

Exact solution for the electric field associated with charge caps on a leaky dielectric droplet at high electric Reynolds number

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In a recent paper [Peng *et al.*, [Phys. Rev. Fluids](#) **9**, 083701 (2024)], the phenomenon of electric charge cap formation on the surface of a leaky dielectric droplet in an electric field was identified. The surface charge caps, which form at the droplet poles when charge relaxation is faster inside the droplet phase than in the ambient, are stagnant and perfectly conducting. Using asymptotic methods, a nonstandard, inhomogeneous, mixed boundary value problem satisfied by the droplet field potential in this high electric Reynolds number limit was derived by Peng *et al.* and then solved numerically. The present article gives an exact solution to that mixed boundary value problem. This solution resolves fully the square-root singularities in the potential at the edges of the electric caps that can cause convergence difficulties in numerical schemes. Knowledge of the solution in analytical form is valuable because it feeds into a nonlinear problem for the coupled hydrodynamics. Moreover, the theoretical approach here is potentially extendible to finding the fields associated with electric charge cap formation in other settings.

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I. INTRODUCTION

In a recent paper, Peng *et al.* [1] elucidated two novel features of the electrodynamics of a leaky dielectric droplet at high electric Reynolds number when surface charge advection is important. Droplet deformation is not necessary for their occurrence, and both phenomena take place on an undeformed, two-dimensional leaky dielectric circular droplet in a uniform electric field: a so-called blowup singularity involving charge annihilation at the equator of a droplet, and the formation of “electric charge caps” at its poles. The setting of their study is a two-dimensional, steady, circular viscous droplet of leaky-dielectric fluid embedded in a second such fluid with differing material parameters sitting in an imposed electric field that is uniform far from the droplet. A key parameter, denoted by ϖ , was identified, reproduced later in (7), and this parameter determines which of the two phenomena take place. For $-1 < \varpi < 0$, or the so-called antiparallel polarization case, the blowup singularity occurs, while for $\varpi > 0$, the parallel-polarization case, the electric charge caps form.

The present paper offers some analytical results to complement the work of Peng *et al.* [1]. It concerns only the second case $\varpi > 0$, where electric charge caps form. These caps form because charge is advected toward the poles of the droplet, and Peng *et al.* [1] carry out an asymptotic analysis of this problem when the electric Reynolds number is large. At leading order in the

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analysis, a pair of electric and hydrodynamic mixed boundary value problems are shown to govern the system. This is a highly nonlinear regime but, interestingly, the “electric cap problem” is only mildly nonlinear: it resembles a classical mixed boundary value problem at first sight, but the edges of the cap are not known *a priori* but are determined by a global charge balance. Another important feature is that this electric cap problem decouples from the hydrodynamic problem yet provides a forcing for it. Both the electric cap problem and the hydrodynamic problem, are solved numerically in Sec. VII of Peng *et al.* [1].

The purpose of the present article is to show that a full analytical solution can be found for the electric cap problem. The key to the analysis is the use of mathematical tools based on conformal mapping and a special “prime function” object [2] that has recently been shown to have great utility in a wide range of fluid dynamics problems. This includes the quantification of frictional slip lengths for superhydrophobic surfaces [3], finding the pumping speed of a fluid in a thermocapillary-stress-driven pump device [4] and calculating the mobility coefficients for torque-driven superhydrophobic cylinders [5]. It has also been used previously in studying leaky dielectrics: it is a key tool in an analytical model of a pumping protocol for leaky-dielectric fluids that exploits Maxwell stresses in a specially designed flow architecture [6]. The result in the present paper is significant not only in providing a convenient closed-form expression for the electric field potential, but also because it fully resolves square-root singularities of this potential associated with the four cap edges, singularities that can cause difficulties in numerical schemes if not accounted for properly. While no analytical solution to the nonlinear hydrodynamic problem has yet been found, meaning that it is still necessary to use numerical methods for it, that problem is forced by the solution of the electric field problem making it convenient to have available analytical expressions for these forcing terms.

In view of the two phenomena identified by Peng *et al.* [1], and the future investigations they are likely to inspire, it is worth documenting this nontrivial analytical result as a complement to that work, and as another example of the versatility of the prime function machinery [2].

II. ELECTRIC CHARGE CAP PROBLEM

It is unnecessary to repeat all the formulational details of the study of Peng *et al.* [1], whose scope was broad, except to summarize what is necessary to explain the derivation of the analytical solution. Indeed, in addition to borrowing the problem setting and the nondimensionalization, the work here is mainly relevant to results in Sec. VII of Peng *et al.* [1] and readers should refer to the original work for more details of the origins of the mixed boundary value problem to be stated next. For ease of cross reference, the presentation here has emulated the notation of that work as closely as possible.

According to Sec. VII(C) of Peng *et al.* [1], in the limit of high electric Reynolds number, the boundary value problem for the leading order electric field potential inside a circular droplet of unit radius centered at the origin, denoted by Peng *et al.* by $\tilde{\phi}^-$ but denoted here simply by ϕ , is that it is harmonic inside the droplet, i.e.,

$$\nabla^2 \phi = 0, \quad \text{inside the droplet.} \quad (1)$$

The droplet boundary, which is the unit circle, is characterized by two charge caps of equal circumferential length, denoted by $\mathcal{C}$ and $\mathcal{D}$ in Fig. 1, and where the edge of cap $\mathcal{C}$ in the first quadrant is at $\theta = \theta^*$ where θ is the usual polar angle in the droplet. On these caps, the boundary conditions are of Dirichlet type, namely

$$\phi = \begin{cases} -\phi^*, & \text{on } \mathcal{C}, \\ +\phi^*, & \text{on } \mathcal{D}, \end{cases} \quad (2)$$

where the data depends on a so-called cap voltage ϕ^* , a constant to be determined by additional conditions stated later. The rest of the droplet boundary, comprising two disconnected boundary portions denoted by $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ in Fig. 1, are charge depletion zones; the notation $\bar{\mathcal{C}}$ was also used by

