

Effect of split endcaps on the flow dynamics in a tall Taylor-Couette setup

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The effects of axial boundaries or endcaps are of fundamental interest in many Taylor-Couette (TC) flow experiments. A main challenge in those experiments has been to minimize these effects, which can substantially alter the flow structure compared to the axially unbounded idealized case. Therefore, understanding and disentangling the influence of endcaps on the TC flow dynamics is essential for the unambiguous interpretation of experimental results, particularly when other dynamical processes (instabilities) in TC flows are involved. In this paper, we study the hydrodynamic evolution of a quasi-Keplerian TC flow in the presence of split endcaps for experimentally relevant high Reynolds numbers, Re , up to 2×10^5 , which are larger than those considered in related previous studies. At these Re , the flow deviates from the ideal TC flow profile without endcaps, resulting in about 15% deviation in angular velocity at the midheight of the cylinder. Aside from turbulent fluctuations caused by shearing instability near the endcaps, the bulk flow remains nearly axially independent and exhibits overall Rayleigh stability. We characterize the scalings of the Ekman and Stewartson layer sizes with Re as well as examine the effect of the ratio of the outer to inner cylinders' angular velocities on the flow. The implications of these findings for ongoing magnetorotational instability (MRI) experiments based on the similar axially bounded TC setup are also discussed. Specifically, it is shown that when imposing a constant axial magnetic field in all the considered configurations, the flow profile modified by the endcaps lowers the critical threshold for the onset of MRI that in turn can facilitate its emergence and detection in those experiments.

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I. INTRODUCTION

Taylor-Couette (TC) flow between two differentially rotating coaxial cylinders is widely used as a basic model to study a variety of fluid dynamical problems, including instabilities, turbulence, and mixing processes. In a TC setup, the flow profile can be configured by adjusting the rotation rates of the cylinders such that the angular velocity Ω of the fluid between the cylinders approximately matches the Keplerian rotation profile [1–3], $\Omega \propto r^{-3/2}$, where r is the cylindrical radius, which is central in astrophysics as it corresponds to the differential rotation profile of accretion disks. This profile characterized by radially decreasing angular velocity, $\partial\Omega/\partial r < 0$, and increasing specific angular momentum, $\partial(r^2\Omega)/\partial r > 0$, is broadly referred to as a quasi-Keplerian regime. It is hydrodynamically stable according to Rayleigh’s centrifugal criterion [4,5]. Furthermore, using conducting liquid metals as a working fluid, this quasi-Keplerian TC flow enables us, because of its hydrodynamical stability, to study the interplay between magnetic fields and flow, and the resulting various magnetohydrodynamic (MHD) instabilities. In this regard, one of the notable applications of the TC setup is in experimental studies of astrophysically significant instabilities, such as the magnetorotational instability (MRI) [6,7] and the current-driven Tayler instability [8] (see the recent reviews [4,9]). Thus, TC setup is a physically convenient system of choice to investigate numerically and experimentally instabilities of astrophysical interest.

A main challenge in TC experiments has been to minimize the effects of endcaps covering the top and bottom ends of a finite-height TC device. These endcaps can have a significant impact on the flow structure and its overall dynamics [4], in the worst case spoiling the desired hydrodynamic stability. Due to the boundary conditions near the endcaps, the imbalance among the pressure, Coriolis, and viscous forces leads to a poloidal motion of fluid, called secondary Ekman circulation (pumping), which is radially outward or inward, when the bulk flow rotates, respectively, slower or faster than the endcaps, and the formation of associated boundary layers, called Ekman layers. Furthermore, the presence of one or more angular velocity jumps at the endcaps is virtually unavoidable as they rotate with different angular velocities, leading to localized regions of strong shear. The combination of Ekman circulation, redistributing angular momentum, and local shear can modify the primary angular velocity profile, thereby leading to the development of secondary hydrodynamic instabilities, interfering with other possible MHD instabilities in the flow, including MRI. There have been numerous attempts to mitigate these effects in TC experiments in order to ensure the stability of flow and maintain a consistent flow profile as close as possible to the classical, or ideal (i.e., with infinitely long cylinders) TC flow profile [10,11]. Specifically, in the experiments, Ekman circulation is generally reduced by splitting the endcaps into two or more segmented sections. These sections are known as rims, if they are in solid rotation with the lateral cylinder walls, and rings when they rotate with their own angular velocities, which is usually between the rotation velocities of the inner and outer cylinders [4]. Such a differentially rotating segmented endcap design allows for the offset of differences between the fluid velocities in the vicinity of the endcaps, which rotate with the latter (due to the no-slip boundary condition), and the bulk flow, thereby reducing the penetration of Ekman circulation deeper into the flow. In principle, dividing endcaps into as many independent ring segments as possible would be desirable in order to better approach the ideal TC flow profile and minimize Ekman circulation [12], which is, however, quite hard to realize in practice.

One of the main interests in these experiments with finite-height TC flow is due to MRI that requires sufficiently high magnetic Reynolds numbers $Rm = \Omega_{in}d^2/\eta \gtrsim 10$ for excitation [13–16], where Ω_{in} is the inner cylinder’s angular velocity, d is the gap width between the cylinders, and η is the magnetic diffusivity. The liquid metals used in those experiments are characterized by very small magnetic Prandtl numbers $Pm = \nu/\eta \sim 10^{-6}$ – 10^{-5} , where ν is the kinematic viscosity, and therefore the resulting Reynolds numbers $Re = \Omega_{in}d^2/\nu$ should reach values as high as $Re \gtrsim 10^6$. Motivated by the initial proposals [13,17] to study MRI in TC experiments with gallium, several theoretical and numerical works addressed the flow dynamics in the finite-height TC setups, analyzing the effects of endcaps with different configurations. Kageyama *et al.* [10] studied both

numerically and experimentally the hydrodynamic TC flow in a wide gap between the cylinders with small aspect ratio where the endcaps corotate with the outer cylinder and took up to $Re \sim 10^3$ in their simulations. They showed that in this configuration, the azimuthal flow profile was significantly modified from that of the ideal TC profile and strong Ekman circulations were observed. Hollerbach and Fournier [12] studied the effect of both rigid endcaps attached to one of the cylinders and split endcaps with independently rotating rings on the flow structure and dynamics in a rapidly rotating and tall TC setup (height 5 times larger than the gap size) at $10^4 \leq Re \leq 10^6$. To reach such high Re , they assumed infinitesimal differential rotation with a vanishing Rossby number $Ro = (\Omega_{in} - \Omega_{out})/\Omega_{out} = 0$, where Ω_{out} is the angular velocity of the outer cylinder. They showed that the endcaps play a crucial role in the structure of the established flow and rigid endcaps would require very large aspect ratios (more than 100) to substantially mitigate the effects of Ekman circulations. Alternatively, the split-ring design of the endcaps can considerably reduce the Ekman circulation, and a TC setup would require up to ten split rings for maintaining flow stability at experimentally high $Re \sim 10^6$. Their study highlighted the significance of Stewartson layers originating from the discontinuity of the angular velocity at the split rings, which at high enough Ro can be prone to Kelvin-Helmholtz instability and, in particular, complicate the dynamics of MRI when it is present in the flow. Thus, it is important to first understand the TC flow dynamics in the hydrodynamic quasi-Keplerian regime of astrophysical relevance with finite differential rotation ($Ro \sim 1$), as required for MRI, before moving to the MHD case, and this is the main motivation of this work.

Burin *et al.* [18] demonstrated in the Princeton Hydrodynamic Turbulence Experiment (HTX) that splitting the top and bottom endcaps into three rings, where the inner rings rotate with the inner cylinder, the outer rings rotate with the outer cylinder, while the middle rings rotate independently, considerably reduces the Ekman circulation in the bulk flow. In the following experiments [19–22], the optimization of the angular velocity of the independent middle ring was performed, achieving a flow profile that matches the ideal TC flow. Numerical simulations, although at Re much lower than experimental values, confirmed the reduction of Ekman circulations in the three-ring setup [2,3,23]. These simulations also showed that the considered quasi-Keplerian TC flow is overall hydrodynamically stable (except for some turbulence in the thin boundary layers) and hence does not lead to an efficient transport of angular momentum consistent with the earlier experiments [1].

Other TC setups with large aspect ratios and endcaps corotating with the outer cylinder were studied in experiments in the quasi-Keplerian rotation regime [24–26]. However, contrary to the Princeton HTX experiment in both the Maryland [24,25] and Twente [26] experiments, the measurements in the middle part of the flow indicated transport of angular momentum for the quasi-Keplerian flows. Simulations then showed that in these experiments large deviations in the flow profile from the ideal TC one occurred at large Re due to strong Ekman circulations, which apparently gave rise to the instability in the bulk flow and resulting angular momentum transport [3].

