

Non-Markov random telegraph processes with arbitrary nonlocal memory

S. S. Melnyk¹, O. V. Usatenko^{1,2,3} and V. A. Yampol'skii^{1,4}

¹*O. Ya. Usikov Institute for Radiophysics and Electronics NASU, 61085 Kharkiv, Ukraine*

²*Instituto de Física, Benemérita Universidad Autónoma de Puebla, 72570 Puebla, México*

³*Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warsaw, Poland*

⁴*V. N. Karazin Kharkiv National University, 61077 Kharkiv, Ukraine*



(Received 23 January 2025; revised 4 May 2025; accepted 27 June 2025; published 24 July 2025)

We study random dichotomous stationary ergodic processes with nonlocal in-time memory. These processes are characterized by a conditional transition probabilities distribution containing an integral term involving the memory function, which takes into account the past of the process. For the first time, we present a method for solving the integro-differential equation, which connects the arbitrary memory and pair correlation functions in an analytical closed form. Conditions of stationarity for the telegraph process with exponential memory are found. It is shown that the pair correlation function of this process can be of two types: either exponentially decreasing with two different decrements or oscillating with exponentially decreasing amplitude.

DOI: [10.1103/xvck-7n24](https://doi.org/10.1103/xvck-7n24)

I. INTRODUCTION

Markov processes are the most fundamental representations of random systems dynamics. According to the definition, the dynamics of Markov systems is determined only by their current state, representing the *local* memory of the system. Despite this limitation in the description of the dynamics of the systems, the Markov processes cover a wide range of phenomena. These include such concepts as the Wiener and dichotomous processes. The state space of a Wiener process is infinite and, in this sense, is maximal; the state space of dichotomous processes is minimal and consists of only two values.

Until the middle of the last century, the Markov processes were generally accepted and natural models for describing random phenomena [1–3]. However, in the 1960s, the situation changed, and the theory of processes with nonlocal memory began to be increasingly included in modern methods of studying random systems [4–11]. While the discrete non-Markov systems are naturally represented by higher-order Markov chains, the description of continuous-time non-Markov processes can be obtained by means of limiting transition when the distance between discrete random events tends to zero. It is precisely this scheme for describing random dichotomous processes that was implemented in Ref. [12], where the conditional transition probabilities distribution (CTPD) containing an integral term involving the memory function (the kernel of the integral term) of the process was obtained. Let us note here some terminological inconsistencies in the definition of random sequences with memory and processes with memory. If the former are called

higher-order Markov chains, then the latter are called non-Markov processes.

The CTPD defines the correlation properties of the processes with *given memory functions*. The pair correlation functions of the telegraph processes $T_x(t)$ with two exactly solvable examples of memory functions were analytically obtained in Ref. [12]. The authors of this work proposed two extreme forms, short and extended, for the memory function. The short-term memory form considers a single point located at a distance T from the current time. At the other extreme, a finite-length stepwise memory function can be used as an approximation for long-term memory.

The relation between the correlation and memory functions is a rather complex integro-differential equation [see Eq. (4)]. If in Ref. [12] the solution to this equation was presented in a recurrent form with respect to the parameter T , in this work, we propose a general method for solving this equation in a closed analytical form for an arbitrary memory function. In some sense, the method developed here is similar to the Green function method for solving ordinary differential equations. If the kernel of the integral equation is exponential, then this equation can also be solved by the standard Laplace transform method. In this exponential memory case, the condition of process stationarity is found, and it is demonstrated that the pair correlation function can be of two types: either exponentially decreasing or oscillating with exponentially decreasing amplitude.

The structure of this paper is as follows. In Sec. II, we present some general definitions and provide a brief description of the approaches used and the relevant previous results. An equation connecting the memory function and the two-point correlation function of the process is reproduced from Ref. [12] in a slightly different form in Sec. III. In Sec. IV, we present the main result of this paper: the general analytical solution of the equation for the correlation function. Section V contains the particular solution to the integro-differential equation with exponential memory function. The stationarity

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

of the process under study is considered in Sec. VI. The last two sections contain discussions, conclusions, and an outline for further research.

II. BASIC DEFINITIONS

Here we outline the procedure for obtaining the dynamical equation for the telegraph process $\mathcal{T}_x(t)$ with memory, and present the relation between the memory function $m(t)$ and correlation function $K(t)$; the reader can find a more detailed and slightly different way of deriving these equations in Ref. [12].