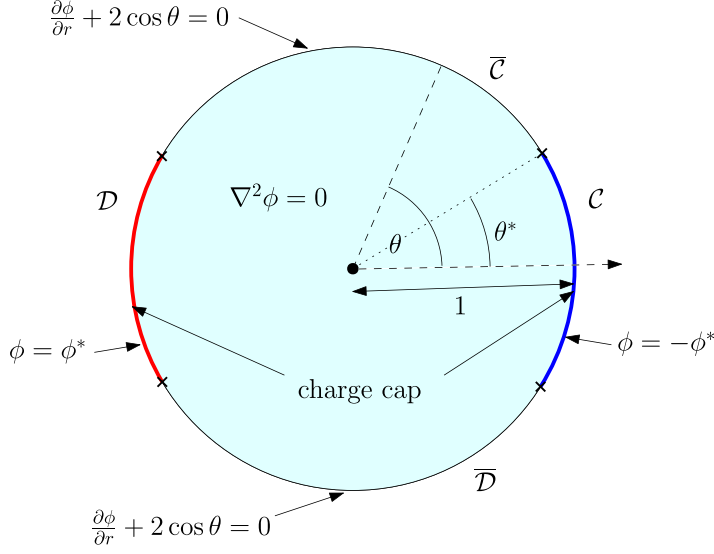


FIG. 1. The mixed boundary value problem for the electric potential ϕ inside a circular leaky dielectric droplet of unit radius in the limit of high electric Reynolds number following Peng *et al.* [1]. The cap voltage ϕ^* and the cap angle θ^* must be found as part of the solution.

Peng *et al.* [1]. Since the surface charge density for a droplet in a far-field uniform electric field is given by

$$q_0 = \frac{\partial \phi}{\partial r} + 2 \cos \theta, \quad (3)$$

then the boundary conditions on these depletion zones, where $q_0 = 0$, are inhomogeneous Neumann conditions on ϕ :

$$\frac{\partial \phi}{\partial r} = -2 \cos \theta, \quad \text{on } \bar{C}, \bar{D}. \quad (4)$$

The potential ϕ is symmetric about the diameter $\theta = 0, \pi$ along the x axis and antisymmetric about the diameter along the y axis $\theta = \pm \pi/2$.

At first sight this appears to be a standard mixed boundary value problem (BVP) with alternating Dirichlet data and Neumann data given on contiguous boundary portions. However, it is nonstandard, first because it contains the unknown cap voltage ϕ_* in the Dirichlet data (2), and, second, because the cap angle θ^* is not known *a priori* but must be determined as part of the solution. The latter feature renders it similar to what has been dubbed a codimension-2 free boundary problem [7], in which a point where the boundary condition changes type must be found as part of the solution. Following Peng *et al.* [1], the parameters ϕ^* and θ^* are determined by the local conditions

$$\lim_{\theta \rightarrow \theta^*} q_0 = 0, \quad \text{or} \quad \lim_{\theta \rightarrow \theta^*} \frac{\partial \phi}{\partial \theta} = 0, \quad (5)$$

and a global total charge condition given by

$$\int_0^{\theta^*} q_0 d\theta = \frac{2\varpi}{1 + \varpi}, \quad (6)$$

where

$$\varpi = \frac{\mathcal{R} - \mathcal{S}}{1 + \mathcal{S}}, \quad \mathcal{R} = \frac{\sigma^-}{\sigma^+}, \quad \mathcal{S} = \frac{\epsilon^-}{\epsilon^+}, \quad (7)$$

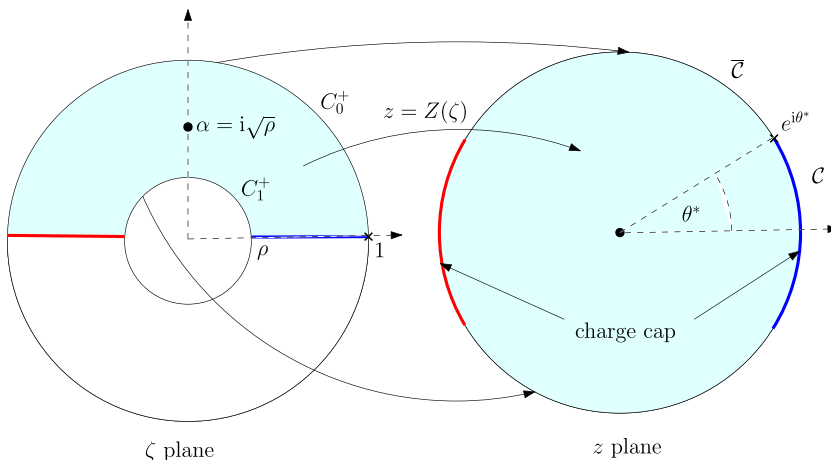


FIG. 2. Conformal mapping from the upper half of a parametric annulus $\rho < |\zeta| < 1$, $\text{Im}[\zeta] \geq 0$ to the physical droplet. The preimage of the center of the droplet is $\alpha = i\sqrt{\rho}$; $Z(\alpha) = 0$. The preimage of the point $e^{i\theta^*}$ at the upper edge of the right hand electric cap is 1: $Z(1) = e^{i\theta^*}$.

with $\sigma^\pm$ and $\epsilon^\pm$ referring to the conductivities and permittivities of the fluid outside (+) and inside (−) the droplet. In their Appendix C, Peng *et al.* [1] use a local analysis to argue that ensuring that the charge density does not blow up as the cap edges are approached implies continuity of both the charge density and $\partial\phi/\partial\theta$ at those cap edges from which the two conditions (5) follow by continuity with the imposed boundary conditions. The analytical solution to be found later corroborates this local analysis of Peng *et al.* [1].

Once the electric field problem inside the droplet has been found, certain reflectional symmetries imply that the field potential outside the droplet follows immediately by invoking these symmetries [1].

