

Signature function: A scale-by-scale measure of turbulent kinetic energy

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(Received 8 May 2026; accepted 9 September 2026; published 22 October 2026)

A real-space measure of the scale-by-scale distribution of turbulent kinetic energy was introduced in [P. A. Davidson, *Turbulence: An Introduction for Scientists and Engineers* (Oxford University Press, Oxford, 2004)], called the signature function. However, this signature function has two weaknesses. First, although measurements suggest that it is positive in fully developed turbulence, non-negative results cannot be guaranteed. Second, its definition involves derivatives, and so its calculation from experimental data is sensitive to noise. Here, we develop a refinement of the original signature function. This version improves on the original in that it can be shown to be strictly positive in isotropic turbulence. Moreover, it is obtained from experimental data through a process of integration, rather than differentiation, and so is less sensitive to noise. We apply this diagnostic to grid turbulence. Here the signature function is shown to recover many of the expected features of a scale-by-scale energy density. For example, it falls off below ten Kolmogorov scales and also for scales that exceed the integral scale. Moreover, it exhibits a $2/3$ law in the inertial subrange, coupled to a bottleneck at the top of the dissipation range. We also apply the signature function to the near-wall region of smooth and rough-wall boundary layers. Although the diagnostic is not guaranteed to stay positive in such a shear layer, in practice it is observed to remain positive. As with grid turbulence, the signature function captures the expected features of a scale-by-scale energy distribution. In particular, it confirms that the energy of the large eddies near the wall scales on the square of the shear velocity.

DOI: [10.1103/6nhp-63v9](#)

I. INTRODUCTION: THE SIGNATURE FUNCTION AS A REAL-SPACE ENERGY DIAGNOSTIC

The search for a real-space diagnostic, say $V(r)$, which provides a scale-by-scale measure of the kinetic energy in isotropic turbulence, analogous to the spherically averaged three-dimensional energy spectrum, $E(k)$, goes back to Townsend's work on shear layers [1]. Here we consider a refinement of a real-space diagnostic developed by the lead author [2,3], called the signature function. This version of the signature function improves on the original in that it can be shown to be strictly positive in isotropic turbulence. Moreover, it is obtained from experimental data through a process of integration, rather than differentiation, and so is less sensitive to noise.

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A. The original signature function

Perhaps the earliest attempt at a real-space diagnostic was Townsend [1], who took the second-order longitudinal structure function,

$$\langle(\Delta v)^2\rangle(r) = \langle(u_x(\mathbf{x} + r\hat{\mathbf{e}}_x) - u_x(\mathbf{x}))^2\rangle = \langle(u'_x - u_x)^2\rangle, \quad (1)$$

to be indicative of the cumulative energy held in a flow below scale r . That is, at any instant, eddies much smaller than r will contribute to either $u_x^2(\mathbf{x} + r\hat{\mathbf{e}}_x)$ or to $u_x^2(\mathbf{x})$, but not both, while eddies much larger than r will induce similar signals at both locations. In short, the structure function acts a bit like a low-pass filter. Since $\frac{3}{4}\langle(\Delta v)^2\rangle(r \rightarrow \infty) = \frac{1}{2}\langle\mathbf{u}^2\rangle$ in isotropic turbulence, this suggests

$$\frac{3}{4}\langle(\Delta v)^2\rangle(r) \approx [\text{energy held in eddies of size } r \text{ or less}], \quad (2)$$

which, in turn, yields the energy density

$$V_T(r) = \frac{d}{dr} \left[\frac{3}{4} \langle(\Delta v)^2\rangle(r) \right]. \quad (3)$$

Here the subscript T indicates that this is Townsend's definition of V , and the factor of $3/4$ is included to ensure that V_T integrates to give the turbulent kinetic energy, $\frac{1}{2}\langle\mathbf{u}^2\rangle$.

Crucially, however, the longitudinal structure function does contain information about eddies whose size is greater than r . That is, eddies much greater than scale r make a contribution to $\langle(\Delta v)^2\rangle$ of order $r^2\langle(\partial u_x/\partial r)^2\rangle$. So, we should add to (2) the term

$$r^2 \times [\text{square of velocity gradients in eddies of size } r \text{ or greater}].$$

This is confirmed in [2], where the spherically averaged energy spectrum in isotropic turbulence,

$$E(k) = \frac{1}{\pi} \int_0^\infty \langle\mathbf{u} \cdot \mathbf{u}'\rangle(r) k r \sin(kr) dr,$$

is shown to be related to the second-order longitudinal structure function by

$$\frac{3}{4}\langle(\Delta v)^2\rangle(r) \approx \int_{\pi/r}^\infty E(k) dk + \frac{r^2}{\pi^2} \int_0^{\pi/r} k^2 E(k) dk. \quad (4)$$

In particular, since

$$\int_0^\infty E(k) dk = \frac{1}{2}\langle\mathbf{u}^2\rangle, \quad \int_0^\infty k^2 E(k) dk = \frac{1}{2}\langle(\nabla \times \mathbf{u})^2\rangle = \frac{1}{2}\langle\boldsymbol{\omega}^2\rangle, \quad (5)$$

$\frac{1}{2}\langle\boldsymbol{\omega}^2\rangle$ being the enstrophy density of the turbulence, we conclude that

$$\frac{3}{4}\langle(\Delta v)^2\rangle(r) \approx [\text{energy of eddies of size } s < r] + \frac{r^2}{\pi^2} [\text{enstrophy in eddies of size } s > r],$$

where we equate the eddy size, s , to π/k . Since we require $V(r)$ and $E(k)$ to play similar roles, but with $k \sim 1/s$, this suggests that we define $V(r)$ such that

$$\frac{3}{4}\langle(\Delta v)^2\rangle(r) = \int_0^r V_{\text{old}}(s) ds + \frac{r^2}{\pi^2} \int_r^\infty \frac{\pi^2 V_{\text{old}}(s)}{s^2} ds, \quad (6)$$

where the subscript "old" indicates that this is related to the old definition of $V(r)$. On differentiating (6), we obtain the original definition of the signature function given in [2]:

$$V_{\text{old}}(r) = -\frac{3r^2}{8} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} [\langle(\Delta v)^2\rangle]. \quad (7)$$

Now, any real-space diagnostic that provides a scale-by-scale measure of energy in isotropic turbulence should have the following properties:

- (i) $\int_0^\infty V(r)dr = \frac{1}{2}\langle \mathbf{u}^2 \rangle$;
- (ii) for a random array of simple eddies of fixed size ℓ_e , $V(r)$ has a sharp peak at around $r \sim \ell_e$, but is small for $r \ll \ell_e$ and $r \gg \ell_e$;
- (iii) $V(r) \geq 0$.

It is readily confirmed that $V_{\text{old}}(r)$ does indeed satisfy

$$\int_0^\infty V_{\text{old}}(s)ds = \frac{1}{2}\langle \mathbf{u}^2 \rangle, \quad (8)$$

along with property (ii). Indeed, it is shown in [2] that

$$rV_{\text{old}}(r) \approx [kE(k)]_{k \approx 9\pi/8r}. \quad (9)$$

Evidently, $V_{\text{old}}(r)$ is closely related to $E(k)$.

Crucially, however, the original definition of the signature function has two major weaknesses. First, although measurements suggest that V_{old} is positive in fully developed turbulence, one cannot ensure that $V_{\text{old}} \geq 0$ under all circumstances. Second, because definition (7) involves two derivatives, the calculation of V_{old} from experimental data is very sensitive to noise. [Definition (7) was applied to grid turbulence in [3], but only after that data was smoothed.] The primary purpose of this paper is to provide an improved definition of $V(r)$ for isotropic turbulence, one that remedies the two weaknesses of V_{old} .

B. Other real-space diagnostics

There have been several alternative strategies to quantify scale-by-scale energy budgets. For example, Danaïla *et al.* [4] consider turbulence which is inhomogeneous and anisotropic, but statistically axisymmetric, and they propose a tool based on the vector structure function, $\langle (\Delta \mathbf{u})^2 \rangle(\mathbf{r}) = \langle (\mathbf{u}' - \mathbf{u})^2 \rangle$, to quantify the anisotropic transfer of energy. This diagnostic is then applied to the problem of colliding turbulent jets.

Yet another strategy is proposed by Hamba [5], who is particularly concerned that any real-space energy density must be non-negative in all circumstances. Consequently, the real-space energy density is defined as the Gaussian filtered one-dimensional transform of $\langle \mathbf{u} \cdot \mathbf{u}' \rangle(r)$. That is to say, we introduce the one-dimensional transform

$$E_1(k) = \frac{1}{\pi} \int_0^\infty \langle \mathbf{u} \cdot \mathbf{u}' \rangle(r) \cos(kr)dr, \quad E_1(k) \geq 0,$$

which is related to $E(k)$ by $E = -k dE_1/dk$ in isotropic turbulence. The real-space energy density is then defined as

$$V_H(r) = \int_0^\infty kE_1(k)kr \exp[-(kr)^2/2]dk, \quad (10)$$

where the subscript H indicates Hamba. These ideas were subsequently extended in [6]. Hamba's approach has connections to our definition of the signature function, as discussed in Appendix B.

C. The structure of the paper

The paper is structured as follows. In Sec. II, we develop a refined version of the signature function which is strictly positive in isotropic turbulence. This is a smoothed version of the old signature function, defined in terms of a weighted integral of the second-order structure function. Here the kinematic properties of the signature function are established, including its relationship to $E(k)$.

Next, in Sec. III, we consider the application of the new signature function to grid turbulence. Here the new diagnostic is shown to recover many of the expected features of a scale-by-scale

energy density. For example, it falls off below ten Kolmogorov scales and also above the integral scale. Moreover, it exhibits a 2/3 law in the inertial subrange, coupled to the usual bottleneck at the top of the dissipation range.

Finally, in Sec. IV, we apply the signature function to the near-wall region of both smooth and rough-wall boundary layers. Although the signature function is not guaranteed to stay positive in such a shear layer, in practice it is observed to remain positive. Moreover, once again, the signature function captures the expected features of a scale-by-scale energy distribution, such as an energy maximum in which the energy of the large eddies scales on the square of the shear velocity.

II. THE DEFINITION OF THE NEW SIGNATURE FUNCTION AND ITS KINEMATIC PROPERTIES

A. The definition of the signature function and its relationship to $E(k)$

Let us start with the definition of the signature function, along with its relationship to the energy spectrum. Consider a smoothed, or filtered, version of $V_{\text{old}}(r)$, defined by

$$rV(r) = \frac{1}{\alpha_4} \int_0^\infty [sV_{\text{old}}(s)] x^4 e^{-x^2} dx, \quad x = \frac{s}{r}, \quad \alpha_4 = \int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}. \quad (11)$$

Here the power x^4 is chosen as it leads to a reasonably sharp filter in the integral. (Lower powers tend to smear the old signature function.) According to the sifting analysis of Appendix A [see (A2) and (A4)], combined with (8) above, definition (11) ensures

$$\int_0^\infty V(r) dr = \int_0^\infty V_{\text{old}}(s) ds = \frac{1}{2} \langle \mathbf{u}^2 \rangle, \quad (12)$$

and

$$rV(r) \approx [sV_{\text{old}}(s)]_{s=\sqrt{5/2}r}. \quad (13)$$

It follows that our $V(r)$ satisfies the first two requirements of any real-space energy density. That is to say, $V(r)$ integrates to give the turbulent kinetic energy, and, like $V_{\text{old}}(r)$, eddies of scale ℓ_e contribute to $V(r)$ at around $r \approx \ell_e$. Moreover, we shall see shortly that, unlike $V_{\text{old}}(r)$, our definition of $V(r)$ satisfies the third requirement of being strictly positive in isotropic turbulence. In short, $V(r)$ satisfies all three requirements of a real-space energy density in isotropic turbulence.