One of the main concepts of our consideration is the higher-order (N -step) Markov sequence. Its CTPD [13–15] describes the probability of finding the r -th element $x_r = a$ after N symbols $x_{r-N}, \dots, x_{r-1}$ in the *binary*, $a = \{0, 1\}$, Markov sequence. This distribution function completely determines the random chain as well as all its statistical properties and can be written in the following form [16–18]:

$$P(x_r = a | x_{r-N}^{r-1}) = p_a + (2a - 1) \sum_{r'=1}^N F(r')(x_{r-r'} - p_1),$$

$$p_0 + p_1 = 1. \quad (1)$$

If the values of $F(r)$ and p_a are such that the function [Eq. (1)] belongs to the interval $(0; 1)$ for any set of x_{r-N}^{r-1} and a , then the constructed sequence is ergodic. In what follows, we consider only this type of sequence.

For such sequences, due to the ergodicity, the parameter p_a plays the role of the one-point probability for the element x_r to take the value a , $p_a = P(x_r = a)$. We use the concise notation $x_{r-N}^{r-1} = x_{r-N}, \dots, x_{r-1}$ for a subsequence of N random symbols. The so-called memory function $F(r')$ describes the strength of influence of the previous symbols $x_{r-r'}$ ($1 \leq r' \leq N$) upon a generating one, x_r , and is related to the correlations in the sequence.

Starting from Eq. (1), we can show that the continuous-time telegraph processes can be considered as a generalization by means of limiting transition of the discrete-time random Markov sequence. Introducing a specific presentation of the function $F(r)$,

$$F(1) = 1 - \beta \Delta t, \quad F(r \geq 2) = m(r \Delta t)(\Delta t)^2, \quad (2)$$

the constants,

$$p_1 = \frac{\lambda_0}{\lambda_0 + \lambda_1}, \quad \beta = \lambda_0 + \lambda_1, \quad (3)$$

and going to the limit $\Delta t \rightarrow dt \rightarrow 0$ in Eqs. (1) and (2), we obtain two equations (for $a = 0$ and $a = 1$) for CTPD of the telegraph process $\mathcal{T}_x(t)$ with memory,

$$dP(x_{t+dt} = a | x_t = 1 - a; x_{t'} < t) = \left[\lambda_{1-a} + (2a - 1) \int_0^\infty m(\tau)(x_{t-\tau} - p_1) d\tau \right] dt. \quad (4)$$

In what follows, we call $m(\tau)$ the *memory function* of the telegraph process $\mathcal{T}_x(t)$. Note that the λ_0 - and λ_1 - terms also describe memory effects. However, unlike the integral nonlocal memory $m(\tau)$, this λ -memory is local.

Similarly to the discrete chain [Eq. (1)], the sufficient condition of the ergodicity of the process is the requirement of positivity of the probabilities in Eq. (4) for any a and prehistory $x_{t-\tau}$. The consequences of this condition are considered in more detail in Sec. VI.

III. EQUATION FOR THE CORRELATION FUNCTION OF THE TELEGRAPH PROCESS WITH MEMORY

An important statistical characteristic of a random process is the correlation function. To get the relation between the memory and the correlation functions, we start by obtaining a similar equation for the random discrete Markov chain with memory using the well-known definition of the correlation function,

$$K_r = \overline{(x_i - p_1)(x_{i+r} - p_1)} = \overline{x_i x_{i+r}} - p_1^2 = \overline{x_0 P(x_r = 1 | x_0)} - p_1^2. \quad (5)$$

It is convenient to write the expression for the CTPD, Eq. (1), in the form

$$P(x_{r+1} = 1 | x_{r-N+1}^r) = p_1 + (1 - \beta \Delta t)(x_r - \bar{x}) + \sum_{r'=1}^N m(r' \Delta t)(x_{r-r'} - \bar{x})(\Delta t)^2. \quad (6)$$

Multiplying Eq. (6) by x_0 and averaging over the ensemble of random sequences, we derive an equation for the correlation function of the random sequence,

$$K_{r+1} = (1 - \beta \Delta t)K_r + \sum_{r'=1}^N m_{r'} K_{r-r'} (\Delta t)^2, \quad r > 0. \quad (7)$$

A relation similar to Eq. (7) was obtained earlier by averaging over the coordinate r along the chain (see Ref. [19]), or by the variational method [18]. The coincidence of the results of two methods for averaging follows from the ergodicity of the sequences under study. It is important to note that for averaging the expression $x_0 P(x_{r+1} = 1 | x_{r-N+1}^r)$ over all variables, other than x_0 , the expression $P(x_{r+1} = 1 | x_{r-N+1}^r)$ becomes the conditional transitional probability distribution $P(x_{r+1} = 1 | x_0)$. This conclusion is true only for a special linear additive type of CTPD [Eqs. (1) and (6)] (see an explanation in the Appendix in Ref. [20]). As a consequence, we get $\overline{x_0 P(x_{r+1} = 1 | x_0)} - p_1^2 = K_{r+1}$.

