Surface nematic uniformity

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An antlike observer confined to a two-dimensional surface traversed by stripes would wonder whether such a striped landscape could be devised in such a way as to appear to be the same wherever they go. Differently stated, this is the problem studied in this paper. In a more technical jargon, we determine all possible *uniform* nematic fields on a smooth surface. It was already known that for such a field to exist, the surface must have constant *negative* Gaussian curvature. Here we show that all uniform nematic fields on such a surface are parallel transported (in Levi-Civita's sense) by special systems of geodesics, which are termed *uniform*. We prove that, for every geodesic on the surface, there are *two* systems of uniform geodesics that include it; they are conventionally called *right* and *left*, to evoke handedness. We found explicitly all uniform fields for Beltrami's pseudosphere. Since both geodesics and uniformity are preserved under isometries, by a classical theorem of Minding, the solution for the pseudosphere carries over all other admissible surfaces, thus providing a general solution to the problem (at least in principle). The proved existence of surface nematic uniform fields suggests the definition of a generalized intrinsic elastic energy for fluid membranes with nematic order, which is but one of the many possible applications of our geometric result.

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I. INTRODUCTION

Owing to its etymology, *nematic* refers to anything related to threads. A line field on a curved surface is a paradigm of nematic systems. This paper is concerned with a general property of unit vector fields $\boldsymbol{n}$ tangent to a surface $\mathcal{S}$, a property that may appear to be predominantly geometric but which has a physical bearing, too. There are several possible physical applications of the geometric theory presented here; they arise whenever a surface, be it rigid or deformable, is inhabited by an order that can be described by a mobile preferred tangent direction $\boldsymbol{n}$. This is, for example, the case of nematic liquid crystal shells, where elongated molecules condensed in an ordered phase cover a curved substrate. Another class of physical examples are fluid biomembranes, which represent idealizations of amphiphilic layers or interfaces in microemulsions (see Ref. [1] for an early review). In the seminal paper by Nelson and Peliti [2] (see also Ref. [3] for a revisitation of some of the original results), it is conjectured that the state of lowest energy of a nematic membrane would be attained when the director $\boldsymbol{n}$ is parallel transported (in the Levi-Civita's sense). On the basis of this intuition, they proposed a successful theory that builds upon an elastic free-energy functional featuring an *intrinsic* measure of nematic distortion.[1] Here by *intrinsic* we mean *invariant* under isometries. Thus, for example, Helfrich's classical model for fluid membranes [8] would *not* be intrinsic, as the elastic free-energy density features the membrane's *mean* curvature, which fails to be an isometric invariant [3]. Deformation and distortion measures (of $\mathcal{S}$ and $\boldsymbol{n}$, respectively) that are not intrinsic are said to be *extrinsic*. A vast literature exists on extrinsic effects in nematic membranes; an incomplete but representative list is in Refs. [9–13]. These effects, which typically couple nematic order and membrane's curvature, are the conceptual analogues of those produced by an external field on the nematic director: They affect equilibrium but from without not from within.[2] A rigorous treatment of the intrinsic elasticity of nematic membranes might illuminate the general structure of their elastic free energy. To this end, we find convenient the general notion of *uniformity*. Besides its own geometric merits, this has also the potential to designate the ground state of nematic distortion: When this cannot be met, geometric frustration is ignited.

Our approach has a scope wider than this specific application may suggest. In a broad sense, it relates to uniform vector fields in isotropic and uniform spaces. These naturally constitute all fields that are invariant under isometries in the appropriate spaces. How many fields could be supported by

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[1]Works on fluid membranes focusing on shape transitions forced by topology can be found in Refs. [4,5], whereas a possible role of curvature is illuminated for example in Ref. [6]. The study [7] on rigid membranes should also be mentioned in this cursory list.

[2]In this respect, also Wulff's functional for droplets with anisotropic surface tension (treated in great detail, for example, in Ref. [14]) is extrinsic.

such manifolds, how different are these fields from each other, and how to construct them explicitly and exhaustively are, in lose language, the issues addressed in this paper. Positively curved manifolds are known to support no such field, and the flat plane supports only the trivial constant fields. Thus, our study will concentrate on surfaces with negative Gaussian curvature; they are all isometric to Beltrami's pseudosphere.

A similar investigation was already successful in three-dimensional space, where not all uniform fields are trivially constant. We start by recalling the outcomes of that study.

There, uniformity simply requires that a nematic field $\boldsymbol{n}$ appears everywhere distorted in the same way. More descriptively, this amounts to say that an observer placed everywhere in space would represent the tensorial descriptor of distortion $\nabla\boldsymbol{n}$ by use of the same (constant) scalars in a *distortion* frame $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n})$ intrinsically associated with the nematic field.

In Ref. [15], following Refs. [16] and [17], $\nabla\boldsymbol{n}$ is represented in the form

$$\nabla\boldsymbol{n} = -\boldsymbol{b}\otimes\boldsymbol{n} + \tfrac{1}{2}T\mathbf{W}(\boldsymbol{n}) + \tfrac{1}{2}S\mathbf{P}(\boldsymbol{n}) + \mathbf{D}, \tag{1}$$

where $S := \mathrm{div}\boldsymbol{n}$ is the *splay*, $T := \boldsymbol{n}\cdot\mathrm{curl}\boldsymbol{n}$ is the *twist*, $\boldsymbol{b} := \boldsymbol{n}\times\mathrm{curl}\boldsymbol{n}$ is the *bend*, $\mathbf{W}(\boldsymbol{n})$ is the skew-symmetric tensor with axial vector $\boldsymbol{n}$, $\mathbf{P}(\boldsymbol{n}) := \mathbf{I} - \boldsymbol{n}\otimes\boldsymbol{n}$ is the projector in the plane orthogonal to $\boldsymbol{n}$, and $\mathbf{D}$ is a symmetric tensor that annihilates $\boldsymbol{n}$, $\mathbf{D}\boldsymbol{n} = \mathbf{0}$, and which admits the following *biaxial* representation:

$$\mathbf{D} = q(\boldsymbol{n}_1\otimes\boldsymbol{n}_1 - \boldsymbol{n}_2\otimes\boldsymbol{n}_2). \tag{2}$$

In (2), $q > 0$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2)$ is a pair of orthogonal unit vectors in the plane orthogonal to $\boldsymbol{n}$, oriented so that $\boldsymbol{n} = \boldsymbol{n}_1\times\boldsymbol{n}_2$.[3] The distortion frame is thus built solely from $\mathbf{D}$.

Letting $\boldsymbol{b} = b_1\boldsymbol{n}_1 + b_2\boldsymbol{n}_2$, we call (S, T, b_1, b_2, q) the distortion *characteristics* of the field $\boldsymbol{n}$ and we say that the latter is *uniform* whenever the former are constant in space.[4] This immediately makes the notion of uniformity in three-dimensional space invariant under isometries in that space.[5]

It was proved in Ref. [15] that the most general uniform nematic field of class C^2 in a flat three-dimensional space is characterized by the conditions

$$S = 0, \quad T = \pm 2q, \quad b_1 = \pm b_2 = b, \tag{3}$$

where $q > 0$ and b are arbitrary scalar parameter.[6] A similar result, perhaps geometrically more attractive, has been proved for *curved* three-dimensional spaces in Refs. [25,26].

[3]The orientation of both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ can be chosen freely, without affecting $\mathbf{D}$, which is uniquely determined by $\nabla\boldsymbol{n}$.

[4]It is argued in Ref. [18] that q should be called the *tetrahedral* splay; we would rather prefer to call it the *octupolar* splay, as done in Ref. [19], to acknowledge the role played by a cubic (octupolar) potential on the unit sphere in representing all distortion characteristics, but T [20,21].

[5]Indeed, an isometry would transform $\boldsymbol{n}$ into $\boldsymbol{n}^* = \mathbf{Q}\boldsymbol{n}$, where $\mathbf{Q}$ is a rotation in the special orthogonal group $\mathsf{SO}(3)$, independent of position, and correspondingly $\nabla\boldsymbol{n}^* = \mathbf{Q}(\nabla\boldsymbol{n})\mathbf{Q}^{\mathsf{T}}$, so that $\boldsymbol{n}^*$ is uniform whenever $\boldsymbol{n}$ is.

[6]This precisely transliterates Meyer's *heliconical* distortion [22], which was observed experimentally in the ground state of *twist-bend* nematic phases [23,24].

On a surface $\mathcal{S}$ the scene changes considerably and the very definition of a uniform nematic field $\boldsymbol{n}$ must be rethought about. Usually, it is the covariant gradient $\nabla_c\boldsymbol{n}$ to be employed as a local tensorial descriptor of distortion: It is indeed intrinsic, but it is unclear why it should be *a priori* preferred. We shall follow an alternative avenue, which in Sec. II will lead us to an independent definition of surface uniformity, also *proved* to be intrinsic and indeed related to the covariant gradient. A number of mathematical preliminaries, needed to follow in full awareness our development in Sec. II, are presented in Appendix A.

A cornerstone in this field is Niv and Efrati's result [27] to the effect that uniform nematic fields are only possible on surfaces with constant *negative* Gaussian curvature (whose value is dictated by the relevant distortion characteristics).[7] These surfaces are also called *pseudospherical*.

Apart from the above necessary condition, nothing else is known (at least to us) about uniform fields on pseudospherical surfaces and, what is more important, their actual existence. In Sec. III, we show that they exist as fields parallel transported by special systems of *uniform* geodesics of $\mathcal{S}$. Thus, parallel transport reappears, but it emerges from a deeper level. We prove that, for every geodesic $\mathcal{C}$ of $\mathcal{S}$, there are *two* such systems that include $\mathcal{C}$. Such a duality propagates to the conveyed uniform fields, in a way reminiscent of the alternative embodied by (3).

In Sec. IV, we construct explicitly all uniform geodesics and conveyed uniform fields for Beltrami's pseudosphere. Since, by Minding's theorem, all surfaces with constant Gaussian curvature are isometric, the solution found for the pseudosphere can (in principle) be extended to all pseudospherical surfaces. An exhaustive characterization of all uniform fields on a pseudospherical surface is afforded by *two* parameters: one is the ratio between bend and splay and the other is a structural rotation angle.

Finally, in Sec. V, we collect our conclusions and some thoughts for further studies. In particular, we make use of the newly characterized notion of nematic uniformity on surfaces to sketch a proposal for an *intrinsic* generalized elastic free-energy functional for fluid membranes, which encompasses as special cases other forms known in the literature. The paper is completed by Appendixes with auxiliary results and an animation showing a typical uniform field on the pseudosphere.

II. SURFACE UNIFORMITY

In this section, we shall start our study by defining when a unit vector field $\boldsymbol{n}$ tangent to $\mathcal{S}$ is said to be *uniform*.

Nematic uniformity has been tackled already in three-dimensional spaces, either flat or curved, where this notion is fairly well understood. For surfaces imbedded in three-dimensional Euclidean space $\mathcal{E}$ (see Appendix A), we need first agree on a definition of uniformity for a tangential unit field $\boldsymbol{n}$. To this end, we assume that a tangent unit vector field $\boldsymbol{n}$ of class at least C^2 is prescribed on a surface $\mathcal{S}$ and we

[7]On a surface $\mathcal{S}$ with zero Gaussian curvature, uniform fields are trivial, as they are the images of constant fields on a region of the flat plane, conveyed by the isometry that folds the latter into $\mathcal{S}$.

write the *gliding laws* in (A9) for the moving frame $(\boldsymbol{n}, \boldsymbol{n}_\perp, \boldsymbol{\nu})$, where $\boldsymbol{\nu}$ is the unit normal orienting $\mathcal{S}$ and $\boldsymbol{n}_\perp$ is chosen so that $\boldsymbol{\nu} = \boldsymbol{n} \times \boldsymbol{n}_\perp$,

$$\begin{aligned} \nabla_s \boldsymbol{n} &= \boldsymbol{n}_\perp \otimes \boldsymbol{c} + \boldsymbol{\nu} \otimes \boldsymbol{d}_1, \\ \nabla_s \boldsymbol{n}_\perp &= -\boldsymbol{n} \otimes \boldsymbol{c} + \boldsymbol{\nu} \otimes \boldsymbol{d}_2, \\ \nabla_s \boldsymbol{\nu} &= -\boldsymbol{n} \otimes \boldsymbol{d}_1 - \boldsymbol{n}_\perp \otimes \boldsymbol{d}_2. \end{aligned} \tag{4}$$

Here ∇_s denotes the *surface* gradient, $\boldsymbol{c}$ is the *spin* connector, and $\boldsymbol{d}_1, \boldsymbol{d}_2$ are the *curvature* connectors of the selected moving frame (see again Appendix A). We further specialize the identity (A10) in the form

$$\boldsymbol{d}_1 \cdot \boldsymbol{n}_\perp = \boldsymbol{d}_2 \cdot \boldsymbol{n}, \tag{5}$$

which guarantees the symmetry of $\nabla_s \boldsymbol{\nu}$.