Now, integrating (11) by parts yields an equivalent definition of our $V(r)$,

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\frac{\partial}{\partial s} \langle (\Delta v)^2 \rangle(s) \right] g(x) ds, \quad x = \frac{s}{r}, \quad g(x) = e^{-x^2} (7x^5 - 2x^7). \quad (14)$$

Integrating by parts a second time tells us that $rV(r)$ can be expressed as a weighted integral of $\langle (\Delta v)^2 \rangle(s)$:

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \langle (\Delta v)^2 \rangle(s) h(x) dx, \quad x = \frac{s}{r}, \quad h(x) = e^{-x^2} (28x^6 - 35x^4 - 4x^8).$$

So, unlike V_{old} , whose evaluation requires the differentiation of experimental data, this signature function can be obtained by integrating such data. The definition is therefore much less sensitive to noise in experimental data. It seems that our definition of $V(r)$ remedies both weaknesses of the original signature function.

In order to relate $V(r)$ to the energy spectrum $E(k)$, we note that (14) can be rewritten in terms of the two-point velocity correlation $\langle \mathbf{u} \cdot \mathbf{u}' \rangle(r)$ using the relationships

$$\frac{\partial}{\partial r} r^4 \frac{\partial}{\partial r} \langle (\Delta v)^2 \rangle = -2r^3 \frac{\partial}{\partial r} \langle \mathbf{u} \cdot \mathbf{u}' \rangle, \quad g(x) = x^4 \frac{d}{dx} [e^{-x^2} (x^2 - 5/2)].$$

On substituting for $g(x)$, integrating by parts, and then substituting for $\langle(\Delta v)^2\rangle(s)$, we find

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty [\langle \mathbf{u} \cdot \mathbf{u}' \rangle(s)] \hat{g}(x) dx, \quad x = \frac{s}{r}, \quad \hat{g}(x) = e^{-x^2} (4x^6 - 20x^4 + 15x^2). \quad (15)$$

Moreover, in isotropic turbulence, $\langle \mathbf{u} \cdot \mathbf{u}' \rangle(r)$ is related to $E(k)$ by

$$\langle \mathbf{u} \cdot \mathbf{u}' \rangle(s) = 2 \int_0^\infty E(k) \frac{\sin ks}{ks} dk, \quad (16)$$

and substituting for $\langle \mathbf{u} \cdot \mathbf{u}' \rangle(s)$ in (15) yields (see Appendix B)

$$rV(r) = \frac{1}{\alpha_3} \int_0^\infty kE(k) [y^3 e^{-y^2}] dy, \quad y = kr/2, \quad \alpha_3 = \int_0^\infty y^3 e^{-y^2} dy = \frac{1}{2}. \quad (17)$$

It follows from (17), as well as from (A4), that $V(r)$ satisfies our three basic requirements:

$$\int_0^\infty V(r) dr = \int_0^\infty E(k) dk = \frac{1}{2} \langle \mathbf{u}^2 \rangle, \quad (18)$$

$$rV(r) \approx [kE(k)]_{k=2\sqrt{2}/r}, \quad (19)$$

$$V(r) \geq 0. \quad (20)$$

We shall put approximation (19) to the test in Sec. II C.

B. Some elementary kinematic properties of the signature function

Several useful relationships follow directly from (17). For example, using the definite integrals (A3), we have

$$\int_0^\infty \frac{4V(r)}{r^2} dr = \int_0^\infty k^2 E(k) dk = \frac{1}{2} \langle \omega^2 \rangle, \quad (21)$$

and

$$\int_0^\infty rV(r) dr = \frac{3\sqrt{\pi}}{2} \int_0^\infty \frac{E(k)}{k} dk = \frac{\langle \mathbf{u}^2 \rangle \ell}{\sqrt{\pi}}, \quad (22)$$

where ℓ is the usual turbulence integral scale, defined by

$$\langle u_x^2 \rangle \ell = (\pi/2) \int_0^\infty k^{-1} E(k) dk.$$

Moreover, integrating (17) yields

$$\int_0^r V(r) dr = \int_0^\infty E(k) [1 - (1 + y^2)e^{-y^2}] dk, \quad y = kr/2, \quad (23)$$

where the function of y in square brackets acts like a high-pass filter. Approximating this function by an idealized filter with a threshold set at $y^2 = 2$ brings us back to (19).

Finally, (17) gives $V(r)$ for small and large r . For example, a low- r expansion yields

$$V(r \rightarrow 0) = \frac{r^3}{8} \int_0^\infty k^4 E(k) dk + O(r^5) = \frac{r^3}{16} \langle (\nabla \times \boldsymbol{\omega})^2 \rangle + O(r^5). \quad (24)$$

Conversely, we can combine (17) with a low- k expansion of $E(k)$,

$$E(k) = \frac{Lk^2}{4\pi^2} + \frac{Ik^4}{24\pi^2} + \dots, \quad L = \int \langle \mathbf{u} \cdot \mathbf{u}' \rangle d\mathbf{r}, \quad I = - \int \mathbf{r}^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle d\mathbf{r}, \quad (25)$$

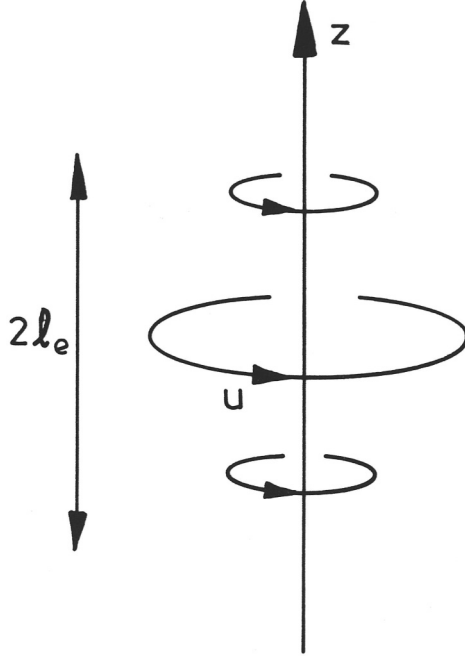


FIG. 1. The spherical eddy given by (27).

where L and I are the Saffman and Loitsyansky integrals, respectively, as discussed in [2]. On substituting for $E(k)$ in (17), and using the definite integrals (A3), we obtain the large- r expansion of $V(r)$:

$$V(r \rightarrow \infty) = \frac{15L}{4\pi^{3/2}r^4} + \frac{35I}{4\pi^{3/2}r^6} + \dots \quad (26)$$

Since L and I are invariants, the form of V at large r is fixed during the decay of turbulence.

C. Simple kinematic examples of the signature function

It is instructive to consider some simple kinematic examples. Consider the spherical eddy,

$$\mathbf{u} = \Omega r \exp[-2\mathbf{x}^2/\ell_e^2] \hat{\mathbf{e}}_\theta, \quad (27)$$

in (r, θ, z) coordinates, as shown in Fig. 1. This expression represents a blob of swirling fluid of characteristic size ℓ_e whose angular velocity falls off as a Gaussian. If such eddies are randomly, but uniformly, distributed in space, and with random orientations, we obtain a statistically isotropic velocity field. It is shown in [2] that the corresponding longitudinal correlation function is

$$\langle u_x(\mathbf{x} + r\hat{\mathbf{e}}_x)u_x(\mathbf{x}) \rangle = \langle u'_x u_x \rangle(r) = \langle u_x^2 \rangle \exp(-r^2/\ell_e^2), \quad (28)$$

from which the second-order structure function and energy spectrum can be calculated:

$$\langle (\Delta v)^2 \rangle(r) = \langle (u'_x - u_x)^2 \rangle = 2\langle u_x^2 \rangle [1 - \exp(-r^2/\ell_e^2)], \quad (29)$$

$$E(k) = \frac{\langle \mathbf{u}^2 \rangle \ell_e}{24\sqrt{\pi}} (k\ell_e)^4 \exp[-\ell_e^2 k^2/4]. \quad (30)$$

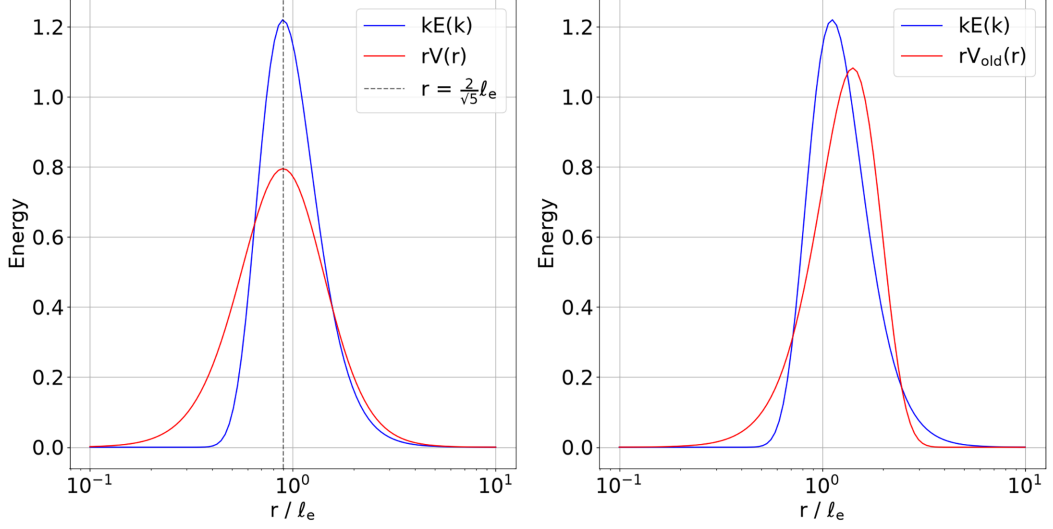


FIG. 2. $rV(r)$ and $kE(k)$ for a random distribution of eddies of size ℓ_e . Left hand side, the signature function $rV(r)$ (shown red) and corresponding energy spectrum $[kE(k)]_{k=2\sqrt{2}/r}$ (in blue). Right hand side, the old signature function rV_{old} (shown red) is compared with $[kE(k)]_{k=9\pi/8r}$ (in blue).

