

New subcritical oblique modes: On an extension of Squire's theorem for spatiotemporally evolving modes

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We revisit Squire's classical theorem, in which he showed that 2D modes are the more unstable ones, i.e., occur at a smaller Reynolds number, compared to 3D modes. Since this only applies to temporal modes, we extend it to spatiotemporal modes. This results in a new class of subcritical modes, where the 3D modes may occur at a smaller Reynolds number than the corresponding 2D modes. Three special features of the new modes become apparent: (i) they exhibit reflection symmetry breaking in the spanwise (z) direction; (ii) the spanwise complex wave number β is proportional to the streamwise wave number α (x direction), but with an imaginary prefactor. Hence, real and imaginary parts of α swap their roles for β , i.e., linking the wave propagation of one direction to the growth in the other direction; and (iii) β_r constitutes a wave number and therefore refers to an associated wavelength in the z direction. As a consequence, a finite box width in the z direction controls the critical 3D Reynolds number, and, in turn, certain perturbations in slender boxes may not be permitted. We have further analyzed the second part of Squire's theorem, i.e., that the homogeneous Squire equation admits no temporally unstable modes, and this is also lost with the new spatiotemporal modes. Finally, it is emphasized that the use of a single new mode of the above-mentioned type is not meaningful, since its spatial part leads to a diverging perturbation energy. Nevertheless, specific combinations of these modes, such as in Briggs' theory, may lead to new subcritical transitions.

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I. INTRODUCTION

The general basis for hydrodynamic stability theory was laid by Orr [1] and Sommerfeld [2], who derived the viscous governing equations for linear stability theory. In temporal linear stability theory, the temporal growth of a single mode is examined, i.e., it grows from an initially infinitesimally small magnitude, and eventually the whole flow becomes unstable, where in reality, at a certain point, the nonlinear effects play a non-negligible role and change the flow behavior. In his famous theorem, Squire [3] elegantly proved that for parallel shear flows, 2D perturbations become unstable

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at smaller Reynolds numbers than 3D perturbations in the case of temporally evolving perturbations. Hence, subsequently the focus of linear stability theory has strongly been on 2D perturbations.

Only decades later, the idea of spatially evolving modes was developed by Watson [4]. The spatially evolving modes were connected to the temporally evolving modes by Gaster [5] in the vicinity of neutral stability. However, in a strict sense, the growth of single spatial modes is unphysical, as perturbations become infinite in the domain and therefore, the perturbation energy is not meaningfully defined, as will be pointed out below.

A step further was the idea of combining both temporally and spatially evolving modes, developed by Betchov and Criminale [6] and later Gaster [7]. The question arises as to whether spatial or spatiotemporal modes can be meaningfully used at all. An elegant solution was provided by Briggs' theory [8], originally developed in the context of wave propagation in plasma physics, which does not consider individual modes but solves an initial value problem, assuming a spatially and temporally localized disturbance of finite extent. For a complete solution of the initial value problem, Briggs has shown that it becomes necessary to allow for spatiotemporal modes. Huerre and Monkewitz [9,10] then transferred the method of Briggs to hydrodynamic linear stability theory. By following the paths of singularities of the system in the complex plane and manipulating the contours associated with the inverse transformation of the mode distributions, they characterized whether a perturbation grows in the whole domain (absolute instability) or is transported by the flow (convective instability). However, this requirement of both a complex wave frequency and complex wave numbers is in contrast to classic temporal linear stability theory and to the assumptions of Squire's theorem. Hence, presently the idea is to extend the results of Squire, which are only valid for temporally evolving modes, to spatiotemporally evolving modes to develop a more general, and arguably more complete theory.

The present ideas were to some extent triggered by the experimental observation that wall-bounded shear flows form oblique structures in the transition from laminar to turbulent flow. Coles and van Atta [11–15] observed spiral structures of alternating laminar and turbulent regions in counterrotating Taylor-Couette flow by suspending metal flakes in oil. By measuring instantaneous velocities with hot-wire arrays, they noticed that the spiral structures move at approximately the mean velocity of the cylinders, and that fluid elements move alternately between the laminar and turbulent regions.