In complete analogy with the flat three-dimensional case, we shall say that a nematic surface field $\boldsymbol{n}$ tangent to $\mathcal{S}$ is *uniform*, whenever the spin connector $\boldsymbol{c}$ has constant components along both $\boldsymbol{n}$ and $\boldsymbol{n}_\perp$, and so the covariant gradient $\nabla_c \boldsymbol{n}$, which, by (A3) and (4), reads as

$$\nabla_c \boldsymbol{n} = \boldsymbol{n}_\perp \otimes \boldsymbol{c}, \tag{6}$$

is uniquely determined in the local distortion frame $(\boldsymbol{n}, \boldsymbol{n}_\perp)$. Thus, equivalently our definition of uniformity requires the covariant gradient of a nematic field to have constant components in a frame moving with the director. In Appendix B, we show directly that the notion of surface uniformity thus introduced is invariant under isometric deformations of the surface $\mathcal{S}$, as it pertains to an intrinsic property. The components of $\boldsymbol{c}$ in the distortion frame can easily be given an intrinsic representation.

Intrinsic distortion measures

Letting $\boldsymbol{c} = c_1 \boldsymbol{n} + c_2 \boldsymbol{n}_\perp$, it easily follows from the first equation in (4) and (6) that

$$c_1 = -b_\perp := -\boldsymbol{n} \times \mathrm{curl}_s \boldsymbol{n} \cdot \boldsymbol{n}_\perp = \mathrm{curl}_s \boldsymbol{n} \cdot \boldsymbol{\nu} = \mathrm{curl}_c \boldsymbol{n} \cdot \boldsymbol{\nu}, \tag{7a}$$

$$c_2 = S := \mathrm{div}_s \boldsymbol{n} = \mathrm{div}_c \boldsymbol{n}, \tag{7b}$$

which identify c_1 with the intrinsic component of the *bend* vector $\boldsymbol{b} = \boldsymbol{n} \times \mathrm{curl}_s \boldsymbol{n}$ (to within a sign) and c_2 with the *splay* S, which is fully intrinsic. Equations (7) also show that our definition of surface uniformity is equivalent to the one adopted in Ref. [27].[8] Since $\boldsymbol{d}_1 = -(\nabla_s \boldsymbol{\nu})\boldsymbol{n}$ and $\boldsymbol{d}_2 = -(\nabla_s \boldsymbol{\nu})\boldsymbol{n}_\perp$, we may also say that the dependence of $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ on the full curvature tensor $\nabla_s \boldsymbol{\nu}$ reveals their *extrinsic* nature.[9] A surface nematic field $\boldsymbol{n}$ has thus no intrinsic *twist*, as $\boldsymbol{n} \cdot \mathrm{curl}_s \boldsymbol{n} = \boldsymbol{d}_1 \cdot \boldsymbol{n}_\perp = \boldsymbol{d}_2 \cdot \boldsymbol{n}$ is purely extrinsic and, by (6), $T = \boldsymbol{n} \cdot \mathrm{curl}_c \boldsymbol{n} \equiv 0$.[10]

In Appendix C, we characterize the connectors of a uniform field; in particular, it is shown that only surfaces with a constant *negative* Gaussian curvature K can bear uniform fields, a result already known from Ref. [27]. In the following section, we shall see how another necessary condition for surface uniformity leads us to identify all possible uniform fields.

III. UNIFORM GEODESICS

As shown in Ref. [27], a surface $\mathcal{S}$ can host a uniform nematic field $\boldsymbol{n}$ only if its Gaussian curvature K obeys

$$K \equiv -(b_\perp^2 + S^2), \tag{8}$$

where $b_\perp$ and S are the bend and splay (constant) *distortion components* of the field. Surfaces with constant negative Gaussian curvature are called *pseudospherical*, following the use of Beltrami, who found their simplest exemplar, the *pseudosphere* (see, for example, Ref. [29, p. 22]). Henceforth, to simplify our development (without affecting its generality), we shall rescale lengths so that $K \equiv -1$. Accordingly, by (7), we write $\boldsymbol{c}$ as

$$\begin{aligned} \boldsymbol{c} &= -\sin\alpha \boldsymbol{n} + \cos\alpha \boldsymbol{n}_\perp, \\ \cos\alpha &= S, \quad \sin\alpha = b_\perp, \end{aligned} \tag{9}$$

so that

$$\tan\alpha = \frac{b_\perp}{S}, \tag{10}$$

which will henceforth be referred to as the *distortion anisotropy*.

Let a uniform field $\boldsymbol{n}$ be prescribed on a pseudospherical surface $\mathcal{S}$. We seek a curve $\mathcal{C}$ that parallel transport $\boldsymbol{n}$ starting from an arbitrary point $\boldsymbol{x} \in \mathcal{S}$. If $\boldsymbol{t}$ denotes the unit tangent vector to $\mathcal{C}$, by (A18), it must be

$$\boldsymbol{n}' \cdot \boldsymbol{n}_\perp \equiv 0, \tag{11}$$

where a prime denotes differentiation in the arc-length parameter along $\mathcal{C}$. Since $\boldsymbol{n}' = (\nabla_s \boldsymbol{n})\boldsymbol{t}$, by (4), (11) is just the same as

$$\boldsymbol{t} \cdot \boldsymbol{c} \equiv 0, \tag{12}$$

which implies that

$$\boldsymbol{t} = \pm(\cos\alpha \boldsymbol{n} + \sin\alpha \boldsymbol{n}_\perp). \tag{13}$$

[8] On a similar terminological note, since surface uniformity entails uniformity of the intrinsic distortion measures, it has also been suggested to call *uniformly distorted* these special nematic surface fields. We prefer to stick with our abridged terminology, for brevity.

[9] However, the reader should not be led to think that the curvature tensor cannot covey intrinsic features. The well-known *theorema egregium* of Gauss indeed shows that the Gaussian curvature K, being an *isometric* invariant (and so depending only on the metric of the imbedded surface), is an intrinsic measure associated with $\nabla_s \boldsymbol{\nu}$ (see, for example, Ref. [28, p. 291]). By (A11b), this also implies that $\boldsymbol{d}_1 \times \boldsymbol{d}_2 \cdot \boldsymbol{\nu}$ is an intrinsic scalar. Said differently, the Gaussian curvature K of $\mathcal{S}$, which is defined in apparently extrinsic terms, belongs indeed to the intrinsic geometry of $\mathcal{S}$, and can accordingly also be measured in purely metric terms as the *angular excess* per unit area [29, p. 140–142].

[10] Here we are guilty of some abuse of notation, as we still denote the bend, splay, and twist of a surface field by the same symbols used for three-dimensional fields. No confusion should, however, arise, as here we are only concerned with surface fields.

By differentiating both sides of (13) along $\mathcal{C}$, again by use of (4), since α is constant, we arrive at

$$\boldsymbol{t}' = \pm[(\boldsymbol{d}\cdot\boldsymbol{t})\cos\alpha + (\boldsymbol{d}_2\cdot\boldsymbol{t})\sin\alpha]\boldsymbol{\nu}, \tag{14}$$

which shows that $\mathcal{C}$ must also be a geodesic of $\mathcal{S}$. Thus, a necessary condition for $\boldsymbol{n}$ to be a uniform field is to be parallel transported along geodesics.

Letting $\boldsymbol{t}_\perp = \boldsymbol{\nu}\times\boldsymbol{t}$, we readily derive from (9) and (13) that

$$\boldsymbol{t}_\perp = \pm(-\sin\alpha\boldsymbol{n} + \cos\alpha\boldsymbol{n}_\perp) = \pm\boldsymbol{c}, \tag{15}$$

which in turn implies that

$$\boldsymbol{n} = \pm(\cos\alpha\boldsymbol{t} - \sin\alpha\boldsymbol{t}_\perp), \quad \boldsymbol{n}_\perp = \pm(\sin\alpha\boldsymbol{t} + \cos\alpha\boldsymbol{t}_\perp). \tag{16}$$

The angle α remains constant along the geodesic $\mathcal{C}$; its value is uniquely determined by the distortion components prescribed for a uniform field. Of course, a uniform field may well fail to exist on $\mathcal{S}$, but if it does exist, it must be parallel transported by a system of geodesics with relative orientation determined by the distortion components. Among all possible systems of geodesics on $\mathcal{S}$, we call *uniform* those that parallel transport a uniform field with constant angle α.

There is a simple criterion to identify all systems of uniform geodesics. Consider first the frame $(\boldsymbol{t},\boldsymbol{t}_\perp,\boldsymbol{\nu})$ associated with a system of uniform geodesics.[11] By computing $\nabla_s\boldsymbol{t}$ from (13) [and, correspondingly, $\nabla_s\boldsymbol{t}_\perp$ from (15)], we easily see that the gliding laws (A9) are obeyed with connectors $(\hat{\boldsymbol{c}},\hat{\boldsymbol{d}}_1,\hat{\boldsymbol{d}}_2)$ given by

$$\hat{\boldsymbol{c}} = \boldsymbol{c}, \quad \hat{\boldsymbol{d}}_1 = \pm(\cos\alpha\boldsymbol{d}_1 + \sin\alpha\boldsymbol{d}_2), \quad \hat{\boldsymbol{d}}_2 = \pm(\cos\alpha\boldsymbol{d}_2 - \sin\alpha\boldsymbol{d}_1), \tag{17}$$

where $(\boldsymbol{c},\boldsymbol{d}_1,\boldsymbol{d}_2)$ are the connectors of the frame $(\boldsymbol{n},\boldsymbol{n}_\perp,\boldsymbol{\nu})$.[12] Since the frames $(\boldsymbol{t},\boldsymbol{t}_\perp,\boldsymbol{\nu})$ and $(\boldsymbol{n},\boldsymbol{n}_\perp,\boldsymbol{\nu})$ have one and the same spin connector $\boldsymbol{c}$, which by (9) has unit length, it follows from (15) that a system of uniform geodesics must satisfy the following condition:[13]

$$\boldsymbol{c}\cdot\boldsymbol{t}_\perp = \boldsymbol{t}_\perp\cdot(\nabla_s\boldsymbol{t})\boldsymbol{t}_\perp = \pm 1. \tag{18}$$

Conversely, if (18) is valid for a system of geodesics, then, as the spin connector $\boldsymbol{c}$ of the frame $(\boldsymbol{t},\boldsymbol{t}_\perp,\boldsymbol{\nu})$ is such that

$$\boldsymbol{c}\cdot\boldsymbol{t} = (\nabla_s\boldsymbol{t})^{\mathsf{T}}\boldsymbol{t}_\perp\cdot\boldsymbol{t} = \boldsymbol{t}_\perp\cdot(\nabla_s\boldsymbol{t})\boldsymbol{t} = \boldsymbol{t}_\perp\cdot\boldsymbol{t}' = 0, \tag{19}$$

it can only be written in either of the following forms:

$$\boldsymbol{c} = \pm\boldsymbol{t}_\perp. \tag{20}$$

It thus follows from (A16a) that the field lines of $\boldsymbol{t}_\perp$ have *constant* geodesic curvature.