The signature function can be found by substituting (29) into definition (14), and then using (A3). The normalized signature function, and corresponding energy spectrum, are

$$\frac{rV(r)}{\frac{1}{2}\langle \mathbf{u}^2 \rangle} = \frac{35(r/\ell_e)^4}{2[1 + (r/\ell_e)^2]^{9/2}}, \quad \frac{kE(k)}{\frac{1}{2}\langle \mathbf{u}^2 \rangle} = \frac{(k\ell_e)^5}{12\sqrt{\pi}} \exp\left[-\frac{(k\ell_e)^2}{4}\right]. \quad (31)$$

The distributions of $rV(r)$ and $[kE(k)]_{k=2\sqrt{2}/r}$, specified by (31), are shown in Fig. 2. As suggested by (19), the two distributions are similar, both exhibiting peaks at $r = 2\ell_e/\sqrt{5}$. Note that the shape of $rV(r)$ is slightly broader than that of $[kE(k)]_{k=2\sqrt{2}/r}$, and this is related to the fact that $V(r)$ is a smoothed-out version of $V_{\text{old}}(r)$, as dictated by (11). Figure 2 also shows $rV_{\text{old}}(r)$ compared to $[kE(k)]_{k=9\pi/8r}$, and it is clear that $rV_{\text{old}}(r)$ is narrower than $rV(r)$.

We note in passing that the one-dimensional energy spectrum, defined as

$$F_{11}(k) = \frac{1}{\pi} \int_0^\infty \langle u'_x u_x \rangle(r) \cos(kr) dr, \quad (32)$$

takes the form of

$$kF_{11}(k) = \frac{1}{6\sqrt{\pi}} \langle \mathbf{u}^2 \rangle k\ell_e \exp[-(k\ell_e)^2/4], \quad (33)$$

for (28). This peaks at $k\ell_e = \sqrt{2}$, rather than at $k\ell_e \approx \pi$, and so artificially shifts energy toward $k = 0$ (large scales). This is a generic problem with $F_{11}(k)$, as discussed in [2, 7].

As a second example, consider Kolmogorov's 2/3 law for the inertial subrange,

$$\langle (\Delta v)^2 \rangle(r) = \beta \varepsilon^{2/3} r^{2/3}, \quad (34)$$

whose spectral equivalent is (see [8])

$$E(k) = 0.760\beta \varepsilon^{2/3} k^{-5/3}. \quad (35)$$

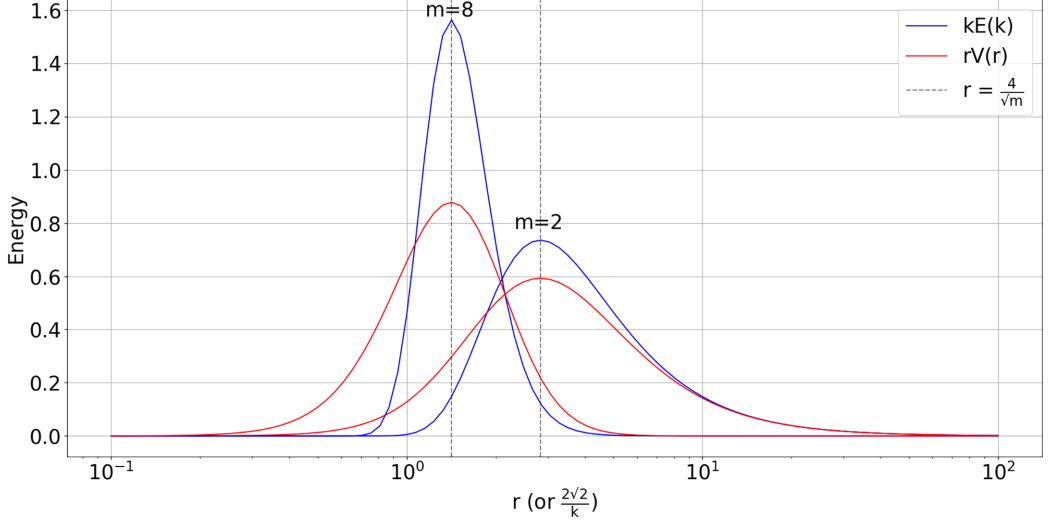


FIG. 3. Log-linear plots of the signature function, $rV(r)$, and corresponding energy spectrum, $[kE(k)]_{k=2\sqrt{2}/r}$, for the power-law spectra given by (38), $m = 2, 8$.

Here β is Kolmogorov's constant, $\beta \approx 2$, and ε is the dissipation rate per unit mass. From definition (14), we have

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\frac{2}{3} \beta \varepsilon^{2/3} s^{-1/3} \right] g(x) ds = \frac{2}{3\sqrt{\pi}} \beta \varepsilon^{2/3} r^{2/3} \int_0^\infty \frac{g(x)}{x^{1/3}} dx = 0.4324 \beta \varepsilon^{2/3} r^{2/3}, \quad (36a)$$

and given that $\beta \approx 2$, this results in

$$rV(r) = 0.432 \beta \varepsilon^{2/3} r^{2/3} \approx 0.865 \varepsilon^{2/3} r^{2/3}. \quad (36b)$$

Moreover, according to (19), the spectral law corresponding to (36a) satisfies

$$E(k) \approx 0.865 \beta \varepsilon^{2/3} k^{-5/3}, \quad (37)$$

which is indeed close to (35).

As our third and final example, consider the normalized power-law spectrum

$$\frac{kE(k)}{\frac{1}{2}\langle \mathbf{u}^2 \rangle} = \frac{2k^m \exp(-k^2)}{((m/2) - 1)!}, \quad m = 2, 4, 6, 8, \dots, \quad (38)$$

where k is normalized by some arbitrary length scale. The corresponding normalized signature function may be found from (17), and if r is normalized by the same length scale as k in (38), then this takes the form

$$\frac{rV(r)}{\frac{1}{2}\langle \mathbf{u}^2 \rangle} = \frac{m(m+2)(r/2)^4}{2[1 + (r/2)^2]^{2+m/2}}, \quad m = 2, 4, 6, 8, \dots \quad (39)$$

The corresponding distributions of $rV(r)$ and $[kE(k)]_{k=2\sqrt{2}/r}$ are shown in Fig. 3 for the cases of $m = 2, 8$. Once again, the two sets of distributions are similar, as suggested by (19). In particular, both sets of distributions peak at the same points. However, the shape of $rV(r)$ is invariably slightly broader than that of $[kE(k)]_{k=2\sqrt{2}/r}$, with a lower peak.

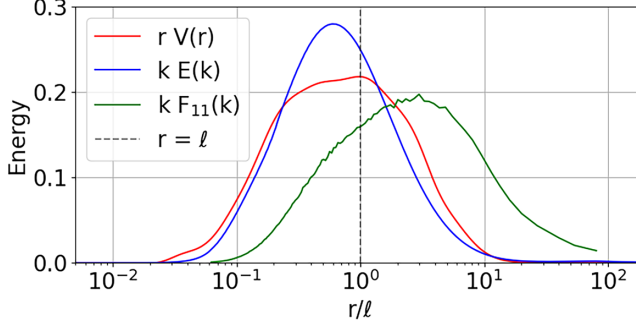


FIG. 4. The distributions of $rV(r)$, $[kE(k)]_{k=2\sqrt{2}/r}$, and $kF_{11}(k)$ for the grid turbulence in [9].

III. THE APPLICATION OF THE SIGNATURE FUNCTION TO GRID TURBULENCE

We now turn to the experimental data. Here measurements almost invariably come in the form of a longitudinal correlation function, $\langle u'_x u'_x \rangle(r)$, and the corresponding one-dimensional spectrum, $F_{11}(k)$. As we have seen in previous work [2], $F_{11}(k)$ offers a poor interpretation of where energy lies in scale space. However, for isotropic turbulence, we may, in principle, use

$$E(k) = k^3 \frac{d}{dk} \left[\frac{1}{k} \frac{dF_{11}}{dk} \right] \quad (40)$$

to calculate $E(k)$ from $F_{11}(k)$, provided, of course, that the one-dimensional spectrum is smooth enough to differentiate. We start our examination of experimental data with grid turbulence which is approximately, but not exactly, isotropic.

Figure 4 shows grid turbulence taken from [9]. This study used several multiscale grids, and the data shown corresponds to grid 2 and to a downstream location of 290 integral scales. The signature function was calculated using definition (14), while $E(k)$ was estimated by smoothing $F_{11}(k)$ and then applying (40). Figure 4 employs log-linear axes and shows $rV(r)$, $[kE(k)]_{k=2\sqrt{2}/r}$, and $[kF_{11}(k)]_{k=2\sqrt{2}/r}$. The distributions of $kE(k)$ and $rV(r)$ do indeed look similar, although the peak in $kE(k)$ sits above that in $rV(r)$, just as it does in Figs. 2 and 3. $kF_{11}(k)$, on the other hand, displaces energy to large scales, as expected. Because of the low Reynolds number, $\text{Re}_M = 4 \times 10^4$, there is no 2/3 law for this data set.

In order to observe a 2/3 law, we have used ONERA grid turbulence data from the Modane S1MA wind tunnel, as documented in [10]. Here the mesh size is large, $M = 0.625$ m, with a Reynolds number based on M of around $\text{Re}_M = 10^6$, and an integral scale of around $\ell = 0.15$ m. Figure 5 shows the signature function for a range of wind-tunnel speeds, where $rV(r)$ is plotted against r/ℓ using log-linear axes. Clearly, $rV(r)$ exhibits a peak close to the integral scale, as it should.

The data for the wind speed of 20 ms^{-1} is replotted in Fig. 6 using log-log axes. The upper panel shows $rV(r)$ plotted against εr , with the theoretical inertial-range law, Eq. (36b), shown for comparison. There is now evidence of a 2/3 law, although this is partially masked by the bottleneck region (a region of excess energy). Such a bottleneck invariably sits at the lower end of the inertial subrange and at the top end of the dissipation range, as discussed in [2]. Finally, for the same wind speed, the lower panel in Fig. 6 shows $rV(r)$ compensated by $r^{2/3}$ and plotted against r/η . There is a rapid fall off of energy below 10η , a bottleneck in the approximate range $r \sim 12 - 70\eta$, and a 2/3 law in the range $r \sim 70 - 200\eta$, although the exponent is slightly less than 2/3. (The same deviation in the power-law exponent is seen in the one-dimensional power spectrum.) Also, the integral scale here corresponds to $\ell \approx 410\eta$, and above $r \sim 600\eta$ there is a rapid fall in energy, as expected. The bottleneck in the range $r \sim 12 - 70\eta$ is consistent with theory and direct numerical simulations, as is the rapid fall off of energy below 10η , as discussed in [2]. (We shall return to the question of the

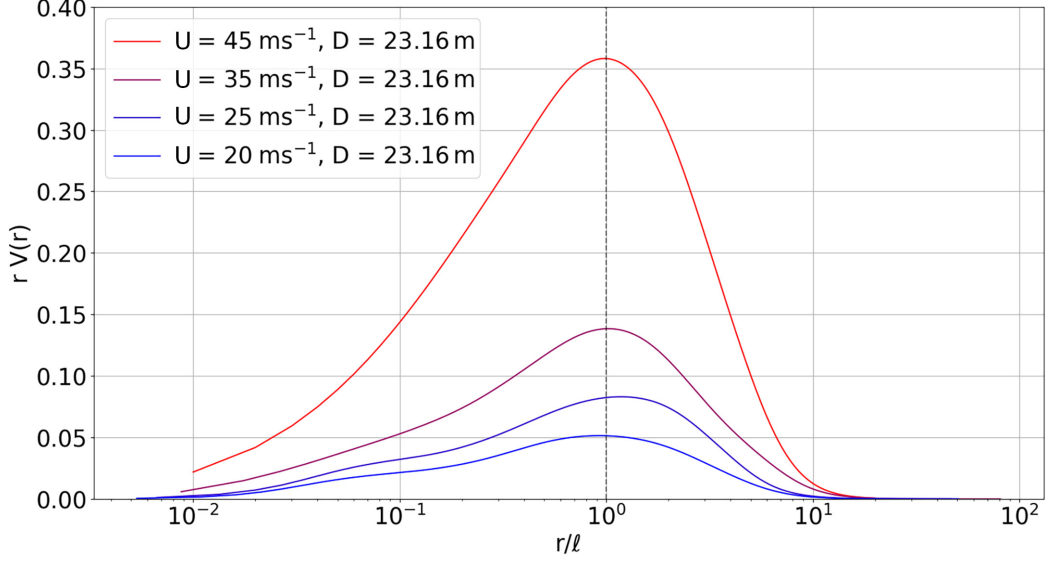


FIG. 5. $rV(r)$ versus r/ℓ for the grid turbulence data in [10]. There is a peak at ℓ .

bottleneck in a subsequent paper, where we apply the signature function to the Karman-Howarth equation.)