For plane Couette flow, oblique stripes of turbulent and laminar regions have been observed by Prigent *et al.* [16,17], using Kalliroscope flakes in the moving belt channel apparatus presented in Daviaud *et al.* [18], where a localized finite perturbation was introduced by a fluid jet. By investigating larger aspect ratios, they found large-wavelength modulations of turbulent and laminar regions at oblique angles, where earlier experiments with smaller aspect ratios only allowed the formation of singular turbulent spots, like in Tillmark and Alfredsson [19] or in Daviaud and coworkers [18,20]. Similarly, oblique patterns in plane Couette flow have also been observed in numerical simulations by Barkley and Tuckerman [21–23]. They employed a tilted domain aligned with the oblique pattern to impose periodic boundary conditions. With this, they were able to sustain these oblique structures for a range of tilt angles and domain sizes. Duguet *et al.* [24] simulated the development of initial noise, transitioning to streaks, then to spot nucleation and finally to oblique stripes in plane Couette flow for a range of Reynolds numbers.

For plane Poiseuille flow, oblique stripes have also been experimentally observed by Tsukahara *et al.* [25] using reflective flakes by slowly reducing the Reynolds numbers of a turbulent flow induced by a turbulent grid. Numerical simulations of plane Poiseuille flow were performed by Tsukahara *et al.* [26] at low Reynolds numbers, who found large-wavelength oblique stripes moving at approximately the mean flow velocity. Tuckerman *et al.* [27] performed DNS in a tilted domain analogous to the plane Couette flow case and found comparable structures to those of plane Couette flow, but with a shorter wavelength.

It is important to note that in all the flows mentioned above, the observed oblique structures were generally *not* at a small amplitude, i.e., in the order of a few % of the laminar base flow, and hence at an amplitude beyond what is usually assumed for linear stability theory. However, it shows that

oblique structures are the predominant flow form under certain conditions, but the underlying causes for this are largely unclear. This question will be discussed in more detail below, at least on the basis of linear theory.

Complex wave numbers in the streamwise and spanwise directions—analyzed in detail in the present work—were studied in the Master’s thesis of Turkac [28] and then again in the dissertation of Yalcin [29], both of which were supervised by the first author. These studies consider single spatial eigenmodes. However, spatial eigenmodes cannot be used individually, as they diverge in space, even initially. With the present work, a jointly formulated comprehensive theory on spatiotemporal oblique modes and subcritical growth by extending Squire’s theorem is now available. It relies on Briggs’ theory on spatiotemporal modes, whereby an intricate combination of modes leads to finite perturbation energy.

II. GOVERNING EQUATIONS AND LINEAR STABILITY THEORY

Let the motion of an incompressible Newtonian fluid be governed by the set of dimensionless Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0, \quad (1a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}, \quad \text{with} \quad \text{Re} := \frac{\varrho U_0 L}{\mu}, \quad (1b)$$

where $\mathbf{u}$ denotes the velocity field, p the pressure field, and Re the Reynolds number, which is composed of the fluid density ϱ , the dynamic viscosity μ , a characteristic velocity U_0 , and a characteristic length scale L .

In linear stability theory, the velocity field is assumed to consist of laminar base velocities fulfilling the Navier-Stokes equations (1), superimposed with velocity perturbations. The momentum equations for these perturbations are then linearized. Let a Cartesian coordinate system with coordinates (x, y, z) for the streamwise, wall-normal, and spanwise directions be introduced with $\mathbf{U} = (U, V, W)$ as the laminar base flow components and $\mathbf{u}' = (u', v', w')$ as the velocity perturbations. In parallel shear flows, the base flow consists only of a streamwise velocity component U depending only on the wall-normal coordinate y , i.e., $\mathbf{U} = (U(y), 0, 0)$. Inserting this into the perturbation equations, a fourth-order differential equation for the wall-normal velocity perturbation v' and a second-order differential equation in terms of the wall-normal vorticity perturbation η' can be generated,

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{\text{Re}} \nabla^4 \right] v' = 0, \quad (2a)$$

