner compared to $\langle \nabla \alpha \cdot \nabla \alpha \rangle$. The high values decrease rapidly as the flow separates from the surface at $x/C \approx 0.2$. The region with the highest concentration of $\langle L \cdot L \rangle - \langle \nabla \alpha \cdot \nabla \alpha \rangle$ is found above and below the tip of the recirculating bubble and in the region of fully developed turbulence downstream of $x/C \approx 1.5$. This spatial distribution explains the deviation between the two lines in the right panel of Fig. 8.

We now proceed with the analysis of the individual terms comprising $\langle L \cdot L \rangle$, see Eq. (32). In panel (a) of Fig. 11, the absence of fluctuations for $x/C < 1$ makes $\langle (\Omega) \times \langle U \rangle \rangle^2$ the only term present in this region. As transition to turbulence starts, the mixed term $\langle (\omega \times \langle U \rangle)^2 \rangle$ increases


 FIG. 12. Streamwise evolution of the components of $\langle \nabla \alpha \cdot \nabla \alpha \rangle$ along line B.

rapidly, overtakes $(\langle \Omega \rangle \times \langle U \rangle)^2$, and peaks at $x/C \approx 1.4$. The turbulent term $\langle (\omega \times u)^2 \rangle$ also reaches a peak at $x/C \approx 1.4$, but its value is half of that of $\langle (\omega \times \langle U \rangle)^2 \rangle$. The other mixed term $\langle (\langle \Omega \rangle \times u)^2 \rangle$ peaks slightly earlier at $x/C \approx 1.3$, but it is weaker. All terms are positive; the only negative term is $2\langle (\omega \times \langle U \rangle) \cdot (\omega \times u) \rangle$. The rest of the terms have negligible contribution.

The terms comprising $\langle \nabla \alpha \cdot \nabla \alpha \rangle$ are shown in Fig. 12. In the laminar section, $\frac{1}{4} \nabla \langle U \rangle^2 \cdot \nabla \langle U \rangle^2$ is the dominant term; all the other terms have negligible values (including $\frac{1}{\rho^2} \nabla \langle P \rangle \cdot \nabla \langle P \rangle$ and $\frac{1}{\rho} \nabla \langle P \rangle \cdot \nabla \langle U \rangle^2$). For $x/C > 1$, the term $\frac{1}{4} \nabla \langle U \rangle^2 \cdot \nabla \langle U \rangle^2$ experiences an increase, eventually reaching values similar to $(\langle \Omega \rangle \times \langle U \rangle)^2$ at $x/C \approx 1.3$. The mixed term $\langle (\nabla \langle U \rangle u)^2 \rangle$ peaks at $x/C \approx 1.4$, achieving $\approx 50\%$ of the value of $\langle (\omega \times \langle U \rangle)^2 \rangle$. The fluctuating terms $\frac{1}{4} \langle \nabla u^2 \cdot \nabla u^2 \rangle$ and $\frac{1}{\rho^2} \langle \nabla p \cdot \nabla p \rangle$ have a similar distribution, with the former having slightly higher values than the latter. After $x/C \approx 1.3$, $\langle (\nabla \langle U \rangle u)^2 \rangle$ is the dominant component of $\langle \nabla \alpha \cdot \nabla \alpha \rangle$. This shows that along line B, the influence of the mean flow persists. The sink (negative) terms are $\langle \nabla (\langle U \rangle u) \cdot \nabla u^2 \rangle$ and $\frac{1}{2} \nabla \langle U \rangle^2 \cdot \nabla \langle u^2 \rangle$. Term $\nabla \langle P \rangle \cdot \nabla \langle U \rangle^2$ makes also a small contribution, but the rest of the terms are negligible. As shown in Fig. 9, the ratio $\langle \nabla \alpha \cdot \nabla \alpha \rangle / \langle L \cdot L \rangle$ is stabilized at $x/C \approx 2$. In this region $\langle \nabla \alpha \cdot \nabla \alpha \rangle \approx \langle (\nabla \langle U \rangle u)^2 \rangle$ and $\langle L \cdot L \rangle \approx \langle (\omega \times \langle U \rangle)^2 \rangle$, i.e., the mixed terms are dominant.

VI. CONCLUSIONS

We have investigated the spatial characteristics of the Lamb vector in a separated, three-dimensional and transitional shear flow around a finite wing. The flow field was obtained from a DNS simulation. The y (i.e., the cross-stream) component of the Lamb vector due to the time-average velocity field was found to be by far the largest one, with two clear peaks of opposite sign in the shear layers that bracket the recirculation zone. This component is responsible for the lift force acting on the wing. The local aerodynamic load distribution was computed by integrating the vertical component of the Lamb vector along the cross-stream direction. Upstream of flow separation, in the shear layers in the pressure and suction sides, the distribution of this component is strongly asymmetric, and this area contributes most to the generation of lift (as expected). Further downstream, however, the contributions of the two shear layers cancel out, resulting in the satisfaction of the Taylor-Sears condition at the trailing edge away from the wing tip.

