

**Accuracy comes at a cost: Optimal localization against a flow**Till Welker<sup>✉\*</sup> and Patrick Pietzonka<sup>✉†</sup>*SUPA, School of Physics and Astronomy, University of Edinburgh, Edinburgh, United Kingdom*

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How much work does it cost for a propelled particle to stay localized near a stationary target, defying both thermal noise and a constant flow that would carry it away? We study the control of such a particle in finite time and find optimal protocols for time-dependent swim velocity and diffusivity, without feedback. Accuracy, quantified via the mean squared deviation from the target, and energetic cost turn out to be related by a tradeoff, which complements the one between precision and cost known in stochastic thermodynamics. We show that accuracy better than a certain threshold requires active driving, which comes at a cost that increases with accuracy. The optimal protocols have discontinuous swim velocity and diffusivity, switching between a passive drift state with vanishing diffusivity and an active propulsion state. If the initial position is fixed, an initial jump of the particle, enabled by a sudden burst of propulsion, can be optimal. This study highlights how a time-dependent diffusivity enhances optimal control and sets benchmarks for cost and accuracy of artificial self-propelled particles navigating noisy environments.

DOI: [10.1103/z6b5-s3z2](https://doi.org/10.1103/z6b5-s3z2)**I. INTRODUCTION**

A human swimmer in a river is fighting against the flow that drags her downstream. To stay in place, she needs to perform work to actively swim against the flow. The same is true for microscopic self-propelled particles in a flow. Being small generally helps to reduce the work required for propulsion, yet thermal fluctuations then pose an additional challenge. In this paper, we find the propulsion and diffusivity that optimize accuracy of localizing a particle in a flow for any given work budget over a fixed time interval. Surprisingly, the best results are obtained for propulsion and diffusivity following nontrivial time protocols.

From a theoretical viewpoint, our study presents a minimal way to analyze the role of accuracy in stochastic thermodynamics [1]. So far, the thermodynamic uncertainty relation establishes a tradeoff between cost and *precision* [2–6]. There, precision is quantified via the mean square deviation of a stochastic variable from its own mean. Accuracy, on the other hand, relates to the mean square deviation from a desired target value [7]. Being precise is not enough to be accurate, since a precisely located particle can still be far away from a desired target. Moreover, the thermodynamic uncertainty relation considers relative uncertainty, comparing the precision of a variable to its squared mean. Here, we instead use accuracy as an absolute measure, which remains finite when the mean deviation from the target is zero.

Tradeoffs between cost and accuracy have been proposed for specific systems with feedback control [8,9]. Here, as a minimal setting, we focus on control that does not require any measurements after the initialization of the system. In the absence of flows, the problem of accurately localizing a Brownian particle at a fixed position is trivial: Resting at a target at zero diffusivity yields perfect accuracy at zero cost. Hence, the simplest nontrivial question is how a particle can be close to a target that moves at a constant velocity. Or, equivalently (after transformation to a co-moving frame [10]), how does a particle localize at a stationary target while being subject to a constant flow?

Beyond its theoretical value, the setup we consider can provide simple benchmarks for the cost of control of active particles in general. To power their self-propulsion, active particles use fuel, which drives them out of equilibrium [11,12]. Individual active particles can carry cargo [13,14] and promise applications in targeted drug delivery [15–17], where accuracy is desirable. As a collective of active particles, active matter behaves in ways not possible at equilibrium: it can phase separate without attraction [18–20] and flock directionally [21,22]. By tuning activity and friction in time and space or imposing boundaries, one can control the direction of flocking [23], stabilize vortices [24], and accelerate self-assembly [25]. Such control of active matter harnesses its microscopically injected energy on a macroscopic scale and must come at a minimal cost, analogously to the results we obtain here for the control of individual active particles.

As a systematic framework to optimize cost and accuracy in small systems, we use optimal control theory [26–28]. There, control parameters are determined such that a predefined objective (travel time, work, accuracy, etc.) is optimized. The resulting optimal protocols can feature discontinuities [26,29–31] and non-monotonicities [31,32], and can depend qualitatively on the noise level [33]. In active matter, optimal control theory has been used to extract work from

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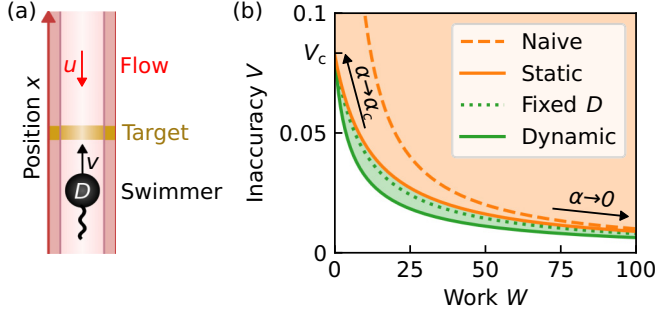


FIG. 1. System and tradeoff. (a) Swimmer with propulsion velocity  $v(t)$  and diffusivity  $D(t)$  in a flow of velocity  $-u$  trying to stay close to a target at  $x = 0$ . (b) Optimal tradeoff between work  $W$  and inaccuracy  $V$  represented by the Pareto front for static (orange line) and for dynamic (green line) protocols for  $v(t)$  and  $D(t)$ . The tradeoff parameter  $\alpha$  controls the emphasis on reducing  $W$  during optimization. The shaded area represents the region accessible for suboptimal protocols. The naive protocol (orange dashed line) describes swimmers staying on average on the target. Protocols with dynamic velocity but fixed diffusivity lead to the Pareto front shown as the green dotted line.

active propulsion [34,35], translate an active droplet [36], move an active particle between two local minima [37], drive transitions between nonequilibrium steady states [38], and transform a potential from an initial to a final shape [34,39]. Optimal control also helps devise strategies to reach a target in a flow. In noise-free systems, an elegant solution exists [40], but it fails once noise is introduced. In response, strategies have been proposed that measure the environment and compensate the effect of noise [41–43], a task that can be accomplished by machine learning [44,45].

We obtain a closed-form solution for the work-accuracy tradeoff of a swimmer localizing in a constant flow, which is arguably the simplest nontrivial accuracy optimisation problem in stochastic thermodynamics. The optimal protocols we find switch between passive drifting and active swimming, with discontinuous changes in velocity and diffusivity. As priority is shifted from high accuracy toward cost saving, the optimum transitions from a partially active to purely passive protocol, analogous to a second-order phase transition. When the initial position is not optimized, an instantaneous jump toward the target at the start of the protocol can be optimal. Crucially, while previous approaches to control active particles treat their diffusivity as fixed, we show that time-dependent control of diffusivity (e.g., by varying shape or volume) significantly reduces cost, or allows for better accuracy at the same cost.

## II. MODEL

We study a propelled particle (“swimmer”) moving along a one-dimensional coordinate  $x$  while being subject to an external flow with velocity  $-u$ , as shown in Fig. 1(a). The swimmer’s propulsion force  $f(t)$  and mobility  $\mu(t)$  can be adjusted by a (generally) time-dependent protocol, running from time  $t = 0$  to the total time  $\tau$ , and the initial position  $x_i$  can be freely chosen (if not stated otherwise). The

mobility is the inverse of the friction coefficient, which is determined by geometric features of the swimmer. A time-dependent mobility can be realized by varying size, shape or orientation, or by binding to and unbinding from other objects advected in the flow. The swimmer’s velocity relative to the flow is then  $v(t) = \mu(t)f(t)$  and its diffusivity is set by the Einstein relation to  $D(t) = k_B T \mu(t)$  at thermal energy  $k_B T$ . On mesoscopic scales, inertia can be neglected, so that the swimmer’s position in the laboratory frame follows the overdamped Langevin equation

$$\dot{x}(t) = v(t) - u + \sqrt{2D(t)}\xi(t), \quad (1)$$

where  $\xi(t)$  is a Gaussian white noise with zero mean and correlations  $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$ . Starting from the initial position  $x_i$ , the probability distribution of the particle position is Gaussian with mean  $\bar{x}(t) = x_i + \int_0^t dt' [v(t') - u]$  and variance  $\sigma^2(t) = 2 \int_0^t dt' D(t')$ . To dedimensionalize the system, we scale time by  $\tau$ , speed by  $u$ , and energy by  $k_B T$ .