III. CONFORMAL MAPPING FROM AN ANNULUS

The key to deriving an analytical solution to this nonstandard mixed BVP for ϕ is use of conformal mapping [2,8]. For a real parameter $0 \leq \rho < 1$, consider the parametric annulus

$$\rho < |\zeta| < 1 \quad (8)$$

shown in Fig. 2. The geometrical parameter ρ will play a key role in the analysis: it naturally parametrizes the one-parameter class of solutions to the mixed BVP, as will be seen. The upper half of the annulus (8), shaded in Fig. 2, can be transplanted to the interior of the fluid droplet by a conformal mapping given by

$$z = Z(\zeta) = -iF(\zeta), \quad \text{where} \quad F(\zeta) = \frac{P(\zeta/\alpha, \rho)P(\zeta\alpha, \rho)}{P(\zeta/\bar{\alpha}, \rho)P(\zeta\bar{\alpha}, \rho)}, \quad \alpha = i\sqrt{\rho} \quad (9)$$

with

$$P(\zeta, \rho) \equiv (1 - \zeta)\hat{P}(\zeta, \rho), \quad \hat{P}(\zeta, \rho) \equiv \prod_{n=1}^{\infty} (1 - \rho^{2n}\zeta)(1 - \rho^{2n}/\zeta). \quad (10)$$

The infinite product defining $\hat{P}(\zeta, \rho)$ is convergent for all $\zeta \neq 0$ provided $0 \leq \rho < 1$. The function $P(\zeta, \rho)$ is closely related to the so-called prime function [2] of the annulus $\rho < |\zeta| < 1$. The infinite product definition (10) can be used to verify the two functional identities,

$$P(1/\zeta, \rho) = -\zeta^{-1}P(\zeta, \rho), \quad P(\rho^2\zeta, \rho) = -\zeta^{-1}P(\zeta, \rho), \quad (11)$$

and, once truncated at a sufficiently large number of terms for the required accuracy, the product can be used to evaluate $P(\zeta, \rho)$. Alternatively, use can be made of a rapidly convergent infinite sum representation of $P(\zeta, \rho)$ described in Chapter 5 of a monograph on the general theory of prime functions [2].

The correspondences under the mapping (9) are as follows; the two functional relations (11) can be used to verify the following statements. The two semicircles denoted by C_0^+ and C_1^+ , which are the halves of the circles $|\zeta| = 1$ and $|\zeta| = \rho$ lying in the upper half ζ plane, and transplanted to $\bar{C}$ and $\bar{D}$, with C_0^+ corresponding to $\bar{C}$. The intervals $[-1, -\rho]$ and $[\rho, 1]$ on the real ζ axis are transplanted to the charge caps on the left and right; the correspondences are color coded in red and blue in Fig. 2. It is easy to see that that mapping (9) has a single simple zero in the upper-half annulus at $\zeta = \alpha = i\sqrt{\rho}$,

$$Z(\alpha) = Z(i\sqrt{\rho}) = 0, \quad (12)$$

so α is the preimage of the center of the droplet. The points $\zeta = \pm 1, \pm \rho$ are the preimages of the four edges of the caps:

$$Z(1) = e^{i\theta^*}, \quad Z(\rho) = e^{-i\theta^*}, \quad Z(-1) = -e^{-i\theta^*}, \quad Z(-\rho) = -e^{i\theta^*}. \quad (13)$$

Since the mapping $Z(\zeta)$ depends only on the single parameter ρ , the first equation in (13) reveals immediately that, if ρ is known, then so is the cap angle θ^* . However, since θ^* is not known *a priori*, neither is ρ .

It is worth mentioning that the function $Z(\zeta)$ in (9) is an example of an unbounded circular slit mapping: see Sec. 5.7 of the monograph [2] where such mappings are discussed. These mappings have this name because, while they transplant the upper half of the annulus to the interior of the droplet, they also transplant the lower half of the same annulus to the unbounded region exterior to the droplet. The entire annulus is therefore transplanted by the mapping (9) to the entire complex plane exterior to two circular-arc slits, $\bar{C}$ and $\bar{D}$ in this case: the semicircle C_0^+ , and its counterpart in the lower-half plane, C_0^- say, have the same image $\bar{C}$ meaning that the circle $|\zeta| = 1$ is transplanted to the “circular slit” $\bar{C}$. Similarly, the circle $|\zeta| = \rho$ is transplanted by the same mapping to the circular slit $\bar{D}$.

IV. SOLUTION OF THE ELECTRIC CAP PROBLEM

Since the electric potential ϕ is harmonic inside the droplet, the complex variable $z = x + iy$ can be introduced and the problem rendered one of finding the function, or complex potential,

$$w(z) = \chi + i\phi \quad (14)$$

which is analytic inside the droplet, where χ is the harmonic conjugate of ϕ there. While analytic inside the droplet $w(z)$ is expected to have square-root singularities at each of the four cap edges. Such singularities are characteristic of transition points between boundary conditions of Dirichlet to Neumann type. Appendix C of Peng *et al.* [1] also discusses these boundary point singularities from the point of view of their matched asymptotic analysis.

With the conformal mapping at hand from Sec. III, the function

$$W(\zeta) \equiv w(Z(\zeta)) = \chi + i\phi \quad (15)$$

can be introduced. It is a composition of the complex potential $w(z)$ with the conformal mapping $Z(\zeta)$ from (9). The objective now is to determine $W(\zeta)$. Once done, and since $Z(\zeta)$ is known, a parametric representation of the complex potential will have been determined.