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{\text{Re}} \nabla^2 \right] \eta' = -U' \frac{\partial v'}{\partial z}, \quad (2b)$$

where the wall-normal vorticity perturbation η' is defined as

$$\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}. \quad (3)$$

Let the functions v' , η' be reformulated by a normal mode approach consisting of a y -dependent amplitude function marked with a tilde and an exponential with a wave frequency ω in time t and

wave numbers α, β in the x and z directions:

$$v'(x, y, z, t) = \tilde{v}(y)e^{i(\alpha x + \beta z - \omega t)}, \quad (4a)$$

$$\eta'(x, y, z, t) = \tilde{\eta}(y)e^{i(\alpha x + \beta z - \omega t)}. \quad (4b)$$

This transforms the spatiotemporal partial differential equations (2) into wall-normal ordinary differential equations in the frequency and wave number domain,

$$\left[(-i\omega + i\alpha U) \left(\frac{d^2}{dy^2} - (\alpha^2 + \beta^2) \right) - i\alpha U'' - \frac{1}{\text{Re}} \left(\frac{d^2}{dy^2} - (\alpha^2 + \beta^2) \right)^2 \right] \tilde{v} = 0, \quad (5a)$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{\text{Re}} \left(\frac{d^2}{dy^2} - (\alpha^2 + \beta^2) \right) \right] \tilde{\eta} = -i\beta U' \tilde{v}, \quad (5b)$$

where the first equation (5a) is called the Orr-Sommerfeld equation (OSE) and the second one (5b) is called the Squire equation (SE). At this point, it is not specified whether ω and α belong to the set of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, giving rise to different physical wave phenomena.

III. TEMPORAL STABILITY: REVISITING SQUIRE'S THEOREM

Squire considered temporally evolving modes with a complex wave frequency $\omega \in \mathbb{C}$, $\omega = \omega_r + i\omega_i$, and real wave numbers $\alpha, \beta \in \mathbb{R}$. Considering the normal mode approach (4), this means that the spatial distribution is assumed to be wavelike, with the frequency ω_r in time and wave numbers α, β in the x and z directions, and the modes can grow in time with positive ω_i or decay in time with negative ω_i .

A. Revisiting the Squire transformation

The key idea of Squire was that the OSE (5a) has a similar structure in 2D and 3D. Comparing the OSE in 2D and 3D by introducing respective indices, i.e., for 2D,

$$\left[(-i\omega_{2D} + i\alpha_{2D}U) \left(\frac{d^2}{dy^2} - \alpha_{2D}^2 \right) - i\alpha_{2D}U'' - \frac{1}{\text{Re}_{2D}} \left(\frac{d^2}{dy^2} - \alpha_{2D}^2 \right)^2 \right] \tilde{v} = 0, \quad (6)$$

and for 3D

$$\left[(-i\omega_{3D} + i\alpha_{3D}U) \left(\frac{d^2}{dy^2} - (\alpha_{3D}^2 + \beta^2) \right) - i\alpha_{3D}U'' - \frac{1}{\text{Re}_{3D}} \left(\frac{d^2}{dy^2} - (\alpha_{3D}^2 + \beta^2) \right)^2 \right] \tilde{v} = 0, \quad (7)$$

it can be observed that both equations have the same form and hence share the same solution space for the following set of invariance transformations:

$$\omega_{2D} \text{Re}_{2D} = \omega_{3D} \text{Re}_{3D}, \quad (8a)$$

$$\alpha_{2D} \text{Re}_{2D} = \alpha_{3D} \text{Re}_{3D}, \quad (8b)$$

$$\alpha_{2D}^2 = \alpha_{3D}^2 + \beta^2. \quad (8c)$$

Probably without knowing it, Squire introduced an equivalence transformation, as it is named in the field of Lie group theory [30]. By introducing a parameter $\phi \in \mathbb{R}^+$ defined as the quotient of the 3D and 2D Reynolds numbers, the 3D quantities can be expressed by the 2D quantities and

vice versa:

$$\phi := \frac{\text{Re}_{3D}}{\text{Re}_{2D}}, \quad (9a)$$

$$\omega_{3D} = \frac{\omega_{2D}}{\phi}, \quad (9b)$$

$$\alpha_{3D} = \frac{\alpha_{2D}}{\phi}, \quad (9c)$$

$$\beta = \pm \sqrt{1 - \frac{1}{\phi^2}} \alpha_{2D}. \quad (9d)$$

For the temporally evolving modes considered by Squire, $\omega \in \mathbb{C}$, $\alpha, \beta \in \mathbb{R}$, this parameter choice implies $\phi > 1$ to keep β real in (9d). This means by definition of (9a) that for a 2D and a 3D flow in the same solution space, the 3D flow has a higher Reynolds number, or in other words, if there is a 2D perturbation associated with a critical (2D) Reynolds number, the corresponding 3D perturbation will be at a higher (3D) Reynolds number.