We aim for the swimmer to be close to a target at position  $x = 0$  over the whole time interval. The ensemble-averaged squared distance from the target at time  $t$  is the second moment  $\langle x(t)^2 \rangle = \sigma^2(t) + \bar{x}^2(t)$  of the particle’s probability distribution. Averaging this quantity over the protocol time leads to

$$\begin{aligned} V &\equiv \int_0^1 dt [\sigma^2(t) + \bar{x}^2(t)] \\ &= \int_0^1 dt [2D(t)(1 - t) + \bar{x}^2(t)], \end{aligned} \quad (2)$$

which we call the “inaccuracy.” Small inaccuracy means good accuracy. The expression in the second line is obtained using integration by parts of the first term. It shows that diffusion impacts  $V$  more strongly at the beginning of the time interval than at its end, because the effect of any fluctuation lasts for the remainder of the protocol time.

The mean work required for swimming is [10]

$$W \equiv \int_0^1 dt f(t)v(t) = \int_0^1 dt \frac{(\dot{\bar{x}}(t) + 1)^2}{D(t)}, \quad (3)$$

where  $v(t) = \dot{\bar{x}}(t) + u$  (with our convention  $u = 1$ ) is the swim velocity. Generating a time-dependent control requires additional work [46]. The work needed to change diffusivity, for example, by changes in shape or volume, is as fundamental as the work required to change the velocity, for example, by changing the internal state. We take this work to generate the control to be paid by the controller and do not consider it for the tradeoff, in line with previous treatments [29,31,34,35].

Equations (2) and (3) already hint at a tradeoff between work and inaccuracy. Work  $W$  can be reduced by a slower swim velocity  $v(t)$  or larger diffusivity  $D(t)$ . Yet, both lead to larger inaccuracy, either through an increased distance between the mean and the target or through an increased variance. We obtain the protocols that optimize the tradeoff between inaccuracy  $V$  and mean work  $W$ , by minimizing the utility functional [47]

$$F_\alpha[D, \bar{x}] = V[D, \bar{x}] + \alpha W[D, \bar{x}] \quad (4)$$

over  $D$  and  $\bar{x}$ . Here, we have made the functional dependence of  $V$  and  $W$  on  $D(t)$  and  $\bar{x}(t)$  explicit. The mean position in the

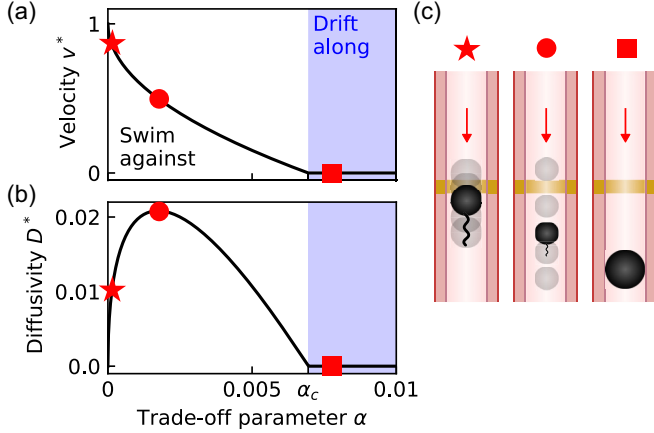


FIG. 2. Static Protocol. (a) Optimal swim velocity and (b) optimal diffusivity as a function of the tradeoff parameter  $\alpha$ . (c) schematic illustration of expensive  $\star$ , intermediate  $\bullet$ , and free  $\blacksquare$  protocols. The solid black particles indicate the mean at the end of the protocol, while the transparent black particles illustrate the diffusive spread. The different sizes of particles and flagella represent the changed mobility (diffusivity) and swim velocity, respectively.

laboratory frame  $\bar{x}(t)$  encapsulates both the controlled force  $f(t) = [\bar{x}(t) + 1]/D(t)$  and the initial condition  $x_i = \bar{x}(0)$ . The tradeoff parameter  $\alpha > 0$  prioritizes accuracy (small  $V$ ) when it is small and fuel saving (small  $W$ ) when it is large. By varying  $\alpha$  we obtain the Pareto front [47,48] for the tradeoff between  $V$  and  $W$  shown in Fig. 1(b). From it, one can read off the best accuracy achievable for a given fuel budget.

### III. STATIC PROTOCOLS

We start by considering protocols with fixed swim velocity  $v$  and diffusivity  $D$ , which are straightforward to implement in experiments and provide a reference for the time-dependent protocols considered below. Inserting the diffusivity  $D$  and the mean  $\bar{x}(t) = x_i + (v - 1)t$  into the utility functional (4) yields the function

$$F_\alpha(D, v, x_i) = D + \frac{(v - 1)^2}{3} + x_i(v - 1) + x_i^2 + \alpha \frac{v^2}{D}. \quad (5)$$

By solving  $\partial F_\alpha / \partial D = 0$  and  $\partial F_\alpha / \partial x_i = 0$ , we obtain the optimal diffusivity  $D^* = \sqrt{\alpha}|v|$  and initial condition  $x_i^* = (1 - v)/2$  that minimize  $F_\alpha$  for given  $v$ . Note that the swimmer is optimally initialized such that its time-averaged mean is zero. Inserting  $D^*$  and  $x_i^*$  in Eq. (5) gives the utility function  $F_\alpha(v) = 2\sqrt{\alpha}|v| + (v - 1)^2/12$ , which has a kink at  $v = 0$ . Depending on  $\alpha$ , this function can be minimal either at the nonzero swim velocity where  $\partial F_\alpha / \partial v$  vanishes or at the kink. We obtain the optimal velocity  $v^* = 1 - 12\sqrt{\alpha}$  for  $\alpha \leq \alpha_c = 1/144$  and  $v^* = 0$  otherwise.

The optimal swim velocity and diffusivity depend characteristically on the tradeoff parameter  $\alpha$ , as shown in Figs. 2(a) and 2(b). With increasing  $\alpha$ , the optimal swim velocity monotonically decreases from the flow velocity  $u = 1$  to 0, while the optimal diffusivity first increases from 0 to about 0.02, before decreasing again to 0. This behavior can be understood by considering the three protocols sketched in Fig. 2(c). For expensive protocols, the diffusivity (and hence mobility)

can be small while actively swimming at a velocity close to  $u$  against the flow. For protocols of intermediate cost, one chooses smaller velocity and larger diffusivity. Beyond  $\alpha_c$ , the optimal velocity and diffusivity are zero. This protocol is free of any cost and avoids thermal fluctuations, but results in a large average deviation as the swimmer passively drifts past the target. The transition at  $\alpha_c$  from active swimming against the flow to passive drifting along is akin to a second-order phase transition with a jump in  $\partial_\alpha v^*$ . This is a common feature of multiobjective optimization problems [48,49].

By inserting the optimal static protocol into Eqs. (2) and (3), we obtain the inaccuracy  $V^* = \sqrt{\alpha}$  and work  $W^* = 1/\sqrt{\alpha} - 12$  for  $\alpha \leq \alpha_c$ . This can be rearranged to

$$V^*(W) = \frac{1}{W + 12}, \quad (6)$$

giving the minimal inaccuracy  $V^*$  for a given fuel budget  $W$  using a static protocol. This optimal tradeoff is shown in Fig. 1(b). Note that the inaccuracy  $V_c = V^*(W = 0) = 1/12$  is finite even if no work is performed, as the particle drifts along with the flow for a finite time. This protocol is much better than the naive protocol of initializing the swimmer at the target  $x_i = 0$ , swimming at the velocity of the flow  $v = 1$ , and tuning the cost through the diffusivity  $D$ . For this naive protocol one gets  $V = D$  and  $W = 1/D$ , such that inaccuracies  $V = 1/W$  diverge as the work goes to zero.