Note that the similar equation for the correlation function is also valid for the autoregressive random sequences (see Yule-Walker equations in Refs. [21–23]).

Rewriting Eq. (7) in the form,

$$\frac{K_{r+1} - K_r}{\Delta t} = -\beta K_r + \sum_{r'=1}^N m_{r'} K_{r-r'} \Delta t, \quad r > 0, \quad (8)$$

and taking the limit $\Delta t \rightarrow dt \rightarrow 0$, we obtain an integro-differential equation for the correlation function $K_r \rightarrow K(t)$ of the random telegraph process $\mathcal{T}_x(t)$,

$$\frac{dK(t)}{dt} + \beta K(t) = \int_0^\infty m(\tau) K(t - \tau) d\tau, \quad t > 0. \quad (9)$$

This equation can be obtained not by passing to the limit $\Delta t \rightarrow 0$ in Eqs. (7) and (8), but by directly using the

continuous time equation [Eq. (4)] describing the telegraph process $\mathcal{T}_x(t)$.

The solution to Eq. (9) is subject to the initial condition

$$K(0) = \overline{x^2} - \bar{x}^2 = \frac{\lambda_0 \lambda_1}{(\lambda_0 + \lambda_1)^2}, \quad (10)$$

and, according to the definition [Eq. (5)], the parity of the correlation function,

$$K(-t) = K(t). \quad (11)$$

The initial condition, Eq. (10), for the correlation function of a telegraph process with memory coincides with the initial condition of a telegraph process without memory, since at the initial moment of time $t = 0$, the past is absent for both processes.

It is necessary to emphasize that Eq. (9) can also be used for solving an inverse problem of finding the unknown memory function $m(t)$ when the correlation function $K(t)$ of the process is given. Once $m(t)$ is calculated, we can generate the telegraph process with an arbitrary prescribed correlation function using the conditional transition probability distribution, Eq. (4), as was done in Ref. [18] for random additive higher-order Markov sequences.

IV. SOLUTION TO THE EQUATION FOR THE CORRELATION FUNCTION

In this section, we obtain the closed-form solution to Eq. (9) for the telegraph process $\mathcal{T}_x(t)$ with arbitrary memory function. First, let us introduce a modified [by the Dirac delta-function $\delta(t)$] memory function,

$$M(t) = m(t) - 2\beta\delta(t), \quad (12)$$

which takes into account both the local and nonlocal memory. Thus, Eq. (9) acquires a simpler form,

$$\frac{dK(t)}{dt} - \int_0^\infty M(\tau)K(t-\tau)d\tau = 0, \quad t > 0. \quad (13)$$

Then, we introduce an auxiliary function $V(t)$, which obeys the same equation [Eq. (13)] as $K(t)$ at positive time t ,

$$\frac{dV(t)}{dt} - \int_0^\infty M(\tau)V(t-\tau)d\tau = 0, \quad t > 0, \quad (14)$$

but satisfies different conditions at $t = 0$ and negative t :

$$V(0) = 1, \quad V(t < 0) = 0, \quad (15)$$

instead of Eqs. (10) and (11).

One can see that the linear form of the correlator,

$$K(t) = \int_0^\infty \chi(t')V(t+t')dt', \quad (16)$$

with an arbitrary function $\chi(t')$, satisfies Eq. (13). Due to the property of the function $V(t < 0) = 0$, the specific form of the weight function $\chi(t') = \chi_0 V(t')$ gives the boundary condition, Eq. (11). The coefficient χ_0 can be found from Eq. (10). So, as a result, we have an expression for $K(t)$ via the function $V(t)$,

$$K(t) = \chi_0 \int_0^\infty V(t')V(t+t')dt', \quad (17)$$

where

$$\chi_0 = K(0) / \int_0^\infty V^2(t)dt = \frac{\lambda_0 \lambda_1}{(\lambda_0 + \lambda_1)^2 \int_0^\infty V^2(t)dt}. \quad (18)$$

Applying the Laplace transform,

$$\widehat{V}(z) = \int_0^\infty \exp(-zt)V(t)dt, \quad (19)$$

to Eq. (14), we obtain the function

$$\widehat{V}(z) = \frac{1}{z - \widehat{m}(z) + \beta}. \quad (20)$$

Applying the reverse Laplace transform to Eq. (20), one can find the function $V(t)$, which, together with Eqs. (17) and (18), present the solution to Eq. (9) in the integral analytical closed form.