Since $w(z)$ is analytic inside the droplet except for expected square-root boundary singularities at the four cap edges, $\pm e^{\pm i\theta^*}$, and since $Z(\zeta)$ is analytic in the upper-half annulus, the composed function $W(\zeta)$ is expected to be analytic in the upper half annulus, including at the four preimages of the cap edges which, as seen from (13), are at $\zeta = \pm 1, \pm \rho$. To see why, and using primes for

derivatives with respect to ζ , i.e., $Z'(\zeta) = dZ/d\zeta$, note that near $\zeta = 1$, say, the conformal mapping has the local behavior

$$z = Z(\zeta) = Z(1) + Z''(1)(\zeta - 1)^2 + \mathcal{O}(\zeta - 1)^3 = e^{i\theta^*} + Z''(1)(\zeta - 1)^2 + \mathcal{O}(\zeta - 1)^3 \quad (16)$$

because, being a slit mapping, $Z'(\zeta)$ vanishes at all four cap edges $\zeta = \pm 1, \pm\rho$. This is because the derivative of a conformal mapping must vanish at a corner of angle 2π , that is, at the edge of a slit. This means that, near $\zeta = 1$, ζ has a local expansion in terms of z of the form

$$\zeta = 1 + a(z - e^{i\theta^*})^{1/2} + \dots, \quad a = 1/\sqrt{Z''(1)} \quad (17)$$

with analogous local behavior at the other three edge preimages $\zeta = -1, \pm\rho$. Therefore, writing the complex potential $W(\zeta)$ as a function of ζ naturally encodes the required square-root behavior of the complex potential at the cap edges.

In view of this implicit advantage in finding $W(\zeta)$, it is convenient to restate the boundary conditions in terms of it. From the Dirichlet condition (2),

$$\text{Im}[W(\zeta)] = \begin{cases} -\phi_*, & \text{for } \zeta \in (\rho, 1), \\ +\phi_*, & \text{for } \zeta \in (-1, -\rho), \end{cases} \quad (18)$$

where $\text{Re}[\cdot]$ and $\text{Im}[\cdot]$ denote, respectively, real and imaginary parts of the quantity contained in square brackets. Letting s denote (nondimensional) arclength around the droplet boundary in the clockwise direction, the Cauchy-Riemann equations imply

$$\frac{\partial \chi}{\partial s} = \frac{\partial \phi}{\partial r}, \quad (19)$$

where the choice $s = -\theta$ can be made. Since

$$\frac{\partial \phi}{\partial r} = \frac{\partial \chi}{\partial s} = -\frac{\partial \chi}{\partial \theta} = -2 \cos \theta, \quad (20)$$

the Neumann conditions (4) on C_0^+ , C_1^+ become, on integration with respect to θ on each boundary portion,

$$\chi = \text{Re}[W(\zeta)] = \begin{cases} 2 \sin \theta, & \text{on } C_0^+, \\ 2 \sin \theta + Q, & \text{on } C_1^+, \end{cases} \quad (21)$$

where one of the integration constants, but not both, has been set to zero without loss of generality. The parameter Q is a constant of integration.

It follows from some theory of the prime function given in detail in the Appendix that the solution for $W(\zeta)$ is

$$W(\zeta) = 2C(K(i\zeta\sqrt{\rho}, \rho) + K(-i\zeta\sqrt{\rho}, \rho) - 1 + D) + \frac{2C + Q}{\log \rho} \left(\log \zeta - \frac{i\pi}{2} \right), \quad (22)$$

where C and D are real constants given, purely in terms of ρ , by

$$C = -\frac{P(-1, \rho)P(+\rho, \rho)}{\hat{P}(+1, \rho)P(-\rho, \rho)}, \quad D = K(-1, \rho) - K(\rho, \rho), \quad (23)$$

and where $K(\zeta, \rho)$ is related to a logarithmic derivative of $P(\zeta, \rho)$:

$$K(\zeta, \rho) \equiv \frac{\zeta}{P(\zeta, \rho)} \frac{\partial P(\zeta, \rho)}{\partial \zeta}. \quad (24)$$

This function has a convenient infinite sum representation readily derived from (10):

$$K(\zeta, \rho) = -\frac{\zeta}{1 - \zeta} + \sum_{n \geq 1} \left(-\frac{\rho^{2n}\zeta}{1 - \rho^{2n}\zeta} + \frac{\rho^{2n}/\zeta}{1 - \rho^{2n}/\zeta} \right). \quad (25)$$

Note that the imaginary part of $W(i)$ vanishes as required to ensure $\phi = 0$ at $\zeta = e^{i\pi/2}$ which corresponds to $\theta = \pi/2$. It also follows from the analysis in the Appendix that the cap voltage ϕ^* is related to C (and hence ρ) and Q by

$$\phi^* = \frac{\pi(C + Q/2)}{\log \rho}. \quad (26)$$

It turns out that Q is the total charge on cap C shown in blue in Fig. 2. To see this, the total charge on C is the following integral of the charge density,

$$\int_{-\theta^*}^{\theta^*} q_0 d\theta = \int_{-\theta^*}^{\theta^*} \left(\frac{\partial \phi}{\partial r} + 2 \cos \theta \right) d\theta = \int_{-\theta^*}^{\theta^*} \left(-\frac{\partial \chi}{\partial \theta} + 2 \cos \theta \right) d\theta, \quad (27)$$

where the Cauchy-Riemann equations (19) have been used. This total charge is therefore

$$(-\chi + 2 \sin \theta)|_{\theta^*} - (-\chi + 2 \sin \theta)|_{-\theta^*}. \quad (28)$$

But (21) says that

$$-\chi + 2 \sin \theta = \begin{cases} 0, & \text{on } C_0^+, \\ -Q, & \text{on } C_1^+, \end{cases} \quad (29)$$

hence, assuming continuity of the charge density at the cap edges, the total charge on cap C is Q . It follows, therefore, from (6) which expresses half the total charge on the cap in terms of ϖ , that

$$\frac{Q}{2} = \frac{2\varpi}{1 + \varpi}. \quad (30)$$

This relation determines Q in terms of ϖ .