We examined also the Lamb vector due to the fluctuating velocity field. It is present in the downstream end of the recirculation zone, above and below the tip, but its strength is smaller compared to the time-average field.

The Helmholtz decomposition was applied to the Lamb vector, and the spatial distributions of the potential and solenoidal components were further analyzed. The former component was found to be dominant in the shear layers, but as the flow transitioned, its contribution decreased steadily. The solenoidal component was maximized above and below the tip of the recirculation zone and downstream of this zone.

Finally, analytical expressions for the Euclidean norms of the Lamb vector and its potential component were derived. The norms contain terms that involve the time-average field only, the fluctuating field only, as well as mixed terms (that arise from the interaction between the mean and fluctuating flow fields). Interestingly, it is found that in the wake of the wing, both norms are dominated by the mixed terms.

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APPENDIX: EUCLIDEAN NORMS OF THE LAMB VECTOR L AND ITS POTENTIAL COMPONENT $\nabla\alpha$

We derive an analytic expression for the square Euclidean norm $\langle L \cdot L \rangle$, where $L = \Omega \times U$. Using Reynolds decomposition, $U = \langle U \rangle + u$, $\Omega = \langle \Omega \rangle + \omega$ and substituting we get

$$L = (\langle \Omega \rangle + \omega) \times (\langle U \rangle + u) = \langle \Omega \rangle \times \langle U \rangle + \langle \Omega \rangle \times u + \omega \times \langle U \rangle + \omega \times u. \quad (\text{A1})$$

Thus the dot product $L \cdot L$ becomes

$$L \cdot L = (\langle \Omega \rangle \times U + \langle \Omega \rangle \times u + \omega \times U + \omega \times u) \cdot (\langle \Omega \rangle \times U + \langle \Omega \rangle \times u + \omega \times U + \omega \times u). \quad (\text{A2})$$

Expanding and taking the time average we get

$$\begin{aligned} \langle L \cdot L \rangle &= (\langle \Omega \rangle \times \langle U \rangle)^2 + \langle (\langle \Omega \rangle \times u)^2 \rangle + \langle (\omega \times \langle U \rangle)^2 \rangle + \langle (\omega \times u)^2 \rangle \\ &\quad + 2\langle (\langle \Omega \rangle \times \langle U \rangle) \cdot \langle \omega \times u \rangle \rangle + 2\langle (\langle \Omega \rangle \times u) \cdot (\omega \times u) \rangle \\ &\quad + 2\langle (\omega \times \langle U \rangle) \cdot (\omega \times u) \rangle + 2\langle (\omega \times \langle U \rangle) \cdot (\langle \Omega \rangle \times u) \rangle. \end{aligned} \quad (\text{A3})$$

Similarly, for the potential term $\nabla\alpha = -\nabla H = -\nabla(\frac{P}{\rho} + \frac{1}{2}U^2)$ we have

$$\nabla\alpha = -\left(\frac{1}{\rho}\nabla(\langle P \rangle + p) + \frac{1}{2}\nabla(\langle U \rangle + u) \cdot (\langle U \rangle + u)\right), \quad (\text{A4})$$

and the dot product $\nabla\alpha \cdot \nabla\alpha$ becomes

$$\begin{aligned} \nabla\alpha \cdot \nabla\alpha &= \left(\frac{1}{\rho}\nabla(\langle P \rangle + p) + \frac{1}{2}\nabla(\langle U \rangle + u) \cdot (\langle U \rangle + u)\right) \\ &\quad \cdot \left(\frac{1}{\rho}\nabla(\langle P \rangle + p) + \frac{1}{2}\nabla(\langle U \rangle + u) \cdot (\langle U \rangle + u)\right). \end{aligned} \quad (\text{A5})$$

Expanding and taking the time average, we obtain

$$\begin{aligned} \langle \nabla\alpha \cdot \nabla\alpha \rangle &= \frac{1}{\rho^2}\nabla\langle P \rangle \cdot \nabla\langle P \rangle + \frac{1}{\rho^2}\langle \nabla p \cdot \nabla p \rangle + \frac{1}{\rho}\nabla\langle P \rangle \cdot \nabla\langle U \rangle^2 + \frac{1}{\rho}\nabla\langle P \rangle \cdot \nabla\langle u^2 \rangle \\ &\quad + \frac{1}{\rho}\langle \nabla p \cdot \nabla(u \cdot \langle U \rangle) \rangle + 2\frac{1}{\rho}\langle \nabla p \cdot \nabla u^2 \rangle + \frac{1}{4}\nabla\langle U \rangle^2 \cdot \nabla\langle U \rangle^2 + \frac{1}{2}\nabla\langle U \rangle^2 \cdot \nabla\langle u^2 \rangle \\ &\quad + \langle \nabla(\langle U \rangle \cdot u) \cdot \nabla u^2 \rangle + \langle (\nabla\langle U \rangle \cdot u)^2 \rangle + \frac{1}{4}\langle \nabla u^2 \cdot \nabla u^2 \rangle. \end{aligned} \quad (\text{A6})$$

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