Let us note that the Fourier transform $\tilde{K}(\omega)$,

$$\tilde{K}(\omega) = \int_{-\infty}^\infty \exp(-i\omega t)K(t)dt, \quad (21)$$

of the solution [Eq. (17)] can be expressed in terms of the Fourier transform $\tilde{V}(\omega)$ of the function $V(t)$:

$$\begin{aligned} \tilde{K}(\omega) &= \chi_0 \tilde{V}(\omega) \tilde{V}(-\omega) \\ &= \frac{\chi_0}{(i\omega - \tilde{m}(\omega) + \beta)(-i\omega - \tilde{m}(-\omega) + \beta)} \\ &= \frac{\chi_0}{|i\omega - \tilde{m}(\omega) + \beta|^2}, \end{aligned} \quad (22)$$

where

$$\tilde{m}(\omega) = \widehat{m}(i\omega), \quad \tilde{m}(-\omega) = \tilde{m}^*(\omega). \quad (23)$$

The possibility of such a representation is related to the fact that the integral in the definition of the correlation function, Eq. (17), can be written in a form $\int_{-\infty}^\infty$ that in turn is related to the vanishing of the function $V(t)$ for negative values of the argument t .

Also note that in a certain sense, the function $V(t)$ in Eq. (16) plays the same role as the Green function in the theory of ordinary differential equations, since this function $V(t)$ represents the solution of the original integro-differential equation [Eq. (9)] in the form of a linear functional, Eq. (16).

Obviously, Eq. (9) and its solution can be generalized for the telegraph processes $x'(t)$ with arbitrary levels a_0 and a_1 instead of the case $a = \{0; 1\}$ by the linear transition $x'(t) = a_0 + (a_1 - a_0)x(t)$. This leads to the same transform for the process parameter $\bar{x}$ and to the correlator

$$K'(t) = (a_1 - a_0)^2 K(t). \quad (24)$$

Using this relation and Eq. (22), one can find the corresponding memory function $m(t)$ and construct the telegraph processes $x'(t)$.

Another way to solve this problem is to rewrite Eq. (4) for CTPD and all consequent relations directly for the telegraph process with $a = \{a_0, a_1\}$, but we do not do this here so as not to complicate the equations.

V. SOLUTION WITH EXPONENTIAL MEMORY FUNCTION

In this Section, as an example, we present the solution to Eq. (9) for the telegraph process $\mathcal{T}_x(t)$ with exponential memory function,

$$m(t) = \alpha \exp(-kt), \quad k > 0, \quad (25)$$

whose Laplace transform is

$$\hat{m}(z) = \alpha / (z + k). \quad (26)$$

Then, from Eq. (20), we get

$$\hat{V}(z) = \frac{z + k}{(z + \beta)(z + k) - \alpha}. \quad (27)$$

The correspondent function $V(t)$ is the sum,

$$V(t) = A_1 \exp(z_1 t) + A_2 \exp(z_2 t), \quad (28)$$

where the poles z_1 and z_2 are the roots of the denominator in Eq. (27):

$$z_{1,2} = \frac{1}{2}[-(k + \beta) \pm \sqrt{D}], \quad D = (k - \beta)^2 + 4\alpha. \quad (29)$$

From the expressions for residues of the function $\hat{V}(z)$ at the points z_1 and z_2 , we find the constants A_1 and A_2 ,

$$A_1 = \frac{z_1 + k}{z_1 - z_2}, \quad A_2 = \frac{z_2 + k}{z_2 - z_1}. \quad (30)$$

From Eqs. (17) and (28), we can conclude that the correlator $K(t)$ has the following form:

$$K(t) = \chi_0 [B_1 \exp(z_1 t) + B_2 \exp(z_2 t)], \quad t \geq 0, \quad (31)$$

where

$$B_1 = -\frac{A_1 A_1}{z_1 + z_1} - \frac{A_1 A_2}{z_1 + z_2}, \quad B_2 = -\frac{A_1 A_2}{z_1 + z_2} - \frac{A_2 A_2}{z_2 + z_2}, \quad (32)$$

and the coefficient χ_0 is given by Eq. (18). Note that the coefficients B_1, B_2, z_1 , and z_2 are, in general, complex. Despite that, due to the symmetric properties of the coefficients A_1, A_2 , the correlator [Eq. (31)] is real, according to the definition. The solution to Eq. (31) can be verified directly by substituting it into Eq. (13).