The value of ρ is still unknown and, hence, so too is the value of C , which depends only on ρ . The cap voltage ϕ^* is also still unknown but, in view of (26) and (30), it will be determined as soon as C , or equivalently, ρ , is found. The value of ρ must follow from the only remaining outstanding conditions stated in (5), the first of which is

$$\lim_{\theta \rightarrow \theta^*} q_0 = 0. \quad (31)$$

The surface charge density q_0 is given by

$$q_0 = \frac{\partial \phi}{\partial r} + 2 \cos \theta = \frac{\partial \chi}{\partial s} + 2 \cos \theta = \operatorname{Re} \left[-iz \frac{dw}{dz} + 2z \right] = \operatorname{Re} \left[Z(\zeta) \left(-i \frac{\zeta W'(\zeta)}{\zeta Z'(\zeta)} + 2 \right) \right], \quad (32)$$

where the chain rule has been used several times. It is instructive to examine how to calculate this quantity. The Appendix shows that an alternative representation of the mapping (9) is

$$z = Z(\zeta) = -iF(\zeta) = -iC(K(\zeta/\bar{\alpha}, \rho) - K(\zeta\bar{\alpha}, \rho) - D), \quad \alpha = i\sqrt{\rho}, \quad (33)$$

so that

$$\zeta Z'(\zeta) = -iC(L(\zeta/\bar{\alpha}, \rho) - L(\zeta\bar{\alpha}, \rho)) = -iC(L(i\zeta/\sqrt{\rho}, \rho) - L(-i\zeta\sqrt{\rho}, \rho)), \quad (34)$$

where

$$L \equiv \zeta \frac{\partial K}{\partial \zeta}, \quad L(\zeta, \rho) = -\frac{\zeta}{(1 - \zeta)^2} - \sum_{n \geq 1} \left(\frac{\rho^{2n} \zeta}{(1 - \rho^{2n} \zeta)^2} + \frac{\rho^{2n} / \zeta}{(1 - \rho^{2n} / \zeta)^2} \right), \quad (35)$$

and where this infinite sum representation follows from (25). Using (22), the derivative $\zeta W'(\zeta)$ can also be written in terms of $L(\zeta, \rho)$ as

$$\zeta W'(\zeta) = 2C(L(i\zeta\sqrt{\rho}, \rho) + L(-i\zeta\sqrt{\rho}, \rho)) + \frac{2C + Q}{\log \rho}. \quad (36)$$

Since $Z'(1) = 0$, as used already in (16), a necessary condition for the limit (31) to exist is $W'(1) = 0$ or, from (36),

$$\frac{2C + Q}{\log \rho} + 2C(L(i\sqrt{\rho}, \rho) + L(-i\sqrt{\rho}, \rho)) = 0. \quad (37)$$

This ensures that the charge density does not blow up as this cap edge is approached; by the symmetries of the system, if condition (37) holds then the same will be true at the three other cap edges. This is the final closure condition since it expresses C , which is a known function of ρ , in terms of Q and ρ . As argued by Peng *et al.* [1], this now guarantees that both the charge density and $\partial\phi/\partial\theta$ are continuous at the cap edges. The fact that the limit (31), which exists if (37) is imposed, is, in fact, zero can be seen by using (32) in the form

$$q_0 = \operatorname{Re} \left[\frac{Z(\zeta)}{\zeta Z'(\zeta)} (-i\zeta W'(\zeta) + 2\zeta Z'(\zeta)) \right], \quad (38)$$

and using the fact that (34) and (36) imply

$$\begin{aligned} -i\zeta W'(\zeta) + 2\zeta Z'(\zeta) &= -i \left(\frac{2C + Q}{\log \rho} + 2C(L(i\zeta/\sqrt{\rho}, \rho) + L(i\zeta\sqrt{\rho}, \rho)) \right) \\ &\rightarrow -i \left(\frac{2C + Q}{\log \rho} + 2C(L(i/\sqrt{\rho}, \rho) + L(i\sqrt{\rho}, \rho)) \right) \end{aligned} \quad (39)$$

as $\zeta \rightarrow 1$. On taking a derivative of the first functional identity in (A3) it can be shown that $L(1/\zeta, \rho) = L(\zeta, \rho)$ implying that the limit (39) is

$$-i \left(\frac{2C + Q}{\log \rho} + 2C(L(-i\sqrt{\rho}, \rho) + L(i\sqrt{\rho}, \rho)) \right) \quad (40)$$

which vanishes if (37) is enforced. Thus, using L'Hopital's rule in (38), $q_0 \rightarrow 0$ near this cap edge confirming, as claimed by Peng *et al.* [1], that (31) holds provided (37) holds. As another check, it was verified by evaluation of the analytical solution that the second condition in (5) also holds.

The closure condition (37) combined with (30), which expresses Q in terms of ϖ , and (23), which expresses C in terms of ρ , mean that

$$\frac{2\varpi}{1 + \varpi} = \frac{P(-1, \rho)P(+\rho, \rho)}{\hat{P}(+1, \rho)P(-\rho, \rho)} (1 + (L(i\sqrt{\rho}, \rho) + L(-i\sqrt{\rho}, \rho)) \log \rho). \quad (41)$$

The parameter ϖ is therefore given in term of ρ by

$$\varpi = \left(\frac{\hat{P}(+1, \rho)P(-\rho, \rho)}{P(-1, \rho)P(+\rho, \rho)} \frac{2}{(1 + (L(i\sqrt{\rho}, \rho) + L(-i\sqrt{\rho}, \rho)) \log \rho)} - 1 \right)^{-1}. \quad (42)$$

Figure 3 shows a graph of ϖ as a function of ρ calculated using (42): it is found to have a monotonic decreasing dependence on ρ , meaning that ρ can serve as a proxy for the ϖ parameter. With ρ taken as the essential parameter characterizing the solution class, all physical quantities of interest are given explicitly by the preceding analysis. The potential field $\phi = \phi(x, y)$ is given parametrically in terms of ζ by the formulas

$$\phi = \operatorname{Im}[W(\zeta)], \quad z = x + iy = Z(\zeta), \quad (43)$$

where $W(\zeta)$ is given explicitly in (22) and $Z(\zeta)$ in (9), the cap voltage ϕ^* is given by (26), or equivalently,

$$\phi^* = \pi \frac{P(-1, \rho)P(+\rho, \rho)}{\hat{P}(+1, \rho)P(-\rho, \rho)} (L(i\sqrt{\rho}, \rho) + L(-i\sqrt{\rho}, \rho)), \quad (44)$$

