Symmetries and equivalence transformations in linear stability theory and their implications for the dispersion relation

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The present work revisits hydrodynamic linear stability theory from the perspective of Lie symmetries. For the linearized Navier-Stokes equations of a two-component shear flow, the expected symmetries and equivalence transformations of the Navier-Stokes group are recovered, while additional ones arise due to linearization. The concept of symmetries and equivalence transformations in space and time is extended to spectral space, where the new definition of a spectral equivalence transformation is introduced. These are not fundamentally new transformations, as they correspond to space-and-time symmetries or equivalence transformations; however, they take on a different form that is used throughout the subsequent analysis. In particular, the Squire transformation is derived from such a symmetry generator of a spectral equivalence transformation with regard to the coupled Orr-Sommerfeld and Squire system, including the vorticity. As another step, these spectral generators are applied to an adjoint-projected residual functional. The variation of this functional is studied via the variation of the differential operator. This approach yields algebraic invariance conditions as well as derivative expressions. At a direct-adjoint eigenpair corresponding to an isolated eigenvalue, these derivatives determine the derivatives of the eigenvalue branch through the standard adjoint sensitivity relation and thus the group velocity. One specific result thereof is that the homogeneous Squire equation does not admit stationary-frame double saddle points in three dimensions (3D), which is a necessary criterion for a potential 3D absolute instability mechanism within the formalism of Briggs's theory.

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I. INTRODUCTION

Continuous symmetry theory studies smooth transformations endowed simultaneously with the properties of a group and a manifold, resulting in a vast number of applications. They are particularly powerful in the study of differential and algebraic equations. The mathematical foundations of continuous symmetry theory were laid by Sophus Lie, who introduced the concept of Lie groups in the late 19th century [1], and Élie Cartan, who put the theory in its modern geometric framing [2].

The earliest general use of symmetry methods in hydrodynamics can be traced back, to the authors' knowledge, to the work of Wilczynski [3]. Bechert applied symmetry methods specifically to investigate wave propagation in gases [4]. A broader conceptual framework for symmetries in hydrodynamics was later presented by Birkhoff, emphasizing similarity and scaling arguments [5]. The group properties of the Navier-Stokes equations were systematically analyzed by Pukhnachev in two dimensions (2D) [6] and by Danilov and by Bytev in 3D [7,8]. Lloyd subsequently characterized the infinitesimal Lie algebra of those 3D group properties [9]. For an overview of Lie reduction techniques for the Navier-Stokes equations to derive invariant solutions, the reader is referred to Fushchych and Popovych [10,11].

A well-known result in the application of symmetry methods to hydrodynamic linear stability theory is Squire's theorem, although Squire did not explicitly frame his analysis in terms of an equivalence transformation. By comparing 2D and 3D temporal normal modes equivalent in the sense of an equivalence transformation, he demonstrated that 2D perturbations always correspond to a lower Reynolds number than their corresponding 3D perturbations [12]. Consequently, 2D perturbations are the critical temporal modes for the onset of instability. By analyzing the homogeneous Squire equation (hSE), Squire was also able to conclude that temporal Squire modes are always damped and that it is sufficient to investigate the Orr-Sommerfeld equation (OSE).

The systematic application of Lie symmetry methods to hydrodynamic linear stability theory is relatively recent. Nold and Oberlack presented a symmetry classification of the linearized incompressible 2D Euler and Navier-Stokes equations for a linear shear flow [13]. Using these symmetries, they systematically derived three invariant ansatz function classes:

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normal modes, Kelvin modes, and a novel family of traveling modes. Those new modes exist in closed form in the inviscid case, conserve energy, move at constant velocity in the cross-stream direction, and are accelerated streamwise by the base flow. In the viscous case, they break down due to damping of high-wave-number contributions. The following paper by Nold *et al.* extended this symmetry classification from a linear shear flow to arbitrary plane shear flows [14]. For inviscid perturbations, additional symmetries arise for linear, logarithmic, algebraic, and exponential base flows. When viscous terms are retained, only linear and quotient-type base flows admit extra symmetries. Using all these additional symmetries, they systematically derived invariant ansatz function classes such as normal modes, Kelvin modes, and several reduced partial differential equations (PDEs) or ordinary differential equations (ODEs) for new modes, whereof some are analytically solvable.

Hau *et al.* applied Lie symmetry analysis to the linearized compressible 2D Euler equations for a homentropic linear shear flow [15]. By constructing optimal systems of subalgebras, i.e., minimal sets of subalgebras such that every other subalgebra is equivalent to one in this set under the group's adjoint action [16,17], they obtained three distinct group-invariant solution families: normal modes, Kelvin modes, and a novel general ansatz function, which unifies the first two and interpolates between them. A key structural result is that the Lie algebra admitted by the compressible equations is a subalgebra of that admitted by the incompressible system, which possesses an additional scaling symmetry. Hence, all invariant solutions derived for the compressible case can be naturally extended to the incompressible case.

Squire's theorem only applies to temporal normal modes. Oberlack *et al.* extended the idea of comparing modes between 2D and 3D to spatiotemporal normal modes. Interestingly, the opposite of Squire's theorem may hold in this case: There may be 3D perturbations with a lower Reynolds number than their corresponding 2D perturbations [18]. However, the investigation of individual of these spatiotemporal modes is meaningless, since the spatial part of their eigenfunction diverges, even initially. Nevertheless, they can still be used inside a framework like Briggs's method in a superposition of modes. Furthermore, Oberlack *et al.* demonstrated that the statement about damped Squire modes only holds true for temporal Squire modes and that for spatiotemporal Squire modes, no such statement can be readily made.

In Sec. II, the governing equations of linear stability are introduced in space and time, and their spectral counterpart. Section III gives an introduction to Lie symmetry theory. Besides symmetries and equivalence transformations, the new definition of a *spectral equivalence transformation* is introduced, which is necessitated by the different nature of a spectral eigenvalue problem formulation in contrast to the physical initial-boundary value problem in space and time. In Sec. IV, the symmetry generators of the governing equations are computed using the GeM package for various settings: (i) space-and-time and spectral (i.e., symmetries and equivalence transformations, and spectral equivalence transformations); (ii) fully coupled and decoupled; and (iii) two-component and one-component shear flow. Furthermore, the well-known Squire transformation is derived by one such generator. In Sec. V, the generators of the spectral equivalence transformations are then applied to an adjoint-projected residual functional. Finally, Sec. VI provides some conclusions.

II. THE LINEARIZED NAVIER-STOKES EQUATIONS FOR A TWO-COMPONENT SHEAR FLOW

The basis of the following analysis are the nondimensional Navier-Stokes equations for an incompressible Newtonian fluid,

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}, \quad \text{with } \mathrm{Re} := \frac{U_0 L}{\nu}, \tag{1b}$$

where $\boldsymbol{u}$, p, and Re denote velocity, pressure, and the Reynolds number, respectively. The Reynolds number comprises a characteristic velocity U_0, a characteristic length scale L and the kinematic viscosity ν.

Let a Cartesian coordinate system with coordinates $\boldsymbol{x} = (x, y, z)$ for the streamwise, wall-normal, and spanwise directions be introduced with the corresponding velocities $\boldsymbol{u} = (u, v, w)$. For the further analysis of linear stability theory, we split the velocity into a laminar base flow velocity $\boldsymbol{U} = (U, V, W)$ and superimposed velocity perturbations $\boldsymbol{u}' = (u', v', w')$ as

$$\boldsymbol{u} = \boldsymbol{U} + \epsilon\boldsymbol{u}' \quad \text{with} \quad \epsilon \ll 1, \tag{2}$$

and, analogously, the pressure, $p = P + \epsilon p'$. Further, we assume to have a two-component shear flow as the laminar base flow,

$$\boldsymbol{U} = (U(y), 0, W(y)). \tag{3}$$

With the above assumptions, using Eq. (1a) and eliminating the pressure term, Eqs. (1b) are linearized into two equations for the perturbation quantities v' and η',

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + W\frac{\partial}{\partial z}\right)\nabla^2 - U''\frac{\partial}{\partial x} - W''\frac{\partial}{\partial z} - \frac{1}{\mathrm{Re}}\nabla^4\right]v' = 0, \tag{4a}$$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + W\frac{\partial}{\partial z} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta' = -U'\frac{\partial v'}{\partial z} + W'\frac{\partial v'}{\partial x}, \tag{4b}$$

where v' is the wall-normal velocity perturbation and η' is the wall-normal vorticity perturbation, defined as

$$\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}. \tag{5}$$

We reformulate the perturbation quantities v' and η' via the normal mode approach, consisting of amplitude functions $\tilde{v}$, $\tilde{\eta}$, dependent on the inhomogeneous y direction, and an exponential with wave frequency ω in time t and respective wave numbers α and β in the homogeneous x and z directions,

$$v'(x, y, z, t) = \tilde{v}(y)e^{i(\alpha x+\beta z-\omega t)}, \tag{6a}$$

$$\eta'(x, y, z, t) = \tilde{\eta}(y)e^{i(\alpha x+\beta z-\omega t)}. \tag{6b}$$

This is closely related to a Fourier transform in space and a Laplace transform in time: The normal mode approach corresponds to considering a single Fourier-Laplace mode, whereas

the Fourier-Laplace transform represents a superposition of such Fourier-Laplace modes.

With Eq. (6), the spatiotemporal PDEs (4a) and (4b) transform into the ODEs for $\tilde{v}$ and $\tilde{\eta}$,

$$\left[(-i\omega + i\alpha U + i\beta W)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right) - i\alpha U'' - i\beta W'' - \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)^2\right]\tilde{v} = 0, \tag{7a}$$

$$\left[(-i\omega + i\alpha U + i\beta W) - \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\right]\tilde{\eta} = (-i\beta U' + i\alpha W')\tilde{v}, \tag{7b}$$

the first of which we call the modified Orr-Sommerfeld equation (mOSE) and the second the modified Squire equation (mSE). The well-known, conventional forms of the two aforementioned equations are obtained when $W = 0$ [19,20]. In general, we assume $\omega, \alpha, \beta \in \mathbb{C}$.

III. SYMMETRY ANALYSIS

Symmetries describe the invariance properties of algebraic or differential equations, the latter of which usually represent the fundamentals of the underlying physics. For classical mechanics, on which the Navier-Stokes equations (1) are based, this is the Galilean group (as described in detail, e.g., by Lévy-Leblond [21]), which forms a 10-dimensional Lie algebra. Specifically, these are three translations in space and one in time, the Galilean transformation with three velocity components and three spatial rotations. These symmetries are also preserved for the linearized Navier-Stokes equations, and additional symmetries arise, as will be seen.

It is noteworthy that, in classical mechanics, the continuous symmetries of the Galilean group are related to conservation laws through Noether's theorem [22]. This correspondence, however, relies on the existence of a variational principle [23], which is not available for the viscous Navier-Stokes equations (1) considered in the present work. However, in classical mechanics, the translations in space correspond to conservation of linear momentum, the translation in time to the conservation of energy, the Galilean transformation to the conservation of the position of the center of mass, and the spatial rotations to the conservation of angular momentum [24].

We may first establish some fundamentals for the symmetry analysis and refer the reader to Bluman *et al.* [23] for a more profound, in-depth presentation of the topic.

A. Fundamentals of symmetries

The basic idea behind symmetries of an equation is that they transform the equation onto itself. Mathematically, the global transformations

$$\boldsymbol{x}^* = \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y}; a), \quad \boldsymbol{y}^* = \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}; a), \quad \text{with} \quad a \in \mathbb{R} \tag{8}$$

refer to a point symmetry (i.e., acting on the independent and dependent variables, with the transformation of derivatives not being independent), if a differential equation is transformed forminvariantly,

$$\mathbf{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \ldots, \boldsymbol{y}^{(p)}) = 0 \quad \Leftrightarrow \quad \mathbf{F}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{y}^{*(1)}, \boldsymbol{y}^{*(2)}, \ldots, \boldsymbol{y}^{*(p)}) = 0. \tag{9}$$

Here $\boldsymbol{y}^{(p)}$ denotes the set of all pth derivatives of $\boldsymbol{y}$, and $\boldsymbol{x}$ and $\boldsymbol{y}$ refer to the independent and dependent variables. If a solution $\boldsymbol{y}(\boldsymbol{x})$ to the differential equation $\mathbf{F} = 0$ is known, then another solution family parametrized by $\boldsymbol{y}^*(\boldsymbol{x}^*)$ can be generated immediately using the transformation rules (8).

In Eq. (8), a denotes the group parameter and indicates that we are focusing on continuous transformations, i.e., so-called Lie groups, which possess group properties (e.g., see Ref. [25] for a comprehensible introduction to the topic). These group properties are particularly important here because they allow the transformations (8) to be rewritten in the so-called infinitesimal form, which can be considered a local version of the transformation. Without loss of generality, we can assume that for $a = 0$, the identity transformation occurs as

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y}; 0), \quad \boldsymbol{y} = \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}; 0). \tag{10}$$

It is particularly important for the further analysis that, under the conditions for Lie groups, the transformations (8) can be written in infinitesimal form using Eqs. (10) as follows:

$$\boldsymbol{x}^* = \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y}; 0) + \left.\frac{\partial \boldsymbol{\phi}}{\partial a}\right|_{a=0} a + \mathcal{O}(a^2) = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, \boldsymbol{y})a + \mathcal{O}(a^2), \tag{11a}$$

$$\boldsymbol{y}^* = \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}; 0) + \left.\frac{\partial \boldsymbol{\psi}}{\partial a}\right|_{a=0} a + \mathcal{O}(a^2) = \boldsymbol{y} + \boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{y})a + \mathcal{O}(a^2). \tag{11b}$$

Lie's remarkable first theorem then states that, based solely on knowledge of the infinitesimals,

$$\boldsymbol{\xi}(\boldsymbol{x}, \boldsymbol{y}) = \left.\frac{\partial \boldsymbol{\phi}}{\partial a}\right|_{a=0}, \quad \boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{y}) = \left.\frac{\partial \boldsymbol{\psi}}{\partial a}\right|_{a=0}, \tag{12}$$

the transformations (8) can be reconstructed. In other words, *the infinitesimal transformations (11) and the global transformations (8) are mathematically equivalent*, see Ref. [23] for a proof of equivalence. The infinitesimal form is used subsequently for deriving the Squire transformation in Sec. IV C 2 and analyzing the adjoint-projected residual functional in Sec. V.