All in all, it would seem that the signature function yields a scale-by-scale energy distribution which is consistent with expectation; falling off below 10η and above ℓ , while exhibiting a $2/3$ law coupled to a bottleneck. It is probably fair to say that, as a measure of the energy distribution, $V(r)$ is

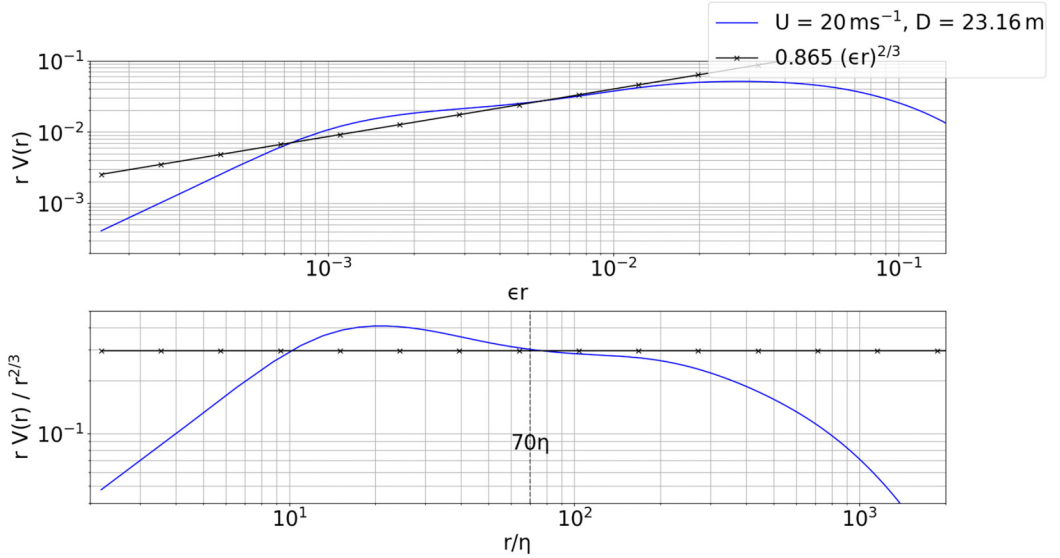


FIG. 6. Data as for in Fig. 5, with wind speed of 20 ms^{-1} . Upper panel: $rV(r)$ plotted against ϵr using log-log axes, with theoretical Eq. (36b) shown for comparison. Bottom panel: $rV(r)$ compensated by $r^{2/3}$ and plotted against r/η using log-log axes.

superior to the ubiquitous one-dimensional energy spectrum, $F_{11}(k)$, which systematically displaces energy toward $k = 0$ (i.e., $r \rightarrow \infty$).

IV. THE SIGNATURE FUNCTION APPLIED TO THE LOG-LAW REGION OF A BOUNDARY LAYER

Our focus so far has been on isotropic turbulence, where we know that $V(r) \geq 0$. However, because the signature function is defined in a strictly one-dimensional sense, in terms of the longitudinal structure function, it can also be applied to a shear layer, such as a boundary layer, provided that there is approximate streamwise homogeneity. Of course, we need to adapt our interpretation of $V(r)$ in such cases, with $V(r)$ now being a scale-by-scale measure of (three times) the streamwise kinetic energy, rather than the full kinetic energy. In any event, it seems worthwhile exploring the form of $V(r)$ in boundary layers.

A. The structure-function log law in boundary layers

Before looking at the data, let us review what we might expect in the near-wall region of a boundary layer. The properties of the near-wall region of a turbulent boundary layer have been thoroughly studied, partly because of the celebrated log law for the mean velocity profile,

$$\frac{\bar{u}_x}{u_*} = A + \frac{1}{\kappa} \ln \frac{u_* y}{\nu} = A + \frac{1}{\kappa} \ln y^+. \quad (41)$$

Here $\bar{u}_x$ is the time-averaged streamwise velocity, u_* the shear velocity, y the distance from the wall, κ is Karman's coefficient ($\kappa \approx 0.4$), and A is an empirical constant. Crucially, as first noted in [11,12], (41) sits alongside a log law for the second-order structure function. This takes the form

$$\frac{\langle (\Delta v)^2 \rangle(r)}{u_*^2} = A_2 + B_2 \ln \frac{\varepsilon r}{u_*^3}, \quad (42)$$

where A_2 and B_2 are empirical coefficients, and $\langle (\Delta v)^2 \rangle(r)$ is the streamwise longitudinal structure function. Measurements suggest $B_2 \approx 2.05$, as discussed in [13].

Now the production of turbulent kinetic energy, $P = u_*^2 \partial \bar{u}_x / \partial y$, roughly balances the local dissipation rate, ε , in the log-law region of a turbulent boundary layer. This yields the estimate $u_*^3 / \kappa y \approx \varepsilon$, or $u_*^3 / \varepsilon \approx \kappa y$. Consequently, (42) is sometimes approximated by

$$\frac{\langle (\Delta v)^2 \rangle}{u_*^2} \approx A'_2 + B_2 \ln \frac{r}{y}, \quad A'_2 = (A_2 - B_2 \ln \kappa), \quad (43)$$

as discussed in [11]. This $\ln(r/y)$ law blends into Kolmogorov's 2/3 law at a streamwise separation of around $r \sim y$, and extends up to $r \sim \delta$, where δ is the boundary layer thickness. The situation is as shown schematically in Fig. 7. Note, however, that there is often a significant mismatch between P and ε in a turbulent boundary layer, with P/ε a function of y . It follows that (42) is the preferred form of the log law, as shown in [12,13].

Both the log law (42) and Kolmogorov's 2/3 law are subregimes of the general scaling

$$\langle (\Delta v)^2 \rangle(r) = F(r, \varepsilon, u_*), \quad \eta \ll r < \delta,$$

from which

$$\frac{\langle (\Delta v)^2 \rangle}{u_*^2} = F\left(\frac{\varepsilon r}{u_*^3}\right), \quad \eta \ll r < \delta. \quad (44)$$

At eddy scales significantly less than y , say $r < y$, the shear velocity ceases to be a relevant parameter and Kolmogorov's 2/3 law follows directly from (44). Conversely, for scales somewhat greater than y , $r > y$, the kinetic energy of an eddy is independent of the scale r , and of order u_*^2 . This requires that F takes the form of a log law, as we now confirm.

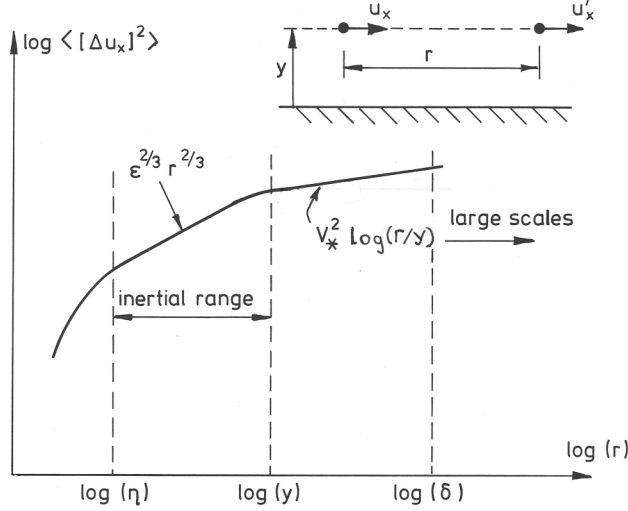


FIG. 7. The $\ln(r/y)$ law blends into Kolmogorov's law at streamwise separation of $r \sim y$.

For the range of scales in which the eddy energy is of order u_*^2 , definition (14) requires

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\frac{\partial}{\partial s} \langle (\Delta v)^2 \rangle(s) \right] g(s/r) ds \sim u_*^2, \quad (45)$$

and since $rV(r)$ is independent of r , it follows that the structure function must take the form

$$\frac{\partial}{\partial s} \langle (\Delta v)^2 \rangle(s) \sim \frac{u_*^2}{s}. \quad (46)$$

This integrates to give a log law, and given the scaling (44), that log law must take the form of (42). Moreover, substituting (42) into (14) yields

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\frac{B_2 u_*^2}{s} \right] g(s/r) ds = \frac{B_2 u_*^2}{\sqrt{\pi}} \int_0^\infty \frac{g(x)}{x} dx = \frac{3}{4} B_2 u_*^2. \quad (47)$$

It follows that the log-law region for the structure function corresponds to an energy plateau of

$$rV(r) = \frac{3}{4} B_2 u_*^2 \approx 1.54 u_*^2, \quad (48)$$

which blends into (36b),

$$rV(r) = 0.432 \beta \varepsilon^{2/3} r^{2/3}, \quad (49)$$

at around $r \sim y$. Indeed, if we make the approximation that $\varepsilon \approx u_*^3 / \kappa y$, then (49) becomes

$$rV(r)/u_*^2 \approx 0.432 \beta (r/\kappa y)^{2/3}, \quad (50)$$

and given that $\beta \approx 2$ and $\kappa \approx 0.4$, (48) and (50) intersect at $r \approx 0.95y$. Finally, we note that the spectral equivalents of (48)–(50) are

$$kF_{11}(k) \sim u_*^2, \quad kF_{11}(k) \sim \varepsilon^{2/3} k^{-2/3} \sim u_*^2 / (ky)^{2/3}. \quad (51)$$

We now put these various predictions to the test, by examining smooth and rough-wall boundary layer data.