In this paper, we focus on the hydrodynamic evolution of the TC flow in preparation for the upcoming DRESHDYN-MRI experiment with liquid sodium [27], which is currently under construction at the Helmholtz-Zentrum Dresden-Rossendorf (HZDR) aiming to detect and study various types of MRI in the laboratory. To reduce the Ekman circulations in this experiment, the endcaps are split into inner and outer rims, each firmly attached to the respective cylinder. Such a configuration of the endcaps was previously analyzed both theoretically and experimentally in the quasi-Keplerian regime. In the early experiments by Wendt [28] and Coles [29], the endcap was split at a midpoint for which no transition to turbulence was reported for the values of Re ranging from 50 to 10^5 . By contrast, Richard and Zahn [30] reanalyzed the data of Wendt [28] and reported turbulence, which was attributed to the finite amplitude instability at large $Re \sim 10^5$. Later this split-ring endcap configuration was studied numerically for the PROMISE experiment by Szklarski [31], who demonstrated an efficient reduction in Ekman circulation if the endcaps are split at a distance $0.4r_{in}$ from the inner cylinder (with radius r_{in}) instead of the midpoint. Another focus of that paper was on the specific role of an axial magnetic field and the emerging Ekman-Hartmann layers. Since the results of this optimization did not depend much on the Hartmann number, this

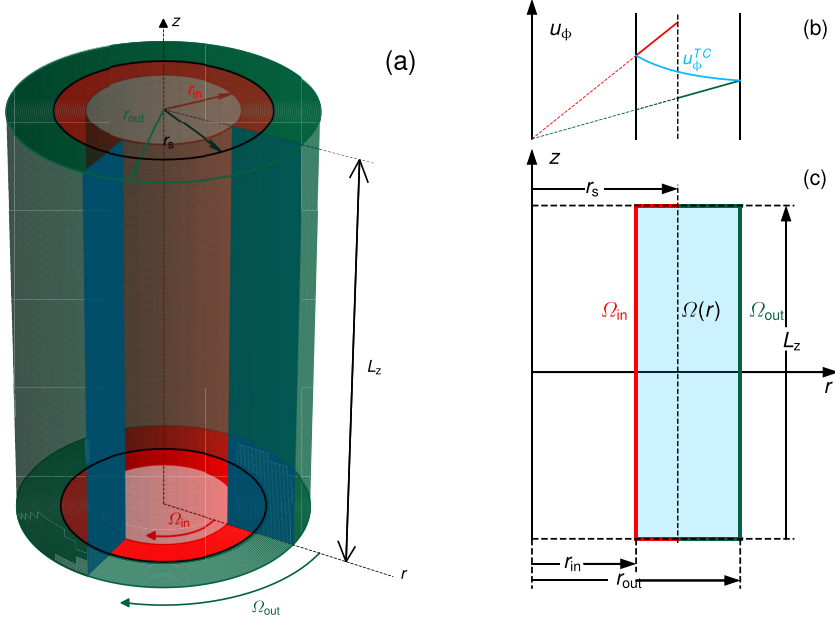


FIG. 1. (a) TC setup with the split endcaps and (c) its 2D section in the (r, z) plane. The outer wall and its endcap rim are in dark green, while the inner wall and its rim are in red. (b) The radial profile of the azimuthal velocities of the rims (dark green and red) plotted together with that of an unbounded TC flow (light blue).

setup was in fact implemented in the PROMISE experiment, confirming the reduction in Ekman circulations in the bulk flow [11]. However, since the PROMISE experiment is limited to $\text{Re} \lesssim 10^4$ by construction, the effectiveness of this endcap configuration (with a ring slit at a radius $1.4r_{\text{in}}$) in reducing Ekman circulations could not be tested for orders of magnitude higher Re needed for MRI in liquid metal TC flows. At such high Re , Ekman and Stewartson layers can become unstable and turbulent, complicating the flow dynamics and affecting MRI modes, which in turn makes it hard to unambiguously identify the latter in the experiments [32,33].

Our main goal is to understand the flow structure and dynamics in the DRESDYN-TC device under the influence of endcaps for a wide range of Reynolds numbers up to $\text{Re} = 2 \times 10^5$ for the quasi-Keplerian rotation first in the purely hydrodynamic regime in the absence of magnetic fields. Specifically, we will characterize the properties of Ekman and Stewartson layers as well as Ekman circulations, arising from these layers as the flow encounters the cylinder walls, as a function of Re . This will in turn form the basis for the subsequent study of the MHD regime of this TC flow with an imposed axial magnetic field when MRI can be active.

The paper is organized as follows. Physical model and numerical setup are given in Sec. II. The main results on the flow structure and dynamics as well as implications for MRI are given in Sec. III and conclusions in Sec. IV.

II. PHYSICAL MODEL

Consider a TC setup axially bounded by the top and bottom endcaps where the inner and outer coaxial cylinders of height L_z have radii r_{in} and r_{out} and rotate, respectively, at angular velocities Ω_{in} and Ω_{out} around the z axis [Fig. 1(a)]. For infinitely long cylinders, this would give rise to the differential rotation of the fluid between the cylinders with the ideal TC angular velocity profile,

$$\Omega_{\text{TC}}(r) = \frac{\Omega_{\text{out}} r_{\text{out}}^2 - \Omega_{\text{in}} r_{\text{in}}^2}{r_{\text{out}}^2 - r_{\text{in}}^2} + \frac{(\Omega_{\text{in}} - \Omega_{\text{out}}) r_{\text{in}}^2 r_{\text{out}}^2}{r_{\text{out}}^2 - r_{\text{in}}^2} \frac{1}{r^2}. \quad (1)$$

TABLE I. List of all the simulations done with the corresponding values of Re , μ , the number of elements in the radial direction, n_r , and the minimum radial resolution, $\Delta r_{\min}$, near the endcaps and cylinder walls.

10^{-3}Re	μ	n_r	$\Delta r_{\min}$
1, 4, 7, 10, 20, 40, 75, 100	0.27	21	10^{-3}
1, 4, 7, 10, 20, 40, 75, 100	0.30	21	10^{-3}
1, 4, 7, 10, 20, 40, 75, 100, 200	0.35	21	10^{-3}
20, 40, 75, 100, 200, 400, 600	0.35	31	4.68×10^{-4}
100, 200, 400, 600	0.35	41	2.69×10^{-4}
1, 10, 100, 200	0.40	21	10^{-3}
1, 10, 100, 200	0.45	21	10^{-3}
1, 10, 20, 40, 100, 200	0.50	21	10^{-3}

The corresponding profile of the azimuthal velocity $u_\phi^{\text{TC}} = \Omega_{\text{TC}} r$ is depicted in Fig. 1(b). The considered DRESHDYN-TC device has endcaps split at a radius $r_s = 1.4r_{\text{in}}$ [Fig. 1(c)], which, as noted above, was shown to result in the efficient reduction of Ekman circulation for the scaled-down PROMISE device but at lower Re [31]. The inner and outer endcap rims rotate with the angular velocities of the inner and outer cylinders, respectively [Figs. 1(a) and 1(c)]. At the top and bottom boundaries, the azimuthal velocity (piecewise) linearly increases with radius on either side of a jump located at r_s [Fig. 1(b)].

In the laboratory frame, a Newtonian incompressible flow is governed by the Navier-Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (2)$$

where $\mathbf{u}$ is the fluid velocity and P is the pressure, while the density ρ and kinematic viscosity ν of the fluid are constant in time and uniform in space. The velocity boundary conditions at the cylinder walls and the endcaps are no-slip.

We nondimensionalize time by Ω_{in}^{-1} , angular velocities by Ω_{in} , length by the gap width between the cylinders, $d = r_{\text{out}} - r_{\text{in}}$, velocity by $\Omega_{\text{in}} d$, and the kinetic energy density $\rho \mathbf{u}^2/2$ by $\rho r_{\text{in}}^2 \Omega_{\text{in}}^2$. The main parameters of the system are the Reynolds number $\text{Re} = \Omega_{\text{in}} d^2/\nu$ and the ratio of the angular velocities of the cylinders $\mu = \Omega_{\text{out}}/\Omega_{\text{in}}$, measuring the degree of differential rotation (shear) of a TC flow and related to the above-mentioned Rossby number via $\text{Ro} = (1 - \mu)/\mu$. The Reynolds number defined here is equal to the inverse of the Ekman number, $\text{Re} = \text{Ek}^{-1}$, used for rapidly rotating flows. However, instead of Ek and Ro we prefer to use Re and μ here, which are commonly adopted in studies of MRI in TC flows [9]. In the DRESHDYN-TC device, the ratio of the cylinder radii is fixed to $r_{\text{in}}/r_{\text{out}} = 0.5$, which is close to its optimal value for MRI excitation [16], and the aspect ratio to $L_z/d = 10$. A full set of the parameters of the DRESHDYN-MRI experiment is given in [15]. In this paper, we assume the quasi-Keplerian rotation regime with $\partial \Omega / \partial r < 0$ and $\partial(r^2 \Omega) / \partial r > 0$, which is linearly stable according to Rayleigh's centrifugal criterion for infinitely long cylinders. Using Eq. (1), these conditions imply $r_{\text{in}}^2/r_{\text{out}}^2 < \mu < 1$.

We solve Eq. (2) using the spectral element code SEMTEX [34,35] in cylindrical coordinates (r, ϕ, z) , which is based on a continuous-Galerkin nodal spectral element method (SEM) in the 2D poloidal (r, z) plane and Fourier expansion in the azimuthal ϕ direction [34,35]. Table I lists all the simulations carried out in this study with the corresponding Reynolds number, ratio of cylinders' angular velocities μ , and number of elements in the radial r direction, n_r , together with the corresponding minimum radial resolution, $\Delta r_{\min}$, near the endcaps and cylinder walls. The number of elements in the axial z direction is fixed to $n_z = 201$ in all the simulations, and the corresponding minimum axial resolution is $\Delta z_{\min} = 4.56 \times 10^{-5}$ near the endcaps. For comparison, the typical Kolmogorov dissipation length for high $\text{Re} = 2 \times 10^5$ is equal to 6×10^{-4} . The order

of the polynomial basis functions in the spectral discretization is fixed to 9. Resolution tests and comparison of the results with those obtained with a different code are discussed in Appendix A.