Let us examine the tradeoff using dimensionful quantities for the inaccuracy  $\tilde{V}$ , the flow velocity  $\tilde{u}$ , and the protocol time  $\tilde{\tau}$ . Achieving the same dimensionful inaccuracy  $\tilde{V} = V\tilde{u}^2\tilde{\tau}^2$  in a faster flow  $\tilde{u}$  or for longer times  $\tilde{\tau}$  requires smaller  $V$  and hence more work (measured in  $k_B T$ ). As an illustrative example we plug in a flow velocity  $\tilde{u} = 10 \mu\text{m/s}$  (typical swimming speed of bacteria) and a time  $\tilde{\tau} = 1 \text{ s}$ , resulting in a characteristic length  $\tilde{l} = \tilde{u}\tilde{\tau} = 10 \mu\text{m}$ . To reach an inaccuracy less than  $\tilde{V}_c = \tilde{l}^2 V_c \simeq (2.9 \mu\text{m})^2$ , active swimming is necessary. To localize the swimmer to  $\tilde{V} = 1 \mu\text{m}^2$ , Eq. (6) dictates a minimal work of  $\tilde{W} = 88 k_B T$ . The optimal swimmer has velocity  $\tilde{v} = 8.8 \mu\text{m/s}$  and diffusivity  $\tilde{D} = 0.88 \mu\text{m}^2/\text{s}$ . For a spherical swimmer in a liquid with viscosity  $\tilde{\eta} = 1 \text{ mPa s}$  (similar to water) the corresponding radius given by the Stokes-Einstein-Sutherland equation is  $\tilde{r} = k_B T / (6\pi\tilde{\eta}\tilde{D}) \simeq 0.25 \mu\text{m}$ . Swimmers of such size can be built, as demonstrated by porous drug-carrying active Janus particles reaching sizes below 100 nm [50].

### IV. DYNAMIC PROTOCOLS

To further improve the work-accuracy tradeoff, one can implement time-dependent protocols for the control parameters of the model. For instance, one can consider a setup where the propulsion force and hence the swim velocity  $v(t)$  are controlled in a time-dependent way, while the shape of the swimmer and hence its diffusivity are time independent. The corresponding Pareto front in Fig. 1(b) (labeled “fixed  $D$ ”) shows that such a dynamic protocol does improve the accuracy compared to the static protocols for any given work budget. However, this improvement is rather small. The calculations leading to this Pareto front are shown in Appendix A; they follow similar steps as in the more general setup we here focus on instead. We consider simultaneous time-dependent

control of both the velocity  $v(t)$  and the diffusivity  $D(t)$  of a swimmer. We now demonstrate that this kind of control leads to a significant advantage in the tradeoff.

The goal of the optimization is again to minimize the utility functional  $F_\alpha$  in Eq. (4), where we now also admit time-dependent protocols  $D(t)$ ,  $\bar{x}(t)$ . Recall the relation of the latter to the swim velocity  $v(t) = \dot{\bar{x}}(t) + 1$ . Straightforward optimization via  $\delta F_\alpha / \delta D = 0$  gives the diffusivity

$$D^*(t) = \sqrt{\frac{\alpha}{2(1-t)}} |\dot{\bar{x}}(t) + 1|. \quad (7)$$

The scaling  $D^* \propto \sqrt{\alpha} |v(t)|$  resembles the static case: a large diffusivity mitigates cost, as required for fast swimming and if  $\alpha$  puts a strong priority on reducing cost. The additional factor  $1/\sqrt{2(1-t)}$  reduces fluctuations at the start of the protocol, and increases them toward the end. This is beneficial, because early fluctuations worsen the overall accuracy for the remainder of the protocol time.

From now on, we use this optimal diffusivity  $D^*(t)$  and search for the optimal protocol for the mean position  $\bar{x}^*(t)$ . Inserting  $D^*(t)$  of Eq. (7) in Eq. (4) yields the functional depending only on  $\bar{x}(t)$

$$F_\alpha[\bar{x}] = \int_0^1 dt L(\bar{x}(t), \dot{\bar{x}}(t), t) \quad (8)$$

with the Lagrangian

$$L(\bar{x}, \dot{\bar{x}}, t) = 2\sqrt{2\alpha(1-t)} |\dot{\bar{x}} + 1| + \bar{x}^2. \quad (9)$$

The corresponding Euler-Lagrange equation is

$$\begin{aligned} 0 &= \frac{\delta F_\alpha}{\delta \bar{x}} = \frac{\partial L}{\partial \bar{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\bar{x}}} \\ &= 2\bar{x} - \frac{d}{dt} 2\sqrt{2\alpha(1-t)} \operatorname{sgn}(\dot{\bar{x}} + 1). \end{aligned}$$

Requiring  $v = \dot{\bar{x}} + 1 > 0$  yields as candidate for the optimal protocol

$$\bar{x}_s(t) = -\sqrt{\frac{\alpha}{2(1-t)}} \quad \text{for } t < t_{\max} = 1 - \frac{\alpha^{1/3}}{2}, \quad (10)$$

which is consistent with the requirement only up to the time  $t_{\max}$ . Here, the swimmer actively swims at velocity  $v_s(t) = \dot{\bar{x}}_s(t) + 1 = 1 - \sqrt{2\alpha(1-t)}^{-3/2}/4 > 0$  against the flow. For further candidates for optimal protocols, note that the Lagrangian has a kink at  $\dot{\bar{x}} + 1 = 0$ , which can be a minimum for a given time  $t$ . Consequently, protocols  $\bar{x}_d(t) = c - t$  with arbitrary offset  $c$ , where the swimmer drifts passively with the flow, can also be optimal in parts of the protocol.

The truly optimal protocol  $\bar{x}^*(t)$  for each tradeoff parameter  $\alpha$  must be pieced together from the swim protocol  $\bar{x}_s(t)$  and the appropriately shifted drift protocols  $\bar{x}_d(t)$ , as shown in Fig. 3(a). Provided that the optimal protocol  $\bar{x}(t)$  has no discontinuous jumps, the most general ansatz for it is that the swimmer starts drifting until it reaches  $\bar{x}_s(t)$  at time  $t_1$ , then it swims until the time  $t_2$ , and afterwards continues drifting until the end of the protocol [note that it can never reach  $\bar{x}_s(t)$  again, because  $v_s(t) > v_d = 0$  for  $t < t_{\max}$ ]. The initial position  $x_i$  and final mean position  $\bar{x}_f$  for this ansatz are determined by  $t_1$  and  $t_2$  as  $x_i = t_1 + \bar{x}_s(t_1)$ ,  $\bar{x}_f = 1 - t_2 + \bar{x}_s(t_2)$ . The assumption that there are no jumps during the protocol is validated

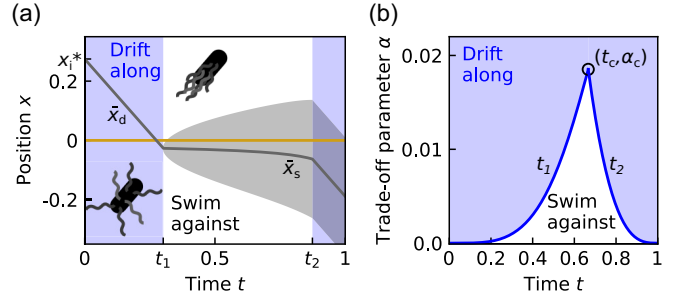


FIG. 3. Dynamic protocol. (a) Optimal protocol for tradeoff parameter  $\alpha = 0.001$ . The grey line is the optimal mean  $\bar{x}^*(t)$ , the grey area is the standard deviation caused by the optimal diffusivity  $D^*(t)$ , and the golden line is the target. The drifting phase with zero mobility is colored in blue and illustrated by a bacterium with spread flagella. The swimming phase is colored in white and illustrated by a bacterium with bundled flagella. (b) Optimal switching times  $t_1$  and  $t_2$  over tradeoff parameters  $\alpha$ . When  $\alpha$  is increased, the cost is reduced by shortening the swimming period. At the critical value  $\alpha_c$ , the start and end of the swim period meet at  $t_c = 2/3$ . Beyond  $\alpha_c$ , it is optimal to always drift along.

in Appendix B by mapping the Lagrangian optimization to a Hamiltonian problem and studying its dynamics.