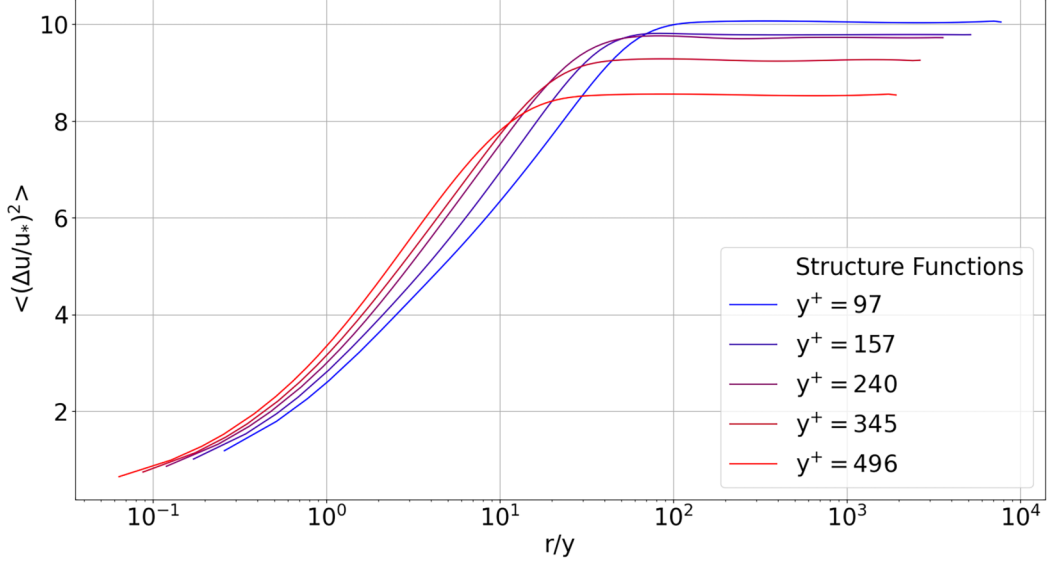


FIG. 8. Smooth-wall data of [13]. Log-linear plots of $\langle(\Delta v)^2\rangle/u_*^2$ versus r/y .

B. The signature function in the log-law region of a boundary layer

We shall consider the zero-pressure-gradient boundary layer data documented in [13], starting with the smooth-wall data at $\text{Re}_\theta = 12\,800$. Figure 8 shows the structure function, $\langle(\Delta v)^2\rangle/u_*^2$, plotted as a function of r/y for five values of y^+ in the range $100 < y^+ < 500$. This confirms the existence of a $\ln(r)$ law, starting at $r \sim y$, consistent with both (42) and (43). However, the collapse

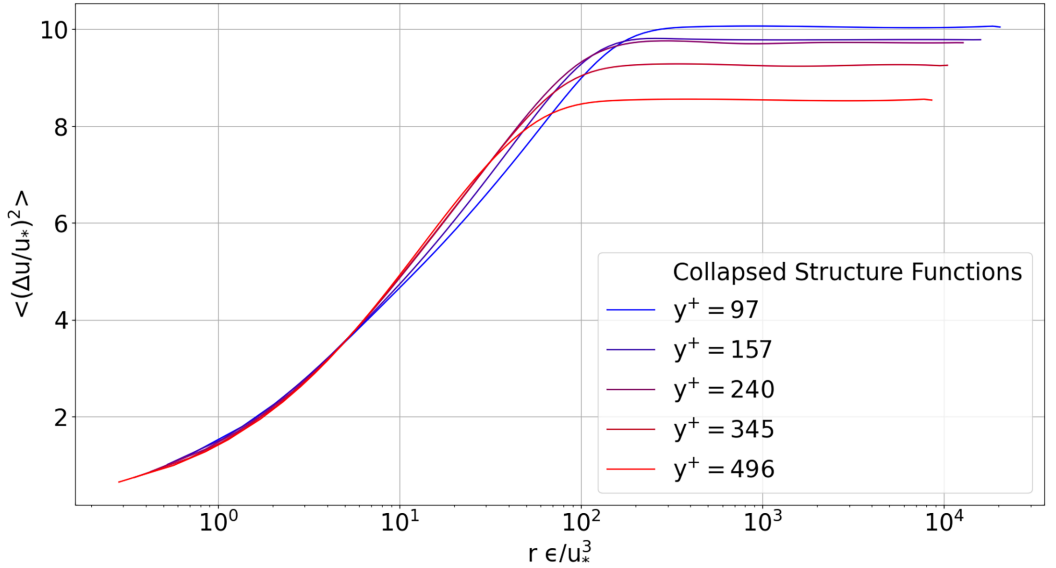


FIG. 9. Smooth-wall data of [13]. Log-linear plots of $\langle(\Delta v)^2\rangle/u_*^2$ versus $r \epsilon / u_*^3$.

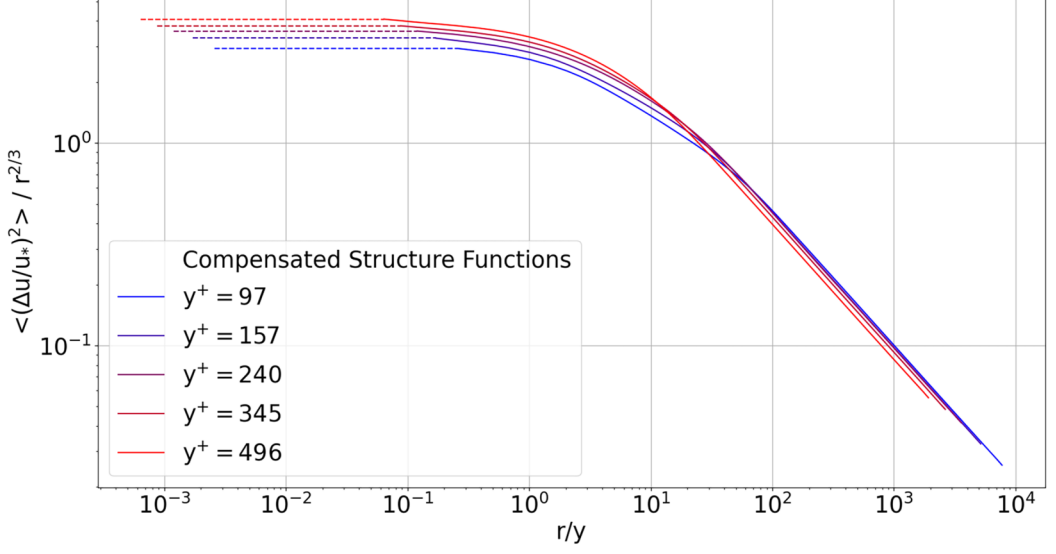


FIG. 10. Smooth-wall data of [13]. Compensated plot of $\langle(\Delta u)^2\rangle/u_*^2$ as a function of r/y .

is poor. This is a consequence of using r/y as a proxy for $\varepsilon r/u_*^3$. That is, we can rewrite (42) and the 2/3 law in the form

$$\frac{\langle(\Delta u)^2\rangle}{u_*^2} = A'_2 + B_2 \ln \frac{r}{y}, \quad A'_2(y) = (A_2 + B_2 \ln(\varepsilon/\kappa P)), \quad (52)$$

$$\langle(\Delta u)^2\rangle(r) = \beta \varepsilon^{2/3} r^{2/3} = \beta (\varepsilon/\kappa P)^{2/3} u_*^2 (r/y)^{2/3}, \quad (53)$$

where $P = u_*^2 \partial \bar{u}_x / \partial y$. A local imbalance between production and dissipation means that, in general, ε/P will be a function of y . Such a dependency prevents the collapse of both (52) and (53). However, when the same data is replotted as a function of $\varepsilon r/u_*^3$, as in Fig. 9, it becomes clear that the true

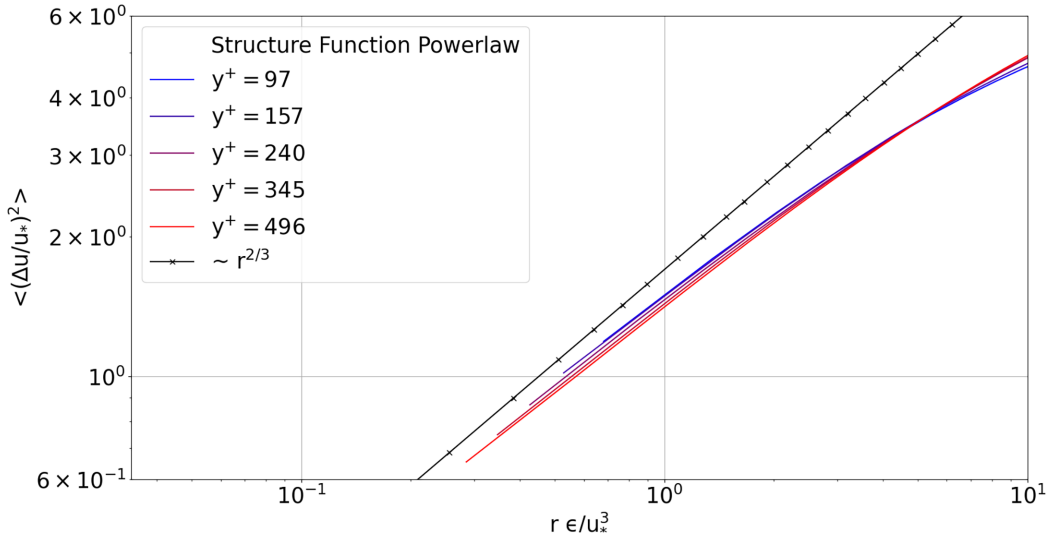


FIG. 11. Smooth-wall data of [13]. Log-log plots of $\langle(\Delta u)^2\rangle/u_*^2$ as a function of $\varepsilon r/u_*^3$.

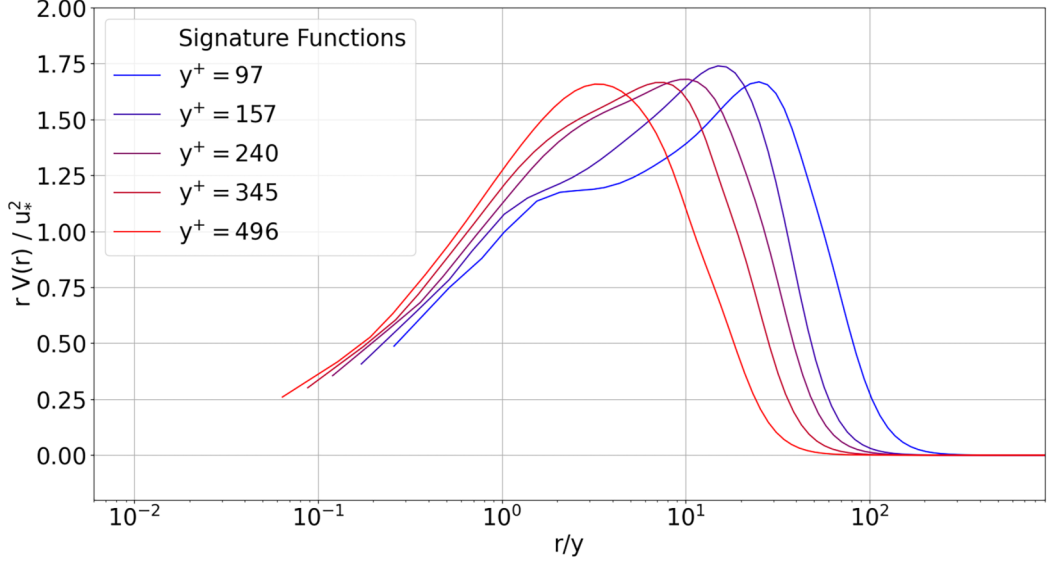


FIG. 12. Smooth-wall data of [13]. Log-linear plots of $rV(r)/u_*^2$ versus r/y .

scaling law is (42), and not (43). This, in turn, confirms that $\langle(\Delta v)^2\rangle(r) = F(r, \varepsilon, u_*)$ for $\eta \ll r < \delta$, rather than $\langle(\Delta v)^2\rangle(r) = F(r, y, u_*)$, a point that is often overlooked in the turbulent boundary layer literature.