In this first hydrodynamic study intended for the DRESHDYN-MRI experiment, we consider only axisymmetric ($\partial/\partial\phi = 0$) perturbations. Restricting to axisymmetric modes offers benefits for computational efficiency, thereby allowing for a more extensive parameter survey at high $\text{Re} \gtrsim 10^5$ considered here. Moreover, in the context of the DRESHDYN-MRI experiment, the axisymmetric mode is of central interest, since it is the most unstable one for MRI [15,36]. Consequently, the primary goal of this study is to understand the evolution of axisymmetric perturbations in a finite-height TC setup having the endcap configuration similar to that in DRESHDYN-TC device, which can be later generalized to nonaxisymmetric perturbations. This approach leads to two distinct scenarios: First, in cases where the axisymmetric flow exhibits instability (turbulence) due to endcaps, it is expected that the nonaxisymmetric flow would similarly display instability given that perturbations resulting from the endcaps lack any modal preference. Second, in those parameter regimes where axisymmetric flow is stable, our future efforts may concentrate on investigating the dynamics of nonaxisymmetric modes. Assuming axisymmetry of the flow to examine the effects of the endcaps on flow dynamics in the Rayleigh-stable regime should establish a lower limit for the instability strength, since nonaxisymmetric modes would create additional degrees of freedom for instability. To support these arguments for axisymmetry of the flow, we carried out additional 3D simulations (see Appendix B), which indicate that when nonaxisymmetric modes with higher azimuthal wave numbers $m \geq 1$ are included, the flow is still dominated by the axisymmetric $m = 0$ mode (at least for $\text{Re} \in \{2, 7.5\} \times 10^4$ that we checked), whose amplitude is orders of magnitude higher than those of the nonaxisymmetric ones (Fig. 15).

III. RESULTS

We study the evolution of the hydrodynamic flow in the Rayleigh-stable regime, taking the ideal TC profile Ω_{TC} [Eq. (1)] as an initial condition, i.e., $\mathbf{u} = \Omega_{\text{TC}} r \mathbf{e}_\phi$ at $t = 0$. Since the latter is a stationary solution in the case of the axially unbounded cylinders, its subsequent evolution proceeds through the adjustment near the endcaps as a result of the no-slip boundary conditions. Specifically, the velocity difference between the fluid attached to and rotating with the endcaps and the bulk azimuthal flow causes imbalance between pressure and centrifugal forces, resulting in the radial motion within thin Ekman layers in the vicinity of the endcaps. This radial motion when turning near the cylinder walls gives rise to poloidal Ekman circulations penetrating deeper into the bulk flow [4,18]. Below we investigate in detail the structure and dynamics of this flow in the saturated state. In Secs. III A–III C, we fix $\mu = \Omega_{\text{out}}/\Omega_{\text{in}} = (r_{\text{out}}/r_{\text{in}})^{-3/2} = 0.35$ (i.e., $\text{Ro} = 1.86$), which corresponds to the exactly Keplerian rotation of the cylinders [9], whereas the dependence on μ is analyzed in Sec. III D.

A. Flow structure and evolution

Figure 2 shows the evolution of the volume-averaged nondimensional kinetic energy $E_{\text{kin}} = (1/2V) \int \mathbf{u}^2 dV$, where V is the total volume of the flow domain between the cylinders. It grows during the initial adjustment phase as a result of the flow acceleration by the inner endcaps, since they rotate faster than the bulk flow [Fig. 1(b)]. Eventually, the flow saturates to a quasisteady state with a nearly constant value, $\check{E}_{\text{kin}}$, of the kinetic energy, which weakly increases with Re . The saturation time scales as $\text{Re}^{1/2}$ (in units of Ω_{in}^{-1}) consistent with the spin-up time scaling in rotating flows with high Re [37]. Although the initial ideal TC profile is independent of Re , the structure of the saturated flow, as shown below, depends on Re . However, the relative difference between $\check{E}_{\text{kin}}$ and $E_{\text{kin,TC}}$ is lower than 10%, as seen from Fig. 2.

Figure 3 shows the structure of the angular velocity $\Omega = u_\phi/r$ in the (r, z) plane and its radial profiles at different axial positions in the saturated state [38]. At high $\text{Re} = 2 \times 10^5$, the deviation of Ω from the ideal one Ω_{TC} , i.e., $\Omega - \Omega_{\text{TC}}$ can be divided into two main—positive (at $r \lesssim r_s$) and

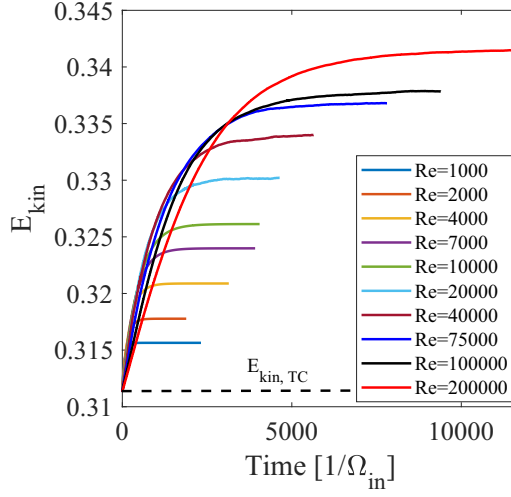


FIG. 2. Evolution of the volume-averaged nondimensional kinetic energy at different Re . The dashed line denotes the initial energy of the ideal TC flow.

negative (at $r \gtrsim r_s$)—parts with the first one being by absolute value larger than the second one. This implies that the azimuthal flow is modified mostly in the inner part $r \lesssim r_s$. At the split radius, the inner rim velocity displays the largest deviation from the TC profile, as shown in Fig. 1(b). On

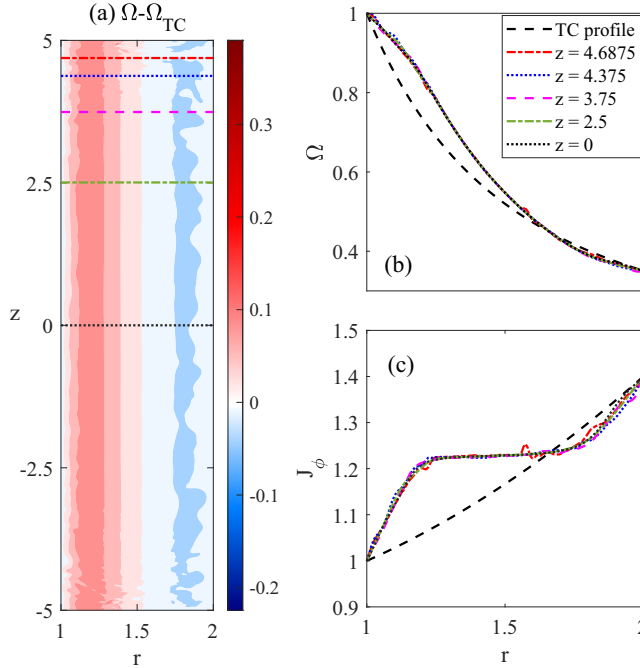


FIG. 3. (a) Deviation, $\Omega - \Omega_{TC}$, of the angular velocity Ω from the ideal one Ω_{TC} in the (r, z) plane. The radial profiles of (b) Ω and (c) the specific angular momentum $J_\phi = ru_\phi$ at different z [marked by the horizontal lines in (a)] in the saturated state at $Re = 2 \times 10^5$.

the other hand, this deviation is nearly uniform along the z axis (i.e., axially independent $\partial/\partial z \approx 0$), as also evident from the almost identical radial profiles of $\Omega(r, z)$ at different z in Fig. 3(b). Note that the same situation with an axially uniform profile of Ω is observed already at $\text{Re} = 10^4$ in Fig. 14 in Appendix A. This agrees with the Taylor-Proudman theorem stating that rapidly rotating flows become uniform along the axis of rotation [37]. By contrast, at small $\text{Re} = 10^3$, the deviations about Ω_{TC} are concentrated mostly near the endcaps, while the bulk of the azimuthal flow is almost unchanged from Ω_{TC} (see Fig. 13 in Appendix A).

To see the impact of changing angular velocity on the angular momentum transport, we show in Fig. 3(c) the radial profiles of the specific angular momentum $J_z = ru_\phi$ at the same axial positions as in Fig. 3(b). It shows that specific angular momentum is significantly increased (decreased) at those radii where the angular velocity Ω is larger (smaller) than Ω_{TC} , implying that the inner rim injects angular momentum, while the outer rim extracts it. Still, it can be seen that the angular momentum transport throughout the bulk of the flow is largely independent of the height, except very close to the endcaps, where Ekman layers are present.

Because of the invariance along the axial z direction, we can focus on the radial profiles at the midheight ($z = 0$). As shown above, the deviation of the flow from the ideal TC one increases with Re . This is also evident in the time-averaged radial profiles of the angular velocity $\langle \Omega \rangle_t$ in the saturated state for different Re depicted in Fig. 4(a) together with the ideal TC profile Ω_{TC} . For $\text{Re} = 10^3$, the flow profile at the midheight of the cylinder is very similar to the ideal one. For higher $\text{Re} \geq 10^4$, Ω is larger than Ω_{TC} at $r \lesssim r_s$, while for $r \gtrsim r_s$ it remains smaller but close to the latter. Note also that these profiles come closer to each other at higher $\text{Re} \geq 10^5$. To further quantify this deviation, Fig. 4(b) shows the time-averaged radial profile of the relative difference $\langle \Omega/\Omega_{\text{TC}} - 1 \rangle_t$, while Fig. 4(d) shows its maximum value over r as a function of Re . The radial profile overall increases in magnitude with Re , mostly in the inner half of the gap width reaching up to 14% for $\text{Re} = 2 \times 10^5$. Accordingly, its maximum also increases with Re more strongly at $\text{Re} \lesssim 10^4$ and then less steeply at $\text{Re} \gtrsim 10^4$ following approximately the power-law $\text{Re}^{0.22}$. This change in the Re dependence can be explained by the fact that we are going towards an asymptotic regime at large Re , where viscosity plays a role only in the thin boundary layers formed near the cylinder walls. This is evident in the local shear profile in Fig. 4(c) as well as in Figs. 7(a) and 7(b), where the boundary layers visible in the radial and axial velocity profiles become thinner as Re increases. By contrast, at smaller Re viscosity is important also in the bulk flow, since boundary layers near the inner and outer cylinder walls are so wide as to interact with each other, and hence the asymptotic scaling does not hold. Finally, the obtained power law is encouraging as it allows an extrapolation to even higher, experimental values of $\text{Re} \sim 10^6$, giving for the relative difference of Ω from Ω_{TC} the value about 16%.