This leads back to Squire's classical finding that 2D modes become unstable at smaller Reynolds numbers. Squire concluded that it is sufficient to consider 2D OSE modes for the stability analysis of the OSE.

B. Revisiting Squire modes

Squire identified a second class of modes relevant to the stability problem: SE modes are modes from the homogeneous SE. The regular (inhomogeneous) SE (5b) is coupled to the OSE and therefore does not contribute any new modes to the stability problem. Here, Squire also only considered temporally evolving modes, i.e., $\omega \in \mathbb{C}$, $\alpha, \beta \in \mathbb{R}$. In order to understand the second part of Squire's theorem, the homogeneous SE

$$\left[(-i\omega + i\alpha U) - \frac{1}{\text{Re}} \left(\frac{d^2}{dy^2} - (\alpha^2 + \beta^2) \right) \right] \tilde{\eta} = 0, \quad (10)$$

is multiplied by the complex conjugate of $\tilde{\eta}$, i.e., $\tilde{\eta}^*$, yielding $\tilde{\eta}\tilde{\eta}^* = |\tilde{\eta}|^2$, and integrated over the domain $\Omega \subset \mathbb{R}$ with respect to the y coordinate:

$$\omega \int_{\Omega} |\tilde{\eta}|^2 dy - \alpha \int_{\Omega} U |\tilde{\eta}|^2 dy - \frac{i}{\text{Re}} \int_{\Omega} \left[\left(\frac{d^2}{dy^2} \tilde{\eta} \right) \tilde{\eta}^* - (\alpha^2 + \beta^2) |\tilde{\eta}|^2 \right] dy = 0. \quad (11)$$

Rearranging the second-derivative term with integration by parts, the boundary term of the integration by parts vanishes, since $\tilde{\eta}$ vanishes at the boundaries. Notably, the integration by parts yields a negative sign for the second-derivative term, and the complex conjugation is commutative with differentiation for real y . When taking the imaginary part of the resulting equation, the integral with the factor of α vanishes because this term is exclusively real. Thus, the remaining integral is rearranged to the right-hand side to obtain

$$\omega_i \int_{\Omega} |\tilde{\eta}|^2 dy = -\frac{1}{\text{Re}} \int_{\Omega} \left[\left| \frac{d}{dy} \tilde{\eta} \right|^2 + (\alpha^2 + \beta^2) |\tilde{\eta}|^2 \right] dy, \quad (12)$$

where the integrands on both sides are exclusively non-negative, so the imaginary part of the wave frequency ω_i must always be nonpositive. Therefore, SE modes never temporally grow, i.e., they are always damped in time. Squire concluded that it is sufficient to consider the OSE modes for the stability analysis.

Overall, combining the results of the Squire transformation and the Squire modes, it is sufficient to analyze the 2D OSE modes when considering temporally evolving modes.

IV. SPATIAL STABILITY

The next logical idea is to examine spatially evolving modes with real wave frequency $\omega \in \mathbb{R}$ and complex wave numbers $\alpha, \beta \in \mathbb{C}$, $\alpha = \alpha_r + i\alpha_i$, and $\beta = \beta_r + i\beta_i$. In contrast to temporally evolving modes, spatially evolving modes can grow in either the positive or negative spatial direction and then decay in the opposite direction. For parallel shear flows, this means that modes can grow both upstream or downstream and in both spanwise directions.

Considering the normal mode approach (4), a positive imaginary part of the wave numbers corresponds to a spatial growth in the negative x and z directions, and a negative imaginary part, respectively, means a spatial growth in the positive x and z directions. This is always accompanied by a spatial decay in the respective opposite directions.