Another simple method of obtaining the solution to Eq. (9) for the telegraph process $\mathcal{T}_x(t)$ with exponential memory function is based on the use of the Laplace transform. However, this method works only in the case of an exponential memory and is not general.

The correlation functions $K(t)$ of the process $\mathcal{T}_x(t)$ with different values of memory constant α are presented in Fig. 1. Let us pay attention to the specific property of the correlation function $K(t)$ of the process with exponential memory. Positive values of the parameter α (the upper curve in Fig. 1) correspond to persistent correlations and result in an upward shift of the curve $K(t)$ compared with the case of the absence of nonlocal memory (the middle curve in Fig. 1). The antipersistent correlations are characterized by negative values of the parameter α . The correspondent function $K(t)$ is presented in Fig. 1 by the lower curve. The points on the upper and bottom curves are the results of numerical simulations. This

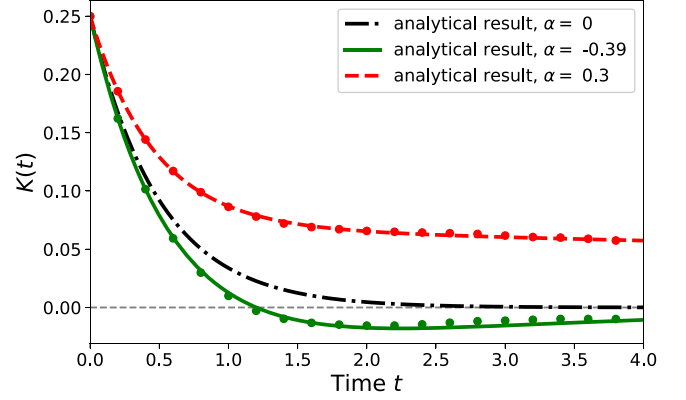


FIG. 1. The correlation functions $K(t)$ of the telegraph processes $\mathcal{T}_x(t)$ with exponential memory function $m(t) = \alpha \exp(-kt)$ for different values of the parameter α . The dashed upper curve ($\alpha = 0.3$) corresponds to the persistent correlations, and the solid bottom line ($\alpha = -0.39$) presents the case of the antipersistent correlations. The dash-dotted middle line corresponds to the process without nonlocal memory ($\alpha = 0$). The parameters used for the generated processes are $\lambda_0 = \lambda_1 = 1, k = 0.2$. The dashed and solid lines present the correlation functions given by Eqs. (31) and (32), and the points on the upper and bottom curves are the results of numerical simulations using Eq. (4). All presented solutions correspond to the same initial condition, Eq. (10). The length of the process time generation is 10^5 , and the time step of generation is $dt = 10^{-2}$.

simulation of process $\mathcal{T}_x(t)$ consists of two steps. First, using Eq. (4) with a small value $dt = 10^{-2}$, we generate the random telegraph process itself, the duration of which is 10^5 . Then we calculate the correlation function by the standard way using the constructed process and Eq. (5).

It is seen from Fig. 1 that the correlation functions obtained from the numerically generated telegraph processes with different constants α are in excellent agreement with the corresponding results for $K(t)$ calculated analytically [Eq. (31)].

VI. STATIONARITY OF THE PROCESS

The stationarity and ergodicity of the process impose the requirement of positivity of the probabilities in Eq. (4) at $a = 0$ and $a = 1$ for any prehistory $x_{t-\tau}$. Therefore, the possible values of parameters α and k in Eq. (25) are constrained by the combinations of parameters λ_0 and λ_1 ,

$$-(\lambda_0 + \lambda_1) \min\left(\frac{\lambda_0}{\lambda_1}, \frac{\lambda_1}{\lambda_0}\right) < \frac{\alpha}{k} < (\lambda_0 + \lambda_1). \quad (33)$$

These inequalities are sufficient conditions for the stationarity of the telegraph process $\mathcal{T}_x(t)$.