By applying (16), with sign chosen equal to the one featuring in (20), it is easy to prove that, for any constant α, the gliding laws of $(\boldsymbol{t},\boldsymbol{t}_\perp,\boldsymbol{\nu})$ deliver the equations

$$S = \mathrm{div}_s\boldsymbol{n} = \cos\alpha \quad \text{and} \quad -b_\perp = \mathrm{curl}_s\boldsymbol{n}\cdot\boldsymbol{\nu} = \sin\alpha, \tag{21}$$

for *both* signs occurring in (20) [and (16)].

Thus, systems of geodesics that satisfy (18) with one sign or the other generate uniform fields characterized by the same distortion components S and $b_\perp$ (not just the same distortion anisotropy). If, starting from a given geodesic $\mathcal{C}$ of $\mathcal{S}$, Eq. (18) has a solution for each choice of sign, then we generate two systems of uniform geodesics that include $\mathcal{C}$ and convey uniform fields with equal distortions. We would then say that this establishes a conjugation by *duality* among uniform fields. Such a duality should not be confused with the usual nematic symmetry, which identifies both $\boldsymbol{n}$ and $-\boldsymbol{n}$ with a headless *director*, although this symmetry is also involved here.

In going from one uniform field to its conjugate, we need first reverse the sign of $\boldsymbol{n}$ on $\mathcal{C}$ and then apply (16) to the solution of the conjugate form of (18) (with the sign reversed). The resulting field $\boldsymbol{n}$ is *not* the opposite of its dual, as both S and $b_\perp$ are preserved, instead of being changed in their opposite, as they would if $\boldsymbol{n}$ were changed into $-\boldsymbol{n}$ also away from $\mathcal{C}$.[14]

Thus, finding all uniform nematic fields on a surface $\mathcal{S}$ with $K\equiv -1$ amounts to find all systems of geodesics that obey (18). By Minding's theorem (see, for example, either Ref. [29, p. 336] or Ref. [30, p. 416]), all surfaces with equal constant Gaussian curvature are (at least locally) isometric.[15] Since both geodesics and uniformity are preserved by isometries, it suffices to find explicitly all systems of uniform geodesics of Beltrami's pseudosphere to find (in principle) those of all pseudospherical surfaces and thus retrace all possible uniform nematic fields on surfaces.

IV. PSEUDOSPHERE

Here we first find all geodesics on the pseudosphere and then determine those that are uniform, according to our definition above. The pseudosphere with $K\equiv -1$ can be represented in isothermal coordinates (u,v) ranging in the domain $\Omega := [0,2\pi]\times(1,+\infty)$ by the mapping $\boldsymbol{r}$ defined as

$$\boldsymbol{r}(u,v) = \frac{1}{v}\cos u\boldsymbol{e}_1 + \frac{1}{v}\sin u\boldsymbol{e}_2 + \left(\ln(v+\sqrt{v^2-1}) - \frac{\sqrt{v^2-1}}{v}\right)\boldsymbol{e}_3, \tag{22}$$

where $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ is a Cartesian frame in three-dimensional translation space $\mathcal{V}$ (see Appendix A). The coordinate moving frame $(\boldsymbol{e}_u,\boldsymbol{e}_v,\boldsymbol{\nu})$ associated with (22) is

$$\boldsymbol{e}_u = -\sin u\boldsymbol{e}_1 + \cos u\boldsymbol{e}_2, \tag{23a}$$

$$\boldsymbol{e}_v = -\frac{1}{v}(\cos u\boldsymbol{e}_1 + \sin u\boldsymbol{e}_2) + \frac{\sqrt{v^2-1}}{v}\boldsymbol{e}_3, \tag{23b}$$

[11] There are two such frames, corresponding to the two possible choices of sign in (13).

[12] So that, in particular, $\hat{\boldsymbol{d}}_1\times\hat{\boldsymbol{d}}_2 = \boldsymbol{d}_1\times\boldsymbol{d}_2 = -\boldsymbol{\nu}$, as required.

[13] It is also clear from being $\boldsymbol{t}_\perp\cdot(\nabla_s\boldsymbol{t})\boldsymbol{t}_\perp = \boldsymbol{t}_\perp\cdot(\nabla_c\boldsymbol{t})\boldsymbol{t}_\perp$ that (18) is an intrinsic condition.

[14] The nematic symmetry is not essential to this reasoning: alternatively, one can reverse the orientation of $\mathcal{C}$, that is, change $\boldsymbol{t}$ into $-\boldsymbol{t}$, in going from one form of (18) to the other (with the sign reversed), so as to leave the trace of $\boldsymbol{n}$ on $\mathcal{C}$ unchanged; the same conclusions as above would follow. The only things that really matter are the *directions* of $\boldsymbol{n}$ and $\boldsymbol{t}$.

[15] The classical proof of Minding's theorem requires that $\mathcal{S}$ be of class at least C^3; for pseudospherical surfaces of class C^2, Minding's theorem has been proved in Ref. [31].

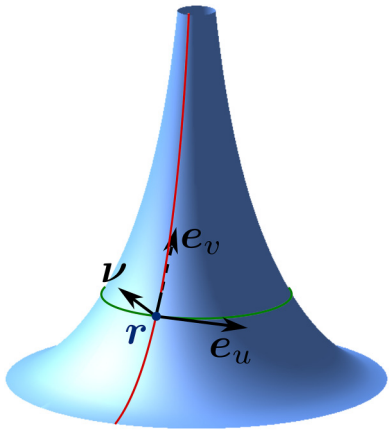


FIG. 1. Beltrami's pseudosphere represented in the parametrization (22). The coordinate frame $(\boldsymbol{e}_u, \boldsymbol{e}_v, \boldsymbol{\nu})$ in (23) glides over it in accordance with (24).

$$\boldsymbol{\nu} = \frac{\sqrt{v^2-1}}{v}(\cos u\boldsymbol{e}_1 + \sin u\boldsymbol{e}_2) + \frac{1}{v}\boldsymbol{e}_3, \tag{23c}$$

and the corresponding gliding laws read as

$$\nabla_s \boldsymbol{e}_u = \boldsymbol{e}_v \otimes \boldsymbol{e}_u - \sqrt{v^2-1}\boldsymbol{\nu} \otimes \boldsymbol{e}_u, \tag{24a}$$

$$\nabla_s \boldsymbol{e}_v = -\boldsymbol{e}_u \otimes \boldsymbol{e}_u + \frac{1}{\sqrt{v^2-1}}\boldsymbol{\nu} \otimes \boldsymbol{e}_v, \tag{24b}$$

$$\nabla_s \boldsymbol{\nu} = \sqrt{v^2-1}\boldsymbol{e}_u \otimes \boldsymbol{e}_u - \frac{1}{\sqrt{v^2-1}}\boldsymbol{e}_v \otimes \boldsymbol{e}_v. \tag{24c}$$

Figure 1 shows a traditional picture of the pseudosphere: It is an unbounded cylindrically-symmetric surface based on a circular rim corresponding to the segment at $v = 1$ on the boundary of Ω. All geodesics can be determined by use of Clairaut's theorem (see, for example, Ref. [30, pp. 230–233]): they are the family of meridians, whose preimages in the (u, v) plane are straight lines parallel to the v axis, and the families of curves whose preimages in the (u, v) plane are circles with centers along the u axis, see Fig. 2. The whole collection of the latter are represented by

$$\begin{aligned} u &= u_0(\varphi) + \rho(\varphi)\cos\vartheta, \\ v &= \rho(\varphi)\sin\vartheta, \end{aligned} \tag{25}$$

where $u_0(\varphi)$ and $\rho(\varphi)$ are smooth functions, ranging in $\mathbb{R}$ and $(1, +\infty)$, respectively, of one parameter, φ, while the other parameter, ϑ, ranges in the admissible interval $I(\varphi)$ (depending on φ) enclosed within

$$\arcsin\frac{1}{\rho} < \vartheta < \pi - \arcsin\frac{1}{\rho}, \tag{26}$$

so that $\rho(\varphi)\sin\vartheta > 1$. A single family of geodesics is identified by taking u_0 constant and letting ρ depend monotonically on φ. Within such a family, one geodesic is singled out by fixing φ and letting ϑ vary in $I(\varphi)$; see Fig. 2.

We can also view (25) as a family of changes of variables, all expressing (u, v) in terms of the *geodesic* coordinates (φ, ϑ); there is one such change of variables for each choice of functions u_0 and ρ. A simple computation shows that the Jacobian of the transformation $(\varphi, \vartheta) \mapsto (u, v)$ does not

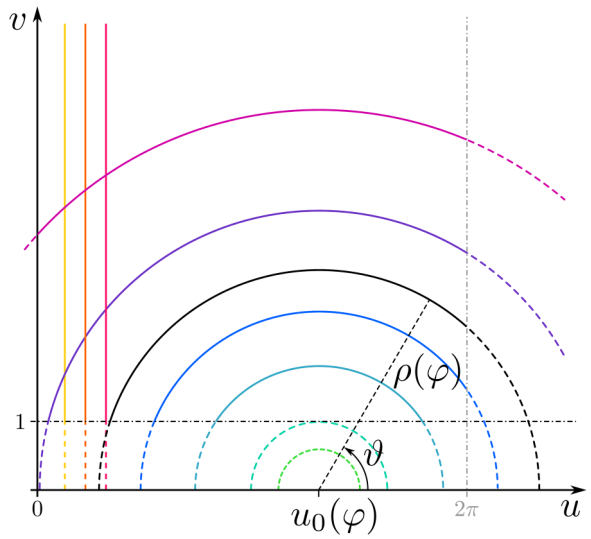


FIG. 2. Preimages of geodesics in the (u, v) plane. The meridians of $\mathcal{S}$ are straight lines parallel to the v axis. All other geodesics are families of circles with centers on the u axis. The particular family represented here is obtained by choosing in (25) $u_0(\varphi)$ as a constant and $\rho(\varphi)$ as a monotonic function of φ, so as to make (27) satisfied. A single circle in this family is drawn for a fixed value of φ and ϑ ranging in the admissible interval $I(\varphi)$.

vanish whenever

$$u_0'\cos\vartheta + \rho' \neq 0, \tag{27}$$

where a prime denotes differentiation with respect to φ. Thus, for each choice of functions u_0 and ρ that satisfy the (local) invertibility condition (27), we can cover the domain Ω with all possible preimages of geodesics (and $\mathcal{S}$ with all possible geodesics). This will play a central role in identifying all systems of uniform geodesics of $\mathcal{S}$.

First, we ask whether meridians are one such system. Since in this case we can choose $\boldsymbol{t} = \pm\boldsymbol{e}_v$, we easily answer the question for the *positive*, as correspondingly $\boldsymbol{t}_\perp = \mp\boldsymbol{e}_u$ and (18) follows as an immediate consequence of (24).

Figure 3 illustrates different instances in which meridians convey a uniform field, with different choices of bend and splay reflected by different values of the angle α in (16). The field lines of $\boldsymbol{n}$ (also shown in Fig. 3) are *loxodromes* of the pseudosphere.[16]

As shown in Appendix D, a geodesic represented by (25) has unit tangent vector $\boldsymbol{t}$ on $\mathcal{S}$ given by[17]

$$\boldsymbol{t} = -\sin\vartheta\boldsymbol{e}_u + \cos\vartheta\boldsymbol{e}_v, \tag{28}$$

so that

$$\boldsymbol{t}_\perp = -\cos\vartheta\boldsymbol{e}_u - \sin\vartheta\boldsymbol{e}_v. \tag{29}$$

To apply (18) to the general system of geodesics represented by (25), we need to compute $\nabla_s\boldsymbol{t}$; it follows from (29) that

$$\nabla_s\boldsymbol{t} = -\sin\vartheta\nabla_s\boldsymbol{e}_u + \cos\vartheta\nabla_s\boldsymbol{e}_v + \boldsymbol{t}_\perp \otimes \nabla_s\vartheta. \tag{30}$$

[16] A loxodrome cuts meridians at a constant angle.