Next, Fig. 10 shows the structure function compensated by $r^{2/3}$. Here the solid lines represent the data, while the dotted lines are an extrapolation of the data to a plateau. It is clear that the data extends into the upper end of the 2/3 law, but does not fully capture the 2/3 law. This is confirmed in Fig. 11, a log-log plot of $\langle(\Delta v)^2\rangle/u_*^2$, which shows the data leaning into a 2/3 law, but terminated part way through that region.

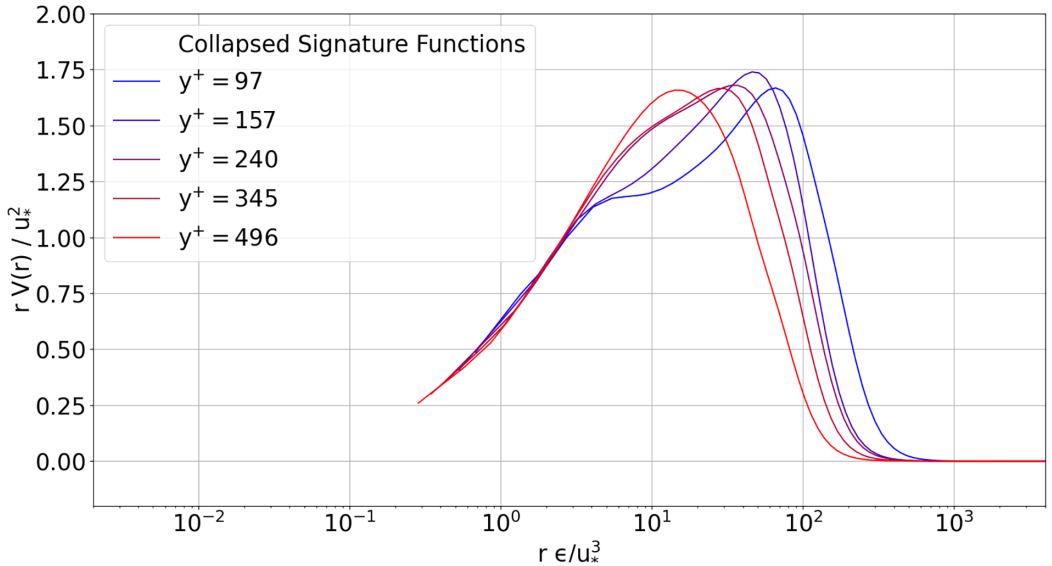
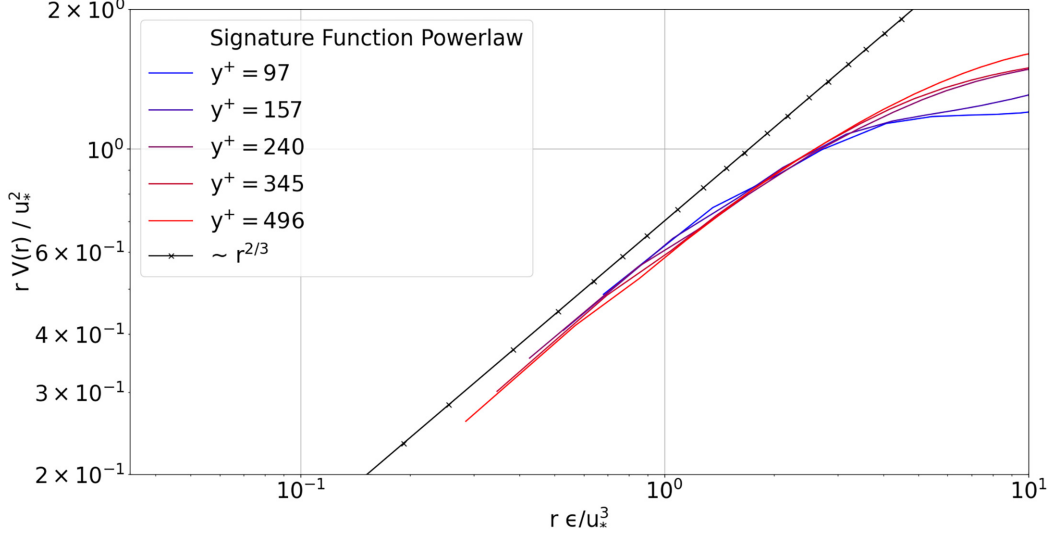
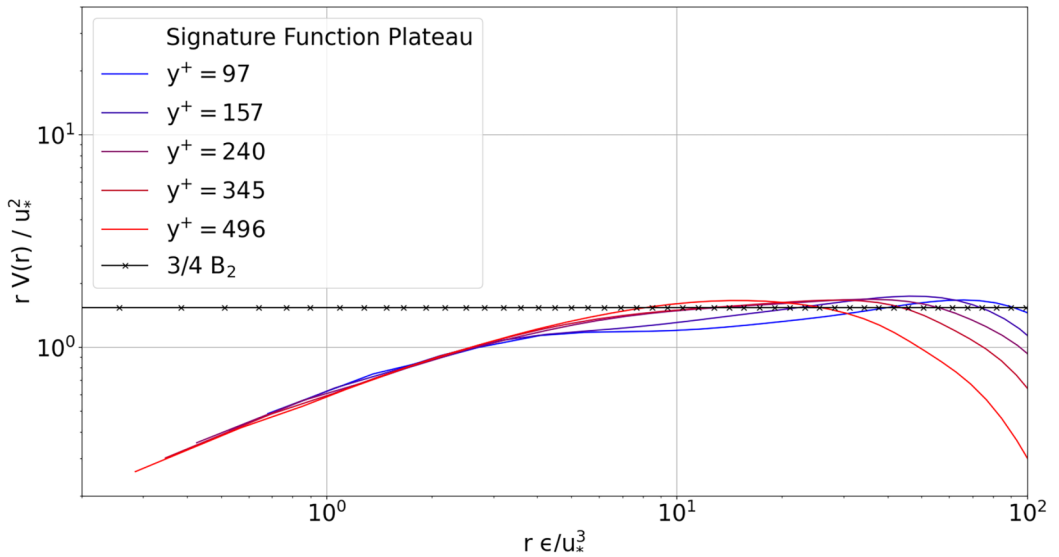


FIG. 13. Smooth-wall data of [13]. Log-linear plots of $rV(r)/u_*^2$ versus $\varepsilon r/u_*^3$.


 FIG. 14. Smooth-wall data of [13]. Log-log plots of $rV(r)/u_*^2$ versus $r\epsilon/u_*^3$ for small r .

Turning now to the signature function, Figs. 12 and 13 show $rV(r)/u_*^2$ as a function of r/y and $r\epsilon/u_*^3$, respectively. As with the structure function, the collapse is better when plotted against $r\epsilon/u_*^3$, at least for $r < 2y$. The curve for $y^+ = 97$ exhibits a plateau followed by a bump. However, the peak progressively moves backwards into the plateau as y^+ increases, eventually merging to form a single maximum. The plateaus lie at $rV(r)/u_*^2 \approx 1.2\text{--}1.6$, consistent with (48), and within the region $1 < r/y < 10$, which is consistent with the $\log(r)$ range in Fig. 8. The peaks, on the other hand, are at $rV(r)/u_*^2 \approx 1.6\text{--}1.7$, and tend to be at larger values of r/y .


 FIG. 15. Smooth-wall data of [13]. Log-log plots of $rV(r)/u_*^2$ versus $r\epsilon/u_*^3$.

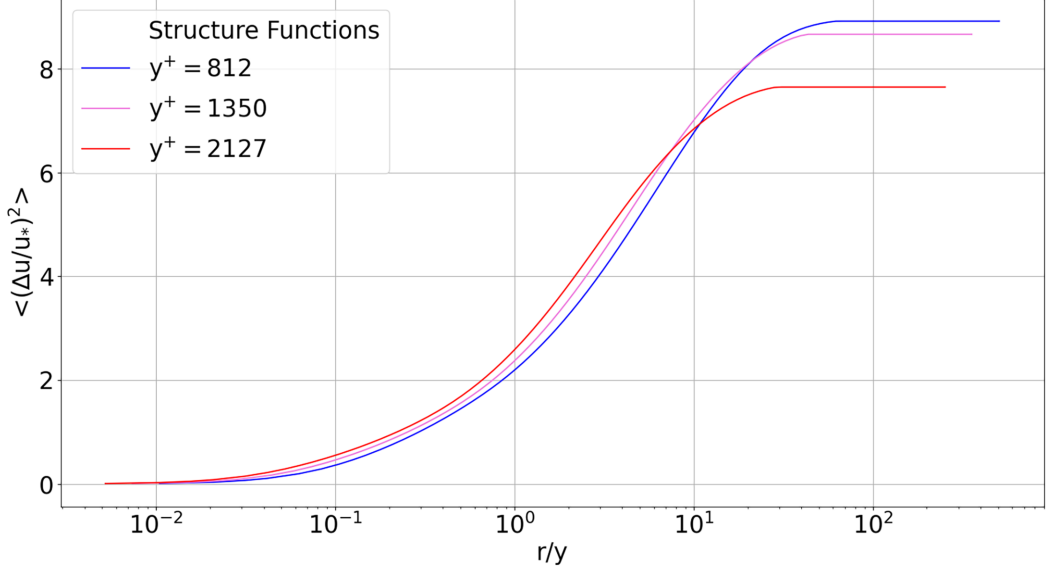


FIG. 16. Rough-wall data of [13]. Log-linear plots of $\langle (\Delta u)^2 \rangle / u_*^2$ versus r/y .

Next, Figs. 14 and 15 show log-log plots of $rV(r)/u_*^2$ as a function of $\varepsilon r/u_*^3$, with Fig. 14 focused on small r and Fig. 15 on larger r . A partial $r^{2/3}$ law is evident in Fig. 14, and an approximate plateau in Fig. 15, at least for the lower values of y^+ .