Note that under these conditions, the real parts of roots z_1, z_2 are negative, $V(t)$ and the correlator $K(t)$ exponentially tends to zero at $t \rightarrow \infty$ (see Fig. 1).

When the discriminant D in Eq. (29) is negative,

$$(k - \beta)^2 + 4\alpha < 0, \quad (34)$$

the roots z_1 and z_2 in Eq. (29) contain imaginary parts, and the correlation function oscillates with exponentially decreasing amplitude (the lower curve in Fig. 1).

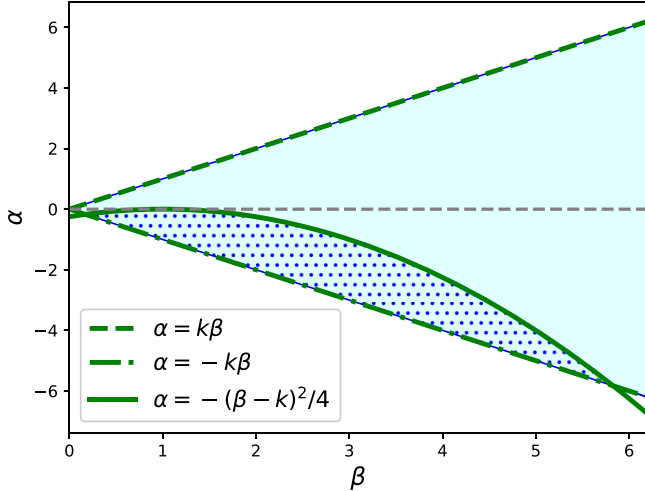


FIG. 2. The colored zone of stationarity for the telegraph process $\mathcal{T}_x(t)$ with exponential memory function $m(t) = \alpha \exp(-kt)$ is located between two straight lines. The region without dots corresponds to the exponentially decreasing correlation function. The dotted region corresponds to the oscillating correlation function with exponentially decreasing amplitude. The equations for lines are shown in the legend; the parameters used here are $\lambda_0 = \lambda_1$, $\beta = \lambda_0 + \lambda_1$, $k = 1$.

Figure 2 shows the stationarity and nonstationarity zones in the (α, β) plane for the telegraph process $\mathcal{T}_x(t)$ at $\lambda_0 = \lambda_1$.

VII. DISCUSSION

In this paper, we studied the properties of a non-Markov version of the telegraph process, whose non-Markov character comes from the evolution equation containing nonlocal memory. This type of memory in the form of an integral presented in the right-hand side of Eq. (9) was previously encountered in Mori's work [5], where a generalized form of the Langevin equation in the stochastic theory of Brownian motion was considered. In Ref. [6], the Langevin equation for the motion of a single particle coupled to a heat bath consisting of a collection of coupled oscillators was also studied and led to the same kind of memory.

Later, the non-Markov random continuous processes described by the Mori-Zwanzig equation were studied in Ref. [21]. As a starting point, the authors used the Markov Gaussian Ornstein-Uhlenbeck process and introduced phenomenologically an integral memory term depending on the past of the process into an expression for the higher-order transition probability function and into the stochastic differential equation. This paper also contains an extensive review of other works where this kind of memory was discussed. It is surprising that, in such seemingly completely unrelated phenomena, the memory manifests itself in equivalent integral forms (in addition to Ref. [21], see also Refs. [1,22,23]).

In the case of the telegraph process $\mathcal{T}_x(t)$, the integral memory comes up very naturally from and is explained by a specific type of conditional transition probability distribution of the additive Markov chain [Eq. (1)]. The contribution to CTPD of this chain contains only linear terms in

the random variables x_r . What is important is that this CTPD solves the very acute problem of representing the CTPD of N -step Markov chains with long-range memory. The finite state space Markov chain of order N is a well-known universal mathematical model to analyze discrete-valued time series with long-range memory in many fields of science. In the general case of *nonadditive* binary Markov chain of the order N , the number $D(N)$ of independent parameters of the chain increases exponentially with respect to N , $D(N) = 2^N$. A random *additive* higher-order Markov chain is characterized by the conditional transition probability distribution, Eq. (1), and contains only $D(N) = N + 1$ independent parameters. In this paper, a similar idea is used for the model of the telegraph process with nonlocal memory. A wide variety of functionals $dP(x_{t+dt} = a | x_t = 1 - a; x_{t' < t})$ in the left-hand side of Eq. (4) is expressed in additive form and depends only on the memory function $m(t)$. This allows us to obtain Eq. (9) for the correlation function and consider its analytical solution.