The infinitesimal form can be used to derive an alternative, infinitesimal invariance condition. The infinitesimal transformations (11) are substituted into the second equation of the global invariance condition (9) and expanded in a Taylor series with respect to the group parameter a to obtain

$$\begin{aligned}&\mathbf{F}(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{y}^{*(1)}, \ldots, \boldsymbol{y}^{*(p)})\\ &\quad = \mathbf{F}(\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, \boldsymbol{y})a + \mathcal{O}(a^2),\ \boldsymbol{y} + \boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{y})a + \mathcal{O}(a^2),\\ &\boldsymbol{y}^{(1)} + \boldsymbol{\eta}^{(1)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}^{(1)})a + \mathcal{O}(a^2),\ \ldots,\\ &\boldsymbol{y}^{(p)} + \boldsymbol{\eta}^{(p)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}^{(1)},\ \ldots,\ \boldsymbol{y}^{(p)})a + \mathcal{O}(a^2))\\ &\quad = \mathbf{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(p)}) + a\mathsf{X}^{(p)}\mathbf{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(p)})\\ &\qquad + \mathcal{O}(a^2) = 0,\end{aligned} \tag{13}$$

where $\mathsf{X}^{(p)}$ denotes the pth prolongation of the generator X, with

$$\mathsf{X} = \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} + \boldsymbol{\eta} \cdot \nabla_{\boldsymbol{y}} = \xi^i \frac{\partial}{\partial x_i} + \eta^j \frac{\partial}{\partial y_j}, \tag{14}$$

and the coefficients of the prolonged generator $\boldsymbol{\eta}^{(p)}$, obtained by the total derivative operator. Because of the derivatives of $\boldsymbol{y}$, the transformations (8) must act consistently on the derivatives of $\boldsymbol{y}$. Since the derivatives are constrained through $\boldsymbol{y}$, the prolongation of the generator ensures that the chain rule is respected. We do not make use of the prolongation directly but note that the computation of symmetries with the GeM package in Sec. IV is based on this.

Using the infinitesimal expansion (13) together with the global invariance condition (9), the infinitesimal invariance condition for differential equations is obtained as

$$\mathsf{X}^{(p)}\mathbf{F}|_{\mathbf{F}=0} = 0. \tag{15}$$

A special case of this invariance condition arises for algebraic equations $\boldsymbol{F} = 0$, such as the adjoint-projected residual functional considered in Sec. V. In that case, there is no distinction between independent and dependent variables, and therefore the prolongation can be substituted by the generator. The infinitesimal invariance condition then reduces to

$$\mathsf{X}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})|_{\boldsymbol{F}=0} = 0. \tag{16}$$

B. Equivalence transformations

Equivalence transformations are generalizations of symmetries in the sense that they do not map an equation to an identical equation but to a similar, i.e., equivalent, equation. Equivalence transformations act on classes of equations that share a general functional form but may differ in their constitutive quantities, i.e., constitutive functions and constitutive parameters.

Equivalence transformations are relevant when the differential equation under consideration contains functions or parameters that characterize material properties, constitutive relations, or external influences.

An equivalence transformation is defined by the global transformation

$$\boldsymbol{x}^* = \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y};a), \quad \boldsymbol{y}^* = \boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y};a), \quad \boldsymbol{K}^* = \boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{K};a), \tag{17}$$

where $\boldsymbol{K}$ denotes constitutive quantities. The transformations of the independent and dependent variables $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are assumed not to depend on $\boldsymbol{K}$, while the transformation of the constitutive quantities $\boldsymbol{\chi}$ may do so.

If $\boldsymbol{K}^* = \boldsymbol{K}$, then the equivalence transformation becomes a symmetry, and therefore every symmetry is considered an equivalence transformation.

In analogy to symmetries, the global invariance condition (9) is extended by $\boldsymbol{K}$ and $\boldsymbol{K}^*$ in the dependencies, respectively. The infinitesimal of the constitutive quantities $\boldsymbol{\kappa}$ is then defined analogously to Eqs. (12) and the generator of the equivalence transformation contains the infinitesimals of Eq. (14) with the additional term $\kappa^k \ \partial/\partial K_k$.

C. Spectral equivalence transformations

The underlying basis of the normal mode approach (6), i.e., the Fourier-Laplace transform, constitutes a nonlocal integral transformation in space and time. As a consequence, the initial-boundary value problem in space and time (4) and the resulting spectral eigenvalue problem (7) are in general not equivalent representations of the same system. In particular, point symmetries of the governing equations in space and time do not carry over to point symmetries of the spectral problem. Instead, the corresponding symmetries may turn out to be nonlocal or distributional. A nonlocal transformation can create new symmetries, but it can also cause symmetries to disappear.

Consequently, we investigate the symmetry properties of both the system of equations in space and time (4) and the spectral system of equations (7) independently of each other.

In the spectral space, the definitions of symmetries and equivalence transformations are not fully adequate for our purposes. By definition of equivalence transformations, the transformations of the independent and dependent variables do not depend on the constitutive quantities. However, we find this definition too narrow for our purposes. In spectral space, the spectral parameters parametrize the eigenvalue problem and thus do not behave like true constitutive parameters, since they are the Fourier-Laplace counterparts of the independent variables.

Shevyakov introduced the notion of a generalized equivalence transformation, where the transformations of the independent and dependent variables can depend on the constitutive quantities [26]. However, allowing all dependencies would be too broad for our purposes.

Consequently, we define a *spectral equivalence transformation* as a generalized equivalence transformation that maps one linear spectral eigenvalue problem onto another within the same class of eigenvalue problems. To preserve the defining structure of such problems, we impose the following admissibility conditions:

(1) The independent variables cannot be transformed. In a spectral eigenvalue problem, they define the domain on which the differential operator acts, determine the location of the boundaries, and serve as the arguments of the constitutive functions. Allowing transformations of the independent variables would generally change the underlying operator and map the problem outside the considered eigenvalue problem class.

(2) The dependent variables, i.e., the eigenfunctions, cannot depend on the constitutive functions. Since the constitutive functions define the differential operator, such a dependence would make the transformation itself operator-dependent. Consequently, different operators would require different transformations, preventing a uniform mapping between the solution spaces of the eigenvalue problem class.

(3) The spectral parameters, i.e., the eigenvalues, and the constitutive parameters are global quantities and may therefore only be transformed among themselves. They cannot depend on local quantities, namely the independent variables, the dependent variables, or the constitutive functions. Otherwise, quantities such as the eigenvalues would become local (e.g., spatially dependent) and would no longer represent the global spectral parameters of an eigenvalue problem. The same applies to constitutive parameters such as the Reynolds number, which must also remain global.

(4) The constitutive functions may depend only on the independent variables and the constitutive quantities. They

cannot depend on the spectral parameters or on the dependent variables, since this would make the differential operator depend on the chosen spectral representation or on the particular solution. Consequently, different spectral modes would, in general, be governed by different operators, violating the concept of a single eigenvalue problem class. This restriction does not apply to constitutive parameters, as they already parametrize the constitutive functions by definition.

These conditions ensure that every admissible transformation maps one member of the considered eigenvalue problem class onto another member of the same class. They preserve the functional decomposition of the eigenvalue problem among the operator, the eigenfunctions, and the global eigenvalues.

IV. SYMMETRIES AND (SPECTRAL) EQUIVALENCE TRANSFORMATIONS OF THE LINEARIZED NAVIER-STOKES EQUATIONS

We use the free, publicly available GeM [27] software package for `Maple` developed by Alexey Shevyakov [28] to compute the symmetries, equivalence transformations, and spectral equivalence transformations of the governing equations. We use the most recent, unreleased version `Gem45_20.mpl`. All `Maple` sheets are available; see the data availability statement.

It is to note that linear PDE systems admit an infinite set of point symmetries because the infinitesimal of the dependent variables $\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{y})$ satisfies the associated linear PDE system trivially, leading to an infinite-dimensional symmetry algebra due to the superposition of the dependent variables [29]. However, one is typically interested only in nontrivial, finite-dimensional symmetries. The Ovsiannikov conjecture states that if the infinitesimal $\boldsymbol{\xi}$ is independent of the dependent variables $\boldsymbol{y}$ and the infinitesimal $\boldsymbol{\eta}$ depends only linearly on $\boldsymbol{y}$, then the Lie algebra of the point symmetries can be decomposed into a finite-dimensional part and an infinite-dimensional Abelian ideal, the latter arising from superposition symmetry [17,29]. In the case this conjecture holds true, the GeM package can exclude the infinite-dimensional component and compute only the nontrivial, finite-dimensional symmetries.

The theorem stating that the Ovsiannikov conjecture is satisfied for scalar linear PDEs of order greater than two [29,30] is not immediately applicable here, as the PDE system (4) comprises a scalar PDE of order four and a scalar PDE of order two, coupled to the former. However, for all cases computed, the Ovsiannikov conjecture holds true.

For linear ODE systems, although trivial symmetries of the same type are present, the associated Lie group of point transformations is always finite-dimensional [29]. Therefore, the Ovsiannikov conjecture does not have to be checked for the ODE system (7).

For the computation of equivalence transformations with the GeM package, the constitutive quantities have to be treated as new dependent variables. As such, at first, they depend on all independent variables, and the actual dependencies must be processed by setting the derivatives of the constitutive quantities to zero, if they do not actually depend on the independent variables. Furthermore, according to the definition of equivalence transformations, the condition has to be set that the transformations for the independent and dependent variables are independent of the constitutive quantities [23,26,29].

For spectral equivalence transformations, the appropriate dependencies must be set according to our definition in Sec. III C.

The entire set of variables and parameters considered thus includes x, y, z, t, v', η', U, W, and Re for Eqs. (4) and α, y, β, ω, $\tilde{v}$, $\tilde{\eta}$, U, W, and Re for the spectral Eqs. (7).

We characterize x, y, z, and t as independent variables and v', η', $\tilde{v}$, and $\tilde{\eta}$ as dependent variables. U and W are constitutive functions dependent on the independent variable y, although a case could be made that these are dependent variables of the laminar base system. However, when focusing only on the perturbation system, we treat them as constitutive functions. Re is a constitutive parameter. The quantities α, β, and ω correspond to the independent variables after the Fourier-Laplace transform. We characterize them as spectral parameters.

When displaying a given symmetry in its global form, it is customary to specify the transformation for each individual variable, even if it is an identity transformation. To keep the notation as compact as possible, we will not give the identity transformations such as $x^* = x$, $y^* = y$, …, but only those parts of the transformation that contain the group parameter. Further, a new group parameter a_i is introduced for each subsequent symmetry, each containing a subscript identifying the transformation.

We investigate the equation system (4) in space and time and the equation system (7) in spectral space. The generators in space and time as well as in spectral space will receive an independent numbering from each other, since they do not correspond one-to-one with one another, as we noted above in Sec. III C. Furthermore, we will also consider the decoupled equations, i.e., Eq. (4a) and the mOSE (7a), as well as the homogeneous Eq. (4b) with the right-hand side set to zero, and the homogeneous modified Squire equation (hmSE), i.e., Eq. (7b), with the right-hand side set to zero. Moreover, we will investigate all those cases also for $W = 0$, i.e., in spectral space the OSE and the hSE.

A. The coupled system of Eqs. (4) and (7)

1. *Symmetries of Eqs. (4)*

According to the definition of symmetry transformations [23], the following dependencies are assumed:

$$\{\xi^x, \xi^y, \xi^z, \xi^t, \eta^{v'}, \eta^{\eta'}\} = \{\xi^x, \xi^y, \xi^z, \xi^t, \eta^{v'}, \eta^{\eta'}\}(x, y, z, t, v', \eta'), \quad (18a)$$

$$\kappa^U = 0, \quad \kappa^W = 0, \quad \kappa^{\mathrm{Re}} = 0. \quad (18b)$$

Using these dependencies, four generators are found. Table I lists the generators found, the associated symmetries, and the global transformations. These symmetries have already been investigated in the context of linear stability theory (e.g., see Refs. [13–15]).

Here X_1 is the translation in the x direction, X_2 in the z direction, and X_3 in time.

X_4 is the scaling of dependent variables. One of the fundamental properties of linear differential equations is the scalability of the dependent variables. If we consider

TABLE I. Symmetries, generators, and associated global transformations.

Symmetry	Generator	Global transformation
Translation in x	$\mathsf{X}_1 = \frac{\partial}{\partial x}$	$x^* = x + a_x$
Translation in z	$\mathsf{X}_2 = \frac{\partial}{\partial z}$	$z^* = z + a_z$
Translation in t	$\mathsf{X}_3 = \frac{\partial}{\partial t}$	$t^* = t + a_t$
Scaling of dependent variables	$\mathsf{X}_4 = v' \frac{\partial}{\partial v'} + \eta' \frac{\partial}{\partial \eta'}$	$v'^* = e^{a_S} v'$, $\eta'^* = e^{a_S} \eta'$

system (4) decoupled, then even two separate scalings are admitted. Therefore v' in Eq. (4a) has its own scaling parameter $a_{Sv'}$, and η' in Eq. (4b) with the right-hand side set to zero has $a_{S\eta'}$. If the equations are considered as a coupled system, then the two-parameter symmetry becomes a one-parameter symmetry with $a_{Sv'} = a_{S\eta'} = a_S$.

We note that from these symmetries X_1, X_2, X_3 and X_4, the normal mode approach (6) can be derived by symmetry reduction [13–15].

2. *Equivalence transformations of Eqs. (4)*

According to the definition of equivalence transformations [23], we set the dependencies of the transformations, i.e.,

$$\{\xi^x, \xi^y, \xi^z, \xi^t, \eta^{v'}, \eta^{\eta'}\} = \{\xi^x, \xi^y, \xi^z, \xi^t, \eta^{v'}, \eta^{\eta'}\}(x, y, z, t, v', \eta'), \tag{19a}$$

$$\{\kappa^U, \kappa^W, \kappa^{\mathrm{Re}}\} = \{\kappa^U, \kappa^W, \kappa^{\mathrm{Re}}\}(x, y, z, t, v', \eta', U, W, \mathrm{Re}). \tag{19b}$$

Using these dependencies, 10 generators are found, of which the generators X_1, X_2, X_3, and X_4 have already been discussed in Sec. IV A 1 as symmetries.