However, individual spatially evolving modes are unphysical, as perturbations grow beyond any bounds in any of the homogeneous directions, or, in other words, become infinite in the domain $x, z \in (-\infty, \infty)$. As a result, the perturbation energy is not defined in a meaningful way even initially at $t = 0$. We will get back to this key point subsequently in the section about the group velocity.

V. SPATIOTEMPORAL STABILITY: EXTENSION OF SQUIRE'S THEOREM

Let us consider spatiotemporally evolving modes with complex wave frequency $\omega \in \mathbb{C}$, $\omega = \omega_r + i\omega_i$, and complex wave numbers $\alpha, \beta \in \mathbb{C}$, $\alpha = \alpha_r + i\alpha_i$, and $\beta = \beta_r + i\beta_i$. For this case, the real wave parameters ω_r , α_r , and β_r describe the harmonics in time and space, while ω_i , α_i , β_i express the growth or decay of the perturbation mode in time and in the x and z directions.

A. Extension of the Squire transformation

Comparing once again the OSE in 2D (6) and 3D (7), both again share the same solution space for the same set of invariance transformations (8) as for the temporally evolving modes, but this time, the third condition (8c) can be split into a real and an imaginary part, yielding

$$\alpha_{2D,r}^2 - \alpha_{2D,i}^2 = \alpha_{3D,r}^2 - \alpha_{3D,i}^2 + \beta_r^2 - \beta_i^2, \quad (13a)$$

$$\alpha_{2D,r}\alpha_{2D,i} = \alpha_{3D,r}\alpha_{3D,i} + \beta_r\beta_i. \quad (13b)$$

By introducing the same parameter ϕ by (9a) as for the temporally evolving modes, the 3D quantities can be expressed again in terms of the 2D quantities and vice versa. Herewith, the transformations for ω and α remain formally the same as in (9b) and (9c), but a quartic equation for β arises,

$$\beta_r^4 - (\alpha_{2D,r}^2 - \alpha_{2D,i}^2) \left(1 - \frac{1}{\phi^2}\right) \beta_r^2 - (\alpha_{2D,r}\alpha_{2D,i})^2 \left(1 - \frac{1}{\phi^2}\right)^2 = 0, \quad (14a)$$

$$\beta_i = \left(1 - \frac{1}{\phi^2}\right) \alpha_{2D,r}\alpha_{2D,i} \frac{1}{\beta_r}, \quad (14b)$$

which has apparently four distinct solutions, and again we have $\phi \in \mathbb{R}^+$:

$$\omega_{3D} = \frac{\omega_{2D}}{\phi}, \quad (15a)$$

$$\alpha_{3D} = \frac{\alpha_{2D}}{\phi}, \quad (15b)$$

$$\beta = \begin{cases} \pm \sqrt{1 - \frac{1}{\phi^2}} \alpha_{2D} & \text{for } \phi > 1 \\ \pm i \sqrt{\frac{1}{\phi^2} - 1} \alpha_{2D} & \text{for } \phi < 1. \end{cases} \quad (15c)$$

$$\beta = \begin{cases} \pm \sqrt{1 - \frac{1}{\phi^2}} \alpha_{2D} & \text{for } \phi > 1 \\ \pm i \sqrt{\frac{1}{\phi^2} - 1} \alpha_{2D} & \text{for } \phi < 1. \end{cases} \quad (15d)$$

Now there are two possible solution branches of β , depending on whether $\phi > 1$ or $\phi < 1$. The physically most interesting case is (15d), i.e., $\phi < 1$, as this would indicate by definition of (9a),

and this is one of the key results of the present analysis, that a 2D flow would have a higher Reynolds number than a 3D flow in the same solution space, or, in other words: If there is a 2D perturbation associated with a critical (2D) Reynolds number, the corresponding 3D perturbation may occur at a smaller, subcritical (3D) Reynolds number, and therefore, 3D modes may be spatiotemporally more unstable at smaller Reynolds numbers.