Inserting the drift-swim-drift ansatz into the utility functional in Eq. (8) yields a function  $F(t_1, t_2)$ . In Appendix C we show that optimizing this function gives an explicit expression  $t_2 = 1 - (2\alpha)^{1/3}$  for the second switch time and an implicit expression  $\alpha = (t_1^4 - t_1^5)/(8 - 8t_1 + 2t_1^2)$  for the first switch time. Hence, the optimal dynamic protocol can be expressed explicitly in terms of  $t_1$ , instead of  $\alpha$ . The resulting optimal switching times are shown in Fig. 3(b). When the tradeoff parameter  $\alpha$  is increased, the cost is reduced by shortening the swim duration. At a critical value  $\alpha_c = 1/54$ , the start and end points of the swimming phase meet at  $t_c = 2/3$ . For  $\alpha > \alpha_c$  it is best to always drift along. This transition is analogous to a second-order phase transition, similar to the result for static protocols, but at a larger value of  $\alpha_c$ .

We can intuitively understand the properties that characterize an optimal protocol, such as the one shown in Fig. 3(a). The swimmer starts upstream and drifts along, with a vanishing mobility (and hence vanishing diffusivity). This saves cost early in the protocol, and avoids fluctuations that would contribute to the inaccuracy over its remaining duration. Soon after the swimmer passes the target, it actively swims against the flow to stay close to it. Swimming at finite cost requires nonzero mobility, and consequently, diffusive fluctuations cannot be avoided. Toward the end of the protocol, the swimmer drifts away with zero mobility, which again avoids cost, while the increasing displacement only affects a short time period.

Inserting the optimal protocols into the functionals for inaccuracy  $V$  [Eq. (2)] and work  $W$  [Eq. (3)] yields the Pareto front shown in Fig. 1(b). These dynamical protocols perform significantly better than the static protocols. Revisiting the example presented above: to reach an inaccuracy of  $V = 0.01$ , the static protocol requires a work of  $88 k_B T$  while the dynamic protocol only requires  $57.3 k_B T$ .

## V. GENERAL INITIAL CONDITIONS

In many scenarios, the initial position  $x_i$  cannot be chosen freely due to practical limitations, such as limited external access to the flow. A fixed initial condition imposes the constraint  $\bar{x}(0) = x_i$  on the optimization in Eq. (8). This constraint preserves the form of the drift and swim protocols, but influences the optimal start time  $t_1$  and end time  $t_2$  of the swim period, and can make an instantaneous jump from  $x_i$  to another position  $x_{0+}$  at the beginning of the protocol beneficial.

To understand the optimal protocols for general initial conditions, we need to explore the role of instantaneous jumps. As discussed above, the Hamiltonian formalism in Appendix B shows that there are no jumps during the optimal protocol. In contrast, initial and final jumps can be optimal. A jump at time  $t'$  from  $x_a$  to  $x_b$  corresponds to a delta peak in the laboratory frame velocity  $\dot{\bar{x}} = (x_b - x_a)\delta(t - t')$ . Integrating the Lagrangian of Eq. (9) over the peak gives the utility contribution

$$\Delta F(x_a \rightarrow x_b, t') = 2\sqrt{2\alpha(1 - t')}|x_b - x_a|. \quad (11)$$

Note that, for finite diffusivity, jumps are unphysical, because the swim velocity enters the work in Eq. (3) quadratically, causing the integral over the delta peak in the velocity to diverge. Yet, choosing the similarly peaked, optimal diffusivity  $D \propto |\dot{\bar{x}} + 1|$  following Eq. (7) eliminates the divergence. Consequently, the jump results not only in a power spike that contributes to the work but also in a diffusivity spike that contributes to the inaccuracy. The jump utility (11) of the final jump vanishes, so that an imposed final position does not affect the optimal protocol for  $t < 1$ . In contrast, the initial jump from the initial position  $x_i$  to the jump target  $x_{0+}$  comes at the utility cost  $\Delta F(x_i \rightarrow x_{0+}, 0) = 2\sqrt{2\alpha}|x_{0+} - x_i|$  and needs to be optimized. The optimal solution balances the additional work and diffusive fluctuations associated with the jump against the reduced offset of the mean.

The position after the initial jump  $x_{0+}$  determines when and if the drifting particle meets the swim solution. The optimal protocol can therefore contain an initial jump or not, and afterwards follow a drift-swim-drift or a pure drift protocol. In Appendix C we analytically find the optimal jump target  $x_{0+}$  for all optimality candidates. Afterwards, we numerically compare the utility of the candidates to find the optimal protocol. Figure 4 shows the optimal protocol for each initial position  $x_i$  and tradeoff parameter  $\alpha$ , revealing four qualitatively distinct regions in parameter space.

When the initial offset from the target is small, it is best not to jump (green and blue regions). In the green region, the drifting particle reaches the swim solution before  $t_2^*$  (optimal end time of the swim period with  $x_i^*$ ). The particle then swims until  $t_2^*$  and drifts afterwards. In the blue region, the drifting particle does not reach the swim solution before  $t_2^*$  and drifts throughout the entire protocol. The optimal initial condition  $x_i^*$  lies in the drift-swim-drift region for  $\alpha < \alpha_c$  and enters the pure drift region for  $\alpha \geq \alpha_c$ .

When the initial offset from the target is large, a jump toward the target becomes optimal (orange and red regions). The jump target lies on the boundary toward the neighboring region. A particle starting downstream of the swim protocol (orange region) jumps onto the swim protocol and swims until

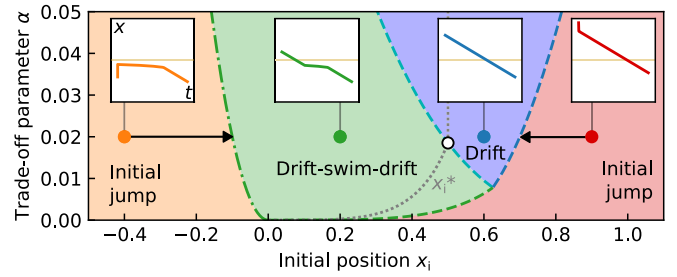


FIG. 4. General initial conditions. Optimal protocol for different fixed initial positions  $x_i$  and tradeoff parameters  $\alpha$ . There are four qualitatively distinct regions, illustrated by the examples in the insets. In the orange and red regions, jumps toward the target (onto the boundary of the neighboring region) are optimal, while in the green and blue regions it is best not to jump. The optimal initial condition  $x_i^*$  is the grey dotted line. It crosses from drift-swim-drift to the drift protocol at  $\alpha_c$ .

$t_2^*$ . For particles starting far upstream (red region), a jump onto the drift-swim-drift protocol is optimal for small  $\alpha$ , while a jump to a drift protocol is better for large  $\alpha$ .

It is important to note that the initial condition  $x_i$  determines whether jumps are beneficial, while the positions of the jump targets are independent of the initial condition. This highlights the difference from optimal state-to-state transformations, like moving a particle in a trap, where the initial jump distance qualitatively depends on the target distance [29].

## VI. CONCLUSION

In this paper, we have revealed a tradeoff between work and accuracy for a microswimmer that aims to localize near a target while being subject to a constant flow. The optimal time-dependent strategies switch between a passive drift state and an active swim state, with discontinuous jumps in swim velocity and diffusivity during the protocol.

Our findings identify swim velocity and diffusivity as powerful control handles for optimal navigation strategies, which so far had focused on the control of swim direction. Our findings provide an analytically traceable and nontrivial benchmark in the search for efficient multiparameter control of active matter [51]. Even if velocity and diffusivity are fixed during the experiment, our optimal static protocols can guide experimentalists when tailoring colloid size and swim speed to the environment's flow speed and protocol time.