Similar to how statistical physics uses new methods of probability theory to solve its problems, the problems of the *curse of dimensionality* (in particular of higher-order Markov chains) are solved using the new field of mathematics known as the concentration of measure [24]. In the last decades, it became clear that the proper utilization of these phenomena in many areas of science might transform the problem of the curse of dimensionality into the problem of the *blessing of dimensionality* [25].

VIII. CONCLUSION

In this paper, the random telegraph processes $\mathcal{T}_x(t)$ with nonlocal in-time memory are studied. One of the most important achievements of this work is the method for solving the integro-differential equation that links an arbitrary memory function and a binary correlation function in an analytical closed form. This method is similar to the method of solving ordinary differential equations using the Green function.

Conditions of stationarity for the telegraph process with exponential memory are found. It is shown that the pair correlation function of this process can be of two types: either exponentially decreasing with two different decrements or oscillating with exponentially decreasing amplitude.

The analytical results are supported by numerical simulations of the corresponding telegraph processes, both with persistent and antipersistent correlations. The correlation functions obtained from the numerically generated telegraph processes with different parameters of the problem under study are in good agreement with the corresponding results obtained analytically.

Future research on the memory-dependent telegraph process should involve finding such important characteristics of the process as its distribution function, the diffusion Fokker-Planck equation governing the distribution function of the process, fluctuation and variance of the process, self-similarity properties for a process with a stepwise memory function [21,26], as well as the application of the theory developed here to specific complex physical phenomena. In particular,

the study of phenomena with a power-law memory function deserves special interest. The solution to this problem may be of particular interest to experts.

In this paper, the parameters λ_0 and λ_1 in Eq. (4) are supposed to be constant. Supposing that λ_0 and λ_1 are dependent on time or introducing the specific form of the memory function, one can describe other important random phenomena, including Levy diffusion, which is a subject of increasing interest over the last two decades [27–30].

We hope that the theory presented here can be applied to the description of different complex phenomena, in particular, meteorological ones, such as seasonal variability in weather prediction and/or 24-hour-ahead temperature forecasts (see Refs. [31,32] in this regard).

ACKNOWLEDGMENTS

O.V.U. is grateful to the Instituto de Física, Benemérita Universidad Autónoma de Puebla for financial support and hospitality. The authors are grateful to F. Pérez-Rodríguez and N. M. Makarov for discussions during the work.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible, and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