[17] Here $\boldsymbol{t}$ is oriented coherently with the positive orientation chosen for ϑ in Fig. 2.

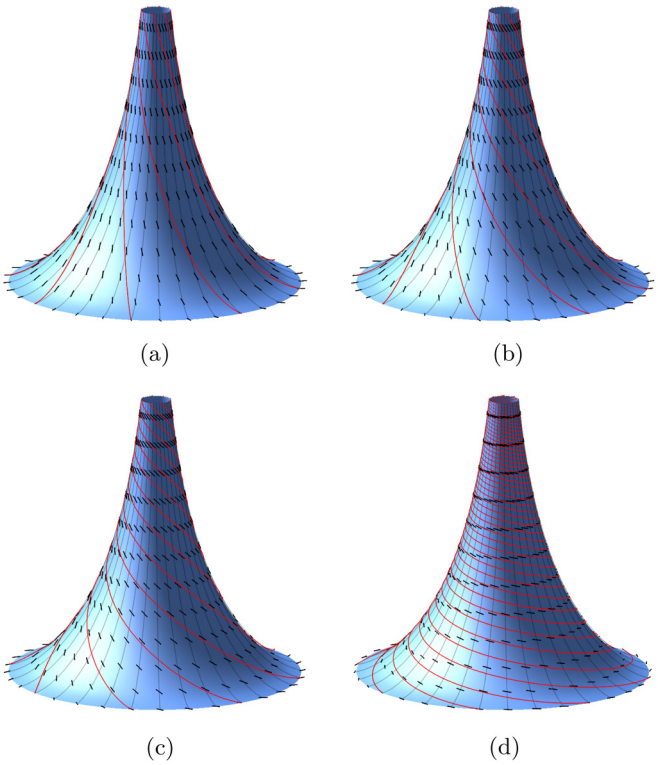


FIG. 3. Meridians are uniform geodesics of the pseudosphere. They convey a uniform field $\boldsymbol{n}$ (represented by headless directors) whose (red) field lines are loxodromes. (a) $\alpha = -\frac{5\pi}{12}$. (b) $\alpha = -\frac{\pi}{3}$. (c) $\alpha = -\frac{\pi}{4}$. (d) $\alpha = -\frac{\pi}{12}$.

Since (see Appendix D),

$$\nabla_s \vartheta = \frac{\sin\vartheta}{u_0' \cos\vartheta + \rho'} \{-\rho' \sin\vartheta \boldsymbol{e}_u + (u_0' + \rho' \cos\vartheta)\boldsymbol{e}_v\}, \tag{31}$$

by (24), (29), and (31), we easily arrive at

$$\boldsymbol{t}_\perp \cdot (\nabla_s \boldsymbol{t})\boldsymbol{t}_\perp = -\frac{u_0' + \rho' \cos\vartheta}{u_0' \cos\vartheta + \rho'}. \tag{32}$$

Thus, (18) is satisfied if and only if

$$u_0' = \pm\rho', \tag{33}$$

that is, whenever

$$u_0 = \pm\rho + m, \tag{34}$$

where m is an arbitrary constant.

Geometrically, (34) represents families of circles in the (u, v) plane with centers on the u axis, all tangent to one another at the point $(m, 0)$, see Fig. 4: for definiteness, we shall call *right* and *left* uniform geodesics those for which (34) holds with either the *plus* or *minus* sign, respectively.[18] The preimages of uniform geodesics in Fig. 4 are drawn only within the domain Ω, where the change of variables in (27) is meaningful. Right and left uniform geodesics are separated by the meridian at $u = m$. Since the lines $u = 0$ and $u = 2\pi$ correspond to the same meridian on the pseudosphere, all uniform fields conveyed by these families of geodesics exhibit a *line defect* along that (singular) meridian.

Loxodromes of the meridians of the pseudosphere are thus the field lines of the only *global* nematic surface uniform fields. Nonetheless, uniform geodesics other than meridians convey *local* nematic surface uniform fields, which exist by their own right everywhere but on a single meridian.[19]

The reader should not be induced to think that a *single* geodesic of $\mathcal{S}$ could be designated as *uniform*. As already apparent from (18), which involves $\nabla_s \boldsymbol{t}$ and not just $\boldsymbol{t}$, the notion of uniformity properly applies to a *system* of geodesics, that is, to the way geodesics are bundled together. This is visually illustrated in Figs. 4(a) and 4(c): They show the same preimage of a geodesic of $\mathcal{S}$, seen in one case as part of a system of right uniform geodesics, and in the other as part of a system of left ones. Thus, if the direction of $\boldsymbol{n}$ is prescribed at a point on $\mathcal{S}$ and an angle α is assigned, which represents constant distortion components, then one would determine through (13) the direction of the tangent $\boldsymbol{t}$ to the local geodesic $\mathcal{C}$ that conveys the uniform field fulfilling these prescriptions. Away from $\mathcal{C}$, however, there are *two* uniform director fields with equal distortion components and the same trace on $\mathcal{C}$, one for each family of uniform geodesics (right and left) to which $\mathcal{C}$ belongs.

The dichotomy between right and left uniform geodesics on the pseudosphere is the visual embodiment of the duality between uniform fields anticipated on analytical grounds in Sec. III. We may assert that such a duality actually persists on all pseudospherical surfaces.[20]

The separating meridian at $u = m$ in Fig. 4(b) is reached by both right and left uniform geodesics in the limit as $\rho \to +\infty$ (and, correspondingly, $u_0 \to \pm\infty$) in (34). Thus, by expanding into a ribbon the separating meridian, right and left uniform geodesics can be further combined, as shown by the examples in Fig. 5.These hybridations produce uniform fields that are of class C^1 (away from the singular meridian), but not C^2, due to the intervening separating meridians that can only be approached asymptotically; they would be ruled out by a request of higher regularity.

The same ostracism would fall on the case shown in Fig. 4(b), for which we illustrate in Fig. 6 the corresponding

[18]These very geodesics were first encountered by Bianchi, who called them *parallel* geodesics (in the sense of Lobatchewski), as they emanate from a point at infinity [that is, the point $(m, 0)$ on the u axis of the half-plane model for hyperbolic two-dimensional space] [32]. They also play a central role in Bianchi's transformation of pseudospherical surfaces that preserves their Gaussian curvature [33]. The circles in the (u, v) plane orthogonal to Bianchi's parallel geodesics [all tangent to the u axis at $(m, 0)$] are called *homocycles* in hyperbolic geometry (see, for example, Ref. [34, p. 50] or Ref. [35, p. 300], for a more recent account). They are preimages under the mapping $\boldsymbol{r}$ in (22) of curves with constant geodesic curvature on the pseudosphere.

[19]A similar, but more general notion of *quasi-uniformity*, which is equally local in nature, has been studied in Refs. [36,37] for both flat and curved surfaces.

[20]Meridians are limiting cases of both right and left uniform geodesics. We may also say that they are self-conjugated, as they satisfy both forms of (18).

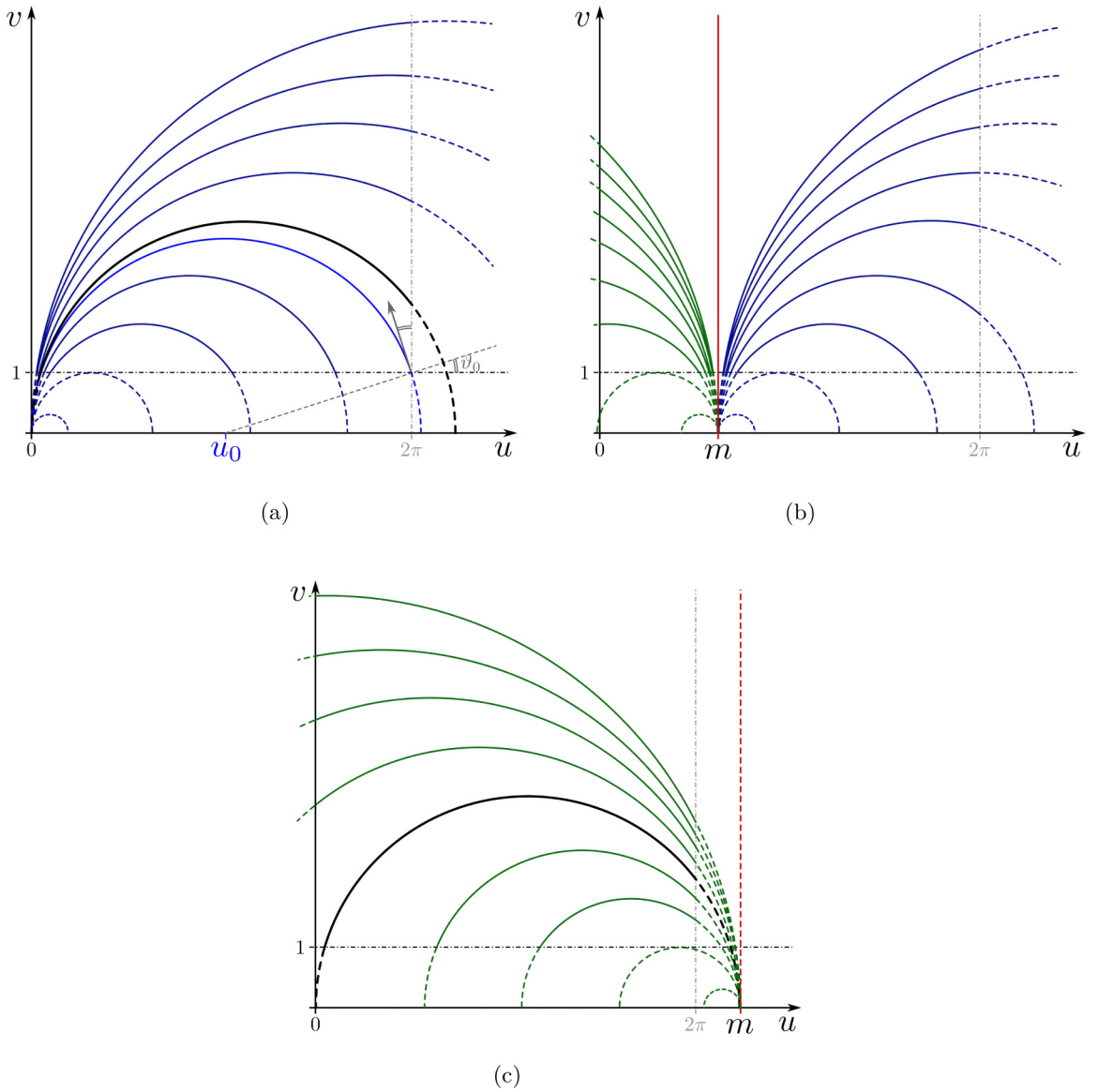


FIG. 4. Preimages of uniform geodesics shown in the (u, v) plane for different values of the parameter m designating the separating meridian (marked in red). Right uniform geodesics are blue, while left uniform geodesics are green. The (exceptionally) black geodesic in panels (a) and (c) is precisely the same curve, belonging to two separate systems of right and left uniform geodesics. (a) Right uniform geodesics, $m = 0$. (b) Right and left uniform geodesics, $0 < m < 2\pi$. (c) Left uniform geodesics, $m > 2\pi$.

uniform geodesics conveying a uniform director field (represented by headless directors) with $\alpha = -\pi/3$.

To illustrate a case of higher regularity, we set $m = 0$, as in Fig. 4(a), and draw via (22) the corresponding family of (right) uniform geodesics on the pseudosphere. Both uniform geodesics and director fields for selected distortion components (again $\alpha = -\pi/3$) are depicted in Fig. 7, where three views are shown; an animation where the pseudosphere rotates about its symmetry axis is provided as a supplementary material accompanying this paper [38].[21]

For $m = 0$, as a measure of defectiveness on the singular meridian, we compute the maximum angular mismatch of directors along it. As is easily seen, this is independent of α, it is attained on the bounding rim, and equals the angle ϑ_0 shown in Fig. 4(a); a simple geometric construction delivers

$$\vartheta_0 = \arccos\left(\frac{4\pi^2 - 1}{4\pi^2 + 1}\right). \tag{35}$$

By choosing $m < 0$ (or $m > 2\pi$, for left uniform geodesics), other, less distorted, uniform fields can easily be generated. In the limit as $m \to -\infty$ (or $m \to +\infty$, for left uniform geodesics), these recover asymptotically the meridians of the pseudosphere.