The proof of principle established here opens the door to new questions. Extending the model to two and three dimensions, accounting for nonuniform or time-dependent flows, and incorporating thermodynamically consistent feedback control are natural next steps. Furthermore, experimental realizations of tunable diffusivity, such as shape or volume changes, or (un)binding from passive objects advected in the flow, should be explored, with an eye on the required additional work. For most cases, there will be practical limitations on the range of control parameters [31]. Addressing these limitations, as well as uncertainties in the initial conditions and the environment, can pave the way toward real-life applications. Even if the idealized full control of diffusivity and swim speed cannot be achieved in those applications, our

results provide theoretical limits on cost and accuracy that serve as ultimate benchmarks.

### ACKNOWLEDGMENT

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### DATA AVAILABILITY

The data that support the findings of this article are openly available [52].

### APPENDIX A: DYNAMIC VELOCITY AND FIXED DIFFUSIVITY

In this section, we discuss the scenario in which the velocity  $v(t)$  can be varied, while the diffusivity  $D$  is fixed throughout the protocol. In this case, the inaccuracy  $V$  and the work  $W$  given in Eqs. (2) and (3) simplify to

$$V = D + \int_0^1 dt \bar{x}^2(t), \quad W = \frac{1}{D} \int_0^1 dt (\dot{\bar{x}}(t) + 1)^2. \quad (\text{A1})$$

The utility functional  $F_\alpha = V + \alpha W$  in Eq. (4) can be optimized with respect to the mean position  $\bar{x}$  by solving the Euler-Lagrange equation

$$0 = \frac{\delta F_\alpha}{\delta \bar{x}} = 2\bar{x}(t) - 2\frac{\alpha}{D}\ddot{\bar{x}}(t).$$

The ansatz for the solution reads

$$\bar{x}(t) = c_1 e^{-\lambda t} + c_2 e^{\lambda t}, \quad (\text{A2})$$

with  $\lambda = \sqrt{D/\alpha}$ . Inserting this in the inaccuracy and working in Eq. (A1) results in a utility function  $F_\alpha(c_1, c_2, D) = V + \alpha W$ . The utility function is optimized with respect to  $c_i$  by solving  $\partial F_\alpha / \partial c_i = 0$ , which yields  $c_1 = e^\lambda / [\lambda(1 + e^\lambda)]$  and  $c_2 = -1 / [\lambda(1 + e^\lambda)]$ . Inserting the optimal  $c_i$  into Eq. (A2) results in the optimal mean trajectory,

$$\bar{x}^*(t) = \frac{1}{\lambda(1 + e^\lambda)} (e^{\lambda(1-t)} - e^{\lambda t}).$$

To obtain the optimal diffusivity, we insert  $c_i$  in the utility function, which yields

$$F_\alpha(D) = D + \frac{\alpha}{D} - \frac{2\alpha^{3/2}}{D^{3/2}} \frac{e^{\sqrt{D/\alpha}} - 1}{e^{\sqrt{D/\alpha}} + 1}.$$

We numerically minimize the function for fixed  $\alpha$  to obtain the optimal  $D$  shown in Fig. 5(a). As for the static protocol shown in Fig. 2, the diffusivity nonmonotonically depends on the tradeoff parameter  $\alpha$ . At a critical value  $\alpha_c$ , there is a discontinuity in  $\partial D^* / \partial \alpha$  analogous to a second-order phase transition. Beyond  $\alpha_c$ , it is best to drift by setting swim velocity and diffusivity to zero. As  $\alpha$  approaches the critical value and the diffusivity approaches 0, we can calculate  $\lim_{D \rightarrow 0} \partial F_\alpha / \partial D = 1 - 1/(120\alpha)$  using L'Hôpital's rule, which yields  $\alpha_c = 1/120$ . Note that the critical value  $\alpha_c$  is larger than for the static protocol but smaller than for the fully dynamic protocol.

Inserting the optimal parameters into  $W$  and  $V$  yields the work-accuracy tradeoff shown in Fig. 1(b). The protocol with

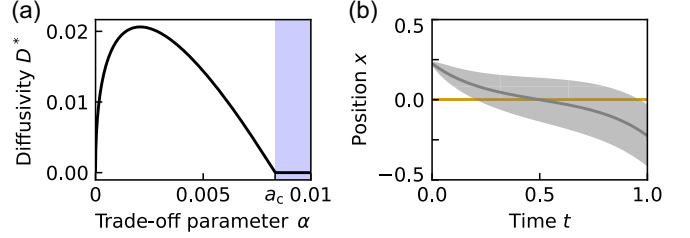


FIG. 5. Dynamic velocity and fixed diffusivity. (a) Optimal diffusivity  $D^*$  as a function of the tradeoff parameter  $\alpha$ . (b) Optimal protocol with dynamic velocity  $v(t)$  but constant diffusivity  $D$  for  $\alpha = 0.001$ . As in Fig. 3(a), the grey line is the optimal mean, the grey area is the standard deviation caused by the optimal diffusivity, and the golden line is the target.

dynamic velocity but static diffusivity performs slightly better than the one in which both parameters are static, but significantly worse than the one in which both are dynamic. This highlights the benefits of tunable diffusivity.

The optimal protocol for  $\alpha = 0.001$  is shown in Fig. 5(b). Remarkably,  $\bar{x}^*(t)$  is symmetric under simultaneous time and parity inversion. To understand why, we adapt the symmetry argument in Ref. [32]. Consider a mean trajectory  $q(t)$  and its time and parity reversed image  $\tilde{q}(t) = -q(1-t)$ . Both work and inaccuracy in Eq. (A1) are invariant under this transformation. Because the functional is quadratic in  $q$  and  $\dot{q}$ , we expect a unique extremum with  $q^*(t) = \tilde{q}^*(t)$ . The invariance of the utility under the inversion breaks for time-dependent diffusivity, as becomes apparent from the Lagrangian in Eq. (9). As a consequence, the optimal protocols for dynamic  $D(t)$  are no longer symmetric under time and parity inversion as shown in Fig. 3(a).

### APPENDIX B: HAMILTONIAN FORMULATION

In this section, we map the Lagrangian optimization problem [minimize  $F_\alpha = \int_0^1 dt L(q, \dot{q}, t)$ ] stated in Eq. (8) to a Hamiltonian problem [solve the equations of motion of the Hamiltonian  $H(q, p, t)$ ].

We start from the Lagrangian

$$L(q, \dot{q}, t) = 2\sqrt{2\alpha(1-t)}|\dot{q} + 1| + q^2,$$

writing the mean position as  $\bar{x} = q$ . The conjugated momentum is  $p = \partial L / \partial \dot{q} = b(t) \text{sgn}(\dot{q} + 1)$  with  $b(t) = 2\sqrt{2\alpha(1-t)}$ . To formulate the Hamiltonian, we require the inverted relation  $\dot{q}(p)$ . For  $|p| < b(t)$  one has  $\dot{q}(p) = -1$ , while for  $|p| \geq b(t)$  the relation is not defined. The Hamiltonian can be computed using a Legendre transform as  $H(q, p, t) = p\dot{q} - L = -p - q^2$  for  $|p| < b(t)$ . The resulting equations of motion  $\dot{q} = \partial_p H$  and  $\dot{p} = -\partial_q H$  can evolve the conjugated momentum  $p$  out of the Hamiltonian's region of definition. To ensure that the Hamiltonian is defined on the relevant range of  $p$ , we add a regularization term  $O(\epsilon)$  to the Lagrangian

$$L(q, \dot{q}, t) = 2\sqrt{2\alpha(1-t)} \left( |\dot{q} + 1| + \frac{\epsilon}{2} (\dot{q} + 1)^2 \right) + q^2.$$

Similar regularization terms have been used to suppress numerical artifacts and to account for the energetic cost of tuning