In the following, we assume that the case (15d) exists, i.e., $\phi < 1$, and we investigate the resulting implications. This means, in particular, that by decomposing (15d) for the real and imaginary parts of β , we find

$$\beta_r = \mp \sqrt{\frac{1}{\phi^2} - 1} \alpha_{2D,i}, \quad (16a)$$

$$\beta_i = \pm \sqrt{\frac{1}{\phi^2} - 1} \alpha_{2D,r}. \quad (16b)$$

From this, we may draw three further important conclusions:

(i) It is interesting to note that the $\pm$ or $\mp$ signs constitute a spanwise reflection symmetry breaking. The original governing equations have no preferred direction in z .

(ii) Interestingly, the real and imaginary parts of α_{2D} swap their roles in the formulation of β , i.e., the real part of β is proportional to the imaginary part of α_{2D} and vice versa, linking the wave propagation of one direction to the growth in the other direction in the (x, z) plane.

(iii) Lastly, the wave number β_r is associated with a wavelength in the z direction. As both in experiments, but in particular in numerical simulations, the box size in the z -direction is usually chosen comparatively small, it means that the box width controls the critical 3D Reynolds number, since the box width limits the largest spanwise wavelength or the smallest β_r , corresponding to a given $\alpha_{2D,i}$. This in turn can determine which perturbations exist, or, in other words, in certain boxes, some perturbations may not be permitted.

B. Extension of the Squire modes

Starting again from the homogeneous SE (10), with the same procedure (11) as for the temporal case, taking the imaginary part of the equation yields this time:

$$\omega_i \int_{\Omega} |\tilde{\eta}|^2 dy - \alpha_i \int_{\Omega} U |\tilde{\eta}|^2 dy = -\frac{1}{\text{Re}} \int_{\Omega} \left[\left| \frac{d}{dy} \tilde{\eta} \right|^2 + (\alpha_r^2 - \alpha_i^2 + \beta_r^2 - \beta_i^2) |\tilde{\eta}|^2 \right] dy, \quad (17)$$

where the second integrand on the left-hand side depends on the laminar base flow U , and, furthermore, the sign of the second term in the integrand on the right-hand side depends on the real and imaginary parts of the wave numbers α and β . Thus, without further assumptions, the signs of the imaginary parts of ω_i and α_i cannot be determined. It is possible that the SE modes grow in time, in space, or in both, or, in other words: The SE modes cannot be neglected in the spatiotemporal stability analysis.

For a temporally neutral mode, i.e., $\omega_i = 0$, we notice that the sign of α_i in (17) can also not be determined for the same reasons.

On a side note, both the classical Squire transformation (9) and the extended Squire transformation (15) are also an equivalence transformation of the homogeneous SE (10) and can therefore also be applied to this equation, whereas the (inhomogeneous) Squire equation (5b) does not have this property. Although SE modes are by definition 3D, as a z and w' dependence is needed for the definition of η' , it might still be possible to analyze the behavior of SE modes through the 2D homogeneous SE and the extended Squire transformation.

Overall, combining the results of the extended Squire transformation and the extension of the SE modes, 2D and 3D OSE modes as well as SE modes have to be analyzed when considering spatiotemporally evolving modes.

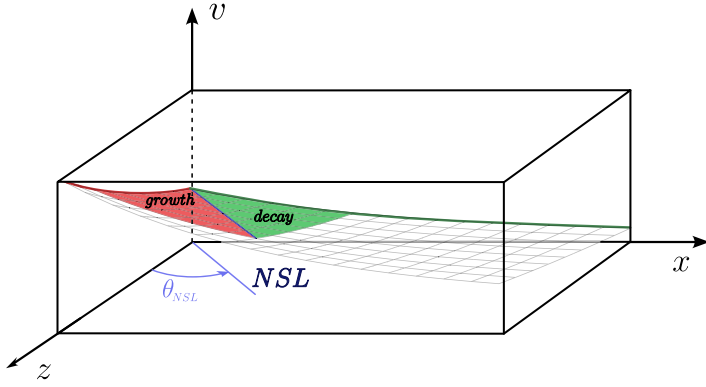


FIG. 1. 3D oblique mode with areas of spatial growth (red), decay (green), the neutral stability line (NSL), and the neutral stability angle θ_{NSL} , shown here as an example for $\alpha_{3\text{D},i} > 0$, $\beta_i < 0$, and $\alpha_{3\text{D},r} = \beta_r = 0$ to avoid oscillations for a better visibility of growth and decay.