[21]Both the pictures shown in Fig. 7 and the animation playing in Ref. [38] were produced with the Maple script in Ref. [39].

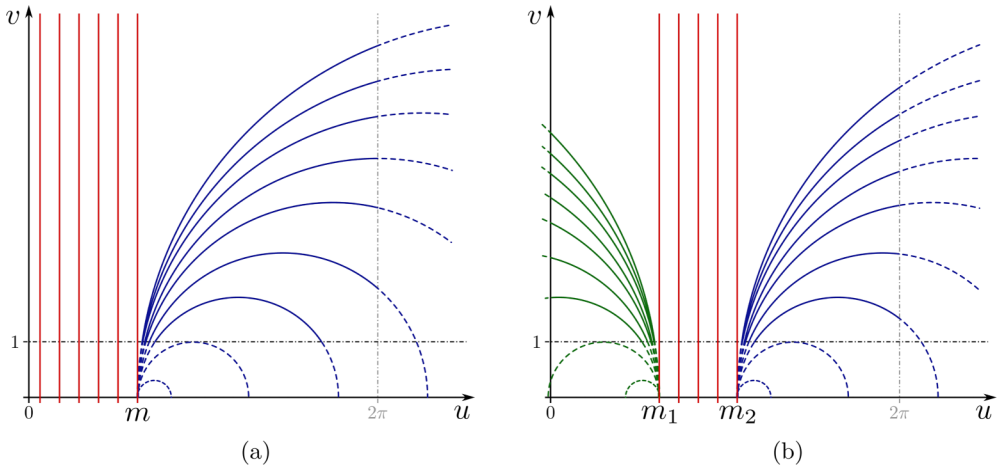


FIG. 5. Hybrid uniform geodesics. (a) Meridians and right uniform. (b) Meridians, right, and left uniform.

V. SUMMARY AND OUTLOOK

We have addressed the problem of determining the most general nematic field on a smooth surface that a two-dimensional observer (intrinsic to the surface and unaware of the dimension along the normal) would see as equally distorted at all points. For these fields, called uniform, we gave a formal definition formulated in the language of moving frames.

A remarkable advance in the study of surface uniformity was the necessary condition proved by Niv and Efrati [27] that requires the hosting surface to be pseudospherical, that is, with constant *negative* Gaussian curvature. However, nothing was known about the actual existence of uniform fields on those surfaces, let alone their structure.

We solved this problem (1) by proving that a uniform field is parallel transported by geodesics and (2) by characterizing all systems of geodesics that can convey a uniform field.

The latter were said to constitute a *system* of uniform geodesics, reflecting more the way they are bundled together than an individual property. For any given geodesic, we found *two* distinct systems of uniform geodesics to which it belongs, all conveying a uniform field.

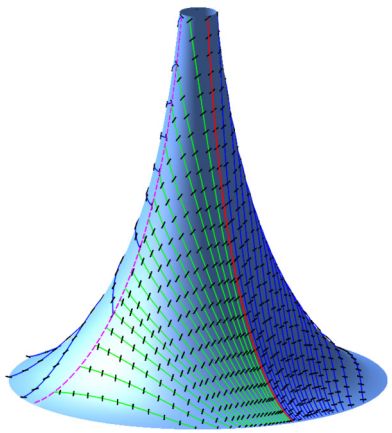


FIG. 6. Uniform geodesics as in Fig. 4(b) conveying a uniform field $\boldsymbol{n}$ as in (16) with $\alpha = -\pi/3$. The separating meridian (at $u = m$) is red, while the singular meridian (at $u = 0$) is (dashed) purple. Right uniform geodesics are blue, while left uniform geodesics are green.

Our general geometric construction was made explicit for Beltrami's pseudosphere, whose uniform fields were characterized completely. These are both without and with defects. The former have the loxodromes of meridians as field lines, all others have a line defect along a singular meridian.

Since, by Minding's classical theorem, all surfaces with the same constant Gaussian curvature are isometric and both geodesics and uniformity are preserved by isometries, the solution for the pseudosphere entrains the solution for all pseudospherical surfaces: It would suffice to carry over (at least locally) the system of uniform geodesics.

Although this last task may fail to be easily accomplished, the general structure of uniform geodesics and the corresponding duality between the generated uniform fields remain valid. An analytic condition was given that characterizes the system of uniform geodesics on a generic surface. Solving it, one can find directly all uniform fields on that surface.

Nematic uniformity touches upon the frustrating power of surfaces. The intuitive idea behind this association is that the ground state of an intrinsic elastic theory based on a surface director order parameter would, if possible, be uniform. Preventing the latter from being realized ignites geometric frustration. To clarify this point and to illustrate a physical application of the theory presented here, perhaps calling for further investigation, we briefly consider an ordered fluid membrane. Imagine that a tangent unit vector field $\boldsymbol{n}$ describes the ordering on an inexensible membrane, whose shape is free to vary. A criterion to build a general elastic theory for this system would be to elect a particular class of uniform fields as ground state and penalize (say, quadratically) all departures from that class. We have learned that uniform fields with prescribed (constant) splay S and bend $b_\perp$ exist on a surface $\mathcal{S}$, provided that the Gaussian curvature K of $\mathcal{S}$ is given by (8). Thus, for S and $b_\perp$ prescribed and for a generic surface $\mathcal{S}$, a minimalist elastic cost involved in the departure of $\mathcal{S}$ and $\boldsymbol{n}$

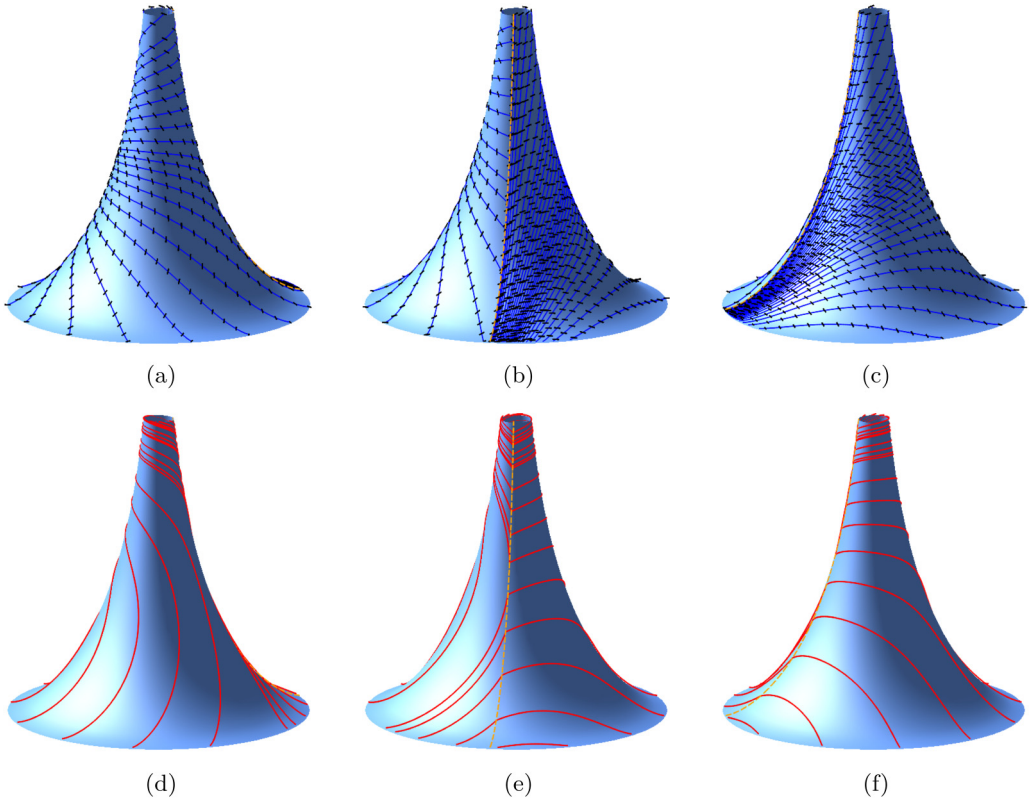


FIG. 7. Three views of the pseudosphere bearing the uniform director field $\boldsymbol{n}$ in (16) with $\alpha = -\pi/3$. The preimages of the conveying geodesics are depicted in Fig. 4(a). Geodesics and directors are shown in panels (a)–(c), while the corresponding field lines of $\boldsymbol{n}$ are shown in panels (d)–(f). Geodesics are blue, nematic field lines are red, and the singular meridian is yellow. A supplementary movie shows all other views [38].

from their ideal ground state would read as

$$\mathcal{F}_{\rm i}[\mathcal{S},\boldsymbol{n}] := \frac{1}{2}\int_{\mathcal{S}} \{k_s(\boldsymbol{c}\cdot\boldsymbol{n}_\perp - S)^2 + k_b(\boldsymbol{c}\cdot\boldsymbol{n} + b_\perp)^2 + k_c(K + S^2 + b_\perp^2)^2\}\mathrm{d}A, \tag{36}$$

or, equivalently, by (7), as

$$\mathcal{F}_{\rm i}[\mathcal{S},\boldsymbol{n}] = \frac{1}{2}\int_{\mathcal{S}} \{k_s(\mathrm{div}_c\boldsymbol{n} - S)^2 + k_b(\mathrm{curl}_c\,\boldsymbol{n}\cdot\boldsymbol{\nu} + b_\perp)^2 + k_c(K + S^2 + b_\perp^2)^2\}\mathrm{d}A, \tag{37}$$

where k_s, k_b, and k_c are positive elastic moduli penalizing splay, bending, and intrinsic curvature, independently, $\boldsymbol{c}$ is the spin connector of the moving frame $(\boldsymbol{n},\boldsymbol{n}_\perp,\boldsymbol{\nu})$, S and $b_\perp$ are constants (which can possibly vanish) designating a uniform target state, and A denotes the area measure. For a deformable surface $\mathcal{S}$, $\mathcal{F}_{\rm i}$ clearly depends on both $\mathcal{S}$ and the director field $\boldsymbol{n}$ defined on it. It follows from (B11) (and from Gauss' *theorema egregium*) that $\mathcal{F}_{\rm i}$ is invariant under isometries of $\mathcal{S}$, and so it is intrinsic. Letting $S = b_\perp = 0$, $k_s = k_b$ and $k_c = 0$ in (37), we readily recover the energy posited in Ref. [2] [see their Eq. (21a)]. Similarly, letting $S = b_\perp = 0$ and $k_c = 0$, we recover the intrinsic part of the energy functional in Ref. [6] [see their Eq. (6)]. In neither of these studies, nor in subsequent ones that we are aware of, was the mismatch in Gaussian curvature penalized as in (37). For arbitrary choices of the parameters featuring in (37), the functional $\mathcal{F}_{\rm i}$ should be studied to ascertain whether the qualitative predictions afforded by the classical intrinsic theories subsumed in it remain valid.[22] It should finally be remarked that, by construction, $\mathcal{F}_{\rm i}$ incorporates only intrinsic energies; extrinsic energies, which are not invariant under isometries of $\mathcal{S}$, could be added to model specific couplings. All these investigations are left for future work. This short account should just serve to illustrate the ability of the geometric result presented here to generate new elastic theories for material surfaces.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [39].

[22] One of these is the ability of planar ordering to expel Gaussian curvature [4].

APPENDIX A: MATHEMATICAL PRELIMINARIES

In this Appendix, to make our development self-contained and to explain the notation employed throughout the paper, we recall a few preliminary results about calculus on surfaces.[23] We use an *absolute* approach to surface calculus,[24] generally inspired by the work of Weatherburn [40,41], whose essential features are also succinctly outlined in Ref. [42].[25]

Let $\mathcal{S}$ be a smooth (at least of class C^2), orientable surface imbedded in three-dimensional space $\mathcal{E}$. We denote by $\boldsymbol{\nu}$ one orientation of the unit normal on $\mathcal{S}$.

