

Spin resonance without a spin: A microwave analog

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An analog of nuclear magnetic resonance is realized in a microwave network with symplectic symmetry. The network consists of two identical subgraphs coupled by a pair of bonds with a length difference corresponding to a phase difference of π for the waves traveling through the bonds. As a consequence, all eigenvalues appear as Kramers doublets. Detuning the length difference from the π condition Kramers degeneracy is lifted, which may be interpreted as a Zeeman splitting of a spin $\frac{1}{2}$ in a magnetic field. The lengths of another pair of bonds are modulated periodically with frequencies of some 10 MHz by means of diodes, thus emulating a magnetic radio-frequency field. This setup enables the realization of well-known NMR phenomena, such as the transformation from the laboratory to the rotating frame and the observation of Lorentzian-shaped resonance lines.

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I. INTRODUCTION

Random matrix theory has proven very successful in the description of the universal features of the spectra of chaotic systems [1,2]. Depending on the presence or absence of time-reversal symmetry (TRS) there are three ensembles, the orthogonal one for systems with TRS and no spin $\frac{1}{2}$, the unitary one for those without TRS, and the symplectic one for systems with TRS and a spin $\frac{1}{2}$. There are numerous experimental studies of the spectral statistics of the first two ensembles, see Ref. [3] for a recent review. But for the realization of the symplectic ensemble one had to wait for many years due to the challenge of implementing a spin $\frac{1}{2}$. The breakthrough finally came from Joyner and coworkers [4] who proposed a graph with a symplectic symmetry without a spin $\frac{1}{2}$. This idea was experimentally realized by us [5,6] and thereafter by others [7–9] in microwave networks. In this paper we take this analogy literally and ask the question: If there is a spin analog in a network with symplectic symmetry, is there, perhaps, also an analog to spin resonance?

In fact already in the midst of the last century Feynman *et al.* [10] noted that every two-level system may be interpreted in terms of a spin- $\frac{1}{2}$ system. Typically, there is one parameter tuning the level splitting that may be interpreted in terms of a magnetic field. As a function of this parameter

it may happen that the levels approach each other and then depart again, exhibiting an anticrossing. It is here where the essential things happen. For this reason two-level systems may be even considered as building blocks of quantum mechanics as a whole. This aspect has been elucidated in a monograph by Dubbers and one of the authors of this paper [11] illustrating the omnipresent importance of two-level systems in quantum mechanics.

A special situation occurs if two parameters x and y are varied at the same time. In this case the energies of the two levels are given in leading order by $E_{\pm} = E_0 \pm \sqrt{ax^2 + 2bxy + cy^2}$, where E_0 is the energy in the absence of the perturbation and a, b, c are parameters depending on the details of the system. Plotting the energy surface in dependence of the two parameters a double cone is obtained, a diabolo, with tips touching at the so-called diabolic point, corresponding to $x = y = 0$. For the special case $a = c$ and $b = 0$ the cross section of the cone is circular. This has been illustrated by Lauber *et al.* [12] in a triangular microwave cavity, where the coordinates of the tip of the triangle acted as parameters to tune the splitting of two neighboring levels. In the spirit of the present work their results may be reinterpreted in terms of a spin $\frac{1}{2}$ interacting with two magnetic fields. There is one difference, however: in the triangle degeneracies occur by coincidence and have to be adjusted by parameter tuning. In the presence of symplectic symmetry all states are generically twofold degenerate.