C. Oblique spatial growth

To illustrate the oblique spatial growth associated with spatiotemporally evolving modes, a neutral stability line (NSL) can be defined in the (x, z) plane by considering the real spatial exponent of the normal mode approach (4). Along a line where the real spatial exponent is zero,

$$\alpha_{3\text{D},i}x + \beta_i z = 0, \quad (18)$$

there is neither growth nor decay. A corresponding neutral stability angle θ_{NSL} (measured from the z axis to be defined in the positive rotation direction for the positive y -direction in a right-hand system) can be defined, with the Squire transformation (15) applied as

$$\tan \theta_{\text{NSL}} = -\frac{\beta_i}{\alpha_{3\text{D},i}} = \mp \sqrt{1 - \phi^2} \frac{\alpha_{2\text{D},r}}{\alpha_{2\text{D},i}}. \quad (19)$$

Figure 1 shows the spatial growth of a spatiotemporally evolving mode. The vertical axis shows the exponential growth or decay of the normal mode approach (4), $\bar{v} = \exp[-(\alpha_{3\text{D},i}x + \beta_i z)]$, on the (x, z) plane with constant y coordinate, illustrated here as an example for $\alpha_{3\text{D},i} > 0$, $\beta_i < 0$, and $\alpha_{3\text{D},r} = \beta_r = 0$ to avoid oscillations for a better visibility of growth and decay. Considering the normal mode approach (4), this choice of $\alpha_{3\text{D},i}$ and β_i means that the mode is decaying in the positive x direction (and growing in the negative x direction) while growing in the positive z direction (and decaying in the negative z direction). This splits the (x, z) plane into a region of overall growth (red) and a region of overall decay (green) along the NSL. Thus, oblique spatial growth can be formulated with this theory.

D. Group velocity

As discussed in the section about spatial stability, individual spatially evolving modes are not physical because their perturbations diverge in an infinite domain even initially at $t = 0$. The same also applies to individual spatiotemporally evolving modes, as they share the same diverging spatial part.

This unphysical behavior can be resolved by considering superpositions of modes, i.e., continuous distributions of modes rather than individual ones. In Briggs's theory, the normal mode approach (4) is replaced by a Laplace transform in time and a Fourier transform in space, leading to the same set of equations (5), but now analyzed in the complex planes of wave frequency and wave numbers. The respective inverse transforms contain contributions from all relevant wave frequencies and wave numbers along chosen integration contours. As the integration contours—especially after

their subsequent deformations—are generally complex, this necessitates the use of both a complex wave frequency and complex wave numbers, or, in other words, spatiotemporally evolving modes. When evaluating these inverse transforms, the integration contours are closed in the complex planes using Jordan’s lemma and the residue theorem in a way that ensures convergence of the integrals, thereby resolving the problem of diverging individual spatial modes.

In the context of Briggs’ theory, a key quantity to assess the spatiotemporal response of a system to an initial disturbance is the group velocity. A saddle point of the dispersion relation, leading to an absolute instability, yields zero group velocity as the necessary condition for such an absolute instability to occur. Further, for a convective instability, the group velocity conveys the velocity an observer has to take to follow the evolving disturbance in a stationary frame of reference. In other words, it is a measure for the propagation velocity of a convective instability.

For practical purposes, we define the group velocity as $\Re(d\omega/d\alpha)$. This definition denotes the derivative of a complex ω according to a complex α , which generally results in a complex expression, of which the real part $\Re(\cdot)$ is taken. This is equal to the definition $\partial\omega_r/\partial\alpha_r$ widely used in the literature, e.g., in the context of Gaster’s transformation [5,7], due to the Cauchy-Riemann equations and the uniqueness of the underlying limit, defining the complex derivative,

$$\frac{d\omega}{d\alpha} = \lim_{h \rightarrow 0} \frac{\omega(\alpha + h) - \omega(\alpha)}{h} = \frac{\partial\omega}{\partial\alpha_r} = \frac{\partial\omega}{\partial(i\alpha_i)}, \quad (20a)$$