Table II lists the additional six new generators, their associated equivalence transformations, and the global transformations. These equivalence transformations have not been investigated in linear stability theory but are well known in physics. They appear (sometimes as symmetries) in different contexts, such as the Galilean group of classical mechanics (e.g., see Ref. [21]) or the Navier-Stokes group for the full nonlinear Navier-Stokes equations (e.g., see Refs. [6–8]).

Here X_5 is the translation in the y direction. It does not appear as a symmetry like the other translations X_1, X_2, and X_3, since the y coordinate is an argument of the laminar base flow $U(y)$, $W(y)$, and thus transforming y transforms the constitutive functions.

X_6 and X_7 are the Galilean transformations in the x and z direction. They are equivalence transformations, since they involve a transformation of U and W, i.e., of constitutive functions. This also applies to the subsequent equivalence transformations. A Galilean transformation in the y direction is not allowed, since it violates the imposed dependencies $U(y)$ and $W(y)$.

X_8 is the rotation in the (x, z) plane. Rotations in the (x, y) and (y, z) plane are not allowed, since they violate the imposed dependencies $U(y)$ and $W(y)$.

X_9 is the scaling in space, and X_{10} is the scaling in time. The full nonlinear Navier-Stokes equations already admit these two scalings, so they are preserved during linearization. Requiring the Reynolds number to remain invariant imposes the constraint $2a_{Sx} = a_{St}$, reducing the two-parameter scaling group to a one-parameter one, with $a_{Sx} = a_{Sns}$, and $a_{St} = 2a_{Sns}$. For the Euler equations, i.e., $1/\mathrm{Re} = 0$ in Eq. (1), the constraint is lifted, and space and time can be scaled independently of each other.

3. *Spectral equivalence transformations of Eqs. (7)*

For spectral equivalence transformations according to our definition in Sec. III C, we set the dependencies of the transformations

$$\{\kappa^\alpha, \kappa^\beta, \kappa^\omega\} = \{\kappa^\alpha, \kappa^\beta, \kappa^\omega\}(\alpha, \beta, \omega, \mathrm{Re}), \tag{20a}$$

$$\xi^y = 0, \tag{20b}$$

$$\{\eta^{\tilde{v}}, \eta^{\tilde{\eta}}\} = \{\eta^{\tilde{v}}, \eta^{\tilde{\eta}}\}(\alpha, y, \beta, \omega, \tilde{v}, \tilde{\eta}, \mathrm{Re}), \tag{20c}$$

$$\{\kappa^U, \kappa^W\} = \{\kappa^U, \kappa^W\}(y, U, W, \mathrm{Re}), \tag{20d}$$

$$\kappa^{\mathrm{Re}} = \kappa^{\mathrm{Re}}(\alpha, \beta, \omega, \mathrm{Re}). \tag{20e}$$

TABLE II. Additional equivalence transformations, generators, and associated global transformations.

Equivalence transformation	Generator	Global transformation
Translation in y	$\mathsf{X}_5 = \frac{\partial}{\partial y}$	$y^* = y + a_y$
Galilean transformation in x	$\mathsf{X}_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial U}$	$x^* = x + a_{Gx}t$, $U^* = U + a_{Gx}$
Galilean transformation in z	$\mathsf{X}_7 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial W}$	$z^* = z + a_{Gz}t$, $W^* = W + a_{Gz}$
Rotation in (x, z)	$\mathsf{X}_8 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} - W\frac{\partial}{\partial U} + U\frac{\partial}{\partial W}$	$x^* = x\cos(a_r) - z\sin(a_r)$, $z^* = x\sin(a_r) + z\cos(a_r)$, $U^* = U\cos(a_r) - W\sin(a_r)$, $W^* = U\sin(a_r) + W\cos(a_r)$
Scaling in space	$\mathsf{X}_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - \eta'\frac{\partial}{\partial \eta'} + U\frac{\partial}{\partial U} + W\frac{\partial}{\partial W} - 2\mathrm{Re}\frac{\partial}{\partial \mathrm{Re}}$	$x^* = e^{a_{Sx}}x$, $y^* = e^{a_{Sx}}y$, $z^* = e^{a_{Sx}}z$, $\eta'^* = e^{-a_{Sx}}\eta'$, $U^* = e^{a_{Sx}}U$, $W^* = e^{a_{Sx}}W$, $\mathrm{Re}^* = e^{-2a_{Sx}}\mathrm{Re}$
Scaling in time	$\mathsf{X}_{10} = t\frac{\partial}{\partial t} - U\frac{\partial}{\partial U} - W\frac{\partial}{\partial W} + \mathrm{Re}\frac{\partial}{\partial \mathrm{Re}}$	$t^* = e^{a_{St}}t$, $U^* = e^{-a_{St}}U$, $W^* = e^{-a_{St}}W$, $\mathrm{Re}^* = e^{a_{St}}\mathrm{Re}$

TABLE III. Spectral equivalence transformations, generators and associated global transformations, with a particular choice of the free function.

Spectral equivalence transformation	Generator	Global transformation
Translation in x	$\tilde{\mathsf{X}}_1 = -i\alpha\left(\tilde{v}\frac{\partial}{\partial\tilde{v}} + \tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}\right)$	$\tilde{v}^* = e^{-i\alpha a_x}\tilde{v}, \quad \tilde{\eta}^* = e^{-i\alpha a_x}\tilde{\eta}$
Translation in z	$\tilde{\mathsf{X}}_1 = -i\beta\left(\tilde{v}\frac{\partial}{\partial\tilde{v}} + \tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}\right)$	$\tilde{v}^* = e^{-i\beta a_z}\tilde{v}, \quad \tilde{\eta}^* = e^{-i\beta a_z}\tilde{\eta}$
Translation in t	$\tilde{\mathsf{X}}_1 = i\omega\left(\tilde{v}\frac{\partial}{\partial\tilde{v}} + \tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}\right)$	$\tilde{v}^* = e^{i\omega a_t}\tilde{v}, \quad \tilde{\eta}^* = e^{i\omega a_t}\tilde{\eta}$
Scaling of dependent variables	$\tilde{\mathsf{X}}_1 = \tilde{v}\frac{\partial}{\partial\tilde{v}} + \tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}$	$\tilde{v}^* = e^{a_S}\tilde{v}, \quad \tilde{\eta}^* = e^{a_S}\tilde{\eta}$
Galilean transformation in x	$\tilde{\mathsf{X}}_2 = \alpha\frac{\partial}{\partial\omega} + \frac{\partial}{\partial U}$	$\omega^* = \omega + a_{Gx}\alpha, \quad U^* = U + a_{Gx}$
Galilean transformation in z	$\tilde{\mathsf{X}}_3 = \beta\frac{\partial}{\partial\omega} + \frac{\partial}{\partial W}$	$\omega^* = \omega + a_{Gz}\beta, \quad W^* = W + a_{Gz}$
Rotation in (x, z)	$\tilde{\mathsf{X}}_4 = -\beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} - W\frac{\partial}{\partial U} + U\frac{\partial}{\partial W}$	$\alpha^* = \alpha\cos(a_r) - \beta\sin(a_r), \quad \beta^* = \alpha\sin(a_r) + \beta\cos(a_r),$ $U^* = U\cos(a_r) - W\sin(a_r), \quad W^* = U\sin(a_r) + W\cos(a_r)$
Scaling in time	$\tilde{\mathsf{X}}_5 = -\omega\frac{\partial}{\partial\omega} - U\frac{\partial}{\partial U} - W\frac{\partial}{\partial W} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}$	$\omega^* = e^{-a_{St}}\omega, \quad U^* = e^{-a_{St}}U,$ $W^* = e^{-a_{St}}W, \quad \mathrm{Re}^* = e^{a_{St}}\mathrm{Re}$

Using these dependencies, five generators are found,

$$\tilde{\mathsf{X}}_1 = f_1(\alpha, \beta, \omega, \mathrm{Re})\left(\tilde{v}\frac{\partial}{\partial\tilde{v}} + \tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}\right),$$

$$\tilde{\mathsf{X}}_2 = f_2(\mathrm{Re})\left(\alpha\frac{\partial}{\partial\omega} + \frac{\partial}{\partial U}\right), \; \tilde{\mathsf{X}}_3 = f_3(\mathrm{Re})\left(\beta\frac{\partial}{\partial\omega} + \frac{\partial}{\partial W}\right),$$

$$\tilde{\mathsf{X}}_4 = f_4(\mathrm{Re})\left(-\beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} - W\frac{\partial}{\partial U} + U\frac{\partial}{\partial W}\right),$$

$$\tilde{\mathsf{X}}_5 = f_5(\mathrm{Re})\left(-\omega\frac{\partial}{\partial\omega} - U\frac{\partial}{\partial U} - W\frac{\partial}{\partial W} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\right). \quad (21)$$

Table III lists the spectral equivalence transformations, categorized according to their counterparts in space and time. The generators (with a particular choice of the free function) and their associated global transformations are shown. These spectral equivalence transformations have not been investigated in linear stability theory, but they are well known in physics. In Sec. V, their implications for linear stability theory will be thoroughly investigated.

It is not surprising that the free functions $f_1(\alpha, \beta, \omega, \mathrm{Re})$ and $f_i(\mathrm{Re})$ appear. This results from the vector space properties of the Lie algebra and the fact that the spectral and constitutive parameters α, β, ω, and Re do not appear with derivatives in the differential equations. Equivalence transformations always admit the scaling of the generator by an arbitrary function of the (constitutive) parameters [31]. Consequently, the generators can be scaled by free functions, provided that the dependencies allow it according to the definition of spectral equivalence transformations given in Sec. III C. Usually, the generators reduce to the spectral formulation of the transformation in space and time if the free functions are set to 1, except for the translation symmetries. The arbitrary functions f_i are assumed to be sufficiently smooth on the domain on which the corresponding generators are defined.

$\tilde{\mathsf{X}}_1$ represents a general scaling of the dependent variables. This is the only generator, where the free function incorporates the spectral parameters α, β, ω. From $\tilde{\mathsf{X}}_1$, multiple spectral equivalence transformations can be derived: For $f_1(\alpha, \beta, \omega, \mathrm{Re}) = -i\alpha$, $-i\beta$, or $i\omega$, this becomes the spectral counterpart of the translation in the x direction X_1, the z direction X_2, and in time X_3, respectively. The three translations lead to a scaling of the eigenfunctions in spectral space. For $f_1(\alpha, \beta, \omega, \mathrm{Re}) = 1$, this becomes the spectral counterpart of the scaling of dependent variables X_4.

The generators $\tilde{\mathsf{X}}_2$ and $\tilde{\mathsf{X}}_3$ for $f_2(\mathrm{Re}) = 1$ and $f_3(\mathrm{Re}) = 1$ represent the spectral counterparts of the Galilean transformation in the x and z directions X_6 and X_7, respectively. A different formulation of the Galilean transformation in spectral space is provided in Appendix A, highlighting the similarity between the translations and the Galilean transformation. In fact, they can be viewed as concretizations of the same generalization, i.e., the generalized Galilean transformation.

The generator $\tilde{\mathsf{X}}_4$ for $f_4(\mathrm{Re}) = 1$ represents the spectral counterparts of the rotation in the (x, z) plane X_8.

$\tilde{\mathsf{X}}_5$ becomes the spectral counterpart of the scaling in time X_{10} for $f_5(\mathrm{Re}) = 1$.

These are exactly the generalizations of the eight equivalence transformations found in space and time, with two equivalence transformations missing, namely the translation in the y direction X_5 and the scaling in space X_9. As these transformations transform the y coordinate, they do not satisfy our definition of spectral equivalence transformations from Sec. III C.

B. The decoupled system

Generally, the decoupled equations possess the same symmetries and (spectral) equivalence transformations as the coupled system; however, they have additional spectral equivalence transformations, which do not have an obvious physical interpretation. Further details can be found in Appendix B 1.

C. The coupled system of Eqs. (4) and (7) with $W = 0$

1. Symmetries and equivalence transformations of Eqs. (4) with $W = 0$

The symmetries found are exactly the same as in Table I. The equivalence transformations found are the same as in

Tables I and II, except that the Galilean transformation in the z direction X_7 and the rotation in the (x, z) plane X_8 are missing, which have disappeared as expected due to $W = 0$. Furthermore, X_9 and X_{10} are still present, but their W terms are missing.

2. *Spectral equivalence transformations of Eqs. (7) with W = 0*

Four generators are found,

$$\tilde{\mathsf{X}}_1, \tilde{\mathsf{X}}_2,$$

$$\tilde{\mathsf{X}}_{15} = f_{15}(\mathrm{Re})\left(\alpha\frac{\partial}{\partial\alpha} - \frac{\alpha^2}{\beta}\frac{\partial}{\partial\beta} - \frac{(\alpha^2+\beta^2)\tilde{\eta}}{\beta^2}\frac{\partial}{\partial\tilde{\eta}} - U\frac{\partial}{\partial U}\right),$$

$$\tilde{\mathsf{X}}_{16} = f_{16}(\alpha, \beta, \omega, \mathrm{Re})\left(-\alpha\frac{\partial}{\partial\alpha} + \frac{\alpha^2}{\beta}\frac{\partial}{\partial\beta} - \omega\frac{\partial}{\partial\omega} + \frac{(\alpha^2+\beta^2)\tilde{\eta}}{\beta^2}\frac{\partial}{\partial\tilde{\eta}} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\right), \quad \text{for } \beta \neq 0. \tag{22}$$

For the further derivation, we assume that $\beta \neq 0$ so that the generators (22) are well defined.

As in space and time, the Galilean transformation in the z direction $\tilde{\mathsf{X}}_3$ and the rotation in the (x, z) plane $\tilde{\mathsf{X}}_4$ are missing because of $W = 0$. The new generators $\tilde{\mathsf{X}}_{15}$, X_{16} represent mixed generalized scalings. They form a basis $\{\tilde{\mathsf{X}}_{15}, \tilde{\mathsf{X}}_{16}\}$ for expressing other transformations, as subsequently shown.