To investigate the effect of angular velocity deviation on the stability of the flow, in Fig. 4(c) we show the radial profile of the time-averaged local shear parameter $\langle q \rangle_t = -\langle \partial \ln \Omega / \partial \ln r \rangle_t$ at the midheight $z = 0$ in the saturated state, which is also known as q parameter in the TC literature [26]. This parameter plays an important twofold role: it determines the local centrifugal stability of the flow according to Rayleigh's criterion and, in the MHD regime, sets the growth rate of MRI. The marginal Rayleigh stability threshold $q_c = 2$, as follows from the condition $\partial(r^2\Omega)/\partial r = 0$, is also shown in this figure for reference. The flow is locally stable if $q \leq q_c$ and unstable if $q > q_c$. It is seen that all the profiles at the midheight are in fact stable. For lower $\text{Re} = 10^3$, the profile of $\langle q \rangle_t$ almost linearly decreases with r and is still quite similar to that of the ideal TC one. However, for higher $\text{Re} \geq 10^4$, it considerably changes and takes on a humplike shape with a maximum around r_s that is quite close to but less than the critical q_c and further extends into a plateau for $\text{Re} \geq 10^5$. Thus, the bulk flow remains Rayleigh-stable, reaching marginal stability at $r \approx r_s$. This general trend holds across all the simulations. Below we will examine the corresponding structure of the flow in these two—lower and higher Re —cases.

The split endcaps not only affect the azimuthal velocity, changing its shear, but also generate radial and axial motions, as seen in the poloidal snapshots for $\text{Re} = 10^4$ in Fig. 5. The axial velocity u_z is highest near the endcaps close to r_s and extend with typical patterns of Ekman circulations into

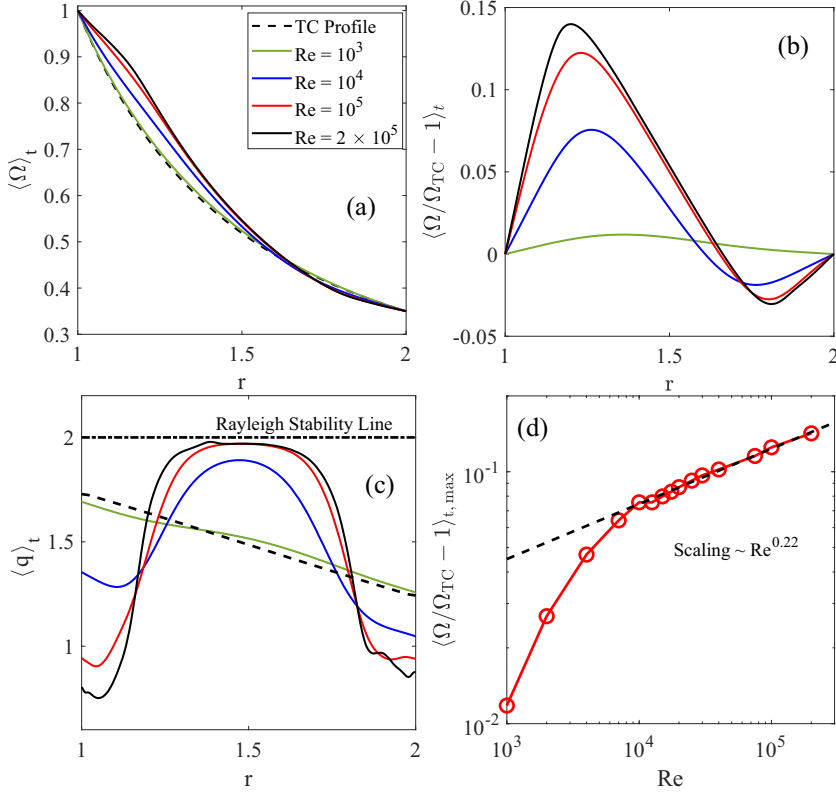


FIG. 4. Time-averaged radial profiles of (a) the angular velocity $\langle \Omega \rangle_t$, (b) its relative deviation from the ideal TC profile, $\langle \Omega / \Omega_{TC} - 1 \rangle_t$, and (c) the local shear parameter $\langle q \rangle_t$ (see text) for $Re \in \{10^3, 10^4, 10^5, 2 \times 10^5\}$ at the midheight $z = 0$ in the saturated state. The time averages here and throughout the paper are performed over the time intervals 1620–1717, 3217–3342, 9393–9542, and 11 694–11 750 (in units of Ω_{TC}^{-1}) with the corresponding sampling periods 0.35, 0.45, 0.27, and 0.24 for $Re \in \{10^3, 10^4, 10^5, 2 \times 10^5\}$, respectively. For reference, shown are the q profile for the ideal TC flow (dashed) and Rayleigh-stability threshold $q_c = 2$ (dot-dashed). (d) The maximum relative deviation of Ω along r at $z = 0$ vs Re , which follows $Re^{0.22}$ at high $Re \gtrsim 10^4$.

the bulk flow. By contrast, the radial velocity u_r is predominantly localized close to the endcaps, forming thin stable Ekman boundary layers there (see zoomed-in area in the u_r plot), and is relatively weak in the bulk flow. Both u_r and u_z are much smaller than the total azimuthal velocity u_ϕ , but are comparable to its perturbation about the initial TC profile, $u_\phi - u_\phi^{TC}$. Note the symmetric and antisymmetric characteristics of the radial and axial velocities around $z = 0$, respectively. Such a degree of symmetry indicates that the perturbed flow at this Reynolds number still remains laminar.

Figure 5 shows the distribution of the relative shear $q - q_c$ with respect to the marginal stability value $q_c = 2$ in the (r, z) plane in the saturated state, indicating the locally stable ($q \leq q_c$, blue) and unstable ($q > q_c$, red) regions. In particular, the Ekman layer as well as the vertical free shear, or Stewartson layer emanating from the top and bottom endcaps at r_s [12,39], remain both stable, despite higher shear $q > q_c$ of the latter one, because viscosity at this value of $Re = 10^4$ still appears to be large enough to prevent disruption of these layers. As a result, the shear layers extends deeper in the flow.

The structure of the flow dramatically changes at higher $Re \gtrsim 10^4$, as seen in Fig. 6 for $Re = 2 \times 10^5$. Ekman layers in the vicinity of the endcaps are much thinner but still remain stable and laminar, as seen from the radial velocity plot [see also Fig. 8(a)]. By contrast, the Stewartson

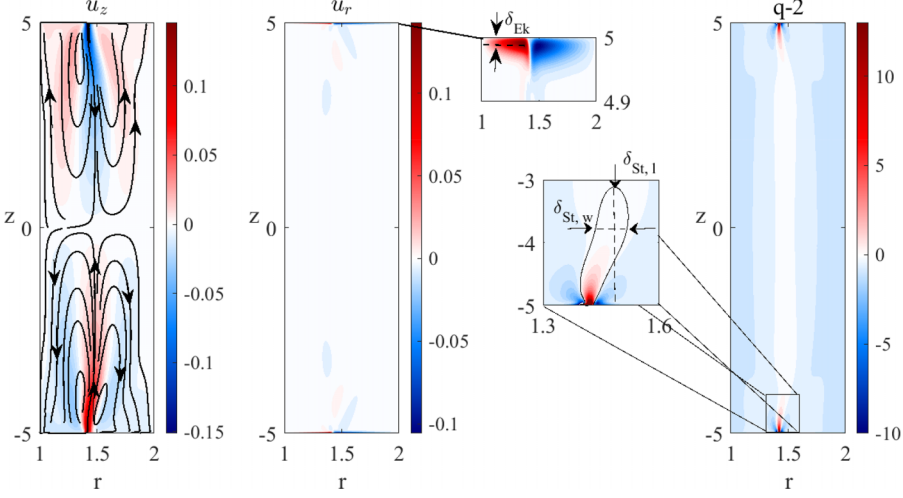


FIG. 5. Snapshots of the axial u_z (left) and radial u_r (middle) velocities in the (r, z) plane for $\text{Re} = 10^4$ in the saturated state, exhibiting symmetry between the upper ($z > 0$) and lower ($z < 0$) parts of the domain for this Re . Ekman circulation direction is shown by the streamlines of the poloidal velocity overplotted on the map of u_z . Stable Ekman layers (with thickness δ_{Ek}) at the endcaps are clearly seen in the zoomed-in inset of u_r . Right panel shows the corresponding local shear parameter in the (r, z) plane relative to the critical value $q_c = 2$ of Rayleigh stability, $q - q_c$. Zoomed-in insets in this panel illustrate the vertical free Stewartson layers originating from the split radius r_s near the endcaps and characterized by the high shear $q \geq q_c$ (red). Their width $\delta_{\text{St},w}$ and length $\delta_{\text{St},l}$ are defined, respectively, as the maximum radial and axial extent of the zero level curve of $q - q_c$ (solid black).

layers have now much higher shear $q \gg q_c$ (thin red areas in the insets of the right panel of Fig. 6) and hence are unstable giving rise to turbulence at such high Re (and hence lower viscosity), which is mostly localized near the endcaps and weakens in intensity as it penetrates deeper into the bulk flow (see also Fig. 9).