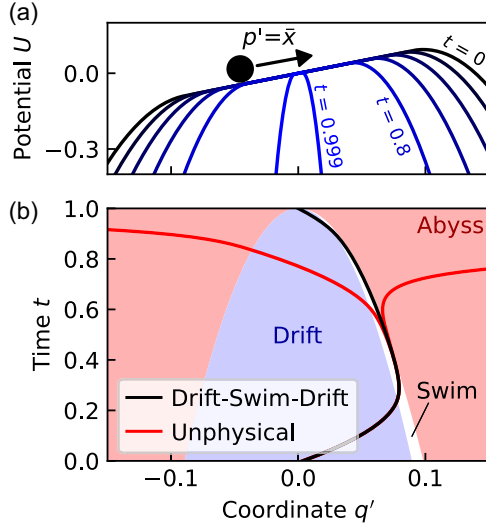


FIG. 6. Hamiltonian description. (a) The Lagrangian optimization problem can be mapped to a ball with momentum  $p' = \bar{x}$  in a time-dependent potential. The linear part of the potential corresponds to the drift protocol, and the nonlinear region between the linear region and the location of the maximal potential corresponds to the swim protocol. The abyss beyond drift and swim protocols results in divergences for  $\epsilon \rightarrow 0$ . (b) Swim, drift, and abyss region. The black line is the solution to the Hamiltonian problem with  $p'(0) = x_i$  and  $q'(1) = 0$  [for which  $q'(0) \approx 0.00357$ ]. The solution is sensitive to initial conditions, as shown by the red curves, generated with  $q'(0) = 0.00355, 0.00360$ . Here we chose  $\alpha = 0.001$  and  $\epsilon = 0.1$  and integrate the dynamics using the Julia package `DifferentialEquations` [53].

the control parameters [51]. For  $\epsilon \rightarrow 0$ , the original Lagrangian of Eq. (9) is recovered. The conjugated momentum is  $p = \partial L / \partial \dot{q} = b(t)[\text{sgn}(\dot{q} + 1) + \epsilon(\dot{q} + 1)]$ . The Hamiltonian follows as the Legendre transform

$$H(q, p, t) = p\dot{q} - L = \frac{[|p| - b(t)]^2}{2\epsilon b(t)} \theta(|p| - b(t)) - p - q^2,$$

and is defined for all  $q, p$ . While this regularization eliminates instantaneous jumps, they would be recovered as diverging velocities for  $\epsilon \rightarrow 0$ . In the following, we show that such divergences do not occur during the optimal protocol.

To get an intuition for the Hamiltonian, it is instructive to exchange momentum and coordinate ( $p \rightarrow q', q \rightarrow p'$ , and  $H \rightarrow -H'$ ). We obtain a Hamiltonian with the mechanical interpretation of a pointlike ball in a time-dependent potential,

$$H'(q', p', t) = -\frac{[|q'| - b(t)]^2}{2\epsilon b(t)} \theta(|q'| - b(t)) + q' + p'^2.$$

The first terms are a time-dependent potential  $U(q', t)$ , shown in Fig. 6(a), while the last term corresponds to a kinetic energy  $p'^2/2m$  with mass  $m = 1/2$ . After the mapping, the mean position of the original problem corresponds to the momentum  $p' = \bar{x}$ . Therefore, the initial momentum is set by the initial position  $p'(0) = x_i$ . But, for a unique solution, we require an additional constraint, which we find by studying the potential  $U(q', t)$ .

In the region between  $-b(t)$  and  $b(t)$ , the potential has a constant slope, such that  $\dot{x} = p' = -\partial_{q'} U = -1$ . This region corresponds to the drift protocol, where the swimmer drifts along with the flow. Beyond this region, the potential additionally decreases quadratically with a prefactor of order  $O(1/\epsilon)$ . This gives a monotonic potential for  $q' < 0$ , while for  $q' > 0$  there is a maximum at  $q_{\max} = b(t)(1 + \epsilon)$ . Between the connection point  $b(t)$  and the maximum  $q_{\max}$  there is a region of a width of order  $O(\epsilon)$ , in which the acceleration  $\ddot{x} = \dot{p}' = -\partial_{q'} U$  is between  $-1$  and  $0$ . This region corresponds to the swim protocol where the particle actively swims against the flow. The regions of drift and swim protocol are shown in Fig. 6(b). Beyond  $-b(t)$  and  $q_{\max}(t)$  the particle is accelerated outward with acceleration of order  $O(1/\epsilon)$ . Because  $-b(t)$  and  $q_{\max}(t)$  constantly move toward  $0$ , a particle crossing the boundary outwards can never come back, and in the limit  $\epsilon \rightarrow 0$ , the acceleration  $O(1/\epsilon)$  results in unphysical divergences. Therefore, a physical solution is confined to the interval  $q' \in [-b(t), q_{\max}(t)]$ . For  $t = 1$ , we have  $-b(1) = q_{\max}(1) = 0$ , and consequently  $q'(1) = 0$ . Now we have a fully defined Hamiltonian problem with fixed initial momentum  $p'(0) = x_i$  and final coordinate  $q'(1) = 0$ . The solution for  $\alpha = 0.001$ ,  $\epsilon = 0.1$  is shown in Fig. 6(b). Such a solution always exists because (1) there exist trajectories over- and undershooting  $q'(1) = 0$ , and (2) the final position depends continuously on the initial condition for our system.

Solving this problem for  $\epsilon \rightarrow 0$  gives the optimal mean particle trajectory  $\bar{x}(t) = \lim_{\epsilon \rightarrow 0} p'(t)$  of the original problem. Note that in the drift and swim regions the acceleration is finite, while the outer region results in unphysical divergences for  $\epsilon \rightarrow 0$ . In contrast, a solution with a finite jump during the protocol would require an instance of diverging acceleration followed by finite acceleration of the ball. This is not possible in our system. Therefore, the optimal solution cannot have jumps in  $\bar{x}(t)$  during the protocol.

The Hamiltonian formulation also tells us that there can be at most one swim phase. For the ball to leave the swim region toward the drift region, its velocity needs to be more negative than the velocity of the connection point  $\dot{b}(t)$ . In the drift region, the ball feels a constant (negative) acceleration, while the acceleration of the connection point  $\dot{b}(t)$  gets monotonically more negative (and hence also the acceleration of the maximum  $\ddot{q}_{\max}(t)$ ). If the connection point reaches the ball again, it therefore must have a more negative velocity and more negative acceleration than the ball. And when the ball reenters the swim region, its acceleration becomes even less negative and the maximum  $q_{\max}(t)$  surpasses the ball in a time of order  $O(\epsilon)$ . As soon as the ball surpasses the maximum, it cannot come back, and the result diverges for  $\epsilon \rightarrow 0$ . Consequently, a physical solution can have at most one swim phase.

### APPENDIX C: CONSTRUCTING THE OPTIMAL PROTOCOL

We insert the drift-swim-drift ansatz in the utility functional (8), using the drift protocols  $\bar{x}_{d,1}(t) = t_1 + \bar{x}_s(t_1) - t$ ,  $\bar{x}_{d,2}(t) = t_2 + \bar{x}_s(t_2) - t$ , and the swim protocol  $\bar{x}_s$  given in

Eq. (10). Integration then gives

$$F_\alpha(t_1, t_2) = F_1(t_1) + F_2(t_2) \quad \text{for } t_1 \leq t_2 \quad (\text{C1})$$

with

$$F_1(t_1) = \frac{\alpha}{2} \frac{t_1}{1-t_1} - \sqrt{\frac{\alpha}{2(1-t_1)}} t_1^2 + \frac{t_1^3}{3} + \frac{4\sqrt{2\alpha}}{3} (1-t_1)^{3/2} - \frac{\alpha}{2} \ln(1-t_1),$$

$$F_2(t_2) = \frac{1}{3} (1-t_2)^3 - \frac{4\sqrt{2\alpha}}{3} (1-t_2)^{3/2} + \frac{\alpha}{2} \ln(1-t_2).$$