For example, for $f_{15}(\mathrm{Re}) = f_{16}(\alpha, \beta, \omega, \mathrm{Re}) = f_5(\mathrm{Re})$, the linear combination $\tilde{\mathsf{X}}_{15} + \tilde{\mathsf{X}}_{16}$ becomes the generalized scaling in time $\tilde{\mathsf{X}}_5$ in Eqs. (21) (with the W term missing, due to $W = 0$), as

$$\tilde{\mathsf{X}}_{15} + \tilde{\mathsf{X}}_{16} = \tilde{\mathsf{X}}_5 = f_5(\mathrm{Re})\left(-\omega\frac{\partial}{\partial\omega} - U\frac{\partial}{\partial U} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\right). \tag{23}$$

$\tilde{\mathsf{X}}_{16}$ on its own is the generator of the famous Squire transformation. We will derive it here for the coupled system, while it is usually derived for the decoupled equations, so the OSE on its own (or the hSE). In that case, the $\tilde{\eta}$ term vanishes, but this has no influence on the other transformations. With $f_{16}(\alpha, \beta, \omega, \mathrm{Re}) = 1$, this yields the generator

$$\tilde{\mathsf{X}}_{16} = -\alpha\frac{\partial}{\partial\alpha} + \frac{\alpha^2}{\beta}\frac{\partial}{\partial\beta} - \omega\frac{\partial}{\partial\omega} + \frac{(\alpha^2+\beta^2)\tilde{\eta}}{\beta^2}\frac{\partial}{\partial\tilde{\eta}} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}, \quad \text{for } \beta \neq 0, \tag{24}$$

which forms the basis for the Squire transformation. In the next step, each infinitesimal is integrated with respect to the group parameter a_{16}, subject to the identity conditions at $a_{16} = 0$. This leads to the following set of transformations:

$$\alpha(a_{16}) = \alpha^* e^{-a_{16}}, \tag{25a}$$

$$\beta(a_{16}) = \pm\sqrt{\alpha^{*2} + \beta^{*2} - \alpha^{*2}e^{-2a_{16}}}, \tag{25b}$$

$$\omega(a_{16}) = \omega^* e^{-a_{16}}, \tag{25c}$$

$$\tilde{\eta}(a_{16}) = \pm\tilde{\eta}^* e^{a_{16}}\frac{\sqrt{\alpha^{*2} + \beta^{*2} - \alpha^{*2}e^{-2a_{16}}}}{\beta^*}, \quad \text{for } \beta^* \neq 0, \tag{25d}$$

$$\mathrm{Re}(a_{16}) = \mathrm{Re}^* e^{a_{16}}. \tag{25e}$$

Here the quantities with an asterisk denote the initial point of the transformation orbit at $a_{16} = 0$, corresponding to the original 3D problem. The quantities without an asterisk describe the transformed state along the transformation orbit.

Since the infinitesimal generator is defined only for $\beta \neq 0$, the corresponding group action is defined on the open set of $\beta(a_{16}) \neq 0$. Nevertheless, the integrated transformations (25) extend continuously to the limiting end point $\beta = 0$, which is attained at the particular limiting value a_{Sq} of the group parameter a_{16}, corresponding to the classical Squire transformation. Since the 2D problem is naturally defined by $\beta = 0$, the parameter a_{Sq} is therefore determined from the condition $\beta(a_{\mathrm{Sq}}) = 0$, which yields

$$a_{\mathrm{Sq}} = -\frac{1}{2}\ln\left(\frac{\alpha^{*2} + \beta^{*2}}{\alpha^{*2}}\right), \quad \text{for } \alpha^*, \beta^*, \alpha^{*2} + \beta^{*2} \neq 0. \tag{26}$$

Note that the complex logarithm $\ln(\cdot)$ is multivalued, so with the factor of $-1/2$, the group parameter a_{Sq} is defined modulo πi. A constraint arises for a_{Sq} because the Reynolds number relation (25e) has to be real and positive, which requires $e^{a_{\mathrm{Sq}}} \in \mathbb{R}^+$. Therefore, we define

$$a_{\mathrm{Sq}} \equiv -\frac{1}{2}\mathrm{Ln}\left(\frac{\alpha^{*2} + \beta^{*2}}{\alpha^{*2}}\right) \mod 2\pi i, \quad \text{for } \alpha^*, \beta^*, \alpha^{*2} + \beta^{*2} \neq 0, \tag{27}$$

where $\mathrm{Ln}(\cdot)$ denotes the principal branch of the complex logarithm, where the argument of the logarithm has to be real and positive as well. We introduce the notational shorthand ϕ_{Sq} for the square root of that argument,

$$\phi_{\mathrm{Sq}} := \sqrt{\frac{\alpha^{*2} + \beta^{*2}}{\alpha^{*2}}}, \quad \text{for } \alpha^*, \beta^*, \alpha^{*2} + \beta^{*2} \neq 0, \tag{28}$$

where the square root $\sqrt{\cdot}$ is taken on the principal branch, consistent with the choice of the group parameter in the Squire transformation. The domain restrictions apply to all subsequent expressions involving ϕ_{Sq} and will therefore not be repeated explicitly. Thus, the transformations (25) become, for $a_{16} \to a_{\mathrm{Sq}}$,

$$\alpha = \phi_{\mathrm{Sq}}\alpha^*, \tag{29a}$$

$$\beta = 0, \tag{29b}$$

$$\omega = \phi_{\mathrm{Sq}}\omega^*, \tag{29c}$$

$$\tilde{\eta} = 0, \tag{29d}$$

$$\mathrm{Re} = \frac{1}{\phi_{\mathrm{Sq}}}\mathrm{Re}^*. \tag{29e}$$

This is the Squire transformation, or expressed in terms of the 3D quantities [12,18] as

$$\alpha^* = \frac{1}{\phi_{\mathrm{Sq}}}\alpha, \tag{30a}$$

$$\beta^* = \pm\sqrt{1 - \frac{1}{\phi_{\mathrm{Sq}}^2}}\,\alpha, \tag{30b}$$

$$\omega^* = \frac{1}{\phi_{\mathrm{Sq}}}\omega, \tag{30c}$$

$$\tilde{\eta}^* \text{ is solution of (31)}, \tag{30d}$$

$$\mathrm{Re}^* = \phi_{\mathrm{Sq}}\mathrm{Re}, \tag{30e}$$

TABLE IV. Spectral generator components.

Coupled system	$\tilde{\mathsf{X}}_1$	$\tilde{\mathsf{X}}_2$	$\tilde{\mathsf{X}}_3$	$\tilde{\mathsf{X}}_4$	$\tilde{\mathsf{X}}_5$
mOSE	$\tilde{\mathsf{X}}_1$	$\tilde{\mathsf{X}}_2$	$\tilde{\mathsf{X}}_3$	$\tilde{\mathsf{X}}_6, \tilde{\mathsf{X}}_7$	$\tilde{\mathsf{X}}_8, \tilde{\mathsf{X}}_9$
hmSE	$\tilde{\mathsf{X}}_1$	$\tilde{\mathsf{X}}_2$	$\tilde{\mathsf{X}}_3$	$\tilde{\mathsf{X}}_{10}, \tilde{\mathsf{X}}_{11}$	$\tilde{\mathsf{X}}_{12}, \tilde{\mathsf{X}}_{13}, \tilde{\mathsf{X}}_{14}$
Coupled system, $W = 0$	$\tilde{\mathsf{X}}_1$	$\tilde{\mathsf{X}}_2$			$\tilde{\mathsf{X}}_{15}, \tilde{\mathsf{X}}_{16}$
OSE	$\tilde{\mathsf{X}}_1$	$\tilde{\mathsf{X}}_2$			$\tilde{\mathsf{X}}_{15}, \tilde{\mathsf{X}}_{16}$
hSE	$\tilde{\mathsf{X}}_1$	$\tilde{\mathsf{X}}_2$			$\tilde{\mathsf{X}}_{15}, \tilde{\mathsf{X}}_{17}, \tilde{\mathsf{X}}_{18}$

with the 3D (forced) Squire equation (SE) expressed in 2D quantities and the parameter ϕ_{Sq} as

$$\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\right]\tilde{\eta}^* = \mp i\sqrt{\phi_{\mathrm{Sq}}^2 - 1}\,\alpha U'\tilde{v}. \tag{31}$$

As is evident, the Squire transformation is not the complete one-parameter equivalence transformation group generated by $\tilde{\mathsf{X}}_{16}$ but rather the continuous extension of a particular transformation orbit to the end point within the equivalence group, where the parameter a_{Sq} is chosen such that $\beta(a_{\mathrm{Sq}}) = 0$. The parameter ϕ_{Sq} is a function of the parameter a_{Sq} as $\phi_{\mathrm{Sq}} = e^{-a_{\mathrm{Sq}}}$.

This ϕ_{Sq} is a free parameter. There are three 2D quantities α, ω, Re involving ϕ_{Sq}, but five 3D quantities α^*, β^*, ω^*, $\tilde{\eta}^*$, and Re^* involving ϕ_{Sq} in the transformations (29) and (30). It is evident that the mapping from 3D to 2D is unique, while the mapping from 2D to 3D is not unique and parametrized by the free parameter ϕ_{Sq} and Eq. (31).

We did not find this Squire transformation in space and time with the GeM package, as the map along the orbit (25) involves the spectral parameters, which do not appear in the equations in space and time, as they act nonlocally.

D. The decoupled system with $W = 0$

Generally, the decoupled equations possess the same symmetries and (spectral) equivalence transformations as the coupled system; however, they have additional spectral equivalence transformations, which do not have an obvious physical interpretation. Further details can be found in Appendix B2.

E. Overview of generator components

Table IV summarizes the spectral generators and their components when they are split. The generator components become their associated generator when setting their free functions identical and adding them.

V. IMPLICATIONS OF SPECTRAL EQUIVALENCE TRANSFORMATIONS FOR THE DISPERSION RELATION

Here we apply the symmetry generators of the spectral equivalence transformations presented in Sec. IV to an adjoint-projected residual functional to form algebraic invariance conditions (16). We leave the asterisk notation aside in this section, as we are not interested in the transformations themselves.

The derivations are generally performed for the mOSE (7a) and the hmSE, because the coupled mSE does not lead to an eigenvalue problem. Furthermore, we also analyze the setting of $W = 0$, i.e., the OSE and the hSE.

For this procedure, we make use of the adjoint spectral equations. The governing spectral equations are formulated as the differential operator $\mathcal{L}$ acting on the spectral eigenfunction $\tilde{q} \in \{\tilde{v}, \tilde{\eta}\}$ as

$$\mathcal{L}\tilde{q} = 0, \tag{32}$$

where $\mathcal{L}_{\mathrm{mOSE}}\tilde{v} = 0$ is given in Eq. (7a) and $\mathcal{L}_{\mathrm{hmSE}}\tilde{\eta} = 0$ is given in Eq. (7b) with the right-hand side set to zero.

Subsequently, an overbar $\overline{(\ldots)}$ denotes complex conjugation, while the dagger symbol $(\ldots)^\dagger$ denotes the Hermitian adjoint of an operator, associated with the sesquilinear inner product

$$\langle \phi, \psi \rangle = \int_\Omega \overline{\phi}\psi \,\mathrm{d}y, \tag{33}$$

where ϕ and ψ are test functions and Ω is the y domain, $\Omega \subset \mathbb{R}$.

The Hermitian adjoint $\mathcal{L}^\dagger$ of the operator is obtained via repeated integration by parts to shift derivatives with respect to y from ψ onto ϕ, while homogeneous boundary conditions eliminate boundary terms, from

$$\int_\Omega \overline{\phi}(\mathcal{L}\psi)\,\mathrm{d}y = \int_\Omega \overline{(\mathcal{L}^\dagger\phi)}\psi \,\mathrm{d}y. \tag{34}$$

The solution to the adjoint eigenvalue problem associated with $\mathcal{L}^\dagger$ is called the adjoint eigenfunction $\tilde{q}^\ddagger$ and is obtained from

$$\mathcal{L}^\dagger\tilde{q}^\ddagger = 0. \tag{35}$$

Specifically, the adjoint equations $\mathcal{L}^\dagger_{\mathrm{mOSE}}\tilde{v}^\ddagger = 0$ and $\mathcal{L}^\dagger_{\mathrm{hmSE}}\tilde{\eta}^\ddagger = 0$ are therefore given by

$$\left[(i\overline{\omega} - i\overline{\alpha}U - i\overline{\beta}W)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\overline{\alpha}^2 + \overline{\beta}^2)\right) - 2i\overline{\alpha}U'\frac{\mathrm{d}}{\mathrm{d}y} - 2i\overline{\beta}W'\frac{\mathrm{d}}{\mathrm{d}y} - \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\overline{\alpha}^2 + \overline{\beta}^2)\right)^2\right]\tilde{v}^\ddagger = 0, \tag{36}$$

and

$$\left[(i\overline{\omega} - i\overline{\alpha}U - i\overline{\beta}W) - \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\overline{\alpha}^2 + \overline{\beta}^2)\right)\right]\tilde{\eta}^\ddagger = 0. \tag{37}$$

We introduce a projected residual functional $\mathcal{F}$ associated with the eigenvalue problem,

$$\mathcal{F} = \int_\Omega \overline{\Phi}(\mathcal{L}\tilde{q})\,\mathrm{d}y = 0. \tag{38}$$

When evaluated on the complete set of test functions Φ, it is equivalent to the original eigenvalue problem.