A major role is played below by the notion of *surface gradient*, which we introduce with the aid of smooth curves $\boldsymbol{x}(t)$ on $\mathcal{S}$. A scalar field $\varphi : \mathcal{S} \to \mathbb{R}$ is differentiable on $\mathcal{S}$ whenever we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\boldsymbol{x}(t)) = \nabla_s\varphi \cdot \dot{\boldsymbol{x}}, \tag{A1}$$

where a superimposed dot $\dot{}$ denotes differentiation with respect to the parameter t and $\dot{\boldsymbol{x}}$ is a vector along the tangent to $\boldsymbol{x}(t)$. Similarly, for a vector field $\boldsymbol{v} : \mathcal{S} \to \mathcal{V}$, where $\mathcal{V}$ is the translation space associated with $\mathcal{E}$,[26]

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(\boldsymbol{x}(t)) = (\nabla_s\boldsymbol{v})\dot{\boldsymbol{x}}, \tag{A2}$$

where the second-rank tensor $\nabla_s\boldsymbol{v}$, which annihilates the normal $\boldsymbol{\nu}$, is the surface gradient of $\boldsymbol{v}$. In particular, $\nabla_s\boldsymbol{\nu}$ is the *curvature tensor* of $\mathcal{S}$: It is a symmetric tensor, whose eigenvalues are the *principal* curvatures of $\mathcal{S}$.[27]

Letting $\mathbf{P}(\boldsymbol{\nu}) := \mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ represent the projection onto the local tangent plane $\mathcal{T}$,[28] we define the *covariant* gradient $\nabla_c\boldsymbol{v}$ as

$$\nabla_c\boldsymbol{v} := \mathbf{P}(\boldsymbol{\nu})\nabla_s\boldsymbol{v}. \tag{A3}$$

It reduces $\nabla_s\boldsymbol{v}$ to an action onto $\mathcal{T}$, and so it can be considered as being totally inherent to the surface (or intrinsic), insensitive to the surrounding space (see also Ref. [29, p. 240]).[29]

The surface curl of $\boldsymbol{v}$, $\mathrm{curl}_s\boldsymbol{v}$, is defined by the identity

$$2\mathrm{skw}(\nabla_s\boldsymbol{v})\boldsymbol{u} = [\nabla_s\boldsymbol{v} - (\nabla_s\boldsymbol{v})^{\mathsf{T}}]\boldsymbol{u} = \mathrm{curl}_s\boldsymbol{v} \times \boldsymbol{u} \quad \forall\, \boldsymbol{u} \in \mathcal{V}, \tag{A4}$$

where skw denotes the skew-symmetric part of a second-rank tensor, a superscript $^{\mathsf{T}}$ the transposition of a tensor, and $\times$ the vector product in $\mathcal{V}$. Equation (A4) simply says that $\mathrm{curl}_s\boldsymbol{v}$ is the axial vector associated with $2\mathrm{skw}(\nabla_s\boldsymbol{v})$. In a parallel way, we define the *covariant* curl of $\boldsymbol{v}$, $\mathrm{curl}_c\boldsymbol{v}$, as the axial vector associated with $2\mathrm{skw}(\nabla_c\boldsymbol{v})$. Likewise, the surface and *covariant* divergence of $\boldsymbol{v}$ are defined as

$$\mathrm{div}_s\boldsymbol{v} := \mathrm{tr}(\nabla_s\boldsymbol{v}) \quad \text{and} \quad \mathrm{div}_c\boldsymbol{v} := \mathrm{tr}(\nabla_c\boldsymbol{v}), \tag{A5}$$

respectively, where tr denotes the trace of a second-rank tensor.

As also recalled in Ref. [42], if $\boldsymbol{h}$ is a *tangential* vector field, that is, such that $\boldsymbol{h} \cdot \boldsymbol{\nu} \equiv 0$, then there exists a scalar field φ on $\mathcal{S}$ such that $\boldsymbol{h} = \nabla_s\varphi$, if and only if

$$\mathrm{skw}(\nabla_s\boldsymbol{h}) = \mathrm{skw}((\nabla_s\boldsymbol{\nu})\boldsymbol{h} \otimes \boldsymbol{\nu}). \tag{A6}$$

Similarly, letting a second-rank tensor field $\mathbf{H}$ be defined on $\mathcal{S}$ so that $\mathbf{H}\boldsymbol{\nu} \equiv \mathbf{0}$, there exists a vector field $\boldsymbol{v}$ on $\mathcal{S}$ such that $\mathbf{H} = \nabla_s\boldsymbol{v}$, if and only if

$$\mathrm{skw}(\nabla_s\mathbf{H}) = \mathrm{skw}(\mathbf{H}(\nabla_s\boldsymbol{\nu}) \otimes \boldsymbol{\nu}), \tag{A7}$$

where $\nabla_s\mathbf{H}$ is a third-rank tensor and skw acts as follows on a generic *triadic* tensor $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3$,

$$2\mathrm{skw}(\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3) := \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3 - \boldsymbol{a}_1 \otimes \boldsymbol{a}_3 \otimes \boldsymbol{a}_2. \tag{A8}$$

We employ the method of *moving frames*.[30] An orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$, where $\boldsymbol{\nu} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$,[31] glides over $\mathcal{S}$ according to the laws

$$\begin{aligned} \nabla_s\boldsymbol{e}_1 &= \boldsymbol{e}_2 \otimes \boldsymbol{c} + \boldsymbol{\nu} \otimes \boldsymbol{d}_1, \\ \nabla_s\boldsymbol{e}_2 &= -\boldsymbol{e}_1 \otimes \boldsymbol{c} + \boldsymbol{\nu} \otimes \boldsymbol{d}_2, \\ \nabla_s\boldsymbol{\nu} &= -\boldsymbol{e}_1 \otimes \boldsymbol{d}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{d}_2, \end{aligned} \tag{A9}$$

where the vector fields $(\boldsymbol{c}, \boldsymbol{d}_1, \boldsymbol{d}_2)$ are everywhere tangent to $\mathcal{S}$; these are the *connectors* of the moving frame. More precisely, $\boldsymbol{c}$ is the *spin* connector, and $\boldsymbol{d}_1, \boldsymbol{d}_2$ are the *curvature* connectors.[32] Since the curvature tensor $\nabla_s\boldsymbol{\nu}$ is symmetric, the

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[23] The experienced reader would likely jump ahead.

[24] That is, not relying on coordinates, as we abhor indices at least as much as Cartan did, see Ref. [29, p. xvii].

[25] The work of Weatherburn was preceded by the introduction of a general vector method in differential geometry by the Italian school that originated from Levi-Civita (see Refs. [43–46] for the relevant historical sources and [47] for a more recent application to soft matter science).

[26] Our notation for $\mathcal{E}$ and $\mathcal{V}$ is the same as in Ref. [48, p. 324], where these geometric structures are further illuminated, especially in connection with their role in formulating modern continuum mechanics.

[27] The tensor $-\nabla_s\boldsymbol{\nu}$ is often also called the *shape operator* (see, for example, Ref. [28, p. 203]).

[28] Here the *dyadic* product $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2$ of vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is defined as the second-rank tensor whose action on a generic vector $\boldsymbol{u}$ is specified as $(\boldsymbol{a}_1 \otimes \boldsymbol{a}_2)\boldsymbol{u} = (\boldsymbol{a}_2 \cdot \boldsymbol{u})\boldsymbol{a}_1$, where $\cdot$ denotes the inner product in $\mathcal{V}$.

[29] In differential geometry, are called *intrinsic* the properties of a surface that are independent of its imbedding in three-dimensional space, and *extrinsic* those properties that depend on that. As effectively put in Ref. [29, p. 11],

> [Intrinsic geometry] means the geometry that is knowable to tiny, antlike, intelligent (but two-dimensional!) creatures living *within* the surface.

Clearly, whenever the normal to the surface features in one of its properties, that is likely to be extrinsic.

[30] This method was first introduced by Cartan [49] and is extensively used in the book [28] to illustrate the differential geometry of spaces, surfaces, and curves. A more recent, rather comprehensive account, far more formal than needed here, can be found in the book [50].

[31] We only consider *positively* oriented, orthonormal frames with one unit vector coincident with $\boldsymbol{\nu}$.

[32] We call them connectors because they connect the frame at one point to the frame at a nearby point. They are further expounded in Refs. [42,51,52], where they are erroneously referred to as *Cartesian* (instead of *Cartanian*). See also Ref. [53] for a recent application to shell theory.

curvature connectors must obey the identity

$$\boldsymbol{d}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{d}_2 \cdot \boldsymbol{e}_1. \tag{A10}$$

In particular, the third equation in (A9) implies that

$$2H := \mathrm{tr}(\nabla_s \boldsymbol{v} = -(\boldsymbol{d}_1 \cdot \boldsymbol{e}_1 + \boldsymbol{d}_2 \cdot \boldsymbol{e}_2), \tag{A11a}$$

$$K := \det(\nabla_s \boldsymbol{v}) = \boldsymbol{d}_1 \times \boldsymbol{d}_2 \cdot \boldsymbol{v}, \tag{A11b}$$

where H and K are the *mean* and *Gaussian* curvatures of $\mathcal{S}$, respectively.[33] In this paper, we apply (A9) to a variety of different moving frames, each entailing its own set of connectors.

Given a smooth curve $\mathcal{C}$ on $\mathcal{S}$, let $\boldsymbol{t}$ be the unit tangent vector to $\mathcal{C}$. The moving frame $(\boldsymbol{t}, \boldsymbol{t}_\perp, \boldsymbol{v})$, where $\boldsymbol{t}_\perp := \boldsymbol{v} \times \boldsymbol{t}$, is the *Darboux* frame of $\mathcal{C}$; it satisfies the equations

$$\begin{aligned} \boldsymbol{t}' &= \kappa_g \boldsymbol{t}_\perp + \kappa_n \boldsymbol{v}, \\ \boldsymbol{t}_\perp' &= -\kappa_g \boldsymbol{t} - \tau_g \boldsymbol{v}, \\ \boldsymbol{v}' &= -\kappa_n \boldsymbol{t} + \tau_g \boldsymbol{t}_\perp, \end{aligned} \tag{A12}$$

where a prime $'$ denotes differentiation with respect to the arc-length coordinate s along $\mathcal{C}$, κ_g is the *geodesic* curvature, κ_n the *normal* curvature, and τ_g the *geodesic* torsion of $\mathcal{C}$.[34] Equations (A12) can be given a more concise (and perhaps more telling) form,

$$\boldsymbol{t}' = \boldsymbol{\Omega} \times \boldsymbol{t}, \quad \boldsymbol{t}_\perp' = \boldsymbol{\Omega} \times \boldsymbol{t}_\perp, \quad \boldsymbol{v}' = \boldsymbol{\Omega} \times \boldsymbol{v}, \tag{A13}$$

by introducing the *spin vector* $\boldsymbol{\Omega}$ defined as

$$\boldsymbol{\Omega} := -\tau_g \boldsymbol{t} - \kappa_n \boldsymbol{t}_\perp + \kappa_g \boldsymbol{v}. \tag{A14}$$

By applying (A9) to the frame $(\boldsymbol{t}, \boldsymbol{t}_\perp, \boldsymbol{v})$, assuming that $\mathcal{C}$ is a field line with tangent $\boldsymbol{t}$ that crosses everywhere at right angles a field line $\mathcal{C}_\perp$ with tangent $\boldsymbol{t}_\perp$, we can write

$$\boldsymbol{t}' = (\nabla_s \boldsymbol{t})\boldsymbol{t} = (\boldsymbol{c} \cdot \boldsymbol{t})\boldsymbol{t}_\perp + (\boldsymbol{d}_1 \cdot \boldsymbol{t})\boldsymbol{v}, \tag{A15a}$$

$$\boldsymbol{t}_\perp' = (\nabla_s \boldsymbol{t}_\perp)\boldsymbol{t}_\perp = -(\boldsymbol{c} \cdot \boldsymbol{t}_\perp)\boldsymbol{t} + (\boldsymbol{d}_2 \cdot \boldsymbol{t}_\perp)\boldsymbol{v}. \tag{A15b}$$

Combining these equations with (A12), and treating in exactly the same way the Darboux frame $(\boldsymbol{t}_\perp, -\boldsymbol{t}, \boldsymbol{v})$ associated with $\mathcal{C}_\perp$, characterized by the differential measures $\kappa_g^\perp$, $\kappa_n^\perp$, and $\tau_g^\perp$, we arrive at the following representation for the connectors:[35]

$$\boldsymbol{c} = \kappa_g \boldsymbol{t} + \kappa_g^\perp \boldsymbol{t}_\perp, \tag{A16a}$$

$$\boldsymbol{d}_1 = \kappa_n \boldsymbol{t} + \tau_g^\perp \boldsymbol{t}_\perp, \tag{A16b}$$

$$\boldsymbol{d}_2 = -\tau_g \boldsymbol{t} + \kappa_n^\perp \boldsymbol{t}_\perp. \tag{A16c}$$

For (A10) to be valid (with $\boldsymbol{e}_1 = \boldsymbol{t}$ and $\boldsymbol{e}_2 = \boldsymbol{t}_\perp$), it must then be

$$\tau_g + \tau_g^\perp = 0, \tag{A17}$$

which is a known identity (see, for example, Ref. [54, p. 488]).