We now turn to the rough-wall data of [13], which corresponds to square-bar roughness at $\text{Re}_\theta = 34\,700$ with a roughness height of $k^+ = 103$. Figure 16 shows the normalized structure function, $\langle (\Delta u)^2 \rangle / u_*^2$, plotted as a function of r/y for y^+ in the range $800 < y^+ < 2200$. This confirms the existence of a $\ln(r)$ law, starting at $r \sim y$, consistent with both (42) and (43). The same data is

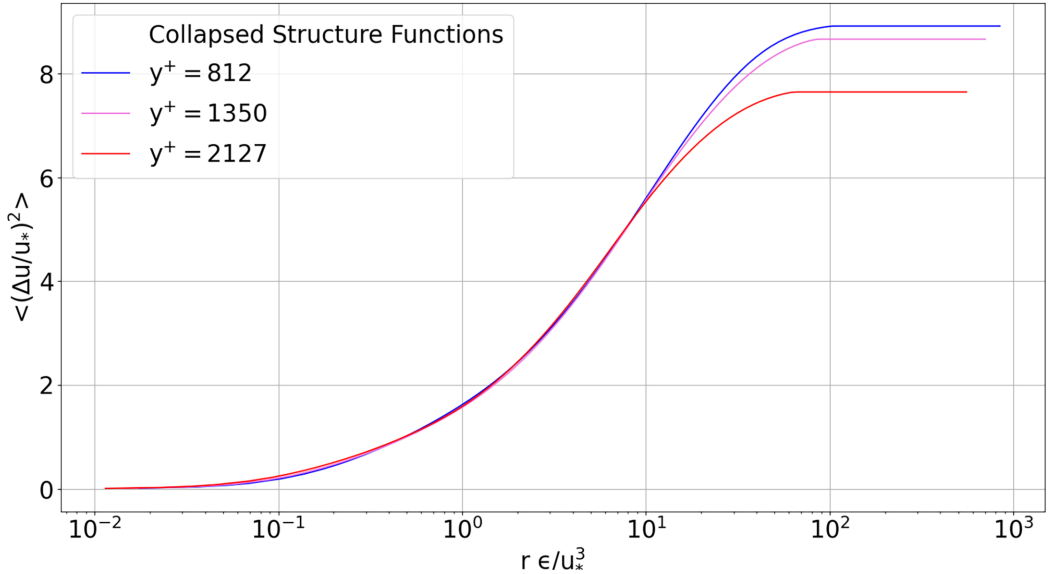
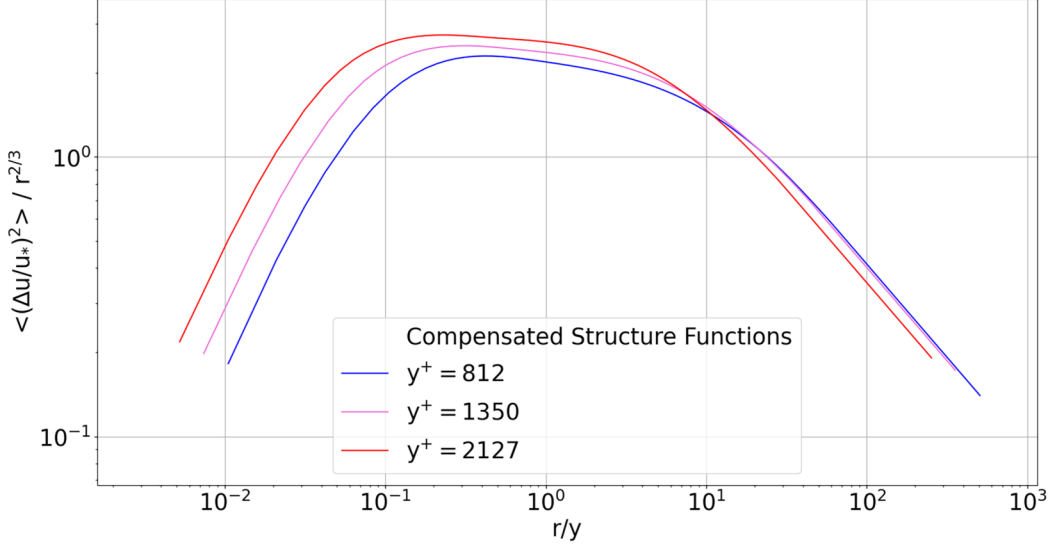


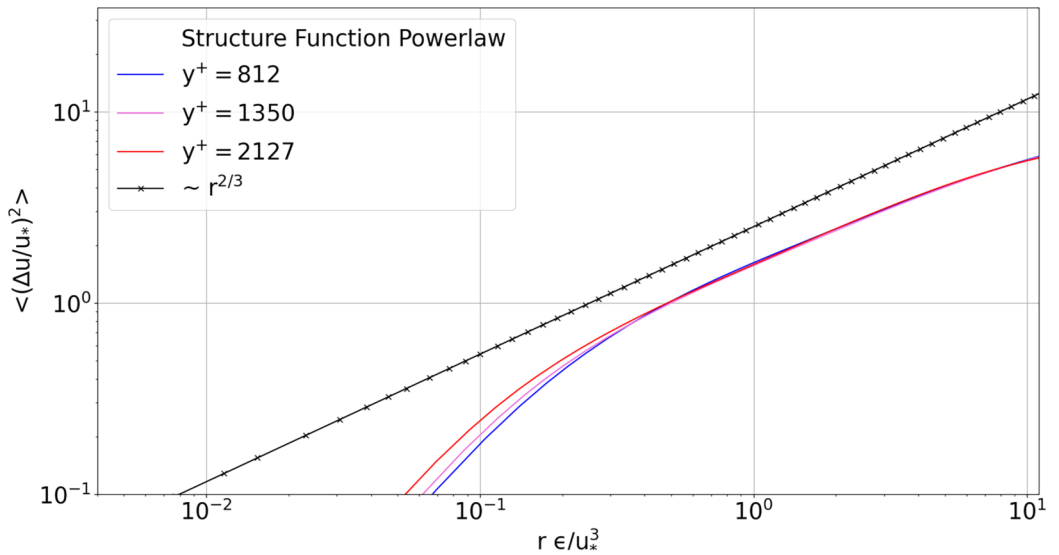
FIG. 17. Rough-wall data of [13]. Log-linear plots of $\langle (\Delta u)^2 \rangle / u_*^2$ versus $\varepsilon r / u_*^3$.

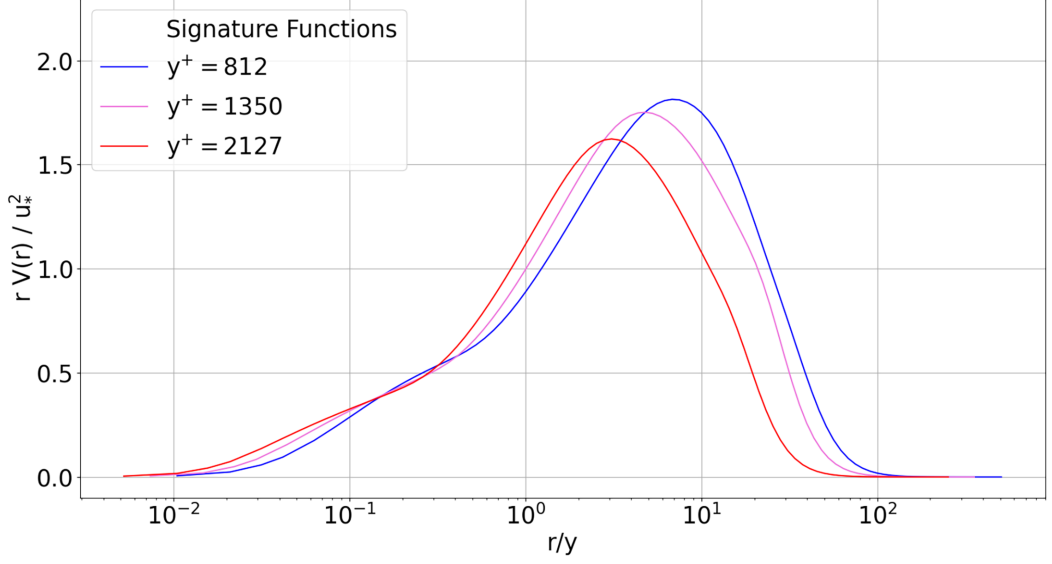

 FIG. 18. Rough-wall data of [13]. Compensated plot of $\langle (\Delta v)^2 \rangle / u_*^2$ as a function of r/y .

replotted as a function of $\varepsilon r / u_*^3$ in Fig. 17, and it is clear that, as for the smooth-wall data, the true scaling law is (42).

Next, Fig. 18 show the structure function compensated by $r^{2/3}$, as a function of r/y . It is clear that the data exhibits a $2/3$ law and, unlike the smooth-wall data, extends completely through the $2/3$ region. This $r^{2/3}$ law is confirmed in Fig. 19, which is a log-log plot of $\langle (\Delta v)^2 \rangle / u_*^2$ versus $\varepsilon r / u_*^3$. Note that the slope is slightly shallower than $2/3$.

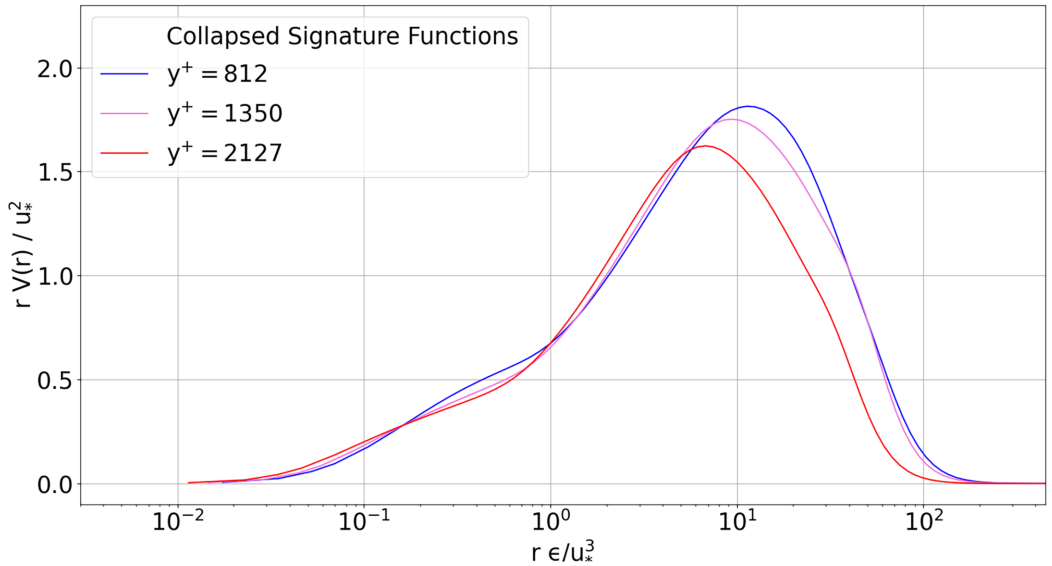
Turning to the signature function, Fig. 20 shows $rV(r)/u_*^2$ as a function of r/y , while Fig. 21 gives $rV(r)/u_*^2$ as a function of $\varepsilon r / u_*^3$. The collapse is clearly better when plotted against $\varepsilon r / u_*^3$, at least for $r < 4y$. Given that $y^+ > 800$, the smooth-wall data suggests that there should not be a