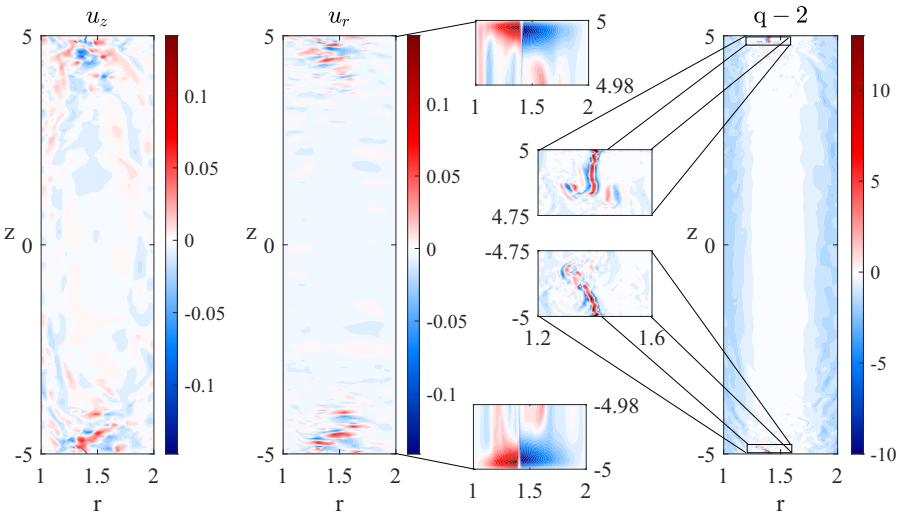


FIG. 6. Same as in Fig. 5 but for a turbulent regime at $\text{Re} = 2 \times 10^5$. Turbulence arises from the instability of the Stewartson layers and penetrates into the bulk flow, gradually weakening in intensity.

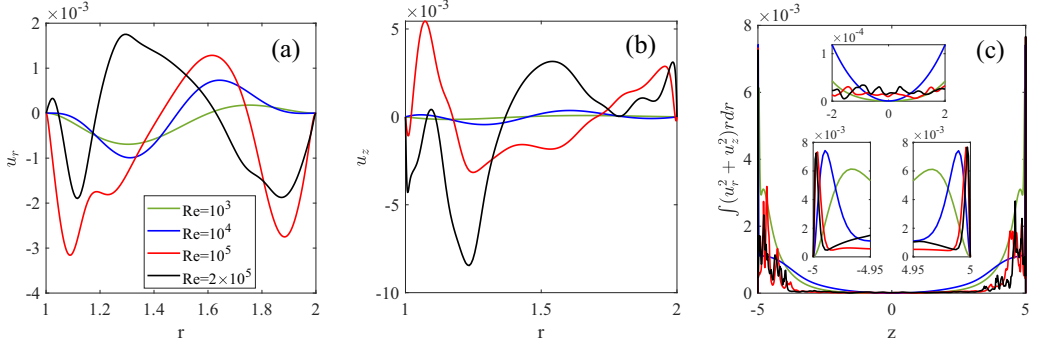


FIG. 7. Radial profiles of (a) the radial u_r and (b) axial u_z velocities at $z = 0$ in the saturated state for different Re . (c) Radially integrated squared poloidal velocity $\int (u_r^2 + u_z^2) r dr$ as a function of z for the same Re .

So far we have characterized the structures of the azimuthal velocity, its shear q and the overall poloidal flow in the saturated state. Let us now examine in a more quantitative manner the radial and axial profiles of the instantaneous radial and axial velocities at different Re in the saturated state, which are shown in Fig. 7. As expected, both u_r and u_z increase by absolute value with increasing Re , but exhibit different behavior along r and z . It is seen in Figs. 7(a) and 7(b) that their variation with r becomes more irregular and stronger for higher Re , forming boundary shear layers with steep radial gradients (shear) near the inner and outer cylinder walls. This behavior is due to the presence of turbulence at $Re \geq 10^5$, as seen in Fig. 6, which is most intensive near the endcaps but also extends down to the midheight.

Figure 7(c) shows the axial profile of the radially averaged squared poloidal velocity, $\int_{r_{in}}^{r_{out}} (u_r^2 + u_z^2) r dr$. Note that for all the considered Re , strong shear is observed near the top and bottom endcaps (see zoomed-in insets) corresponding to thin Ekman layers discussed above, which increases with increasing Re . For lower $Re = 10^3$, when turbulence is still absent, the poloidal velocity gradually decreases from its maximum values near the endcaps to almost zero at $z = 0$, spanning thus the whole height. By contrast, as Re increases further, the poloidal velocity increases near the endcaps and becomes fluctuating due to turbulence there at $Re \geq 10^5$, but rapidly decays away from the endcaps to very small values at $|z| \gtrsim 4$ and becomes almost independent of height at these z . This indicates that although Ekman circulations impact the overall flow, they become more and more localized towards the endcaps as Re increases.

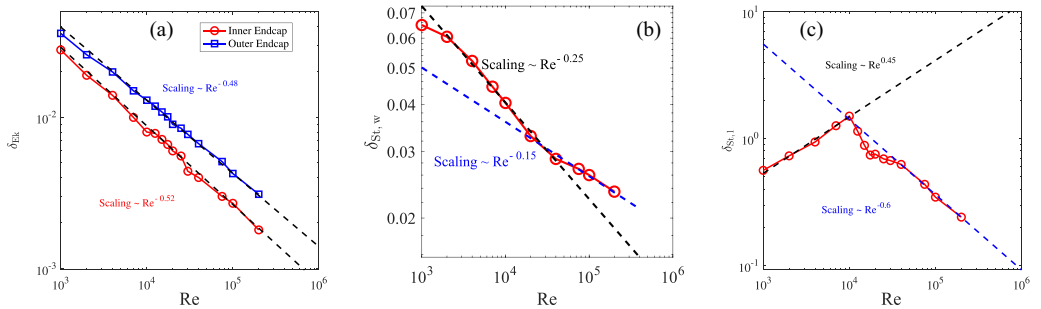


FIG. 8. (a) Ekman boundary layer thickness δ_{Ek} near the inner (red) and outer (blue) endcap rims as well as (b) the width $\delta_{St,w}$ and (c) length $\delta_{St,l}$ of the Stewartson layer as a function of Re . Dashed lines show the power-law fits.

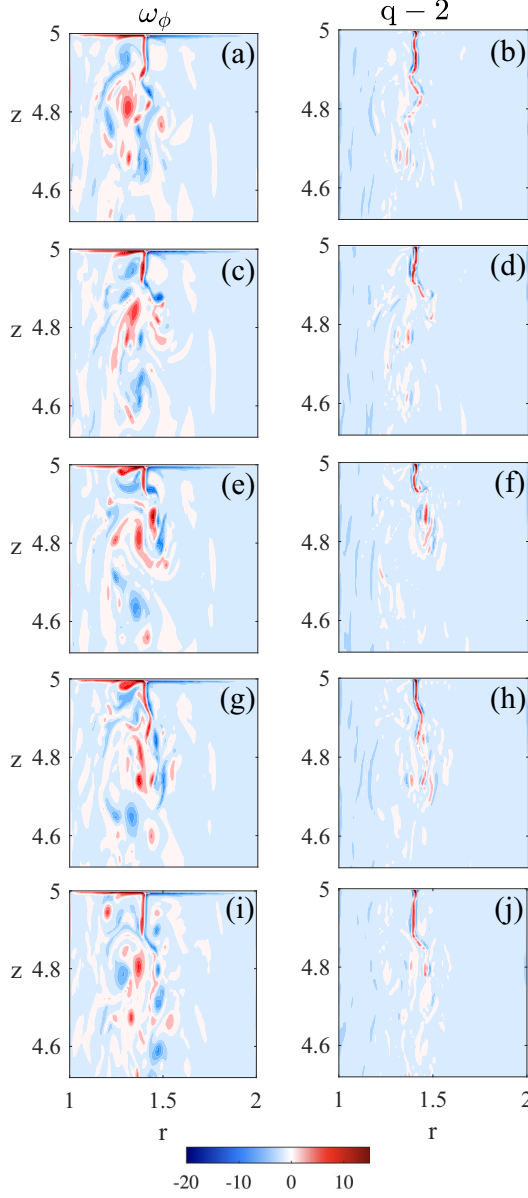


FIG. 9. Azimuthal component of vorticity $\omega_\phi = (\nabla \times \mathbf{u})_\phi$ (left) and relative local shear, $q - q_c$ (right) in the saturated state for $\text{Re} = 2 \times 10^5$ at different time moments increasing from top to bottom: (a), (b) at $t = 11\,563$, (c), (d) at $t = 11\,575$, (e), (f) at $t = 11\,578$, (g), (h) at $t = 11\,580$, and (i), (j) at $t = 11\,582$.