For a unit vector $\boldsymbol{n}(s)$ tangent to $\mathcal{S}$ prescribed along $\mathcal{C}$ as a function of s, we say that it is *parallel transported* along $\mathcal{C}$ if

$$\boldsymbol{n}' = \boldsymbol{\Omega}_\parallel \times \boldsymbol{n}, \tag{A18}$$

where the spin vector $\boldsymbol{\Omega}_\parallel$ is such that $\boldsymbol{\Omega}_\parallel \cdot \boldsymbol{v} \equiv 0$.[36] Equivalently, (A18) prescribes $\boldsymbol{n}'$ to be parallel to $\boldsymbol{v}$. Letting $\boldsymbol{t}$ be the unit vector tangent to $\mathcal{C}$, we can represent $\boldsymbol{n}$ as

$$\boldsymbol{n} = \cos\gamma \boldsymbol{t} + \sin\gamma \boldsymbol{t}_\perp, \tag{A19}$$

where $\gamma = \gamma(s)$ is a smooth function. With the aid of (A12), we can easily compute

$$\boldsymbol{n}' = (\gamma' + \kappa_g)\boldsymbol{n}_\perp + (\kappa_n \cos\gamma - \tau_g \sin\gamma)\boldsymbol{v}, \tag{A20}$$

which shows that $\boldsymbol{n}$ is parallel transported along $\mathcal{C}$ whenever

$$\gamma' + \kappa_g = 0. \tag{A21}$$

In particular, if $\kappa_g \equiv 0$, which is the case if $\mathcal{C}$ is a geodesic of $\mathcal{S}$, then γ is constant along $\mathcal{C}$.

A surface $\mathcal{S}$ can also be represented by use of coordinates (u, v) as the image of a mapping $\boldsymbol{r} : \Omega \to \mathcal{E}$, where Ω is a domain in $\mathbb{R}^2$. For a surface of class C^2, coordinates (u, v) can be chosen so as to be, at least locally, *isothermal*,[37] that is, such that

$$\boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0 \quad \text{and} \quad |\boldsymbol{r}_u| = |\boldsymbol{r}_v|, \tag{A22}$$

where $\boldsymbol{r}_u := \partial_u \boldsymbol{r}$ and $\boldsymbol{r}_v := \partial_v \boldsymbol{r}$. Letting

$$\boldsymbol{e}_u := \frac{\boldsymbol{r}_u}{|\boldsymbol{r}_u|} \quad \text{and} \quad \boldsymbol{e}_v := \frac{\boldsymbol{r}_v}{|\boldsymbol{r}_v|}, \tag{A23}$$

we orient $\mathcal{S}$ so that $\boldsymbol{v} = \boldsymbol{e}_u \times \boldsymbol{e}_v$. The moving frame $(\boldsymbol{e}_u, \boldsymbol{e}_v, \boldsymbol{v})$ is subject to (A9), where the appropriate connectors will be denoted as $(\boldsymbol{c}, \boldsymbol{d}_u, \boldsymbol{d}_v)$.

APPENDIX B: ISOMETRIC INVARIANCE

Here we prove that a nematic field $\boldsymbol{n}$ uniform on a surface $\mathcal{S}$ remains uniform when conveyed by any isometric deformation of $\mathcal{S}$.

To this end, we recall that a deformation $\boldsymbol{y} : \mathcal{S} \to \mathcal{E}$, which is of class C^2 and produces the surface $\mathcal{S}^* = \boldsymbol{y}(\mathcal{S})$ as image of $\mathcal{S}$, is an isometry whenever its surface gradient can be represented as

$$\nabla_s \boldsymbol{y} = \mathbf{R}\mathbf{P}(\boldsymbol{v}), \tag{B1}$$

where $\mathbf{R}$ is a member of the special orthogonal group $\mathsf{SO}(3)$, generally depending on position over $\mathcal{S}$.[38] Suppose that $\boldsymbol{n}$ is a uniform field on $\mathcal{S}$ and let $\boldsymbol{n}^*$ be the unit vector field conveyed on $\mathcal{S}^*$ by $\boldsymbol{y}$, so that $\boldsymbol{n}^* = (\nabla_s \boldsymbol{y})\boldsymbol{n}$. Thus, $\mathbf{R}$ can be

[33] By $\mathrm{tr}(\nabla_s \boldsymbol{v})$ and $\det(\nabla_s \boldsymbol{v})$, we mean the sum and the product of the principal curvatures of $\mathcal{S}$, respectively.

[34] Equations (A12) differ by the sign of τ_g from the traditional form presented in most textbooks on differential geometry (see, for example, Ref. [54, p. 264]).

[35] Since by (A14) κ_g is the component of $\boldsymbol{\Omega}$ along $\boldsymbol{v}$, (A16a) fully justifies calling $\boldsymbol{c}$ the *spin* connector.

[36] This definition of parallel transport along a curve is that of Levi-Civita [55], as interpreted in the kinematic analogy of Persico [56]. $\boldsymbol{\Omega}_\parallel$ can also be characterized as the *least* spin vector $\boldsymbol{\Omega}$ that makes $\boldsymbol{n}$ glide along $\mathcal{C}$ as part of the moving frame $(\boldsymbol{n}, \boldsymbol{n}_\perp, \boldsymbol{v})$, see Ref. [47].

[37] For surfaces of class C^1, isothermal coordinates can fail to exist. The existence of isothermanl coordinates for C^2 surfaces is established by a classical theorem revisited in Ref. [57].

[38] Equation (B1) is a special form of the polar decomposition theorem for the deformation of surfaces proved in Ref. [58]: It requires that the surface stretching tensor $\mathbf{U}$ be the projection $\mathbf{P}(\boldsymbol{v})$.

represented as

$$\mathbf{R} = \boldsymbol{n}^* \otimes \boldsymbol{n} + \boldsymbol{n}^*_\perp \otimes \boldsymbol{n}_\perp + \boldsymbol{\nu}^* \otimes \boldsymbol{\nu}, \tag{B2}$$

where $\boldsymbol{n}^*_\perp = \boldsymbol{\nu}^* \times \boldsymbol{n}^*$ and $\boldsymbol{\nu}^*$ is the unit normal to $\mathcal{S}^*$ so that $(\boldsymbol{n}^*, \boldsymbol{n}^*_\perp, \boldsymbol{\nu}^*)$ is oriented as $(\boldsymbol{n}, \boldsymbol{n}_\perp, \boldsymbol{\nu})$. The moving frame $(\boldsymbol{n}^*, \boldsymbol{n}^*_\perp, \boldsymbol{\nu}^*)$ obeys on $\mathcal{S}^*$ the laws

$$\begin{aligned}
\nabla^*_s \boldsymbol{n}^* &= \boldsymbol{n}^*_\perp \otimes \boldsymbol{c}^* + \boldsymbol{\nu}^* \otimes \boldsymbol{d}^*_1,\\
\nabla^*_s \boldsymbol{n}^*_\perp &= -\boldsymbol{n}^* \otimes \boldsymbol{c}^* + \boldsymbol{\nu}^* \otimes \boldsymbol{d}^*_2,\\
\nabla^*_s \boldsymbol{\nu}^* &= -\boldsymbol{n}^* \otimes \boldsymbol{d}^*_1 - \boldsymbol{n}^*_\perp \otimes \boldsymbol{d}^*_2,
\end{aligned} \tag{B3}$$

where ∇^*_s denotes the surface gradient on $\mathcal{S}^*$ and the connectors $(\boldsymbol{d}^*_1, \boldsymbol{d}^*_2)$ satisfy

$$\boldsymbol{d}^*_1 \cdot \boldsymbol{n}^*_\perp = \boldsymbol{d}^*_2 \cdot \boldsymbol{n}^*, \tag{B4}$$

which like (5) ensures that $\nabla^*_s \boldsymbol{\nu}^*$ is symmetric.

Our aim is to prove that $\boldsymbol{c}^*$ has constant components in the frame $(\boldsymbol{n}^*, \boldsymbol{n}^*_\perp, \boldsymbol{\nu}^*)$ whenever $\boldsymbol{c}$ has constant components in the frame $(\boldsymbol{n}, \boldsymbol{n}_\perp, \boldsymbol{\nu})$. We shall reach this conclusion as a consequence of the requirement of integrability in (A7) applied to (B1).

Setting

$$\mathbf{H} := \nabla_s \boldsymbol{y} = \boldsymbol{n}^* \otimes \boldsymbol{n} + \boldsymbol{n}^*_\perp \otimes \boldsymbol{n}_\perp, \tag{B5}$$

we first compute

$$\begin{aligned}
\nabla_s \mathbf{H} = {} & \boldsymbol{n}^* \otimes [\boldsymbol{n}_\perp \otimes \mathbf{P}(\boldsymbol{\nu})(\boldsymbol{c} - \mathbf{R}^\mathsf{T}\boldsymbol{c}^*) + \boldsymbol{\nu} \otimes \boldsymbol{d}_1]\\
& + \boldsymbol{n}^*_\perp \otimes [\boldsymbol{n} \otimes \mathbf{P}(\boldsymbol{\nu})(\mathbf{R}^\mathsf{T}\boldsymbol{c}^* - \boldsymbol{c}) + \boldsymbol{\nu} \otimes \boldsymbol{d}_2]\\
& + \boldsymbol{\nu}^* \otimes [\boldsymbol{n} \otimes \mathbf{P}(\boldsymbol{\nu})\mathbf{R}^\mathsf{T}\boldsymbol{d}^*_1 + \boldsymbol{n}_\perp \otimes \mathbf{P}(\boldsymbol{\nu})\mathbf{R}^\mathsf{T}\boldsymbol{d}^*_2],
\end{aligned} \tag{B6}$$

where we have used the identity

$$\nabla_s \boldsymbol{u}^* = \nabla^*_s \boldsymbol{u}^* \mathbf{R} \mathbf{P}(\boldsymbol{\nu}), \tag{B7}$$

valid, by the chain rule, for a generic differentiable vector field $\boldsymbol{u}^*$ on $\mathcal{S}^*$, and we have applied several times both (4) and (B3). Similarly, we compute

$$\mathbf{H}(\nabla_s \boldsymbol{\nu}) \otimes \boldsymbol{\nu} = -\boldsymbol{n}^* \otimes \boldsymbol{d}_1 \otimes \boldsymbol{\nu} - \boldsymbol{n}^*_\perp \otimes \boldsymbol{d}_2 \otimes \boldsymbol{\nu}. \tag{B8}$$