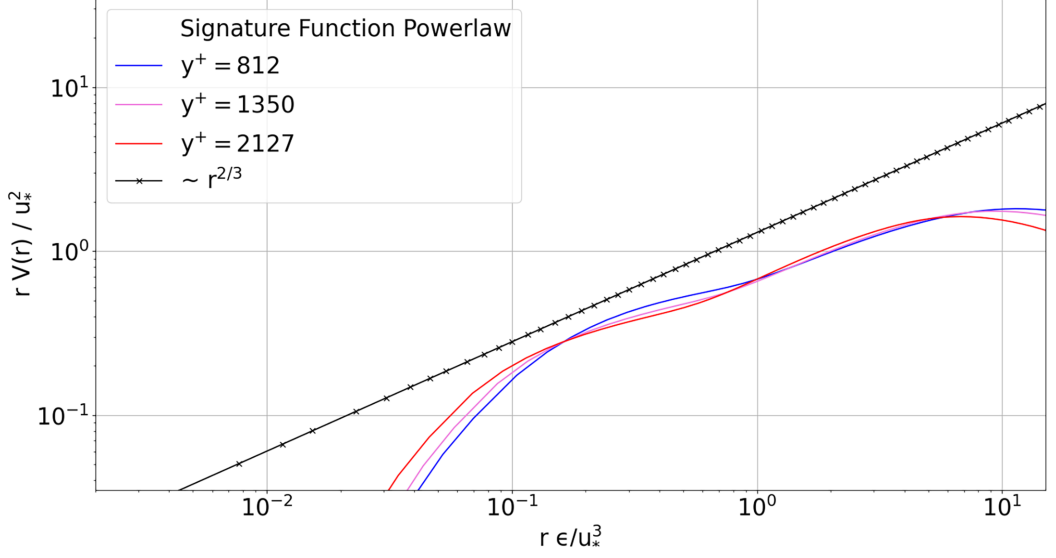

 FIG. 19. Rough-wall data of [13]. Log-log plots of $\langle (\Delta v)^2 \rangle / u_*^2$ as a function of $\varepsilon r / u_*^3$.

FIG. 20. Rough-wall data of [13]. Log-linear plots of $rV(r)/u_*^2$ versus r/y .

plateau, merely a single maximum, and this is indeed the case. The peaks lie in the range $rV(r)/u_*^2 \approx 1.5\text{--}1.8$, which is similar to the smooth-wall case.

Finally, Figs. 22 and 23 show log-log plots of $rV(r)/u_*^2$ as a function of $\varepsilon r/u_*^3$, with Fig. 22 centered on small r and Fig. 23 on larger r . An $r^{2/3}$ law is evident in Fig. 22, although the slope is slightly shallower than $2/3$, as is the case in Fig. 19. The peaks in Fig. 23 lie close to $3B_2/4 = 1.54$, but are typically slightly higher.

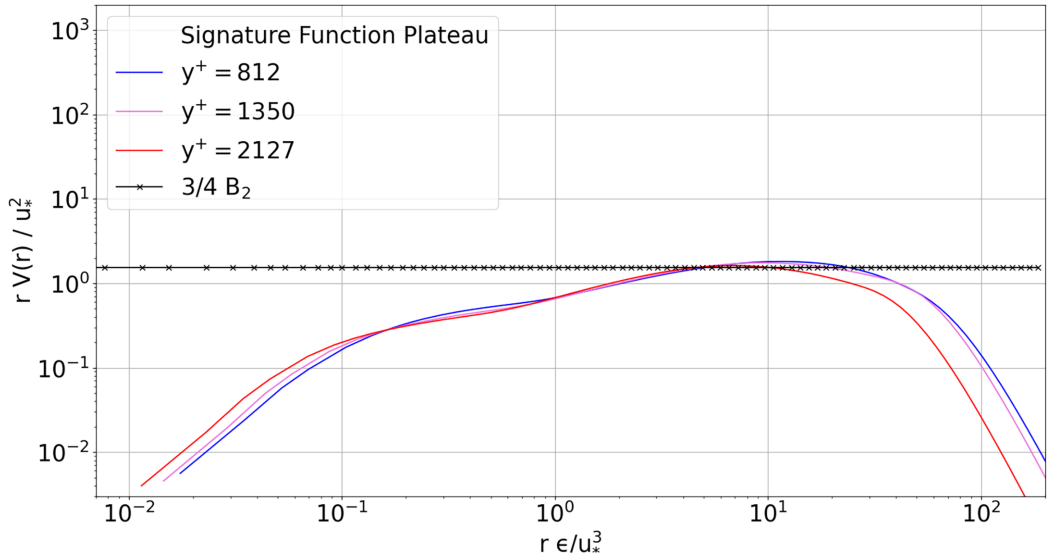
FIG. 21. Rough-wall data of [13]. Log-linear plots of $rV(r)/u_*^2$ versus $\varepsilon r/u_*^3$.


 FIG. 22. Rough-wall data of [13]. Log-log plots of $rV(r)/u_*^2$ versus $\epsilon r/u_*^3$ for small r .

V. CONCLUSIONS

Our version of the signature function improves on the original in that it can be shown to be strictly positive in isotropic turbulence. Moreover, it is obtained from experimental data through a process of integration, rather than differentiation, and so it is less sensitive to noise. Crucially, simple kinematic examples confirm that $rV(r) \approx [kE(k)]_{k=2\sqrt{2}/r}$, and so in isotropic turbulence $V(r)$ and $E(k)$ distribute energy in similar ways in scale space.

When applied to grid turbulence, our signature function recovers many of the expected features of a scale-by-scale energy density. For example, it falls off below ten Kolmogorov scales and also for


 FIG. 23. Rough-wall data of [13]. Log-log plots of $rV(r)/u_*^2$ versus $\epsilon r/u_*^3$.

scales substantially above the integral scale. Moreover, it exhibits a $2/3$ law in the inertial subrange, coupled to a bottleneck at the top of the dissipation range. Its behavior in near-wall turbulence is also in line with expectations. For example, the trend shown in Fig. 13, of the plateau

$$rV(r) = \frac{3}{4}B_2u_*^2 \approx 1.5u_*^2$$

adjacent to a bump for $y^+ < 100$, but a single maximum at $rV(r) \approx 1.7u_*^2$ for larger y^+ , is also observed in premultiplied one-dimensional spectra, $kF_{11}(k)$. (See, for example, Fig. 11 in [14] and Fig. 6 in [15].)

In summary, our scale-by-scale energy density is simple to implement (all that is needed is the longitudinal structure function), it distributes energy in isotropic turbulence in a manner similar to $E(k)$, and it yields energy distributions in grid turbulence and boundary layers that conform to expectation. In a follow-up paper, we shall apply this signature function to the Karman-Howarth equation, and thus move from kinematics to dynamics. This will allow us to examine energy fluxes and energy dissipation in real-space, rather than in Fourier space.

ACKNOWLEDGMENTS

The authors thank A. Cha and F. Smith for helpful discussions on the use of the signature function in processing the Modane SIMA wind-tunnel data.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: THE SIFTING OF A SMOOTH FUNCTION USING A NARROW FILTER

Let $f(s)$ be a relatively smooth function, and $F(r)$ be generated from $f(s)$ according to

$$F(r) = \frac{1}{\alpha_m} \int_0^\infty f(s)x^m e^{-x^2} dx, \quad x = \frac{s}{r}, \quad \alpha_m = \int_0^\infty x^m e^{-x^2} dx. \quad (\text{A1})$$

Here m is an integer, with $m \geq 3$, the filter $x^m e^{-x^2}$ is narrow for $m \gg 1$, and (A1) ensures

$$\int_0^\infty F(r) \frac{dr}{r} = \int_0^\infty f(s) \frac{ds}{s}. \quad (\text{A2})$$

For moderate values of m , $F(r)$ can be thought of as a smoothed version of $f(s)$. However, for large m (a narrow filter), the process of generating $F(r)$ is reminiscent of sifting $f(s)$ using a delta function. That is, $f(s)$ is replaced by $F(r)$, with s/r corresponding to the peak in $x^m e^{-x^2}$, i.e., $s/r = x_{\text{peak}} = \sqrt{m/2}$. A rough approximation to $F(r)$ may be obtained by assuming that, in the vicinity of x_{peak} , $f(s)$ can be approximated by a linear function of s . It is readily confirmed that, when $f(s)$ in (A1) is replaced by a linear function, we obtain

$$F(r) = f(s = \lambda_m r), \quad \lambda_m = \alpha_{m+1}/\alpha_m.$$

Moreover, given the definite integrals,

$$\alpha_{2n+1} = \frac{1}{2}n!, \quad (n \geq 1), \quad \alpha_{2n} = \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}}, \quad (n \geq 2), \quad (\text{A3})$$

we find $\alpha_{m+1}/\alpha_m \approx \sqrt{(m + \frac{1}{2})/2}$, to within an accuracy of 0.5% (for $m \geq 3$). It follows that $\lambda_m \approx \sqrt{(m + \frac{1}{2})/2}$. Of course, using a linear approximation to $f(s)$ for all x can be justified only when the filter is narrow (i.e., m is large). Nevertheless, we shall use the estimate

$$F(r) \approx f(s = \lambda_m r), \quad \sqrt{m/2} < \lambda_m < \sqrt{(m+1)/2}, \quad m \geq 3. \quad (\text{A4})$$

APPENDIX B: THE RELATIONSHIP BETWEEN THE SIGNATURE FUNCTION AND $E(k)$

From (15) and (16) we have

$$rV(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\int_0^\infty E(k) \frac{\sin 2xy}{xy} dk \right] \hat{g}(x) dx, \quad y = \frac{kr}{2}, \quad \hat{g}(x) = e^{-x^2} (4x^6 - 20x^4 + 15x^2),$$

from which

$$rV(r) = \frac{2}{\sqrt{\pi}r} \int_0^\infty \frac{E(k)}{k} \left[\int_0^\infty \sin(2xy) e^{-x^2} (4x^5 - 20x^3 + 15x) dx \right] dk.$$

Given the definite integral

$$\int_0^\infty \sin(2xy) e^{-x^2} (4x^5 - 20x^3 + 15x) dx = 2\sqrt{\pi} y^5 e^{-y^2},$$

we obtain the simple relationship

$$rV(r) = 2 \int_0^\infty kE(k) [y^3 e^{-y^2}] dy. \quad (\text{B1})$$

Combining this with (A4), we find $rV(r) \approx [kE(k)]_{k=2\sqrt{2}/r}$. We can also write this as

$$V(r) = \int_0^\infty kE(k) [y^3 \exp(-y^2)] dk, \quad y = kr/2.$$

By way of comparison, Hamba [5] proposed the real-space energy density

$$V_H(r) = 2 \int_0^\infty kE_1(k) [y \exp(-2y^2)] dk, \quad y = kr/2,$$

where $E_1(k)$ one-dimensional transform of $\langle \mathbf{u} \cdot \mathbf{u}' \rangle(r)$, which satisfies $E = -kdE_1/dk$ in isotropic turbulence.

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