B. Ekman and Stewartson layer scalings

In all studies of TC flows with axial boundaries, the Ekman boundary layer and its stability play an important role in the dynamics of the entire flow. The well-known scaling $\text{Re}^{-0.5}$ of a stable (laminar) Ekman layer thickness with Re [37], which follows from a balance between the viscous and Coriolis forces, is widely discussed for TC flows [12,40,41]. In the present setup, the endcaps divided into two rims rotating, respectively, with the inner and outer cylinders give rise to two distinct Ekman layers near each rim (see insets in u_r plots of Figs. 5 and 6). We define the Ekman

layer thickness δ_{Ek} as an axial distance from the endcap to the location of the first local maximum of the time-averaged radial velocity u_r (see Fig. 5). Figure 8(a) shows the Ekman layer thickness as a function of Re that follows a power-law dependence $\delta_{\text{Ek}} \sim \text{Re}^{-0.52}$ and $\delta_{\text{Ek}} \sim \text{Re}^{-0.48}$ near the inner and outer endcap rims, respectively, which hold for all the considered Re from the laminar to turbulent regimes. This is an indirect indication that these layers remain stable even for very high Re , as also seen in Figs. 5 and 6. These scalings are close to the expected classical ones for the laminar Ekman layer noted above, while a slight deviation from the latter can be attributed to a specific geometry of our setup as well as to the proximity to the Stewartson layers interacting with the Ekman layers near r_s . Note also that due to the difference in the angular velocities of the rims, the thickness of the boundary layer near the inner rim is somewhat smaller than that near the outer rim, corresponding to the higher effective Re at the inner rim than at the outer one.

Let us now characterize the properties of the free Stewartson shear layers emanating from the endcaps at r_s due to the jump between the angular velocities of the inner and outer rims. The width $\delta_{\text{St},w}$ and axial length $\delta_{\text{St},l}$ of these layers are defined, respectively, as the maximum radial and axial extent of the zero level curve of $q - q_c$ originating from the split radius r_s at the endcaps, as shown in the inset of the right panel of Fig. 5. The dependence of $\delta_{\text{St},w}$ on Re is shown in Fig. 8. In the stable (laminar) regime at $\text{Re} \lesssim 10^4$, the Stewartson layer is stationary and its width exhibits a power-law scaling $\delta_{\text{St},w} \sim \text{Re}^{-0.25}$, which is consistent with the classical scaling based on the balance between Coriolis and viscous forces across the thin radial extent of this layer [12,39,42]. On the other hand, in the turbulent regime at higher $\text{Re} \gtrsim 10^4$, the Stewartson layer is unstable and fluctuating (Figs. 6 and 9), so its width and axial length (see below) are determined by first averaging $q - q_c$ in time and then, as in the laminar regime, finding the dimensions of its zero level curve near the endcaps in the (r, z) plane. As a result, we obtain the less steep scaling of the width $\delta_{\text{St},w} \sim \text{Re}^{-0.15}$. This deviation of the turbulent Stewartson layer scaling from that of the laminar one can be attributed to the enhanced effective turbulent diffusivity. At the same time, the boundary layers at the cylinder walls are thin at high Re , leaving the Stewartson layer more space to expand radially in the predominantly inviscid bulk flow. At any rate, the transition of the Stewartson layer from a stationary (stable) to highly dynamic (turbulent) state results in a change in its scaling properties.

Stewartson layers penetrate deeper into the bulk flow, as seen in the axial velocity map in Figs. 5 and 6. Previous studies [42] conducted in a different configuration—a disk of a finite radius rotating coaxially with an infinite medium surrounding the disk—showed that the length $\delta_{\text{St},l}$ to which these layers extend into the flow increases linearly with Re . Figure 8(c) shows this length in the present TC setup, which behaves with Re differently because of the differences in the system geometry. Specifically, it increases as $\delta_{\text{St},l} \sim \text{Re}^{0.45}$ for $\text{Re} \leq 10^4$, but then decreases as $\delta_{\text{St},l} \sim \text{Re}^{-0.6}$ for $\text{Re} > 10^4$ due to the emerging turbulence and vortex shedding caused by the instability of Stewartson layers at these Re (Figs. 6 and 9), which in fact results in the reduction of $\delta_{\text{St},l}$ (see Sec. III C). Finally, we note that the values of δ_{Ek} , $\delta_{\text{St},w}$ and $\delta_{\text{St},l}$ at even higher $\text{Re} \in \{4, 6\} \times 10^5$, as we also checked (not shown), are consistent with the power-law scalings obtained for the saturated state at $\text{Re} \leq 2 \times 10^5$ that are shown in Fig. 8.

C. Vortex shedding

The flow is nearly stationary at smaller $\text{Re} \lesssim 10^4$ once it settles down in a saturated state (Fig. 5), whereas at higher Re , as discussed above, it is very turbulent (Fig. 6), involving the formation and evolution of vortices in the vicinity of the endcaps. To trace these vortices in more detail, in Fig. 9 we show the azimuthal component of vorticity $\omega_\phi = (\nabla \times \mathbf{u})_\phi$ together with the relative shear $q - q_c$ in the (r, z) plane for $\text{Re} = 2 \times 10^5$ at different times in the saturated state. The vortices primarily emerge at the inner side near r_s , due to the interplay of the Ekman pumping and unstable, high-shear Stewartson layers and migrate towards the midheight. These vortices disrupt the Stewartson layer, limiting its axial length $\delta_{\text{St},l}$. Specifically, consider a typical large vortex at time moment $t = 11\,563$ [red spot near the point $(r, z) = (1.3, 4.8)$] surrounded by smaller-scale vortices [Fig. 9(a)]

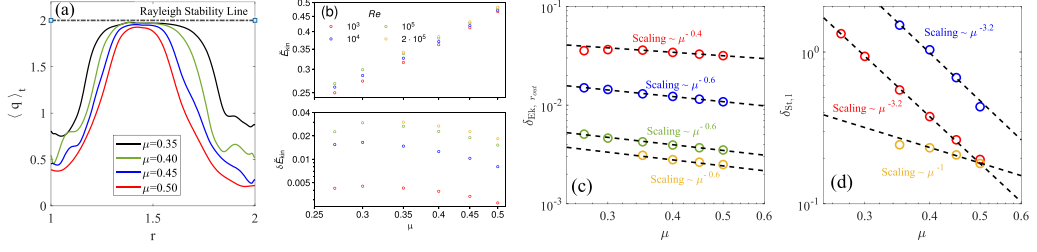


FIG. 10. (a) Radial profile of the time-averaged shear parameter $\langle q \rangle_t$ in the saturated state at the midheight $z = 0$ for different μ and $Re = 2 \times 10^5$. (b) Dependence of the volume-averaged kinetic energy of the total flow $\bar{E}_{kin}$ and its deviation $\delta \bar{E}_{kin}$ (see text) on μ at different Re , (c) Ekman layer thickness, δ_{Ek} , near the outer endcap rim and (d) Stewartson layer length, $\delta_{St,l}$, vs μ for different $Re = 10^3$ (red), 10^4 (blue), 10^5 (green) and 2×10^5 (yellow). The dashed lines in (c) and (d) show the power-law fits.

corresponding to the site of the dynamic (“flapping”) Stewartson layer [Fig. 9(b)]. As this vortex evolves, at $t = 11\,575$, it gets distorted [Fig. 9(c)] and so does the tail of Stewartson layer [Fig. 9(d)]. The deformed vortex then breaks up into a number of smaller-scale vortices propagating away from the endcaps and gradually decaying [Figs. 9(e) and 9(g)]. These smaller-scale vortices in turn disrupt (cut off) the Stewartson layer at a certain axial distance—the layer length $\delta_{St,l}$ from the endcaps [Figs. 9(f) and 9(h)], significantly altering the scaling of the latter from $\delta_{St,l} \sim Re^{0.45}$ increasing with Re in the laminar regime to the decreasing $\delta_{St,l} \sim Re^{-0.6}$ in the turbulent regime [Fig. 8(c)]. After that this cycle of vortex formation, evolution, breakup, and migration to the midheight recurs [Figs. 9(i) and 9(j)]. To quantify the time scale of such a vortex shedding process, we did Power Spectral Density (PSD) analysis for ω_ϕ in time at each (r, z) and found the frequency $f = 5.05$ at which PSD reaches a maximum in the (r, z) plane. The corresponding period $T = f^{-1} = 0.2$ can be interpreted as the period of such a vortex shedding cycle.

D. Effect of varying the angular velocity ratio μ

So far, we have focused on the effect of endcaps at various Re for the quasi-Keplerian rotation of the cylinders with $\mu = 0.35$. The above analysis clearly shows that at high enough $Re \gtrsim 10^4$ the shear layers induce turbulence, which is most intense near the endcaps and decreases into the bulk flow. The shear q at the midheight for such large Re always stays below, but close to, the marginal stability limit $q_c = 2$ [Fig. 4(c)]. Since μ can be varied in MRI-experiments, here we explore how the above results change with μ .

For the ideal TC setup without endcaps, the flow becomes overall more and more stable as μ increases away from the Rayleigh-stability threshold, which for this parameter gives $\mu_c = r_{in}^2/r_{out}^2 = 0.25$ from Eq. (1). Figure 10(a) shows the radial profile of the time-averaged shear parameter $\langle q \rangle_t$ at the midheight in the saturated state at different μ and $Re = 2 \times 10^5$. Indeed, it is seen that the flow becomes more stable, since shear q decreases (at a given radius) with increasing μ . However, q at the split radius r_s always remains close to, but slightly lower than, the Rayleigh-stability threshold due to the endcap effect regardless of the μ value.