Now, requiring (A7) to hold amounts to require that three skew-symmetric tensors vanish, one for each left component of $\nabla_s \mathbf{H}$ in the frame $(\boldsymbol{n}^*, \boldsymbol{n}^*_\perp, \boldsymbol{\nu}^*)$. These identities eventually require that the corresponding axial vectors vanish. The integrability condition is thus reduced to the equations

$$\begin{aligned}
\boldsymbol{n} \times \mathbf{P}(\boldsymbol{\nu})(\boldsymbol{c} - \mathbf{R}^\mathsf{T}\boldsymbol{c}^*) &= \mathbf{0},\\
\boldsymbol{n}_\perp \times \mathbf{P}(\boldsymbol{\nu})(\boldsymbol{c} - \mathbf{R}^\mathsf{T}\boldsymbol{c}^*) &= \mathbf{0},\\
\boldsymbol{n}_\perp \cdot \mathbf{R}^\mathsf{T}\boldsymbol{d}^*_1 &= \boldsymbol{n} \cdot \mathbf{R}^\mathsf{T}\boldsymbol{d}^*_2.
\end{aligned} \tag{B9}$$

While the last one, by (B2), is equivalent to (B4), and so it is automatically satisfied, the first two are equivalent to

$$\boldsymbol{c} - \mathbf{R}^\mathsf{T}\boldsymbol{c}^* = \lambda \boldsymbol{\nu}, \tag{B10}$$

for some scalar λ. Since $\boldsymbol{c} \cdot \boldsymbol{\nu} = 0$, (B10) and (B2) imply that $\lambda = -\boldsymbol{c}^* \cdot \boldsymbol{\nu}^* = 0$, and so

$$\boldsymbol{c}^* = \mathbf{R}\boldsymbol{c} = -b_\perp \boldsymbol{n}^* + S\boldsymbol{n}^*_\perp, \tag{B11}$$

which proves that the field $\boldsymbol{n}^*$ is uniform with the same intrinsic components of bend and splay as $\boldsymbol{n}$. Thus, as desired, we conclude that the notion of uniformity is isometrically invariant: a field uniform on a surface $\mathcal{S}$ will be conveyed into a field uniform on all surfaces isometric to $\mathcal{S}$.

APPENDIX C: ANALYTIC CHARACTERIZATION OF UNIFORMITY

This Appendix collects conditions for surface uniformity that involve the connectors of a moving frame. They are necessary and sufficient but hard to resolve; they are recorded here for possible later use, mainly to provide an alternative, independent proof of the known fact that a uniform field can only be exhibited by a surface with constant negative Gaussian curvature.

In tune with the proof of isometric invariance of uniformity presented in Appendix B, here we employ systematically the (necessary and sufficient) integrability condition in (A7) to characterize the connectors of all moving frames $(\boldsymbol{n}, \boldsymbol{n}_\perp, \boldsymbol{\nu})$ compatible with a uniform unit vector field $\boldsymbol{n}$. In accordance with (7), we start by writing $\boldsymbol{c}$ in (4) as

$$\boldsymbol{c} = -b_\perp \boldsymbol{n} + S\boldsymbol{n}_\perp. \tag{C1}$$

To apply (A7) to $\mathbf{H} = \nabla_s \boldsymbol{n}$, we first insert (C1) in (4) and, since both $b_\perp$ and S are constant, we obtain that

$$\begin{aligned}
2\mathrm{skw}(\nabla^2_s \boldsymbol{n}) = {} & \boldsymbol{n}_\perp \otimes [(b^2_\perp + S^2)(\boldsymbol{n}_\perp \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n}_\perp) + b_\perp(\boldsymbol{d}_1 \otimes \boldsymbol{\nu} - \boldsymbol{\nu} \otimes \boldsymbol{d}_1)\\
& - S(\boldsymbol{d}_2 \otimes \boldsymbol{\nu} - \boldsymbol{\nu} \otimes \boldsymbol{d}_2) + (\boldsymbol{d}_2 \otimes \boldsymbol{d}_1 - \boldsymbol{d}_1 \otimes \boldsymbol{d}_2)]\\
& + \boldsymbol{\nu} \otimes [b_\perp(\boldsymbol{d}_2 \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{d}_2) - S(\boldsymbol{d}_2 \otimes \boldsymbol{n}_\perp - \boldsymbol{n}_\perp \otimes \boldsymbol{d}_2) + (\nabla_s \boldsymbol{d}_1) - (\nabla_s \boldsymbol{d}_1)^\mathsf{T}],
\end{aligned} \tag{C2a}$$

$$\begin{aligned}
2\mathrm{skw}[(\nabla_s \boldsymbol{n})(\nabla_s \boldsymbol{\nu}) \otimes \boldsymbol{\nu}] = {} & \boldsymbol{n}_\perp \otimes [b_\perp(\boldsymbol{d}_1 \otimes \boldsymbol{\nu} - \boldsymbol{\nu} \otimes \boldsymbol{d}_1) - S(\boldsymbol{d}_2 \otimes \boldsymbol{\nu} - \boldsymbol{\nu} \otimes \boldsymbol{d}_2)]\\
& - \boldsymbol{\nu} \otimes [(\boldsymbol{d}_1 \cdot \boldsymbol{n})(\boldsymbol{d}_1 \otimes \boldsymbol{\nu} - \boldsymbol{\nu} \otimes \boldsymbol{d}_1) + (\boldsymbol{d}_2 \cdot \boldsymbol{n})(\boldsymbol{d}_2 \otimes \boldsymbol{\nu} - \boldsymbol{\nu} \otimes \boldsymbol{d}_2)],
\end{aligned} \tag{C2b}$$

where use of (5) has also been made.

We then require that the right-hand sides of Eqs. (C2) be equal to one another. This amounts to two equalities between axial vectors, one for each nonvanishing left component of the third-rank tensors on (C2),

$$(b^2_\perp + S^2)\boldsymbol{\nu} + \boldsymbol{d}_1 \times \boldsymbol{d}_2 = \mathbf{0}, \tag{C3}$$

$$\mathrm{curl}_s \boldsymbol{d}_1 = (\boldsymbol{c}_\perp \cdot \boldsymbol{d}_2)\boldsymbol{\nu} - (\boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + \boldsymbol{d}_2 \otimes \boldsymbol{d}_2)\boldsymbol{n}_\perp, \tag{C4}$$

where $\boldsymbol{c}_\perp := \boldsymbol{\nu} \times \boldsymbol{c} = -S\boldsymbol{n} - b_\perp \boldsymbol{n}_\perp$. It readily follows from (A11b) that (C3) can be written in the equivalent form,

$$b^2_\perp + S^2 = -K, \tag{C5}$$

which requires $\mathcal{S}$ to be a surface with constant *negative* Gaussian curvature related to the bend and splay components of the uniform field $\boldsymbol{n}$, as already proved by a different method in Ref. [27].

By applying the same integrability condition to $\mathbf{H} = \nabla_s \boldsymbol{v}$, we obtain again (C5), accompanied this time by another equation, which supplements (C4),

$$\operatorname{curl}_s \boldsymbol{d}_2 = -(\boldsymbol{c}_\perp \cdot \boldsymbol{d}_1)\boldsymbol{v} + (\boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + \boldsymbol{d}_2 \otimes \boldsymbol{d}_2)\boldsymbol{n}. \tag{C6}$$

It is not necessary to require explicitly the integrability of $\nabla_s \boldsymbol{n}_\perp$, as this is already implicit in (C4), (C5), and (C6), once we set $\boldsymbol{n}_\perp = \boldsymbol{v} \times \boldsymbol{n}$.

Equations (C4), (C5), and (C6) constitute a system of necessary and sufficient conditions for the existence of a surface uniform field phrased in terms of the connectors of a moving frame. Of course, taken by itself, (C5) is only necessary: It prescribes the class of background surfaces upon which the first-order partial differential equations (C4) and (C6) should be integrated. The latter integration is not an easy task: Both (C4) and (C6) can indeed be made to depend only on the fields $\boldsymbol{d}_1, \boldsymbol{d}_2$ (and, of course, $\boldsymbol{v}$), by use of the equations,

$$\boldsymbol{n} = (\nabla_s \boldsymbol{v})^{-1} \boldsymbol{d}_1, \quad \boldsymbol{n}_\perp = (\nabla_s \boldsymbol{v})^{-1} \boldsymbol{d}_2, \tag{C7}$$

as, by (C5), the curvature tensor $\nabla_s \boldsymbol{v}$ is invertible on the local tangent plane.[39] Once the fields $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ that solve (C4) and (C6) are known, the first equation in (C7) then delivers the desired uniform field $\boldsymbol{n}$.

The method illustrated in the main text to find all surface uniform fields has luckily followed a different, more geometric avenue.

APPENDIX D: GRADIENT IN GEODESIC COORDINATES

In this Appendix, we show how to express the surface gradient on the pseudosphere $\mathcal{S}$ of a differentiable function $\psi(\varphi, \vartheta)$ that depends on the geodesic coordinates (φ, ϑ) defined in (25).

We start by considering a curve $t \mapsto \boldsymbol{p}(t)$ generated on $\mathcal{S}$ by taking $\varphi = \varphi(t)$ and $\vartheta = \vartheta(t)$,

$$\boldsymbol{p}(t) = \boldsymbol{r}(u(t), v(t)), \tag{D1}$$

where the functions $u(t)$ and $v(t)$ are obtained by composing $(\varphi(t), \vartheta(t))$ with (25). By the chain rule and use of (A23), since for $\boldsymbol{r}$ in (22)

$$|\boldsymbol{r}_u| = |\boldsymbol{r}_v| = \frac{1}{v}, \tag{D2}$$

we readily see that

$$\dot{\boldsymbol{p}} = \frac{1}{\rho \sin\vartheta}\big\{[(u_0' + \rho' \cos\vartheta)\dot{\varphi} - \rho \sin\vartheta\,\dot{\vartheta}]\boldsymbol{e}_u + (\rho' \sin\vartheta\,\dot{\varphi} + \rho \cos\vartheta\,\dot{\vartheta})\boldsymbol{e}_v\big\}, \tag{D3}$$

where a superimposed dot denotes differentiation with respect to the parameter t. By letting $\dot{\varphi} = 0$ in (D3), we reobtain (28) in the main text.

Moreover, by inserting (D3) into the identity

$$\dot{\psi} = \frac{\partial \psi}{\partial \varphi}\dot{\varphi} + \frac{\partial \psi}{\partial \vartheta}\dot{\vartheta} = \nabla_s \psi \cdot \dot{\boldsymbol{p}}, \tag{D4}$$

by the arbitrariness of $\dot{\varphi}$ and $\dot{\vartheta}$, we derive the following equations:

$$\frac{1}{\rho \sin\vartheta}(u_0' + \rho' \cos\vartheta)\psi_u + \frac{\rho'}{\rho}\psi_v = \frac{\partial \psi}{\partial \varphi}, \tag{D5a}$$

$$-\psi_u + \cot\vartheta\,\psi_v = \frac{\partial \psi}{\partial \vartheta}, \tag{D5b}$$

for the components (ψ_u, ψ_v) of $\nabla_s \psi$ in the frame $(\boldsymbol{e}_u, \boldsymbol{e}_v)$. The solution of the linear system in (D5) is given by

$$\psi_u = \frac{\sin\vartheta}{u_0' \cos\vartheta + \rho'}\left(\rho \cos\vartheta \frac{\partial \psi}{\partial \varphi} - \rho' \sin\vartheta \frac{\partial \psi}{\partial \vartheta}\right), \tag{D6a}$$

$$\psi_v = \frac{\sin\vartheta}{u_0' \cos\vartheta + \rho'}\left(\rho \sin\vartheta \frac{\partial \psi}{\partial \varphi} + (u_0' + \rho' \cos\vartheta)\frac{\partial \psi}{\partial \vartheta}\right), \tag{D6b}$$

which for $\psi \equiv \vartheta$ deliver (31) in the main text.

[39] It readily follows from (C7) that the identity (5) is automatically satisfied.

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