We write the variation of this functional $\delta\mathcal{F}$ as an eigenvalue-sensitivity relation [32–37],

$$\delta\mathcal{F} = \int_\Omega \delta\overline{\Phi}(\mathcal{L}\tilde{q})\,\mathrm{d}y + \int_\Omega \overline{\Phi}(\delta\mathcal{L}\tilde{q})\,\mathrm{d}y + \int_\Omega \overline{\Phi}(\mathcal{L}\delta\tilde{q})\,\mathrm{d}y. \tag{39}$$

Evaluated at one direct-adjoint eigenpair $\tilde{q}$ solving Eq. (32) and $\Phi = \tilde{q}^{\ddagger}$ solving Eq. (35), the first term cancels by construction, and the third term cancels by Eq. (35), using the definition of the adjoint operator (34) and repeated integration by parts with vanishing boundary terms. Thus, only the second term remains as

$$\delta\mathcal{F} = \int_\Omega \overline{\tilde{q}^{\ddagger}}(\delta\mathcal{L}\tilde{q})\,\mathrm{d}y. \tag{40}$$

This means that the variation of the adjoint-projected residual functional $\delta\mathcal{F}$ is given by the variation of the differential operator $\delta\mathcal{L}$. This variation of the differential operator with respect to a quantity appears either as a partial derivative when the quantity is a variable, e.g., α, β, ω, and Re, or as a functional derivative if the quantity itself is a function, e.g., $U(y)$, $W(y)$.

Subsequently, we always imply the evaluation of the functional $\mathcal{F}$ at a direct-adjoint eigenpair.

The free functions of the generators cancel in the formulation of algebraic invariance conditions (16), so they are omitted.

In this section, we only show the application of the generators of the homogeneous translations and scaling of the dependent variables $\tilde{\mathsf{X}}_1$, the Galilean transformation in the x direction $\tilde{\mathsf{X}}_2$, and the scaling in time $\tilde{\mathsf{X}}_5$. The application of the generators $\tilde{\mathsf{X}}_3$ and $\tilde{\mathsf{X}}_4$ can be found in Appendix C.

A. $\tilde{\mathsf{X}}_1$: Homogeneous translations in space and time and scaling of dependent variables

The three homogeneous translation generators in the x, z direction and in time and the scaling of dependent variables lead to the scaling of the eigenfunctions in spectral space $\tilde{\mathsf{X}}_1$. However, since the differential operator is independent of the eigenfunctions, the application of the generator to the adjoint-projected residual functional leads to a trivial result.

B. $\tilde{\mathsf{X}}_2$: Galilean transformations in the x direction

We apply the generator $\tilde{\mathsf{X}}_2$ of the Galilean transformation in the x direction to the adjoint-projected residual functional to obtain

$$\tilde{\mathsf{X}}_2\mathcal{F} = \alpha\frac{\partial\mathcal{F}}{\partial\omega} + \int_\Omega \frac{\delta\mathcal{F}}{\delta U(y)}\,\mathrm{d}y = 0. \tag{41}$$

The partial derivative and the variation of the differential operator in a Fréchet-derivative-manner are given as

$$\frac{\partial\mathcal{F}}{\partial\omega} = \int_\Omega \overline{\tilde{q}^{\ddagger}}\left(\frac{\partial\mathcal{L}}{\partial\omega}\tilde{q}\right)\mathrm{d}y, \tag{42a}$$

$$\delta\mathcal{L} = \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial U}\delta U(y) + \frac{\partial\mathcal{L}}{\partial U''}\delta U''(y)\right)\mathrm{d}y. \tag{42b}$$

Starting with the hmSE, where the variation of the differential operator is as follows:

$$\delta\mathcal{L}_{\mathrm{hmSE}} = i\alpha\int_\Omega \delta U(y)\,\mathrm{d}y, \tag{43}$$

the terms in Eq. (41) are given as

$$\alpha\frac{\partial\mathcal{F}_{\mathrm{hmSE}}}{\partial\omega} = -i\alpha\int_\Omega \overline{\tilde{\eta}^{\ddagger}}\tilde{\eta}\,\mathrm{d}y, \tag{44a}$$

$$\int_\Omega \frac{\delta\mathcal{F}_{\mathrm{hmSE}}}{\delta U(y)}\,\mathrm{d}y = i\alpha\int_\Omega \overline{\tilde{\eta}^{\ddagger}}\tilde{\eta}\,\mathrm{d}y. \tag{44b}$$

Evidently, the terms (44) cancel, so the algebraic invariance condition (41) becomes trivial.

Continuing with the mOSE, the variation of the differential operator is given as

$$\delta\mathcal{L}_{\mathrm{mOSE}} = i\alpha\int_\Omega \left[\delta U(y)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right) - \delta U''(y)\right]\mathrm{d}y. \tag{45}$$

For the $\delta U''(y)$ term in Eq. (45), integration by parts is performed twice, where the boundary terms vanish because $\tilde{v}$ and its adjoint $\tilde{v}^{\ddagger}$, as well as their derivatives $\mathrm{d}\tilde{v}/\mathrm{d}y$ and $\mathrm{d}\tilde{v}^{\ddagger}/\mathrm{d}y$, vanish at the boundaries. This yields

$$\delta\mathcal{L}_{\mathrm{mOSE}} = i\alpha\int_\Omega \left[\overline{\tilde{v}^{\ddagger}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^{\ddagger}}\tilde{v})\right]\delta U(y)\,\mathrm{d}y. \tag{46}$$

With that, the terms in Eq. (41) are given as

$$\alpha\frac{\partial\mathcal{F}_{\mathrm{mOSE}}}{\partial\omega} = -i\alpha\int_\Omega \overline{\tilde{v}^{\ddagger}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v}\,\mathrm{d}y, \tag{47a}$$

$$\int_\Omega \frac{\delta\mathcal{F}_{\mathrm{mOSE}}}{\delta U(y)}\,\mathrm{d}y = i\alpha\int_\Omega \left[\overline{\tilde{v}^{\ddagger}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^{\ddagger}}\tilde{v})\right]\mathrm{d}y. \tag{47b}$$

When the terms (47) are substituted into Eq. (41), only the second term of Eq. (47b) remains with a flipped sign as

$$\int_\Omega \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^{\ddagger}}\tilde{v})\,\mathrm{d}y = 0, \tag{48}$$

but this term vanishes as well, since after the integration, $\tilde{v}$, $\tilde{v}^{\ddagger}$, $\mathrm{d}\tilde{v}/\mathrm{d}y$, and $\mathrm{d}\tilde{v}^{\ddagger}/\mathrm{d}y$ vanish at the boundaries. Thus, the algebraic invariance condition (41) becomes trivial, too.

C. $\tilde{\mathsf{X}}_5$: Scaling in time

We apply the generator $\tilde{\mathsf{X}}_5$ of the scaling in time to the adjoint-projected residual functional to obtain

$$\tilde{\mathsf{X}}_5\mathcal{F} = -\omega\frac{\partial\mathcal{F}}{\partial\omega} - \int_\Omega U\frac{\delta\mathcal{F}}{\delta U(y)}\,\mathrm{d}y - \int_\Omega W\frac{\delta\mathcal{F}}{\delta W(y)}\,\mathrm{d}y + \mathrm{Re}\frac{\partial\mathcal{F}}{\partial\mathrm{Re}} = 0, \tag{49}$$

where the new, required derivative is obtained as

$$\frac{\partial\mathcal{F}}{\partial\mathrm{Re}} = \int_\Omega \overline{\tilde{q}^{\ddagger}}\left(\frac{\partial\mathcal{L}}{\partial\mathrm{Re}}\tilde{q}\right)\mathrm{d}y. \tag{50}$$

Starting with the hmSE, the terms in Eq. (49) are given as

$$-\omega\frac{\partial\mathcal{F}_{\mathrm{hmSE}}}{\partial\omega} = i\omega\int_\Omega \overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y, \tag{51a}$$

$$-\int_\Omega U\frac{\delta\mathcal{F}_{\mathrm{hmSE}}}{\delta U(y)}\,\mathrm{d}y = -i\alpha\int_\Omega U\overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y, \tag{51b}$$

$$-\int_\Omega W\frac{\delta\mathcal{F}_{\mathrm{hmSE}}}{\delta W(y)}\,\mathrm{d}y = -i\beta\int_\Omega W\overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y, \tag{51c}$$

$$\mathrm{Re}\frac{\partial\mathcal{F}_{\mathrm{hmSE}}}{\partial\mathrm{Re}} = \frac{1}{\mathrm{Re}}\int_\Omega \overline{\tilde\eta^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-(\alpha^2+\beta^2)\right)\tilde\eta\,\mathrm{d}y. \tag{51d}$$

The algebraic invariance condition (49) thus becomes

$$i\omega\int_\Omega \overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y - i\alpha\int_\Omega U\overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y - i\beta\int_\Omega W\overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y + \frac{1}{\mathrm{Re}}\int_\Omega \overline{\tilde\eta^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-(\alpha^2+\beta^2)\right)\tilde\eta\,\mathrm{d}y = 0. \tag{52}$$

For the mOSE, an algebraic invariance condition (C9) can be analogously derived, as can be seen in Appendix C 3.

These two algebraic invariance conditions (52) and (C9) reduce to the adjoint-projected residual functional (38),

$$\tilde{\mathsf{X}}_5\mathcal{F} = -\int_\Omega \overline{\tilde q^\ddagger}(\mathcal{L}\tilde q)\,\mathrm{d}y = -\mathcal{F} = 0. \tag{53}$$

This happens because in the differential operators of the mOSE and the hmSE, the time dimension is linear in the denominator for every term, i.e., ω, U, U'', and $1/\mathrm{Re}$. The generator of the scaling in time $\tilde{\mathsf{X}}_5$ from Eqs. (21) differentiates with respect to these quantities and multiplies them again, yielding the exact differential operators again. The invariance of the adjoint-projected residual functional under this transformation therefore links the temporal dynamics of the mode with the advective and diffusive mechanisms, like the original governing differential equations do.

The generator associated with the scaling in time produces algebraic invariance conditions equal to the Fredholm solvability condition of the eigenvalue problem [38], namely that the adjoint projection of the differential operator acting on the eigenfunction must vanish for a nontrivial solution to exist. In other words, *the Fredholm solvability condition of the eigenvalue problem emerges naturally from the invariance of the adjoint-projected residual functional under the scaling in time*.

Squire modes: The second part of Squire's theorem

We outline the energy argument of the second part of Squire's theorem briefly [12,18]. First, the hSE, i.e., Eq. (7b), with $W = 0$ and the right-hand side set to zero, is multiplied by the complex conjugate $\overline{\tilde\eta}$, with $\tilde\eta\overline{\tilde\eta} = |\tilde\eta|^2$, integrated over the y domain Ω, and multiplied by (-1) to obtain

$$i\omega\int_\Omega |\tilde\eta|^2\,\mathrm{d}y - i\alpha\int_\Omega U|\tilde\eta|^2\,\mathrm{d}y + \frac{1}{\mathrm{Re}}\int_\Omega\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}\tilde\eta\right)\overline{\tilde\eta} - (\alpha^2+\beta^2)|\tilde\eta|^2\right]\mathrm{d}y = 0. \tag{54}$$

After that, the second derivative is rewritten with integration by parts with vanishing boundary terms to

$$i\omega\int_\Omega |\tilde\eta|^2\,\mathrm{d}y - i\alpha\int_\Omega U|\tilde\eta|^2\,\mathrm{d}y - \frac{1}{\mathrm{Re}}\int_\Omega\left[\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde\eta\right|^2 + (\alpha^2+\beta^2)|\tilde\eta|^2\right]\mathrm{d}y = 0. \tag{55}$$

As Squire assumed temporal modes $\omega\in\mathbb{C}$, $\alpha,\beta\in\mathbb{R}$, solving for the imaginary part of the wave frequency, ω_i, eliminates the second integral, as α, U, and $|\tilde\eta|^2$ are real. Bringing the third integral with the Reynolds number to the right-hand side leads to

$$\omega_i\int_\Omega |\tilde\eta|^2\,\mathrm{d}y = -\frac{1}{\mathrm{Re}}\int_\Omega\left[\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde\eta\right|^2 + (\alpha^2+\beta^2)|\tilde\eta|^2\right]\mathrm{d}y. \tag{56}$$

Since all integrands are nonnegative, ω_i must be nonpositive, meaning that temporal Squire modes never grow in time, $\omega_i\leqslant 0$.

Oberlack *et al.* extended this procedure analogously to spatiotemporal modes, where no such statement about Squire modes can be obtained [18], meaning that they can grow in time, space, or both. The statement about damped Squire modes exclusively applies to temporal modes.

Now we return to the application of the generators to the adjoint-projected residual functional. The expression for the hmSE (52) simplifies for the hSE, i.e., for the setting of the Squire transformation, with $W = 0$, to

$$i\omega\int_\Omega \overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y - i\alpha\int_\Omega U\overline{\tilde\eta^\ddagger}\tilde\eta\,\mathrm{d}y + \frac{1}{\mathrm{Re}}\int_\Omega\left[\overline{\tilde\eta^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}\tilde\eta\right) - (\alpha^2+\beta^2)\overline{\tilde\eta^\ddagger}\tilde\eta\right]\mathrm{d}y = 0. \tag{57}$$

Equation (57) is structurally very similar to Eq. (54). However, the statement concerning damped temporal Squire modes cannot be inferred from Eq. (57), because the inner products contain terms such as $\overline{\tilde\eta^\ddagger}\tilde\eta$. This is in general not the same as $\tilde\eta\overline{\tilde\eta} = |\tilde\eta|^2$, in general not positive and does not in general reproduce the classical energy argument by Squire. However, Eq. (57) is a valid identity structurally similar to the result of Squire.

D. $\tilde{\mathsf{X}}_6$–$\tilde{\mathsf{X}}_{18}$: Generator components

The algebraic invariance conditions (52) and (C9) arise from the generator of the scaling in time $\tilde{\mathsf{X}}_5$.

When splitting the generator of the scaling in time $\tilde{\mathsf{X}}_5$, the algebraic invariance conditions can be associated with the generator component $\tilde{\mathsf{X}}_{16}$, i.e., the Squire transformation. As the results from the other generator components of $\tilde{\mathsf{X}}_5$ are trivial, the generator for the scaling in time $\tilde{\mathsf{X}}_5$ inherits the algebraic invariance conditions from its component $\tilde{\mathsf{X}}_{16}$. The same holds true for $\tilde{\mathsf{X}}_8$, $\tilde{\mathsf{X}}_9$, $\tilde{\mathsf{X}}_{14}$, and $\tilde{\mathsf{X}}_{17}$.