$$\frac{\partial\omega}{\partial\alpha_r} = \frac{\partial(\omega_r + i\omega_i)}{\partial\alpha_r} = \frac{\partial\omega_r}{\partial\alpha_r} + i \frac{\partial\omega_i}{\partial\alpha_r} \quad (20b)$$

for arbitrary $h \in \mathbb{C}$. The group velocity is retrieved from the analytic dispersion relation $\mathcal{D}_{2D}$ via a Taylor series expansion around a point that fulfills the dispersion relation $\mathcal{D}_{2D} = 0$, i.e.,

$$d\mathcal{D}_{2D} = 0 = \frac{\partial\mathcal{D}_{2D}}{\partial\omega_{2D}} d\omega_{2D} + \frac{\partial\mathcal{D}_{2D}}{\partial\alpha_{2D}} d\alpha_{2D}, \quad (21a)$$

$$\frac{d\omega_{2D}}{d\alpha_{2D}} = - \frac{\frac{\partial\mathcal{D}_{2D}}{\partial\alpha_{2D}}}{\frac{\partial\mathcal{D}_{2D}}{\partial\omega_{2D}}}. \quad (21b)$$

Applying the extension of the Squire transformation (15) also to the group velocity leads to

$$\frac{d\omega_{3D}}{d\alpha_{3D}} = \frac{d\omega_{2D}}{d\alpha_{2D}}, \quad (22a)$$

$$\frac{d\omega_{3D}}{d\beta} = \pm \frac{1}{i\sqrt{1-\phi^2}} \frac{d\omega_{2D}}{d\alpha_{2D}}, \quad (22b)$$

from which the real part will be taken:

$$\Re\left(\frac{d\omega_{3D}}{d\alpha_{3D}}\right) = \Re\left(\frac{d\omega_{2D}}{d\alpha_{2D}}\right), \quad (23a)$$

$$\Re\left(\frac{d\omega_{3D}}{d\beta}\right) = \pm \frac{1}{\sqrt{1-\phi^2}} \Im\left(\frac{d\omega_{2D}}{d\alpha_{2D}}\right). \quad (23b)$$

Note that dividing by i in (22b), taking the real part of this expression translates to taking the imaginary part, as in (23b). This is analogous to (16), where the roles of the real and imaginary part of α_{2D} are changed in the formulation of β . Here, the imaginary part of the 2D group velocity determines the real part of the z component of the 3D group velocity.

VI. CONCLUSIONS AND OUTLOOK

By extending Squire’s classical theorem to spatiotemporally evolving modes, i.e., considering a complex wave frequency and complex wave numbers at the same time, we were able to derive relationships between the 2D and 3D parameters of the eigenvalue problem.

Interestingly, in this parameter space, there may be unstable perturbation modes with a lower Reynolds number in 3D than in 2D, meaning that they would appear first in 3D. That said, the stability analyses can still be conducted in 2D for simplicity's sake and then scaled to 3D according to (15). Moreover, in that parameter space, it is possible to formulate perturbation modes with oblique spatial growth in the (x, z) plane.

Furthermore, the SE modes, stemming from the homogeneous SE, cannot be neglected in the spatiotemporal stability analyses, because, unlike in the case of temporally evolving modes, the sign of the growth factors cannot be determined.

However, it should be emphasized once again that the subcritical spatiotemporal modes may not be considered as individual modes, as their total perturbation energy is not bounded. Rather, we reckon that these modes can only be used in special combinations, i.e., the present extension of Squire's theorem may be especially useful, e.g., in the context of Briggs' method. In this theory, a spatiotemporal Dirac impulse or a time-periodic spatial Dirac impulse and their temporal evolution in the sense of an initial value problem are investigated. The forcings are typically located on the right-hand side of the OSE or the homogeneous SE. The equivalence transformation of the new subcritical oblique modes is preserved, but it is affected insofar as the scaling factor ϕ rescales the amplitude of the forcings.

The question of how the new modes will develop for large times, i.e., when they have left the linear regime, will be particularly fascinating. It is at least conceivable that nonlinear structures with oblique character will also arise from the linear modes.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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