The switching times  $t_1$  and  $t_2$  can be optimized separately, as long as  $t_1 \leq t_2$ . To obtain the optimal  $t_2$ , we solve  $\partial_{t_2} F_\alpha = 0$  resulting in two real solutions  $t_2^+ = t_{\max}$  and  $t_2^* = 1 - (2\alpha)^{1/3}$ . Inserting in Eq. (C1) reveals  $F_2(t_2^*) = a[\ln(2\alpha) - 3]/6 < F_2(t_2^+)$ , such that  $t_2^*$  is optimal. The optimal  $t_1$  is the solution to

$$0 = \partial_{t_1} F_\alpha = \partial_{t_1} F_1 = \frac{2-t_1}{2(1-t_1)^2} \alpha + t_1^2 - \left( 2\sqrt{2(1-t_1)} + \sqrt{\frac{2}{1-t_1}} t_1 + [2(1-t_1)]^{-3/2} t_1^2 \right) \sqrt{\alpha}.$$

Note that  $\partial_{t_1} F_\alpha$  is a second-order polynomial in  $\sqrt{\alpha}$ . The implicit solution  $\alpha^* = \frac{t_1^4 - t_1^5}{8 - 8t_1 + 2t_1^2}$  is the only one satisfying  $t_1 \leq t_2^*$ .

#### APPENDIX D: PROTOCOLS FOR GENERAL INITIAL CONDITIONS

If the initial condition  $x_i$  is fixed, a jump  $x_i \rightarrow x_{0+}$  at the start can be beneficial. The position  $x_{0+}$  determines the time  $t_1$  when the drifting particle meets the swim protocol at  $x_{0+}(t_1) = t_1 + \bar{x}_s(t_1)$ , as shown in Fig. 7(a). Finding the optimal  $x_{0+}$  is consequently equivalent to finding the optimal  $t_1$ . Adding the jump penalty of Eq. (11) to the utility in Eq. (C1) gives

$$F_\alpha(t_1, t_2 | x_i) = F_1(t_1) + F_2(t_2) + 2\sqrt{2\alpha} |x_i - x_{0+}|.$$

As long as  $t_1 \leq t_2$ , the optimal  $t_2^*$  remains unchanged compared to Appendix C. For  $t_1 > t_2^*$ , the times  $t_1$  and  $t_2$  can no longer be optimized separately. However, in this case the optimal  $t_2$  is always  $t_2 = t_1$  because  $F_2(t_2)$  is monotonically growing on the interval  $(t_2^*, t_{\max})$ . Consequently, the swim protocol vanishes, and the particle only drifts. This scenario will be treated later and here we only need to consider  $t_1 < t_2^*$  and we can optimize  $t_1$  separately.

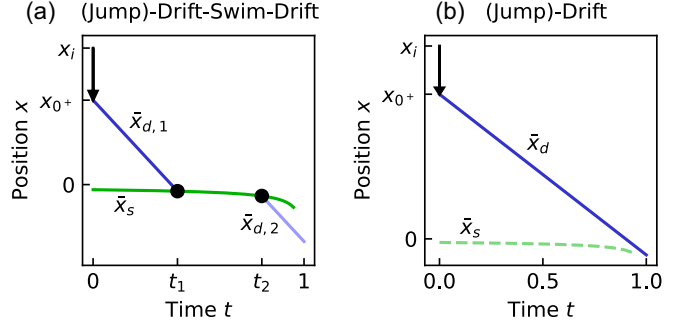


FIG. 7. Constructing the optimal protocol. Mean trajectory starting from  $x_i$ . (a) Ansatz if the target of the jump  $x_{0+}$  is such that the drifting particle meets the swim protocol  $\bar{x}_s$  at  $t_1 \leq t_2^*$ . Afterwards, it swims until time  $t_2^*$  and then drifts. (b) Ansatz if  $x_{0+}$  is such that the swim protocol is not reached before  $t_2^*$  and the particle drifts throughout the protocol. Here  $\alpha = 10^{-3}$ .

To find the optimal  $t_1$ , we use the derivative

$$\partial_{t_1} F_\alpha = -(2\sqrt{2\alpha} + \alpha(1-t_1)^{3/2})S + \partial_{t_1} F_1$$

with the direction of the jump  $S(x_i, t_1) = \text{sgn}(x_i - x_{0+})$  defined such that  $\text{sign}(0) = 0$ . As in Appendix C this is a second-order polynomial in  $\sqrt{\alpha}$  such that  $\partial_{t_1} F_\alpha = 0$  can be solved for  $\alpha$ , which gives an implicit solution  $\alpha = f(t_1 | S)$ . Other optimality candidates are the boundaries of the interval  $t_1 = 0$  and  $t_1 = t_2^*$ , representing swimming from the start and no swimming, respectively. If the initial condition is not optimized, the function also has a kink at  $x_i = x_{0+}$ , corresponding to no jump, which is also a candidate for the optimum. To find the optimal (jump)-swim-drift-swim protocol, the utility of the candidates needs to be compared.

The drift protocol does not necessarily meet the swim protocol in the interval  $[0, t_2^*]$ . Therefore we separately consider the pure drift protocol  $\bar{x} = x_{0+} - t$  after the initial jump  $x_i \rightarrow x_{0+}$ , as shown in Fig. 7(b). The corresponding utility function reads

$$F_\alpha(x_{0+} | x_i) = 2\sqrt{2\alpha} |x_i - x_{0+}| + x_{0+}^2 - x_{0+} + 1/3.$$

The optimality candidates are  $x_{0+} = x_i$  (no jump), where the function has a kink, and  $x_{0+} = \frac{1}{2} + S\sqrt{2\alpha}$  for which  $\partial_{x_{0+}} F_\alpha = 0$ .

To find the best protocol for general initial conditions  $x_i \neq x_i^*$ , the utilities of the optimality candidates of the jump-drift-swim-drift protocol and the jump-drift protocols are compared numerically. The resulting optima are shown in Fig. 4.

- [1] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, *Rep. Prog. Phys.* **75**, 126001 (2012).
- [2] A. C. Barato and U. Seifert, Thermodynamic uncertainty relation for biomolecular processes, *Phys. Rev. Lett.* **114**, 158101 (2015).
- [3] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Dissipation bounds all steady-state current fluctuations, *Phys. Rev. Lett.* **116**, 120601 (2016).

- [4] C. Hyeon and W. Hwang, Physical insight into the thermodynamic uncertainty relation using Brownian motion in tilted periodic potentials, *Phys. Rev. E* **96**, 012156 (2017).
- [5] J. M. Horowitz and T. R. Gingrich, Proof of the finite-time thermodynamic uncertainty relation for steady-state currents, *Phys. Rev. E* **96**, 020103(R) (2017).
- [6] J. M. Horowitz and T. R. Gingrich, Thermodynamic uncertainty relations constrain non-equilibrium fluctuations, *Nat. Phys.* **16**, 15 (2020).