Interestingly, for the mOSE, the same algebraic invariance condition (C9) also appears when applying both individual components $\tilde{\mathsf{X}}_6$ and $\tilde{\mathsf{X}}_7$ of the rotation $\tilde{\mathsf{X}}_4$ to the adjoint-projected residual functional separately. When combining them to obtain the actual rotation $\tilde{\mathsf{X}}_4$, they cancel each other

TABLE V. Results of the application of the spectral generators to the adjoint-projected residual functional.

$\tilde{\mathsf{X}}_1$		Trivial and trivial
$\tilde{\mathsf{X}}_2$		Trivial and trivial
$\tilde{\mathsf{X}}_3$		Trivial and trivial
$\tilde{\mathsf{X}}_4$		Trivial and trivial
	$\tilde{\mathsf{X}}_6$	(C9)
	$\tilde{\mathsf{X}}_7$	(C9)
	$\tilde{\mathsf{X}}_{10}$	Trivial
	$\tilde{\mathsf{X}}_{11}$	Trivial
$\tilde{\mathsf{X}}_5$		(52) and (C9)
	$\tilde{\mathsf{X}}_8$	(C9)
	$\tilde{\mathsf{X}}_9$	(C9)
	$\tilde{\mathsf{X}}_{12}$	Trivial
	$\tilde{\mathsf{X}}_{13}$	Trivial
	$\tilde{\mathsf{X}}_{14}$	(52)
	$\tilde{\mathsf{X}}_{15}$	Trivial and trivial
	$\tilde{\mathsf{X}}_{16}$	(C10)
	$\tilde{\mathsf{X}}_{17}$	(57)
	$\tilde{\mathsf{X}}_{18}$	Trivial

out, so that the result for the actual rotation is trivial again. This happens because the algebraic invariance conditions (52) and (C9), i.e., the Fredholm solvability condition, are an intrinsic property of the transformation group algebra.

E. Overview of derived algebraic invariance conditions

Table V summarizes the results of the application of the spectral generators to the adjoint-projected residual functional. Note that Eq. (57) corresponds to Eq. (52) with $W = 0$, and that Eq. (C10) to Eq. (C9) with $W = 0$. Trivial means that the resulting equation is $0 = 0$. If there are two results, then the first one refers to the hmSE or hSE, and the second one to the mOSE or OSE.

F. Derivatives of the adjoint-projected residual functional

A further application of the adjoint-projected residual functional (38) and symmetry generators is to obtain derivative expressions of the eigenvalues with respect to spectral or constitutive parameters. For the direct-adjoint eigenpair, these derivatives are equivalent to those determined through implicit differentiation of the dispersion relation $\mathcal{D} = 0$ at isolated eigenvalues, together with the standard assumptions of differentiability of the operator family and eigenvectors, and compatible domains [32,36].

For the purpose of this section, when applying the generators to the adjoint-projected residual functional, the sought derivatives are not rewritten using the adjoint-projected residual functional (38), but the other terms are. In this section, we only analyze a one-component shear flow so that $W = 0$.

1. Derivatives in two dimensions

For 2D cases with $\beta = 0$, the algebraic invariance condition of the Galilean transformation in the x direction (41) is rearranged to yield

$$\frac{\partial \mathcal{F}}{\partial \omega} = -\frac{1}{\alpha}\int_\Omega \frac{\delta \mathcal{F}}{\delta U(y)}\,\mathrm{d}y, \tag{58}$$

and specifically with Eqs. (44b), (47b), and (48) we find

$$\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \omega} = -i\int_\Omega \overline{\tilde{\eta}^\ddagger}\tilde{\eta}\,\mathrm{d}y, \tag{59a}$$

$$\frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \omega} = -i\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}\,\mathrm{d}y. \tag{59b}$$

We can normalize $\tilde{\eta}^\ddagger$ and $\tilde{v}^\ddagger$ at will and choose specifically

$$\int_\Omega \overline{\tilde{\eta}^{\ddagger\sharp}}\tilde{\eta}\,\mathrm{d}y = i \quad \Rightarrow \quad \frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \omega} = 1, \tag{60a}$$

$$\int_\Omega \overline{\tilde{v}^{\ddagger\sharp}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}\,\mathrm{d}y = i \quad \Rightarrow \quad \frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \omega} = 1, \tag{60b}$$

where the sharp symbol $(\ldots)^\sharp$ denotes the normalized adjoint eigenfunctions. In formalisms such as Briggs's method, expressions like the group velocity $\mathrm{d}\omega/\mathrm{d}\alpha$ appear when implicitly differentiating, so Eqs. (60) are a sensible choice of normalization. However, this breaks down at exceptional points or points of multiple eigenvalues, where $\partial\mathcal{D}/\partial\omega = 0$. At such points, the proposed normalization is not available [32,36].

Subsequently, we assume that the saddle points to be investigated in Briggs's method are double roots of α at an isolated algebraically simple eigenvalue ω, so that $\partial\mathcal{D}/\partial\omega \neq 0$ and the normalization (60) is applicable.

Continuing with the algebraic invariance condition of the scaling in time (49), rearranging yields

$$\frac{\partial \mathcal{F}}{\partial Re} = \frac{\omega}{\mathrm{Re}}\frac{\partial \mathcal{F}}{\partial \omega} + \frac{1}{\mathrm{Re}}\int_\Omega U\frac{\delta \mathcal{F}}{\delta U(y)}\,\mathrm{d}y, \tag{61}$$

and specifically with Eqs. (44a), (51b), (47a), and (C7b)

$$\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \mathrm{Re}} = -\frac{i\omega}{\mathrm{Re}}\int_\Omega \overline{\tilde{\eta}^\ddagger}\tilde{\eta}\,\mathrm{d}y + \frac{i\alpha}{\mathrm{Re}}\int_\Omega U\overline{\tilde{\eta}^\ddagger}\tilde{\eta}\,\mathrm{d}y, \tag{62a}$$

$$\begin{aligned}\frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \mathrm{Re}} = &-\frac{i\omega}{\mathrm{Re}}\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}\,\mathrm{d}y \\ &+ \frac{i\alpha}{\mathrm{Re}}\int_\Omega U\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}\,\mathrm{d}y,\end{aligned} \tag{62b}$$

with $\mathrm{Re} \neq 0$. With the same normalization as above, this yields

$$\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \mathrm{Re}} = \frac{\omega}{\mathrm{Re}} + \frac{i\alpha}{\mathrm{Re}}\int_\Omega U\overline{\tilde{\eta}^{\ddagger\sharp}}\tilde{\eta}\,\mathrm{d}y, \tag{63a}$$

$$\frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \mathrm{Re}} = \frac{\omega}{\mathrm{Re}} + \frac{i\alpha}{\mathrm{Re}}\int_\Omega U\overline{\tilde{v}^{\ddagger\sharp}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\tilde{v}\,\mathrm{d}y. \tag{63b}$$

For the hSE, since the adjoint differential operator is the same as the complex conjugate differential operator, $\mathcal{L}^\dagger_{\mathrm{hSE}} = \overline{\mathcal{L}_{\mathrm{hSE}}}$, the adjoint eigenfunction may be chosen proportional to the complex conjugate of the direct eigenfunction. The inner

product can then be written with the established normalization as

$$\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \mathrm{Re}} = \frac{\omega}{\mathrm{Re}} - \frac{\alpha}{\mathrm{Re}} \frac{\int_\Omega U \tilde{\eta}^2 \,\mathrm{d}y}{\int_\Omega \tilde{\eta}^2 \,\mathrm{d}y}, \tag{64}$$

which no longer requires any knowledge of the adjoint eigenfunction. Here $\tilde{\eta}^2$ is a bilinear product and is therefore generally complex-valued.

For the OSE, the normalization constant K of the adjoint eigenfunction in $\tilde{v}^{\ddagger\sharp} = K\tilde{v}^{\ddagger}$ can be determined explicitly from the normalization condition (60b), as

$$K = -\frac{i}{\int_\Omega \tilde{v}^{\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \overline{\alpha}^2\right)\overline{\tilde{v}} \,\mathrm{d}y}, \tag{65}$$

for a nonzero denominator.

The derivative expressions (60) and (63) can be used, e.g., in Briggs's method, where the inverse Fourier-Laplace transform of a perturbation quantity can be asymptotically approximated using the method of steepest descent and the residue theorem. An algebraic, necessary criterion for this asymptotic approximation is a saddle point of the Fourier-Laplace exponent with respect to α [35,39–42]. When applying the extended Squire transformation (30) [18] to the spectral parameters, the saddle points of the Fourier-Laplace phase can now refer to the wave frequency ω or the Reynolds number Re via ϕ_{Sq} in Eq. (30e).

For the asymptotic evaluation, the saddle point of the Fourier-Laplace exponent always depends on the observer ray $x/t = V$. It satisfies $\mathrm{d}\omega/\,\mathrm{d}\alpha = V$, provided such a value exists. A special saddle point is the stationary-frame saddle point associated with the origin $x/t = 0$, as this specific one is relevant to the characterization of absolute instabilities, with $\mathrm{d}\omega/\,\mathrm{d}\alpha = 0$. Since $\mathrm{d}\omega/\,\mathrm{d}\alpha = -(\partial\mathcal{D}/\partial\alpha)/(\partial\mathcal{D}/\partial\omega)$, this means specifically for the dispersion relation $\mathcal{D} = 0$ that $\partial\mathcal{D}/\partial\alpha = 0$. Therefore, a stationary-frame saddle point is algebraically characterized by a double root of the dispersion relation with respect to α.

2. Derivatives in three dimensions

In 3D cases with a nonzero wave number $\beta \neq 0$, the generator component $\tilde{\mathsf{X}}_{15}$, component of the scaling in time $\tilde{\mathsf{X}}_5$, is applied to the adjoint-projected residual functional as

$$\tilde{\mathsf{X}}_{15}\mathcal{F} = \alpha\frac{\partial \mathcal{F}}{\partial \alpha} - \frac{\alpha^2}{\beta}\frac{\partial \mathcal{F}}{\partial \beta} - \int_\Omega U \frac{\delta \mathcal{F}}{\delta U(y)} \,\mathrm{d}y = 0. \tag{66}$$

Note that the $\tilde{\eta}$ term has been dropped from the above equation, as explained in Sec. V A. Solving this equation for the β term leads to

$$\frac{\partial \mathcal{F}}{\partial \beta} = \frac{\beta}{\alpha}\frac{\partial \mathcal{F}}{\partial \alpha} - \frac{\beta}{\alpha^2}\int_\Omega U \frac{\delta \mathcal{F}}{\delta U(y)} \,\mathrm{d}y, \tag{67}$$

and specifically, with Eqs. (C5a), (51b), (C6a), and (C7b),

$$\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \beta} = \frac{2\beta}{\mathrm{Re}}\int_\Omega \overline{\tilde{\eta}^{\ddagger\sharp}}\tilde{\eta} \,\mathrm{d}y, \tag{68a}$$

$$\begin{aligned}\frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \beta} &= 2i\beta\omega\int_\Omega \overline{\tilde{v}^{\ddagger\sharp}}\tilde{v} \,\mathrm{d}y - 2i\alpha\beta\int_\Omega U\overline{\tilde{v}^{\ddagger\sharp}}\tilde{v} \,\mathrm{d}y \\ &\quad + \frac{4\beta}{\mathrm{Re}}\int_\Omega \overline{\tilde{v}^{\ddagger\sharp}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\alpha^2+\beta^2\right)\right)\tilde{v} \,\mathrm{d}y,\end{aligned} \tag{68b}$$

with $\mathrm{Re} \neq 0$. We use the same normalization [with a β term in Eq. (69b) for 3D cases], namely

$$\int_\Omega \overline{\tilde{\eta}^{\ddagger\sharp}}\tilde{\eta} \,\mathrm{d}y = i \quad \Rightarrow \quad \frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \omega} = 1, \tag{69a}$$

$$\int_\Omega \overline{\tilde{v}^{\ddagger\sharp}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\alpha^2+\beta^2\right)\right)\tilde{v} \,\mathrm{d}y = i \quad \Rightarrow \quad \frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \omega} = 1, \tag{69b}$$

and with the discussed properties of the saddle points, the derivative expressions become

$$\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \beta} = \frac{2i\beta}{\mathrm{Re}}, \tag{70a}$$

$$\frac{\partial \mathcal{F}_{\mathrm{OSE}}}{\partial \beta} = \beta\left(2i\omega\int_\Omega \overline{\tilde{v}^{\ddagger\sharp}}\tilde{v} \,\mathrm{d}y - 2i\alpha\int_\Omega U\overline{\tilde{v}^{\ddagger\sharp}}\tilde{v} \,\mathrm{d}y + \frac{4i}{\mathrm{Re}}\right), \tag{70b}$$

with

$$K = -\frac{i}{\int_\Omega \tilde{v}^{\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\overline{\alpha}^2 + \overline{\beta}^2)\right)\overline{\tilde{v}} \,\mathrm{d}y}, \tag{71}$$

if the denominator is nonzero.

In the 3D formalism of Briggs's method, established by Brevdo, a double saddle point of the Fourier-Laplace phase with respect to both α and β is an algebraic, necessary criterion for the asymptotic approximation [43]. To find such points, the derivative expressions (70) can be used.

As both Eqs. (70) are linear in β, the algebraic double-root condition $\partial\mathcal{D}/\partial\beta = 0$ is always satisfied for the trivial $\beta = 0$. Consequently, every stationary-frame single saddle point in α automatically satisfies the criterion of a stationary-frame double saddle point established by Brevdo [43]. We refer to such a point as a *trivial stationary-frame double saddle point*.

In particular, Squire modes (SE modes) admit only trivial stationary-frame double saddle points, since Eq. (70a) can only trivially become zero with $\beta = 0$. As this is equivalent to stationary-frame single saddle points only in α, a necessary criterion for the 2D asymptotic approximation via Briggs's method at the origin, the SE modes can potentially only give rise to absolute instability mechanisms in 2D. Phrased in another way, the group velocity with respect to β can only trivially become zero with $\beta = 0$, as

$$\frac{\partial \omega}{\partial \beta} = -\frac{\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \beta}}{\frac{\partial \mathcal{F}_{\mathrm{hSE}}}{\partial \omega}} = -\frac{2i\beta}{\mathrm{Re}}. \tag{72}$$

Conversely, the OSE modes may also admit *nontrivial stationary-frame double saddle points*, i.e., stationary-frame saddle points simultaneously in α and in $\beta \neq 0$, which is a necessary criterion for a 3D asymptotic approximation via Briggs's method at the origin. Thus, the OSE modes can potentially give rise to 2D and 3D absolute instability mechanisms.