The volume-averaged kinetic energy of the total flow in the saturated state, $\bar{E}_{kin}$, as a function of μ at different Re is shown in the top panel of Fig. 10(b). It increases linearly with μ for all the considered Re , which can be attributed to the mean azimuthal flow. It is also evident from the bottom panel of Fig. 10(b) that the maximum deviation of $\bar{E}_{kin}$ from the kinetic energy, $E_{kin,TC}$, of the classical TC flow, i.e., $\delta \bar{E}_{kin} = \bar{E}_{kin} - E_{kin,TC}$, is reached for μ between 0.3 and 0.35, while for larger μ the flow tends to be more stable. This is further supported by the following analysis of the Ekman and Stewartson layer properties with respect to μ .

Figure 10(c) shows the scaling of the Ekman layer thickness with μ for different Re at the outer endcap rim, while at the inner rim it remains nearly unchanged with μ , since only Ω_{out}

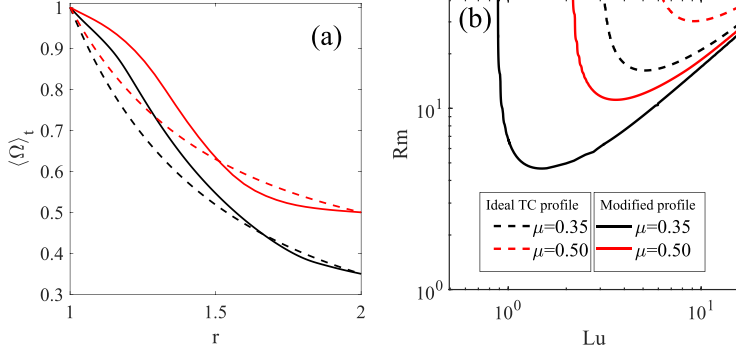


FIG. 11. (a) Radial profile of the time-averaged angular velocity $\langle \Omega \rangle_t$ in the saturated state at $z = 0$ for $\mu = 0.35$ (black) and 0.5 (red) for the ideal TC flow given by Eq. (1) (dashed) and in the presence of endcaps (solid) at $Re = 2 \times 10^5$. (b) Marginal stability curves of MRI in the (Lu, Rm) plane (the area above these curves is MRI-unstable) obtained for the Ω profiles from panel (a). Note that the modification of the flow profiles due to endcaps significantly reduces the critical Lu and Rm for the onset of MRI.

is increased in μ . It is seen that for smaller $Re = 10^3$, δ_{Ek} decreases with μ as a power law $\mu^{-0.4}$, whereas for larger $Re \gtrsim 10^4$, this decrease becomes steeper $\mu^{-0.6}$. This change in the scaling can be attributed to the change from the nearly ideal TC profile at $Re = 10^3$, which is still dominated by viscosity, to the modified, profile of Ω in the asymptotic regime at high $Re \gtrsim 10^4$ (Figs. 3, 13, and 14), where the effect of viscosity is primarily confined within the boundary layers. As for the Stewartson layer, it is seen in Fig. 10(d) that its length decreases with increasing μ as $\delta_{St,l} \sim \mu^{-3.2}$ in the laminar regime at $Re \lesssim 10^4$ and, as expected, is smaller and less steep $\delta_{St,l} \sim \mu^{-1}$ in the turbulent regime at $Re \gtrsim 10^4$, implying more stability at higher μ due to reduced shear. In the envisioned DRESDYN-MRI experiment, the stability of the base flow before switching on a magnetic field is important to unambiguously identify MRI. Increasing μ , say, to $\mu = 0.5$ in this experiment (but still not too much to suppress MRI) might be a viable possibility to ensure hydrodynamic stability of the flow.

E. Implications for MRI

As shown above, the structure and dynamics of a finite-height TC flow at large Re can be strongly affected by the Ekman circulations and unstable shear layers. In particular, we have also shown that higher μ increases hydrodynamic stability of the flow, which can be favorable for the identification of MRI. On the other hand, larger μ is challenging as it significantly increases the critical values of Lundquist $Lu = B_{0z}d/\eta\sqrt{\rho\mu_0}$ and magnetic Reynolds $Rm = \Omega_{in}d^2/\eta$ numbers for the onset of MRI in experiments [15], where B_{0z} is the imposed constant axial magnetic field and μ_0 the magnetic permeability of vacuum. Since in the upcoming DRESDYN-MRI experiments, the technically reachable maximum values of these two numbers are $Lu_{\max} = 10$ and $Rm_{\max} = 40$, it is interesting to examine whether MRI can be achieved in these experiments for the radial profile of the angular velocity modified from the ideal TC one due to the endcap effects despite higher μ .

For this purpose, following our previous studies [15, 43, 44] and a related recent study on the Princeton MRI experiment [45], we carry out linear stability analysis of the corresponding MHD problem for two different radial profiles of the angular velocity—an ideal TC profile Ω_{TC} and the actual (modified) profile resulting from the endcap effect taken from the above simulations at $Re = 2 \times 10^5$ and $\mu \in \{0.35, 0.5\}$ [Fig. 11(a)]—subject to the axial field B_{0z} . We then compare the onset criterion of MRI for these two equilibria. Assuming axisymmetric perturbations to have a standard modal form $\propto \exp(\gamma t + k_z z)$, we solve linearized MHD equations with no-slip for velocity and insulating for magnetic field boundary conditions at the cylinder walls to find the MRI growth rate γ . For simplicity, periodic boundary conditions are adopted in the z direction, because at large Re

TABLE II. Critical values of (Lu, Rm) for the onset of MRI for both the ideal and modified TC profiles.

μ	Ideal TC profile	Modified TC profile
0.35	(5.9, 16.2)	(1.5, 4.6)
0.5	(9.3, 30.3)	(3.6, 11.5)

the angular velocity is almost uniform in z (Fig. 3). The details of this eigenvalue problem, linear MHD equations, and the numerical procedure to solve them are described in [15].

The marginal stability (i.e., $\gamma = 0$) curves in the (Lu, Rm) plane obtained for the ideal and modified TC profiles are shown in Fig. 11(b). It is seen that the critical Lu and Rm for the onset of MRI appear to increase, i.e., imply more stability, with increasing μ (see also [15]). The main result is that the modified profile of the angular velocity yields considerably lower critical values of Lu and Rm than those for the ideal TC profile Ω_{TC} (see Table II). This is consistent with the recent results of Wang *et al.* [45–47] who reported MRI in the Princeton TC experiment at much (about 3 times) lower Lu and Rm than dictated by 1D linear stability analysis for an ideal TC flow. While this is favorable also for the DRESDYN-MRI experiments, since MRI may set in at lower Lu and Rm achievable with less efforts and energy expenses, it may query the comparability of the experimental results with the original problem of MRI in Keplerian flows. Interestingly, though, the critical values of Lu and Rm for the ideal TC flow at the Keplerian $\mu = 0.35$ could be nearly recovered in the saturated flow of the real experiment at $\mu = 0.5$, as seen in Fig. 11(b).

IV. CONCLUSIONS

In this paper, we conducted 2D axisymmetric hydrodynamic study of the flow in a finite-height Taylor-Couette setup whose cylinders are covered with top and bottom split endcaps for a range of Reynolds numbers $Re \in [10^3, 2 \times 10^5]$ and the ratio of cylinders' angular velocities $\mu = \Omega_{out}/\Omega_{in} \in [0.27, 0.5]$ (i.e., Rossby numbers $Ro \in [1, 2.7]$) in the quasi-Keplerian rotation regime. We showed that the flow at $Re \lesssim 10^4$ is laminar and characterized by large-scale poloidal Ekman circulations due to the endcaps penetrating deeper down to the cylinder midheight. In this regime, the Ekman layers attached to the endcaps and free Stewartson shears layer emanating from the junction of endcap rims are stable and stationary. By contrast, at higher $Re \gtrsim 10^4$, Ekman layers remain stable, whereas the Stewartson layers become unstable, get distorted and develop turbulence accompanied by the vortex shedding process. This turbulence and resulting poloidal motion is strongest near the endcaps, and weakens away from the endcaps, reaching a relatively low, axially uniform level in

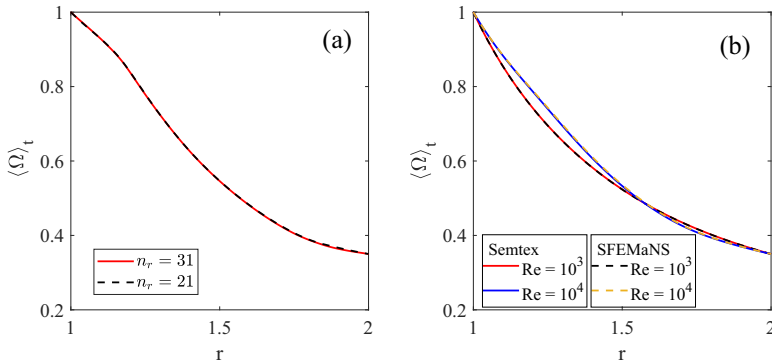


FIG. 12. (a) Time-averaged radial profile of the angular velocity, $\langle \Omega \rangle_t$, in the saturated state at midheight for $Re = 2 \times 10^5$, $\mu = 0.35$ and $n_r = 21$ (black dashed) and 31 (red), while $n_z = 201$ is fixed. (b) Comparison of $\langle \Omega \rangle_t$ in the saturated state at midheight obtained from the SEMTEX and SFEMaNS codes for different Re .