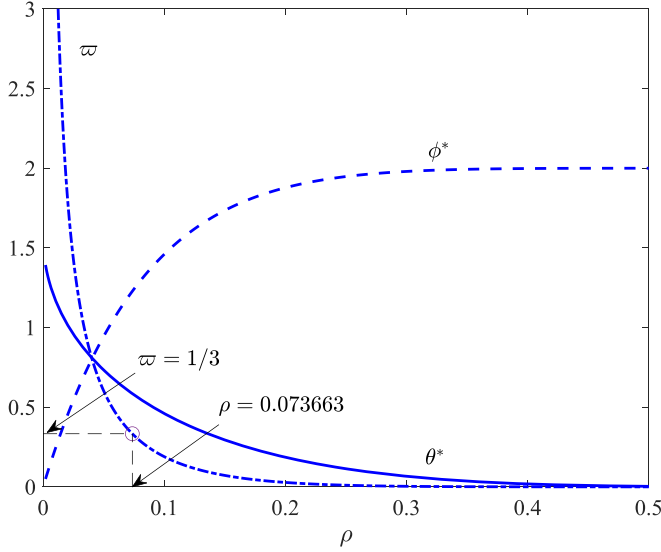


FIG. 3. Graph of ϖ , cap voltage ϕ^* , and cap angle θ^* as functions of ρ as given by the explicit formulas (42), (44), and (45). The point $\rho = 0.073663$ corresponding to the choice $\varpi = 1/3$ used in Fig. 13 of Peng *et al.* [1], and in Fig. 5 here, is indicated. The monotonic relationship between ϖ and ρ means that either one can be used to uniquely parametrize the class of solutions.

and the cap angle θ^* follows from

$$\theta^* = -i \log Z(1) = -i \log \left(-i \frac{P(-i/\sqrt{\rho}, \rho) P(+i\sqrt{\rho}, \rho)}{P(+i/\sqrt{\rho}, \rho) P(-i\sqrt{\rho}, \rho)} \right). \quad (45)$$

The functions $P(\zeta, \rho)$, $K(\zeta, \rho)$, and $L(\zeta, \rho)$ are readily evaluated using (10), (25), and (35), respectively.

If ϖ is known, on the other hand, the only numerical step is use of a one-dimensional Newton iteration, say, to invert (42) for the relevant value of ρ , which is a simple matter, then all other quantities of physical interest will be given explicitly by the formulas set out above.

Figure 3 also shows graphs of the cap voltage ϕ^* and cap angle θ^* as functions of ρ , with the value $\rho = 0.073663$ (correct to six decimal places), corresponding to the choice $\varpi = 1/3$ used in Fig. 13 of Peng *et al.* [1], clearly marked. Indeed, with all quantities known as explicit functions of the parameter ρ , it is a simple matter to reproduce Fig. 13(b) of Peng *et al.* [1] as done in Fig. 4 which shows the graphs of cap voltage ϕ^* and cap angle θ^* plotted now as functions of ϖ .

As a further check on the analysis, Fig. 5 reproduces Figs. 13(c) and 13(d) of Peng *et al.* [1] showing, for the fixed value $\varpi = 1/3$, the surface charge density q_0 as a function of angle θ for $\theta \in (0, \pi/2)$ plotted now on the basis of formula (32), and the surface value of the potential ϕ , also as a function of θ , calculated using (43). Indeed, the authors of the original work [1] have shared data from their numerical calculations with the present author and these have been cross checked against the values from the analytical solutions found here.

V. DISCUSSION

Since the aim of this paper is to show how certain mathematical tools, based on use of the so-called prime function [2], can be used to find an analytical solution to a nonstandard mixed boundary value problem arising as the leading order asymptotic problem in the study of charge caps on a leaky dielectric droplet, it is worth commenting on the generality and flexibility of the methods used. Other types of forcing of the droplet—perhaps by a distribution of point charges rather than an imposed uniform field—might well give rise, at high electric Reynolds numbers, to situations

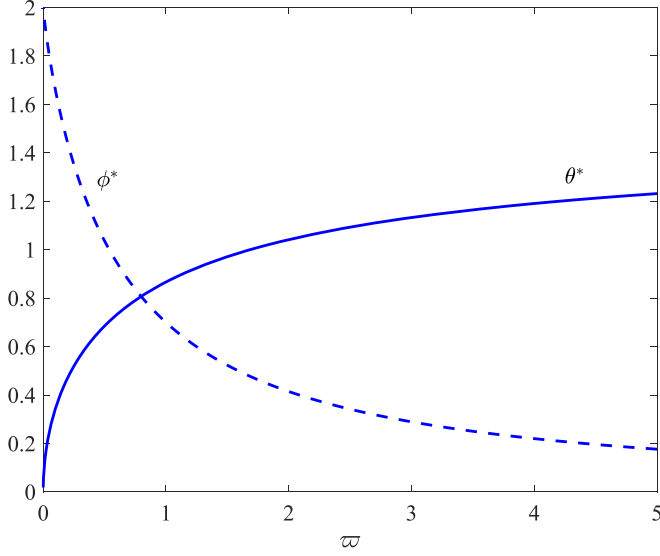


FIG. 4. Reproduction, based on an analytical solution, of Fig. 13(b), of Peng *et al.* [1].

involving two charge caps that are no longer of the same circumferential length, or even different numbers of charge caps of differing lengths.

Supposing still that there are only two caps, the particular choice $\alpha = i\sqrt{\rho}$ for the preimage of the center of the droplet in Fig. 2 corresponds to a symmetric case when the charge caps are of equal circumferential length. If α is kept on the imaginary axis but shifted toward C_0^+ or C_1^+ , the formula for $Z(\zeta)$ given in (9) is also the relevant mapping, but now for two charge caps with different circumferential lengths: in other words, $|\alpha| = \sqrt{\rho}$ is the symmetric case of equal-length caps. The same mapping (9) is relevant in more general circumstances when the charge caps are not necessarily the same size, as might occur for other background fields. As mentioned earlier, the theoretical importance of the particular prime function of the annulus used here is exemplified by its use in quite disparate physical problems ranging from studies of the frictional properties of superhydrophobic surfaces [3] to modeling fluid pump designs [4,6].