VI. CONCLUSION AND OUTLOOK

We emphasize three central insights of this work: (i) the computation of the generators of equivalence transformations

and spectral equivalence transformations in linear stability theory; (ii) a method of applying symmetry generators to an adjoint-projected residual functional, resulting in algebraic invariance conditions; and (iii) the same method resulting in analytic expressions of derivatives for our functional. These derivative expressions yield the same eigenvalue-branch derivatives as implicit differentiation of a local dispersion relation.

First, we computed the generators of symmetries and equivalence transformations of the governing equations in space and time, as well as the generators of spectral equivalence transformations of the spectral equations. This was done for the coupled and decoupled systems of the governing equations, and for two- and one-component base shear flows. All known symmetries were recovered, including translation in space and time, Galilean transformation, rotation and scaling of the dependent variables and of space and time, in their formulation in space and time and in their spectral formulation, if admitted. Furthermore, we calculated the Squire generator and derived the global Squire transformation. Additional (spectral) equivalence transformations have been found and characterized.

Second, we have formulated an adjoint-projected residual functional. The variation of this functional is studied through the variation of the differential operator, which can be carried out analytically in the general case. By applying the symmetry generators to this functional, we derived an algebraic invariance condition equal to the Fredholm solvability condition of the eigenvalue problem. However, in most cases, this procedure resulted in a trivial statement. For the hSE, that algebraic invariance condition is similar to, but distinct from, the second part of Squire's theorem, i.e., the statement that temporal Squire modes never grow.

Third, using the adjoint-projected residual functional formulation and applying symmetry generators to it, we derived analytic derivative expressions for selected constitutive parameters. These derivative expressions yield the same eigenvalue-branch derivatives as implicit differentiation of a local dispersion relation. They can be used, for example, in Briggs-type analyses. In particular, it follows from these that the hSE only admits trivial stationary-frame double saddle points, since the group velocity with respect to the spanwise wave number can only trivially become zero. This means that the hSE cannot give rise to 3D absolute instability mechanisms via Briggs's method.

We strongly believe that, for other equations and settings in linear stability theory, such as specific base shear flows and associated additional symmetry generators, a systematic application of Lie symmetry theory could provide valuable insights.

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DATA AVAILABILITY

The computational materials supporting the results of this study are available from Ref. [44]. The repository contains the Maple worksheet files used to compute all symmetry generators reported in this article with the GeM software package.

APPENDIX A: GALILEAN TRANSFORMATION IN DIFFERENT FORMULATION

In the spectral formulation of the spatial translation symmetries $\tilde{\mathsf{X}}_1$, the eigenfunction is scaled. The Galilean transformations are a generalization of the translations in space, i.e., translations that are linear in time. However, the notation in $\tilde{\mathsf{X}}_2$ and $\tilde{\mathsf{X}}_3$ is fundamentally different. There the scaling has been absorbed into the transformed wave frequency ω^*, since the wave frequency ω scales linearly in time, too. Alternatively, it can remain as a scaling of the eigenfunction like

$$\begin{aligned} &x^* = x + a_{Gx}t, \quad U^* = U + a_{Gx}, \\ &\mathsf{X}_{Gx} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial U}, \end{aligned} \tag{A1a}$$

$$\begin{aligned} &\tilde{v}^* = e^{-i\alpha a_{Gx}t}\tilde{v}, \quad \tilde{\eta}^* = e^{-i\alpha a_{Gx}t}\tilde{\eta}, \quad U^* = U + a_{Gx}, \\ &\tilde{\mathsf{X}}_{Gx} = -i\alpha t\tilde{v}\frac{\partial}{\partial \tilde{v}} - i\alpha t\tilde{\eta}\frac{\partial}{\partial \tilde{\eta}} + \frac{\partial}{\partial U}, \end{aligned} \tag{A1b}$$

and

$$\begin{aligned} &z^* = z + a_{Gz}t, \quad W^* = W + a_{Gz}, \\ &\mathsf{X}_{Gz} = t\frac{\partial}{\partial z} + \frac{\partial}{\partial W}, \end{aligned} \tag{A2a}$$

$$\begin{aligned} &\tilde{v}^* = e^{-i\beta a_{Gz}t}\tilde{v}, \quad \tilde{\eta}^* = e^{-i\beta a_{Gz}t}\tilde{\eta}, \quad W^* = W + a_{Gz}, \\ &\tilde{\mathsf{X}}_{Gz} = -i\beta t\tilde{v}\frac{\partial}{\partial \tilde{v}} - i\beta t\tilde{\eta}\frac{\partial}{\partial \tilde{\eta}} + \frac{\partial}{\partial W}. \end{aligned} \tag{A2b}$$

The disadvantage of this alternative formulation is that the time t then appears as a factor in the spectral formulation. However, this is consistent with the notation used for the spectral formulation of the translations.

APPENDIX B: SYMMETRIES AND (SPECTRAL) EQUIVALENCE TRANSFORMATIONS IN DIFFERENT SETTINGS

1. The decoupled system

a. Symmetries of Eq. (4a) and of the homogeneous Eq. (4b).

The generators found are exactly the same as in Table I, with X_4 split into two separate generators, each corresponding to the single dependent variable of the respective equation, i.e., v' or η', respectively.

b. Equivalence transformations of Eq. (4a) and of the homogeneous Eq. (4b).

For Eq. (4a), the generators found are exactly the same as in Tables I and II and specifically X_4 with only the v' term and X_9 without the η' term.

For the homogeneous Eq. (4b), we observe from a computational perspective that different sequences of the symmetry components in `Maple` using the commands `DEtools[rifsimp]` and `pdsolve` produce different generators. Some of these generators are more aesthetically pleasing and easier to interpret, since generators can be combined linearly anyway. However, they should be equivalent, and the total number of generators should remain the same.

For the specific sequence of the symmetry components here in this section, we had to bypass the Ovsiannikov conjecture procedure in the GeM package.

Thus, 12 generators are found in total. These are all the known X_1 through X_{10}, and specifically X_4 with only the η' term, as in Tables I and II. Furthermore, the two new generators are

$$\mathsf{X}_{11} = \frac{\partial}{\partial \eta'}, \quad \mathsf{X}_{12} = y\frac{\partial}{\partial \eta'}, \tag{B1}$$

representing a translation in η' and an additive transformation in η', linear in y, respectively.

These equivalence transformations originate from bypassing the Ovsiannikov conjecture and are both finite-dimensional remnants of the infinite-dimensional superposition symmetry. The homogeneous Eq. (4b) is linear and homogeneous, and its highest derivative is of second order. The Laplacian operator annihilates linear and constant functions, while the temporal derivative and the advection terms in x and z restrict admissible additive transformations to functions that are either constant or linear in y, resulting in the generators X_{11} and X_{12}. These equivalence transformations only occur on an operator level. Upon imposing physically meaningful boundary conditions on the initial-boundary value problem, such as $\eta'|_{\partial\Omega} = 0$, both transformations are eliminated, where $\partial\Omega$ is the boundary of the domain.

c. Spectral equivalence transformations of Eq. (7a) and of the homogeneous Eq. (7b).

For Eq. (7a), seven generators are found in total. These are $\tilde{\mathsf{X}}_1$ with only the $\tilde{v}$ term, $\tilde{\mathsf{X}}_2$ and $\tilde{\mathsf{X}}_3$, as in Eqs. (21). The four new generators are

$$\begin{aligned}
\tilde{\mathsf{X}}_6 &= f_6(\mathrm{Re})\Big(\frac{\alpha}{\alpha^2+\beta^2}\Big(-\alpha\beta\frac{\partial}{\partial\alpha} \\
&\quad + \alpha^2\frac{\partial}{\partial\beta} - \beta\omega\frac{\partial}{\partial\omega} + \beta\mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\Big) - W\frac{\partial}{\partial U}\Big), \\
\tilde{\mathsf{X}}_7 &= f_7(\mathrm{Re})\Big(\frac{\beta}{\alpha^2+\beta^2}\Big(-\beta^2\frac{\partial}{\partial\alpha} \\
&\quad + \alpha\beta\frac{\partial}{\partial\beta} + \alpha\omega\frac{\partial}{\partial\omega} - \alpha\mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\Big) + U\frac{\partial}{\partial W}\Big), \\
\tilde{\mathsf{X}}_8 &= f_8(\mathrm{Re})\Big(\frac{\alpha}{\alpha^2+\beta^2}\Big(\beta^2\frac{\partial}{\partial\alpha} \\
&\quad - \alpha\beta\frac{\partial}{\partial\beta} - \alpha\omega\frac{\partial}{\partial\omega} + \alpha\mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\Big) - U\frac{\partial}{\partial U}\Big), \\
\tilde{\mathsf{X}}_9 &= f_9(\mathrm{Re})\Big(\frac{\beta}{\alpha^2+\beta^2}\Big(-\alpha\beta\frac{\partial}{\partial\alpha} \\
&\quad + \alpha^2\frac{\partial}{\partial\beta} - \beta\omega\frac{\partial}{\partial\omega} + \beta\mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\Big) - W\frac{\partial}{\partial W}\Big),
\end{aligned} \tag{B2}$$

with $\alpha^2+\beta^2 \neq 0$. They represent mixed generalized scalings. The generators form a basis $\{\tilde{\mathsf{X}}_6, \tilde{\mathsf{X}}_7, \tilde{\mathsf{X}}_8, \tilde{\mathsf{X}}_9\}$ to express other transformations.

For example, for $f_6(\mathrm{Re}) = f_7(\mathrm{Re}) = f_4(\mathrm{Re})$, the linear combination $\tilde{\mathsf{X}}_6 + \tilde{\mathsf{X}}_7$ results in the generalized rotation in the (x, z) plane $\tilde{\mathsf{X}}_4$, given in Eqs. (21), as

$$\tilde{\mathsf{X}}_6 + \tilde{\mathsf{X}}_7 = \tilde{\mathsf{X}}_4 = f_4(\mathrm{Re})\Big(-\beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} - W\frac{\partial}{\partial U} + U\frac{\partial}{\partial W}\Big). \tag{B3}$$

Thus, the rotation generator can be split into two components $\tilde{\mathsf{X}}_6$ and $\tilde{\mathsf{X}}_7$, which in general can be scaled independently of each other.

Furthermore, for $f_8(\mathrm{Re}) = f_9(\mathrm{Re}) = f_5(\mathrm{Re})$, the linear combination $\tilde{\mathsf{X}}_8 + \tilde{\mathsf{X}}_9$ yields the generalized scaling in time $\tilde{\mathsf{X}}_5$, given in Eqs. (21), as

$$\begin{aligned}
\tilde{\mathsf{X}}_8 + \tilde{\mathsf{X}}_9 &= \tilde{\mathsf{X}}_5 \\
&= f_5(\mathrm{Re})\Big(-\omega\frac{\partial}{\partial\omega} - U\frac{\partial}{\partial U} - W\frac{\partial}{\partial W} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\Big).
\end{aligned} \tag{B4}$$

Thus, the generator of the scaling in time can be split into two components $\tilde{\mathsf{X}}_8$ and $\tilde{\mathsf{X}}_9$, which in general can be scaled independently of each other.

Ergo, we have derived generalizations of all five generators given in Eqs. (21), in the sense that the free functions of their base components can be scaled independently. Additionally, two additional linearly independent generators can be constructed from the basis $\{\tilde{\mathsf{X}}_6, \tilde{\mathsf{X}}_7, \tilde{\mathsf{X}}_8, \tilde{\mathsf{X}}_9\}$, which do not have such an obvious physical interpretation. Clearly, the mOSE on its own has more symmetry properties than the coupled system, as its vorticity generation is not restricted by the mSE.

For the homogeneous Eq. (7b), eight generators are found in total. These are $\tilde{\mathsf{X}}_1$ with only the $\tilde{\eta}$ term, $\tilde{\mathsf{X}}_2$ and $\tilde{\mathsf{X}}_3$, as in Eqs. (21). The five new generators are

$$\begin{aligned}
\tilde{\mathsf{X}}_{10} &= f_{10}(\mathrm{Re})\Big(\alpha\frac{\partial}{\partial\beta} - \frac{2i\alpha\beta}{\mathrm{Re}}\frac{\partial}{\partial\omega} - W\frac{\partial}{\partial U}\Big), \\
\tilde{\mathsf{X}}_{11} &= f_{11}(\mathrm{Re})\Big(-\beta\frac{\partial}{\partial\alpha} + \frac{2i\alpha\beta}{\mathrm{Re}}\frac{\partial}{\partial\omega} + U\frac{\partial}{\partial W}\Big), \\
\tilde{\mathsf{X}}_{12} &= f_{12}(\mathrm{Re})\Big(\alpha\frac{\partial}{\partial\alpha} - \frac{2i\alpha^2}{\mathrm{Re}}\frac{\partial}{\partial\omega} - U\frac{\partial}{\partial U}\Big), \\
\tilde{\mathsf{X}}_{13} &= f_{13}(\mathrm{Re})\Big(\beta\frac{\partial}{\partial\beta} - \frac{2i\beta^2}{\mathrm{Re}}\frac{\partial}{\partial\omega} - W\frac{\partial}{\partial W}\Big), \\
\tilde{\mathsf{X}}_{14} &= f_{14}(\alpha,\beta,\omega,\mathrm{Re})\Big(-\alpha\frac{\partial}{\partial\alpha} - \beta\frac{\partial}{\partial\beta} \\
&\quad + \frac{2i(\alpha^2+\beta^2) - \omega\mathrm{Re}}{\mathrm{Re}}\frac{\partial}{\partial\omega} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\Big),
\end{aligned} \tag{B5}$$

with $\mathrm{Re} \neq 0$. They represent mixed generalized scalings. The generators again form a basis $\{\tilde{\mathsf{X}}_{10}, \tilde{\mathsf{X}}_{11}, \tilde{\mathsf{X}}_{12}, \tilde{\mathsf{X}}_{13}, \tilde{\mathsf{X}}_{14}\}$ to express other transformations.