- [7] Note that the word “accuracy” is sometimes used in the meaning of trueness, considering only the mean deviation from a target. We here follow the standard definition of accuracy (ISO 5725-1) requiring both trueness and precision.
- [8] L. Cocconi, B. Mahault, and L. Piro, Dissipation-accuracy tradeoffs in autonomous control of smart active matter, *New J. Phys.* **27**, 013002 (2025).
- [9] L. Cocconi, B. Mahault, and L. Piro, Nonequilibrium energetics of sensing and actuation by a smart active particle, *arXiv:2603.15602*.
- [10] T. Speck, J. Mehl, and U. Seifert, Role of external flow and frame invariance in stochastic thermodynamics, *Phys. Rev. Lett.* **100**, 178302 (2008).
- [11] A. Zöttl and H. Stark, Emergent behavior in active colloids, *J. Phys.: Condens. Matter* **28**, 253001 (2016).
- [12] G. Gompper, R. G. Winkler, T. Speck, A. Solon, C. Nardini, F. Peruani, H. Löwen, R. Golestanian, U. B. Kaupp, L. Alvarez, *et al.*, The 2020 motile active matter roadmap, *J. Phys.: Condens. Matter* **32**, 193001 (2020).
- [13] L. Baraban, M. Tasinkevych, M. N. Popescu, S. Sanchez, S. Dietrich, and O. Schmidt, Transport of cargo by catalytic Janus micro-motors, *Soft Matter* **8**, 48 (2012).
- [14] M. Popescu, M. Tasinkevych, and S. Dietrich, Pulling and pushing a cargo with a catalytically active carrier, *Europhys. Lett.* **95**, 28004 (2011).
- [15] D. Patra, S. Sengupta, W. Duan, H. Zhang, R. Pavlick, and A. Sen, Intelligent, self-powered, drug delivery systems, *Nanoscale* **5**, 1273 (2013).
- [16] J. R. Baylis, K. Y. Chan, and C. J. Kastrup, Halting hemorrhage with self-propelling particles and local drug delivery, *Thrombosis Res.* **141**, S36 (2016).
- [17] B. Wang, Y. Zhang, and L. Zhang, Recent progress on micro- and nano-robots: Towards *in vivo* tracking and localization, *Quant. Imaging Med. Surg.* **8**, 461 (2018).
- [18] M. E. Cates and J. Tailleur, Motility-induced phase separation, *Annu. Rev. Condens. Matter Phys.* **6**, 219 (2015).
- [19] I. Buttinoni, J. Bialké, F. Kümmel, H. Löwen, C. Bechinger, and T. Speck, Dynamical clustering and phase separation in suspensions of self-propelled colloidal particles, *Phys. Rev. Lett.* **110**, 238301 (2013).
- [20] Y. Fily and M. C. Marchetti, Athermal phase separation of self-propelled particles with no alignment, *Phys. Rev. Lett.* **108**, 235702 (2012).
- [21] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Novel type of phase transition in a system of self-driven particles, *Phys. Rev. Lett.* **75**, 1226 (1995).
- [22] J. Toner and Y. Tu, Flocks, herds, and schools: A quantitative theory of flocking, *Phys. Rev. E* **58**, 4828 (1998).
- [23] M. J. Falk, V. Alizadehyazdi, H. Jaeger, and A. Murugan, Learning to control active matter, *Phys. Rev. Res.* **3**, 033291 (2021).
- [24] H. Reinken, D. Nishiguchi, S. Heidenreich, A. Sokolov, M. Bär, S. H. Klapp, and I. S. Aranson, Organizing bacterial vortex lattices by periodic obstacle arrays, *Commun. Phys.* **3**, 76 (2020).
- [25] J. F. Schubert, S. F. Navas, and S. H. Klapp, Self-assembly and time-dependent control of active and passive triblock Janus colloids, *J. Chem. Phys.* **163**, 044904 (2025).
- [26] S. Blaber and D. A. Sivak, Optimal control in stochastic thermodynamics, *J. Phys. Commun.* **7**, 033001 (2023).
- [27] J. Alvarado, E. G. Teich, D. A. Sivak, and J. Bechhoefer, Optimal control in soft and active matter, *Annu. Rev. Condens. Matter Phys.* **17**, 327 (2026).
- [28] U. Seifert, *Stochastic Thermodynamics* (Cambridge University Press, Cambridge, 2025).
- [29] T. Schmiedl and U. Seifert, Optimal finite-time processes in stochastic thermodynamics, *Phys. Rev. Lett.* **98**, 108301 (2007).
- [30] S. Blaber, M. D. Louwerse, and D. A. Sivak, Steps minimize dissipation in rapidly driven stochastic systems, *Phys. Rev. E* **104**, L022101 (2021).
- [31] M. Baldovin, I. Ben Yedder, C. A. Plata, D. Raynal, L. Rondin, E. Trizac, and A. Prados, Optimal control of levitated nanoparticles through finite-stiffness confinement, *Phys. Rev. Lett.* **135**, 097102 (2025).
- [32] S. A. M. Loos, S. Monter, F. Ginot, and C. Bechinger, Universal symmetry of optimal control at the microscale, *Phys. Rev. X* **14**, 021032 (2024).
- [33] H. J. Kappen, Linear theory for control of nonlinear stochastic systems, *Phys. Rev. Lett.* **95**, 200201 (2005).
- [34] R. Garcia-Millan, J. Schüttler, M. E. Cates, and S. A. M. Loos, Optimal closed-loop control of active particles and a minimal information engine, *Phys. Rev. Lett.* **135**, 088301 (2025).
- [35] K. S. Olsen, R. Goerlich, Y. Roichman, and H. Löwen, Harnessing non-equilibrium forces to optimize work extraction, *Nat. Commun.* **16**, 11031 (2025).
- [36] S. Shankar, V. Raju, and L. Mahadevan, Optimal transport and control of active drops, *Proc. Natl. Acad. Sci. USA* **119**, e2121985119 (2022).
- [37] D. Gupta, S. H. L. Klapp, and D. A. Sivak, Efficient control protocols for an active Ornstein-Uhlenbeck particle, *Phys. Rev. E* **108**, 024117 (2023).
- [38] M. Baldovin, D. Guéry-Odelin, and E. Trizac, Control of active Brownian particles: An exact solution, *Phys. Rev. Lett.* **131**, 118302 (2023).
- [39] L. K. Davis, K. Proesmans, and É. Fodor, Active matter under control: Insights from response theory, *Phys. Rev. X* **14**, 011012 (2024).
- [40] B. Liebchen and H. Löwen, Optimal navigation strategies for active particles, *Europhys. Lett.* **127**, 34003 (2019).
- [41] L. G. Nava, R. Großmann, and F. Peruani, Markovian robots: Minimal navigation strategies for active particles, *Phys. Rev. E* **97**, 042604 (2018).
- [42] C. Calascibetta, L. Biferale, F. Borra, A. Celani, and M. Cencini, Optimal tracking strategies in a turbulent flow, *Commun. Phys.* **6**, 256 (2023).
- [43] L. Piro, A. Vilfan, R. Golestanian, and B. Mahault, Energetic cost of microswimmer navigation: The role of body shape, *Phys. Rev. Res.* **6**, 013274 (2024).
- [44] E. Schneider and H. Stark, Optimal steering of a smart active particle, *Europhys. Lett.* **127**, 64003 (2019).
- [45] S. Muñíos-Landin, A. Fischer, V. Holubec, and F. Cichos, Reinforcement learning with artificial microswimmers, *Sci. Rob.* **6**, eabd9285 (2021).
- [46] A. C. Barato and U. Seifert, Thermodynamic cost of external control, *New J. Phys.* **19**, 073021 (2017).

- [47] P. Ngatchou, A. Zarei, and A. El-Sharkawi, Pareto multi objective optimization, in *Proceedings of the 13th International Conference on Intelligent Systems Application to Power Systems* (IEEE, Piscataway, NJ, 2005), p. 84.
- [48] L. F. Seoane and R. Solé, Multiobjective optimization and phase transitions, in *Proceedings of ECCS 2014: European Conference on Complex Systems* (Springer, Berlin, 2016), p. 259.
- [49] J. C. Sunil, R. A. Blythe, M. R. Evans, and S. N. Majumdar, Minimizing the profligacy of searches with reset, *Phys. Rev. E* **110**, 054122 (2024).
- [50] X. Ma, K. Hahn, and S. Sanchez, Catalytic mesoporous Janus nanomotors for active cargo delivery, *J. Am. Chem. Soc.* **137**, 4976 (2015).
- [51] L. K. Davis, Optimal multi-parameter control of trapped active matter, [arXiv:2603.16778](https://arxiv.org/abs/2603.16778).
- [52] <https://github.com/tillwelker/Accuracy-Comes-at-a-Cost-Optimal-Localisation-Against-a-Flow>.
- [53] C. Rackauckas and Q. Nie, DifferentialEquations.jl—A performant and feature-rich ecosystem for solving differential equations in Julia, *J. Open Res. Software* **5**, 15 (2017).