Finally, in the event that a different number of charge caps are expected, it is worth remarking that the prime function of the annulus that has been used in this study is just one of a large class

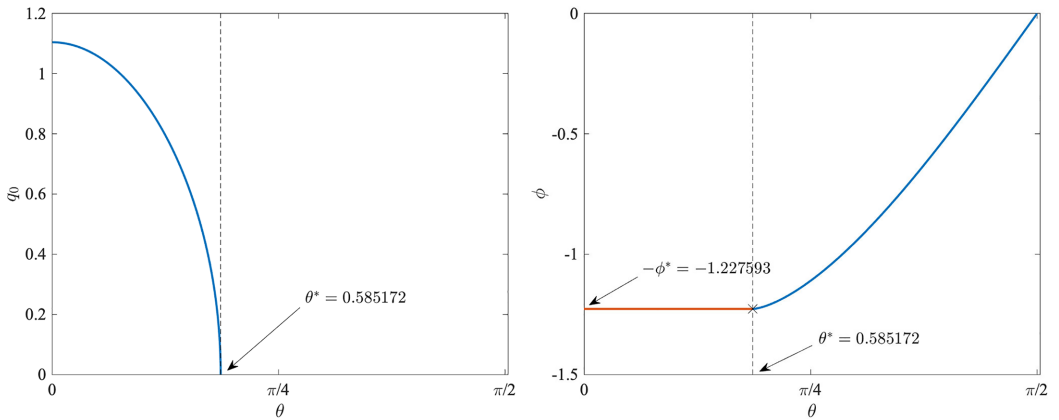


FIG. 5. Reproductions of Figs. 13(c) and 13(d) of Peng *et al.* [1] for $\varpi = 1/3$. The values of q_0 are calculated using formula (32), the values of ϕ using (43).

of such prime functions [2], each relevant to different geometries, or topologies, of the domain of interest. This means that the analysis carried out here is potentially extendible, likely with only quite minor changes in detail, to a circular droplet with just one charge cap, or even three, or more, caps. A recent study by the author [5] that involved finding the slow viscous flow around torque-driven, superhydrophobic cylinders provides a showcase, albeit in a different physical setting, of the utility of this prime function machinery. In that problem, the formulas for the fields of interest, written in terms of the relevant prime functions, were of exactly the same form no matter how many “no-shear slots” characterized the superhydrophobic cylinder.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: SOME RELEVANT FUNCTION THEORY

The reader is referred to Chapter 5 of a monograph by the author [2] for the relevant background on the prime function, but the key aspects needed for the present problem will now be summarized. The notation C_0 is used for the unit circle $|\zeta| = 1$, and C_1 for the circle $|\zeta| = \rho$. A key functional identity is that

$$F(\zeta) = \frac{P(\zeta/\alpha, \rho)P(\zeta\alpha, \rho)}{P(\zeta/\bar{\alpha}, \rho)P(\zeta\bar{\alpha}, \rho)} = C(K(\zeta/\bar{\alpha}, \rho) - K(\zeta\bar{\alpha}, \rho) - D), \quad (\text{A1})$$

where $\alpha = i\sqrt{\rho}$ for two symmetric charge caps and

$$C = -\frac{P(-1, \rho)P(+\rho, \rho)}{\bar{P}(+1, \rho)P(-\rho, \rho)}, \quad D = K(-1, \rho) - K(\rho, \rho) \quad (\text{A2})$$

are real constants depending only on ρ . This identity can be established using the theory of so-called loxodromic functions described in Chapter 11 of [2]. The details of the proof will not be given here, however the validity of the identity (A1) can readily be checked numerically using (10) and (25) to evaluate $P(\zeta, \rho)$ and $K(\zeta, \rho)$.

A simple exercise using (11) reveals that two corresponding functional identities satisfied by $K(\zeta, \rho)$ are

$$K(1/\zeta, \rho) = 1 - K(\zeta, \rho), \quad K(\rho^2\zeta, \rho) = K(\zeta, \rho) - 1. \quad (\text{A3})$$

These will be used extensively in what follows.

The relevant conformal mapping (9) is

$$z = Z(\zeta) = -iF(\zeta). \quad (\text{A4})$$

Let $z = e^{i\theta}$ on either $\bar{C}$ or $\bar{D}$, then

$$2 \sin \theta = \frac{1}{i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{i}(z - \bar{z}) = \frac{1}{i}(Z(\zeta) - \overline{Z(\zeta)}). \quad (\text{A5})$$

On C_0^+ , the preimage of $\bar{C}$, where $\bar{\zeta} = 1/\zeta$, this means that

$$\begin{aligned} 2 \sin \theta &= \frac{1}{i}(Z(\zeta) - \overline{Z(1/\zeta)}) = -F(\zeta) - \overline{F(1/\zeta)} \\ &= -C(K(\zeta/\bar{\alpha}, \rho) - K(\zeta\bar{\alpha}, \rho) + K(1/(\zeta\alpha), \rho) - K(\alpha/\zeta, \rho) - 2D) \\ &= -C(K(\zeta/\bar{\alpha}, \rho) - K(\zeta\bar{\alpha}, \rho) - K(\zeta\alpha, \rho) + K(\zeta/\alpha, \rho) - 2D) \\ &= -(R(\zeta) - S(\zeta)), \end{aligned} \quad (\text{A6})$$

where

$$R(\zeta) = C(K(\zeta/\alpha, \rho) + K(\zeta/\bar{\alpha}, \rho) - 1 - D), \quad S(\zeta) = C(K(\zeta\alpha, \rho) + K(\zeta\bar{\alpha}, \rho) - 1 + D). \quad (\text{A7})$$

Note that, on C_0 ,

$$\begin{aligned} \overline{S(\zeta)} &= C(K(\bar{\alpha}/\zeta, \rho) + K(\alpha/\zeta, \rho) - 1 + D) \\ &= C((1 - K(\zeta/\bar{\alpha}, \rho)) + (1 - K(\zeta/\alpha, \rho)) - 1 + D) \\ &= -C(K(\zeta/\bar{\alpha}, \rho) + K(\zeta/\alpha, \rho) - 1 - D) = -R(\zeta). \end{aligned} \quad (\text{A8})$$