-
- [1] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*, Springer Series in Synergetics, Vol. 13 (Springer Verlag, Berlin, 1985).
 - [2] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981, 1992).
 - [3] W. Horsthemke and R. Lefever, *Noise-Induced Transitions. Theory and Applications in Physics, Chemistry and Biology* (Springer Verlag, Berlin, 1984).
 - [4] R. Zwanzig, Ensemble method in the theory of irreversibility, *J. Chem. Phys.* **33**, 1338 (1960).
 - [5] H. Mori, Transport, collective motion, and Brownian motion, *Prog. Theor. Phys.* **33**, 423 (1965).
 - [6] G. W. Ford, M. Kac, and P. Mazur, Statistical mechanics of assemblies of coupled oscillators, *J. Math. Phys.* **6**, 504 (1965).
 - [7] R. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, New York, 2001).
 - [8] V. Zaburdaev, S. Denisov, and P. Hänggi, Space-time velocity correlation function for random walks, *Phys. Rev. Lett.* **110**, 170604 (2013).
 - [9] A. Rebenshtok, S. Denisov, P. Hänggi, and E. Barkai, Non-normalizable densities in strong anomalous diffusion: Beyond the central limit theorem, *Phys. Rev. Lett.* **112**, 110601 (2014).
 - [10] L. Ferialdi, Exact closed master equation for Gaussian non-Markovian dynamics, *Phys. Rev. Lett.* **116**, 120402 (2016).
 - [11] D. Nickelsen and H. Touchette, Noise correction of large deviations with anomalous scaling, *Phys. Rev. Lett.* **121**, 090602 (2018).
 - [12] S. S. Apostolov, S. S. Melnyk, O. V. Usatenko, V. A. Yampol'skii, P. Grigolini, and A. Krokhin, Random telegraph processes with nonlocal memory, *Phys. Rev. E* **110**, 024303 (2024).
 - [13] A. Raftery, A model for high-order Markov chains, *J. R. Stat. Soc. B* **47**, 528 (1985).
 - [14] W. K. Ching, X. Huang, M. K. Ng, and T. K. Siu, *Markov Chains: Models, Algorithms and Applications* (Springer: Sci Bus Media, New York, 2013).
 - [15] A. Berchtold and A. E. Raftery, The mixture transition distribution model for high-order Markov chains and non-Gaussian time series, *Stat. Sci.* **17**, 328 (2002).
 - [16] O. V. Usatenko and V. A. Yampol'skii, Binary N -step Markov chains and long-range correlated systems, *Phys. Rev. Lett.* **90**, 110601 (2003).
 - [17] O. V. Usatenko, S. S. Apostolov, Z. A. Mayzelis, and S. S. Melnik, *Random Finite-Valued Dynamical Systems: Additive Markov Chain Approach* (Cambridge Scientific Publisher, Cambridge, 2010).
 - [18] S. S. Melnyk, O. V. Usatenko, and V. A. Yampol'skii, Memory functions of the additive Markov chains: Applications to complex dynamic systems, *Physica A* **361**, 405 (2006).
 - [19] S. S. Melnyk, O. V. Usatenko, V. A. Yampol'skii, S. S. Apostolov, and Z. A. Maiselis, Memory functions and correlations in additive binary Markov chains, *J. Phys. A: Math. Gen.* **39**, 14289 (2006).
 - [20] S. S. Melnik and O. V. Usatenko, Entropy of finite random binary sequences with weak long-range correlations, *Phys. Rev. E* **90**, 052106 (2014).
 - [21] S. S. Melnyk, V. A. Yampol'skii, and O. V. Usatenko, Entropy of finite random binary sequences with weak long-range correlations, *Phys. Rev. E* **100**, 052141 (2019).
 - [22] G. U. Yule, VII. On a method of investigating periodicities in disturbed series with special reference to Wolfer's sunspot numbers, *Phil. Trans. R. Soc. Lond. A* **226**, 267 (1927).
 - [23] G. T. Walker, On periodicity in series of related terms, *Proc. R. Soc. Lond. A* **131**, 518 (1931).
 - [24] J. Lafferty, H. Liu, and L. Wasserman, Sparse nonparametric graphical models, *Stat. Sci.* **27**, 519 (2012).
 - [25] A. N. Gorban and I. Y. Tyukin, Blessing of dimensionality: Mathematical foundations of the statistical physics of data, *Phil. Trans. R. Soc. A* **376**, 20170237 (2018).
 - [26] O. V. Usatenko, V. A. Yampol'skii, K. E. Kechedzhy, and S. S. Melnyk, Symbolic stochastic dynamical systems viewed as binary N -step Markov chains, *Phys. Rev. E* **68**, 061107 (2003).
 - [27] G. Margolin and E. Barkai, Aging correlation functions for blinking nanocrystals, and other on-off stochastic processes, *J. Stat. Phys.* **122**, 137 (2006).
 - [28] T. Sandev, L. Kocarev, R. Metzler, and A. Chechkin, Stochastic dynamics with multiplicative noise under resetting, *Chaos Soliton Fract.* **165**, 112878 (2022).
 - [29] F. D. Stefani, J. P. Hoogenboom, and E. Barkai, Beyond quantum jumps: Blinking nanoscale light emitters, *Phys. Today* **62**(2), 34 (2009).

- [30] M. Niemann, H. Kantz, and E. Barkai, Fluctuations of $1/f$ noise and the low-frequency cutoff paradox, [Phys. Rev. Lett.](#) **110**, 140603 (2013).
- [31] P. V. V. Le, J. T. Randerson, and R. Willett, S. Wright, P. Smyth, C. Guilloteau, A. Mamalakis, and E. Foufoula-Georgiou, Climate-driven changes in the predictability of seasonal precipitation, [Nat. Commun.](#) **14**, 3822 (2023).
- [32] I. Price, A. Sanchez-Gonzalez, and F. Alet, T. R. Andersson, A. El-Kadi, D. Masters, T. Ewalds, J. Stott, S. Mohamed, P. Battaglia, R. Lam, and M. Willson, Probabilistic weather forecasting with machine learning, [Nature \(London\)](#) **637**, 84 (2025).