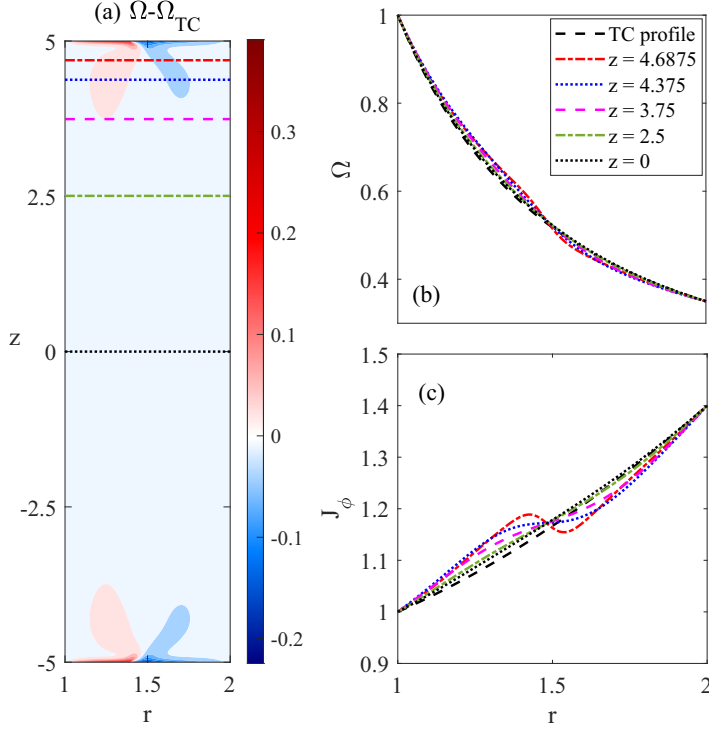
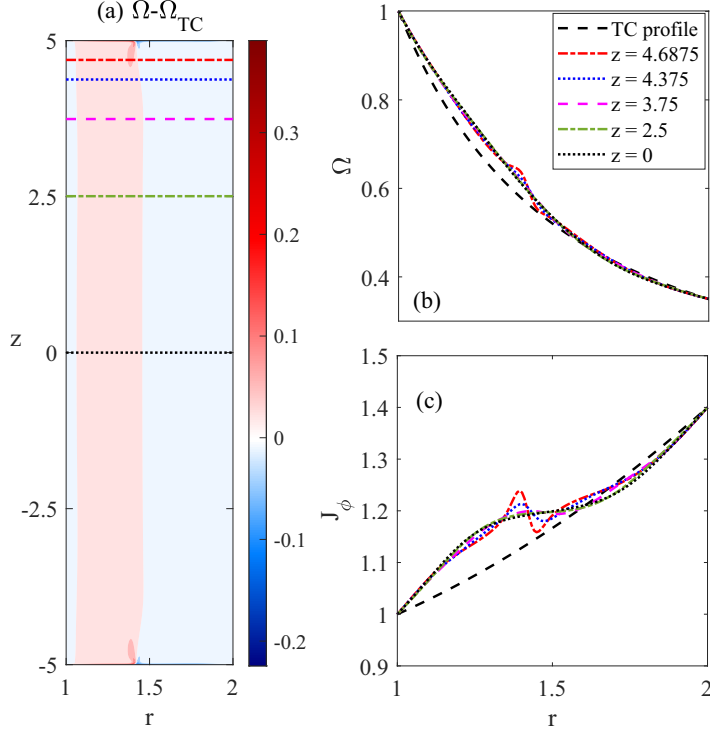


FIG. 13. (a) Deviation, $\Omega - \Omega_{TC}$, of the angular velocity Ω from Ω_{TC} in the (r, z) plane and (b) the radial profile of Ω at different z [marked by horizontal lines in (a)] in the saturated state for $Re = 10^3$. (c) The specific angular momentum $J_z = r^2\Omega$ vs r at the same z as in (b).

the bulk flow. In this regime, the mean angular velocity of the bulk flow is nearly uniform along the axial z direction and Rayleigh-stable. The larger Re is, the more this profile deviates from the ideal (i.e., infinitely long) TC one.

We characterized the structure of the Ekman and Stewartson layers and showed that their sizes exhibit power-law scalings with Re and μ . For the Stewartson layers these scalings have different exponents in the laminar (at lower $Re \lesssim 10^4$) and turbulent (at higher $Re \gtrsim 10^4$) regimes, which can be attributed to small-scale vortices shed near the unstable Stewartson layer at high Re , limiting its length and modifying width. For the same Re , the flow is more stable for larger μ due to the decreased azimuthal velocity shear at the endcap slit.

These results can be important for future MRI-experiments in general and for DRESDYN-MRI experiments in particular, since they suggest that those experiments, which usually require very high $Re \gtrsim 10^6$ for the onset of MRI, could also be conducted at higher μ to prevent any hydrodynamic instability in the system complicating MRI dynamics and its unambiguous identification. Carrying out a preliminary linear stability analysis on the angular velocity profile established in the considered TC setup with endcaps and added axial magnetic field, we showed that the critical Lundquist Lu and magnetic Reynolds Rm numbers for the MRI excitation turn out to be about 3 times lower than those in the ideal, i.e., axially unbounded TC setup. This suggests that MRI can be in fact observed at much lower Lu and Rm in the DRESDYN-MRI experiment than those dictated by the linear analysis for the ideal TC flow profile. Future more general 3D MHD studies in a finite-height TC setup with a background axial field will allow us to understand the dynamics of MRI under the influence of endcaps, thereby preparing a theoretical basis for the DRESDYN-MRI experiment. The present hydrodynamic study is a first step towards such a more complex MHD analysis.


 FIG. 14. Same as in Fig. 13 but for $\text{Re} = 10^4$.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: RESOLUTION AND OTHER CASE STUDIES

To check the validity of our discretization approach, Fig. 12(a) shows the time-averaged angular velocity, $\langle \Omega \rangle_t$, at the midheight as a function of r for $\text{Re} = 2 \times 10^5$ and $\mu = 0.35$ for the two values of $n_r \in \{21, 31\}$. It is clearly seen that $n_r = 21$ can resolve the flow profile quite well even for $\text{Re} \leq 2 \times 10^5$.

To ensure the validity of the adopted numerical approach, in Fig. 12(b) we further compare the flow profiles in the saturated state for $\text{Re} \in \{10^3, 10^4\}$ computed with the SEMTEX code used in this study and spectral/finite element code SFEMaNS [48,49], which has been extensively used for the simulations of the Princeton MRI experiment [45–47,50]. SFEMaNS is well suited to model fluid-boundary interactions in the experimental device. The code solves the Navier-Stokes equations for

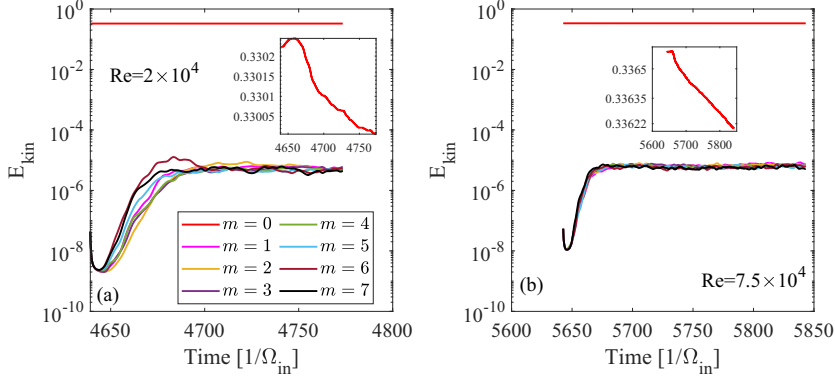


FIG. 15. Evolution of the volume-averaged energy of the axisymmetric $m = 0$ and nonaxisymmetric $m \leq 7$ modes from 3D simulations for (a) $\text{Re} = 2 \times 10^4$ and (b) $\text{Re} = 7.5 \times 10^4$ restarted from the axisymmetric saturated state.

incompressible flow on a mesh in the (r, z) plane. This comparison of the Ω profiles demonstrates a very good agreement between these two numerical approaches.

Figure 13(a) shows the deviation, $\Omega - \Omega_{\text{TC}}$, of the angular velocity Ω from the ideal one Ω_{TC} in the (r, z) plane for lower $\text{Re} = 10^3$. It is seen that this deviation is very small in the bulk flow except some perturbation near the split radius r_s of the endcaps. The similarity between these two angular velocity profiles as well as the corresponding azimuthal ϕ components of angular momentum are also demonstrated in Figs. 13(b) and 13(c), showing their radial profiles at different z . By contrast, at higher $\text{Re} = 10^4$ shown in Fig. 14(a), $\Omega - \Omega_{\text{TC}}$ is larger and nearly uniform in z , similar to that for $\text{Re} = 2 \times 10^5$ in Fig. 3(a). Figure 14(b) shows that Ω is slightly larger than Ω_{TC} at $r \lesssim r_s$ and approaches an ideal TC profile at $r \gtrsim r_s$. Accordingly, the angular momentum increases near the inner part and slightly decreases in the outer part [Fig. 14(c)].

APPENDIX B: EVOLUTION OF NONAXISYMMETRIC MODES

To investigate the role of nonaxisymmetric modes, we did some 3D simulations for $\text{Re} \in \{2, 7.5\} \times 10^4$ with nonzero azimuthal wave numbers $m \neq 0$ by restarting 2D simulations from the saturated axisymmetric $m = 0$ state and then adding several nonaxisymmetric modes with $m \leq 7$. Figure 15 shows the subsequent evolution of the volume-averaged kinetic energy of each m mode. It is clear that the flow is largely dominated by the axisymmetric $m = 0$ mode, while nonaxisymmetric modes saturate to much smaller amplitudes. Note also that in the presence of nonaxisymmetric modes, kinetic energy of the $m = 0$ mode slightly decreases as its energy is transferred to higher m modes.

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