Hence, on C_0 ,

$$2 \sin \theta = -(R(\zeta) - S(\zeta)) = 2\text{Re}[S(\zeta)]. \quad (\text{A9})$$

Similarly, on C_1^+ , the preimage of $\bar{D}$, where $\bar{\zeta} = \rho^2/\zeta$,

$$\begin{aligned} 2 \sin \theta &= \frac{1}{i}(Z(\zeta) - \bar{Z}(\rho^2/\zeta)) = -F(\zeta) - \bar{F}(\rho^2/\zeta) \\ &= -C(K(\zeta/\bar{\alpha}, \rho) - K(\zeta\bar{\alpha}, \rho) + K(\rho^2/(\zeta\alpha), \rho) - K(\rho^2\alpha/\zeta, \rho) - 2D) \\ &= -C(K(\zeta/\bar{\alpha}, \rho) - K(\zeta\bar{\alpha}, \rho) - K(\zeta\alpha, \rho) + K(\zeta/\alpha, \rho) - 2D) \\ &= -(R(\zeta) - S(\zeta)). \end{aligned} \quad (\text{A10})$$

Note that, on C_1 ,

$$\begin{aligned} \overline{S(\zeta)} &= C(K(\rho^2\bar{\alpha}/\zeta, \rho) + K(\rho^2\alpha/\zeta, \rho) - 1 + D) \\ &= C(-K(\zeta/\bar{\alpha}, \rho) - K(\zeta/\alpha, \rho) - 1 + D) \\ &= C(-(K(\zeta/\alpha, \rho) + K(\zeta/\bar{\alpha}, \rho) - 1 - D) - 2) = -R(\zeta) - 2C. \end{aligned} \quad (\text{A11})$$

Hence, on C_1 ,

$$2 \sin \theta = -(R(\zeta) - S(\zeta)) = 2\text{Re}[S(\zeta)] + 2C. \quad (\text{A12})$$

Putting all the pieces together, it has been shown that

$$\text{Re}[2S(\zeta)] = \begin{cases} 2 \sin \theta, & \text{on } C_0, \\ 2 \sin \theta - 2C, & \text{on } C_1, \end{cases} \quad (\text{A13})$$

where $2S(\zeta)$ is analytic and single valued in the annulus $\rho < |\zeta| < 1$. Also,

$$\text{Im}[2S(\zeta)] = \begin{cases} 0, & \text{for } \zeta \in (\rho, 1), \\ 0, & \text{for } \zeta \in (-1, -\rho). \end{cases} \quad (\text{A14})$$

Note also that, using well-known properties of the complex logarithm,

$$\text{Re}\left[2C \frac{\log \zeta}{\log \rho} - \frac{i\pi C}{\log \rho}\right] = \begin{cases} 0, & \text{on } C_0, \\ 2C, & \text{on } C_1, \end{cases} \quad (\text{A15})$$

and that

$$\text{Im}\left[2C \frac{\log \zeta}{\log \rho} - \frac{i\pi C}{\log \rho}\right] = \begin{cases} -\pi C / \log \rho, & \text{for } \zeta \in (\rho, 1), \\ +\pi C / \log \rho, & \text{for } \zeta \in (-1, -\rho), \end{cases} \quad (\text{A16})$$

It follows from (A13) and (A15) that

$$\text{Re}\left[2S(\zeta) + 2C \frac{\log \zeta}{\log \rho} - \frac{i\pi C}{\log \rho}\right] = \begin{cases} 2 \sin \theta, & \text{on } C_0, \\ 2 \sin \theta, & \text{on } C_1, \end{cases} \quad (\text{A17})$$

and that

$$\operatorname{Im}\left[2S(\zeta) + 2C \frac{\log \zeta}{\log \rho} - \frac{i\pi C}{\log \rho}\right] = \begin{cases} -\pi C / \log \rho, & \text{for } \zeta \in (\rho, 1), \\ +\pi C / \log \rho, & \text{for } \zeta \in (-1, -\rho). \end{cases} \quad (\text{A18})$$

Finally, it should be noted that, for any real constant Q ,

$$\operatorname{Re}\left[Q \frac{\log \zeta}{\log \rho} - \frac{i\pi Q}{2 \log \rho}\right] = \begin{cases} 0, & \text{on } C_0, \\ Q, & \text{on } C_1 \end{cases} \quad (\text{A19})$$

and

$$\operatorname{Im}\left[Q \frac{\log \zeta}{\log \rho} - \frac{i\pi Q}{2 \log \rho}\right] = \begin{cases} -\pi Q / (2 \log \rho), & \text{for } \zeta \in (\rho, 1) \\ \pi Q / (2 \log \rho), & \text{for } \zeta \in (-1, -\rho). \end{cases} \quad (\text{A20})$$

It follows that the solution for $W(\zeta)$ satisfying (21), or

$$\operatorname{Re}[W(\zeta)] = \begin{cases} 2 \sin \theta, & \text{on } C_0^+, \\ 2 \sin \theta + Q, & \text{on } C_1^+, \end{cases} \quad (\text{A21})$$

is

$$\begin{aligned} W(\zeta) &= \underbrace{2S(\zeta, \rho) + 2C \frac{\log \zeta}{\log \rho} - \frac{i\pi C}{\log \rho}}_{\text{from (A17)}} + \underbrace{Q \frac{\log \zeta}{\log \rho} - \frac{i\pi Q}{2 \log \rho}}_{\text{from (A19)}} \\ &= 2C(K(i\zeta\sqrt{\rho}, \rho) + K(-i\zeta\sqrt{\rho}, \rho) - 1 + D) + \frac{2C + Q}{\log \rho} \left(\log \zeta - \frac{i\pi}{2}\right) \end{aligned} \quad (\text{A22})$$

as stated in (22). However, since $W(\zeta)$ must also satisfy (18), it follows from (A18) and (A20) that the cap voltage is

$$\phi^* = \frac{\pi(C + Q/2)}{\log \rho}. \quad (\text{A23})$$

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