For example, for $f_{10}(\mathrm{Re}) = f_{11}(\mathrm{Re}) = f_4(\mathrm{Re})$, the linear combination $\tilde{\mathsf{X}}_{10} + \tilde{\mathsf{X}}_{11}$ results in the generalized rotation in the (x, z) plane $\tilde{\mathsf{X}}_4$ in Eqs. (21) again.

Further, for $f_{12}(\mathrm{Re}) = f_{13}(\mathrm{Re}) = f_{14}(\alpha, \beta, \omega, \mathrm{Re}) = f_5(\mathrm{Re})$, the linear combination $\tilde{\mathsf{X}}_{12} + \tilde{\mathsf{X}}_{13} + \tilde{\mathsf{X}}_{14}$ yields the generalized scaling in time $\tilde{\mathsf{X}}_5$ in Eqs. (21) again.

Ergo, we have derived generalizations of all five generators listed in Eqs. (21). Additionally, three additional, linearly independent generators can be constructed from the basis $\{\tilde{\mathsf{X}}_{10}, \tilde{\mathsf{X}}_{11}, \tilde{\mathsf{X}}_{12}, \tilde{\mathsf{X}}_{13}, \tilde{\mathsf{X}}_{14}\}$, which do not have such an obvious physical interpretation. Clearly, the hmSE on its own has more symmetry properties than the coupled system and more than the mOSE on its own.

From a computational perspective, we note that the order of the symmetry components in `Maple` matters, as the commands `DEtools[rifsimp]` and `pdsolve` produce different algebras for different sequences of symmetry components. In principle, they should be equivalent, but in some scenarios, we were genuinely only able to compute seven generators for the homogeneous Eq. (7b), with a term such as $W\partial/\partial W$ missing. This means that there would be no way to linearly combine the generators to formulate a generalized scaling in time. By a different order of symmetry components, this problem has been solved, and Eqs. (B5) have been produced.

2. The decoupled system with $W = 0$

a. Symmetries of Eq. (4a) and of the homogeneous Eq. (4b), with $W = 0$.

For Eq. (4a), the generators found are the same as in Table I except that X_4 only has the v' term.

For the homogeneous Eq. (4b), five generators are found in total. These are all the four generators from Table I, with X_4 only having the η' term. The new generator is

$$\mathsf{X}_{13} = t\frac{\partial}{\partial z} - \frac{z\mathrm{Re}\eta'}{2}\frac{\partial}{\partial \eta'}, \tag{B6}$$

which is a parabolic Galilean transformation in z, with a rescaling of η' compensating the diffusive term, ensuring that the parabolic operator is invariant.

However, it is discarded, as it involves the constitutive quantity Re in the transformation of the dependent variable η', as this is not allowed for symmetries by definition.

b. Equivalence transformations of Eq. (4a) and of the homogeneous Eq. (4b), with $W = 0$.

For Eq. (4a), the generators found are the same as in Tables I and II, but with $W = 0$, so X_7 and X_8 are missing, and in X_9 and X_{10}, the W term is missing. Furthermore, X_4 only has the v' term, and X_9 does not have the η' term.

For the homogeneous Eq. (4b), the nine generators are found in total. These are X_1 through X_6 and X_9 and X_{10}, and specifically X_4 with only the η' term, from Tables I and II, and further X_{13} from Eq. (B6). All nine generators involve a free function $f_i(\mathrm{Re})$.

However, since the transformations of independent and dependent variables cannot depend on the constitutive quantities by definition of an equivalence transformation, the free function must be set to $f_i(\mathrm{Re}) = 1$ for all of them. Therefore, these are the operators from Tables I and II (with the aforementioned changes), i.e., without the free functions, and additionally X_{13}. This again involves the constitutive quantity Re in the transformation of the dependent variable η', and this is not permitted for equivalence transformations by definition.

c. Spectral equivalence transformations of Eq. (7a) and of the homogeneous Eq. (7b) with $W = 0$.

For Eq. (7a), the generators found are the same as in the coupled case. These are $\tilde{\mathsf{X}}_1$ with only the $\tilde{v}$ term and $\tilde{\mathsf{X}}_2$ from Eqs. (21), and $\tilde{\mathsf{X}}_{15}$ and $\tilde{\mathsf{X}}_{16}$ without the $\tilde{\eta}$ term, from Eq. (22).

For the homogeneous Eq. (7b), five generators are found in total. These are $\tilde{\mathsf{X}}_1$ with only $\tilde{\eta}$ term and $\tilde{\mathsf{X}}_2$ from Eqs. (21), and $\tilde{\mathsf{X}}_{15}$ without the $\tilde{\eta}$ term, from Eq. (22). The two new generators are

$$\begin{aligned}\tilde{\mathsf{X}}_{17} &= f_{17}(\alpha, \beta, \omega, \mathrm{Re})\left(-\alpha\frac{\partial}{\partial\alpha} + \frac{2i\alpha^2 - \omega\mathrm{Re}}{\mathrm{Re}}\frac{\partial}{\partial\omega} + \mathrm{Re}\frac{\partial}{\partial\mathrm{Re}}\right),\\ \tilde{\mathsf{X}}_{18} &= f_{18}(\alpha, \beta, \omega, \mathrm{Re})\left(-\beta\frac{\partial}{\partial\beta} + \frac{2i\beta^2}{\mathrm{Re}}\frac{\partial}{\partial\omega}\right),\end{aligned} \tag{B7}$$

with $\mathrm{Re} \neq 0$. They represent mixed generalized scalings. They are a splitting of $\tilde{\mathsf{X}}_{16}$ from Eq. (22), where the components can be scaled independently of each other.

APPENDIX C: APPLICATION OF THE SPECTRAL GENERATORS TO THE ADJOINT-PROJECTED RESIDUAL FUNCTIONAL

1. $\tilde{\mathsf{X}}_3$: Galilean transformations in the z direction

Similarly to Sec. V B, applying the generator $\tilde{\mathsf{X}}_3$ of the Galilean transformation in the z direction to the adjoint-projected residual functional yields

$$\tilde{\mathsf{X}}_3\mathcal{F} = \beta\frac{\partial\mathcal{F}}{\partial\omega} + \int_\Omega \frac{\delta\mathcal{F}}{\delta W(y)}\,\mathrm{d}y = 0, \tag{C1}$$

analogously leading to the same trivial results, with

$$\delta\mathcal{L} = \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial W}\delta W(y) + \frac{\partial\mathcal{L}}{\partial W''}\delta W''(y)\right)\mathrm{d}y. \tag{C2a}$$

$$\delta\mathcal{L}_{\mathrm{hmSE}} = i\beta\int_\Omega \delta W(y)\,\mathrm{d}y, \tag{C2b}$$

$$\delta\mathcal{L}_{\mathrm{mOSE}} = i\beta\int_\Omega \left[\delta W(y)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right) - \delta W''(y)\right]\mathrm{d}y, \tag{C2c}$$

and the same cancellation as in Sec. V B.

2. $\tilde{\mathsf{X}}_4$: Rotation in the (x, z) plane

We apply the generator $\tilde{\mathsf{X}}_4$ of the rotation in the (x, z) plane to the adjoint-projected residual functional to obtain

$$\tilde{\mathsf{X}}_4\mathcal{F} = -\beta\frac{\partial\mathcal{F}}{\partial\alpha} + \alpha\frac{\partial\mathcal{F}}{\partial\beta} - \int_\Omega W\frac{\delta\mathcal{F}}{\delta U(y)}\,\mathrm{d}y + \int_\Omega U\frac{\delta\mathcal{F}}{\delta W(y)}\,\mathrm{d}y = 0, \tag{C3}$$

where the new, required derivatives are obtained as

$$\frac{\partial\mathcal{F}}{\partial\alpha} = \int_\Omega \overline{\tilde{q}^\ddagger}\left(\frac{\partial\mathcal{L}}{\partial\alpha}\tilde{q}\right)\mathrm{d}y, \tag{C4a}$$

$$\frac{\partial\mathcal{F}}{\partial\beta} = \int_\Omega \overline{\tilde{q}^\ddagger}\left(\frac{\partial\mathcal{L}}{\partial\beta}\tilde{q}\right)\mathrm{d}y. \tag{C4b}$$

For the terms involving the base flow velocity in Eq. (C3), we can use the established variations of the functional (42b) and (C2a) and, for the mOSE, apply integration by parts twice as described. However, we also insert the other base flow velocity component into the integral. Any boundary terms will vanish following analogous reasoning. Starting with the hmSE, the terms in Eq. (C3) are given as

$$-\beta\frac{\partial\mathcal{F}_{\mathrm{hmSE}}}{\partial\alpha} = -\beta\int_\Omega \overline{\tilde{\eta}^\ddagger}\left(iU + \frac{2\alpha}{\mathrm{Re}}\right)\tilde{\eta}\,\mathrm{d}y, \tag{C5a}$$

$$\alpha\frac{\partial\mathcal{F}_{\mathrm{hmSE}}}{\partial\beta} = \alpha\int_\Omega \overline{\tilde{\eta}^\ddagger}\left(iW + \frac{2\beta}{\mathrm{Re}}\right)\tilde{\eta}\,\mathrm{d}y, \tag{C5b}$$

$$-\int_\Omega W\frac{\delta\mathcal{F}_{\mathrm{hmSE}}}{\delta U(y)}\,\mathrm{d}y = -i\alpha\int_\Omega W\overline{\tilde{\eta}^\ddagger}\tilde{\eta}\,\mathrm{d}y, \tag{C5c}$$

$$\int_\Omega U\frac{\delta\mathcal{F}_{\mathrm{hmSE}}}{\delta W(y)}\,\mathrm{d}y = i\beta\int_\Omega U\overline{\tilde{\eta}^\ddagger}\tilde{\eta}\,\mathrm{d}y. \tag{C5d}$$

When the terms in Eq. (C5) are substituted into Eq. (C3), they cancel each other out, meaning that the algebraic invariance condition (C3) becomes trivial. Continuing with the mOSE, the terms in Eq. (C3) are given as

$$-\beta\frac{\partial\mathcal{F}_{\mathrm{mOSE}}}{\partial\alpha} = -\beta\int_\Omega \overline{\tilde{v}^\ddagger}\left[iU\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right) + (-i\omega + i\alpha U + i\beta W)(-2\alpha) - iU'' + \frac{4\alpha}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\right]\tilde{v}\,\mathrm{d}y, \tag{C6a}$$

$$\alpha\frac{\partial\mathcal{F}_{\mathrm{mOSE}}}{\partial\beta} = \alpha\int_\Omega \overline{\tilde{v}^\ddagger}\left[iW\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right) + (-i\omega + i\alpha U + i\beta W)(-2\beta) - iW'' + \frac{4\beta}{\mathrm{Re}}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\right]\tilde{v}\,\mathrm{d}y, \tag{C6b}$$

$$-\int_\Omega W\frac{\delta\mathcal{F}_{\mathrm{mOSE}}}{\delta U(y)}\,\mathrm{d}y = -i\alpha\int_\Omega W\left[\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y, \tag{C6c}$$

$$\int_\Omega U\frac{\delta\mathcal{F}_{\mathrm{mOSE}}}{\delta W(y)}\,\mathrm{d}y = i\beta\int_\Omega U\left[\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y. \tag{C6d}$$

When substituting the terms of Eq. (C6) into Eq. (C3), they cancel each other out. For the respective third terms in Eq. (C6a) and in Eq. (C6b), integration by parts is applied twice, analogously to the Galilean transformation case, but here with the other base flow velocity component in the integrand. However, the result is analogous.

Overall, the algebraic invariance condition (C3) becomes trivial, too.

3. Scaling in time

Continuing with the mOSE, the terms in Eq. (49) are given as

$$-\omega\frac{\partial\mathcal{F}_{\mathrm{mOSE}}}{\partial\omega} = i\omega\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v}\,\mathrm{d}y, \tag{C7a}$$

$$-\int_\Omega U\frac{\delta\mathcal{F}_{\mathrm{mOSE}}}{\delta U(y)}\,\mathrm{d}y = -i\alpha\int_\Omega U\left[\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y, \tag{C7b}$$

$$-\int_\Omega W\frac{\delta\mathcal{F}_{\mathrm{mOSE}}}{\delta W(y)}\,\mathrm{d}y = -i\beta\int_\Omega W\left[\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y, \tag{C7c}$$

$$\mathrm{Re}\frac{\partial\mathcal{F}_{\mathrm{mOSE}}}{\partial\mathrm{Re}} = \frac{1}{\mathrm{Re}}\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)^2\tilde{v}\,\mathrm{d}y. \tag{C7d}$$

The algebraic invariance condition (49) thus becomes

$$i\omega\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v}\,\mathrm{d}y - i\alpha\int_\Omega U\left[\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y - i\beta\int_\Omega W\left[\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y + \frac{1}{\mathrm{Re}}\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)^2\tilde{v}\,\mathrm{d}y = 0, \tag{C8}$$

which can be rewritten, using integration by parts twice. This is similar to the Galilean transformation case but with the same base flow velocity component in the integrand. However, the result is analogous,

$$i\omega \int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v}\,\mathrm{d}y - i\alpha \int_\Omega \left[U\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - U''(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y$$
$$- i\beta \int_\Omega \left[W\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - W''(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y + \frac{1}{\mathrm{Re}}\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)^2\tilde{v}\,\mathrm{d}y = 0. \tag{C9}$$

For $W = 0$, this simplifies to

$$i\omega \int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v}\,\mathrm{d}y - i\alpha \int_\Omega \left[U\overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)\tilde{v} - U''(\overline{\tilde{v}^\ddagger}\tilde{v})\right]\mathrm{d}y$$
$$+ \frac{1}{\mathrm{Re}}\int_\Omega \overline{\tilde{v}^\ddagger}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2+\beta^2)\right)^2\tilde{v}\,\mathrm{d}y = 0. \tag{C10}$$

The same considerations as for the algebraic invariance (57) apply here.

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