Another system showing similar features of the energy surface is graphene. In the reciprocal lattice close to the K and K' points valence and conduction band form double cones touching at the Dirac point [13]. This is an immediate consequence of the linear dispersion relation at these points. Here the wave numbers k_x and k_y may be interpreted as magnetic fields. Microwave studies of graphene lattices made up of dielectric resonators have been performed in our group [14,15], though

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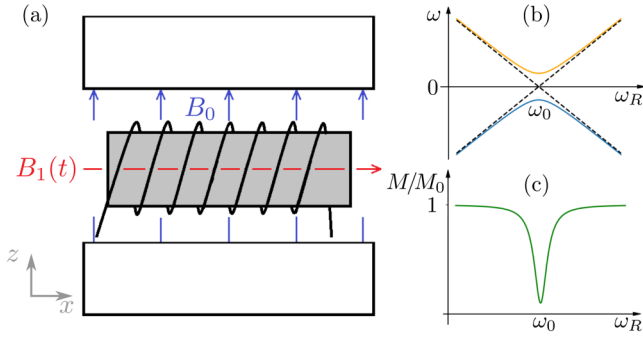


FIG. 1. (a) Sketch of a typical NMR setup. (b) Eigenvalues of a spin $\frac{1}{2}$ system, exposed to a static magnetic field B_0 in z and a radio-frequency field $B_1(t)$ in x direction, in the rotating coordinate system, and (c) magnetization of the sample in dependence of the angular frequency ω_R of the radio-frequency field.

in another context, and extended to other planar hexagonal structures by Wu and Hu [16] and Yang *et al.* [17]. Also the works in superconducting cavities with graphene lattices constructed by metallic scatterers have to be mentioned here [18]. But graphene and graphenelike structures, though showing the needed symmetry, are not suited in the present context: at least several hundred lattice sites have to be realized for the diabolic structure of the energy surface to evolve, making this approach highly inefficient. The ultimate minimal system exhibiting the desired features needs only four lattice sites, as we will see.

This overview, restricted to microwave realizations, shows that there are many options to realize spin equivalents in two-level systems, though the number of possibilities shrinks considerably if we restrict ourselves to systems where degeneracies occur generically and not just by coincidence. Thus the emulation of spin equivalents and equivalents of static magnetic field poses no problems. The main challenge of the present experiment had been the realization of time-dependent fields needed for the simulation of nuclear magnetic resonance (NMR) spectroscopy. There had been efforts to realize time dependencies in microwave billiards and microwave graphs over decades with no success, with the only exception of a microwave realization in our group in a very specific periodically driven system [19]. The demands are hard: for the present work this meant a variation of the bond lengths in the microwave network on the scale of the wavelengths, typically of the order of centimeter, with frequencies up to 100 MHz. The solution of this problem signified the main technical achievement of this work. It opens the door for the study of a completely new class of systems, inaccessible hitherto, namely time-dependent graphs. In this respect we consider the present microwave realization of NMR only as a first and very specific example of a Floquet system.

II. BASICS OF NUCLEAR MAGNETIC RESONANCE

In a typical NMR experiment [20] a probe, containing an ensemble of nuclear spins $\vec{I}$, is exposed to a static magnetic field B_0 , by convention in z direction, and a radio-frequency field $B_1(t) = 2B_1 \cos(\omega_R t)$ in x direction, generated by a coil wrapping the probe, see Fig. 1(a). The system is described by

the time-dependent Schrödinger equation

$$\dot{\psi} = -\frac{i}{\hbar} H_{\text{NMR}} \psi \quad (1)$$

with the Hamiltonian

$$H_{\text{NMR}} = -\hbar[\omega_0 I_z + 2\omega_1 \cos(\omega_R t) I_x], \quad (2)$$

where $\omega_0 = \gamma B_0$ and $2\omega_1 = 2\gamma B_1$ are the Larmor angular frequencies in the static and the radio-frequency field, and γ is the gyromagnetic ratio. The factor 2 has been introduced for later convenience. For a spin $I = \frac{1}{2}$, assumed in the following, the angular momentum operators may be expressed in terms of the Pauli matrices, $\vec{I} = \frac{1}{2}\vec{\sigma}$, with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

The time-dependent part in Eq. (2) may be interpreted in terms of the interaction of a spin with a superposition of two magnetic fields rotating clockwise and counterclockwise in the xy plane. The standard approach to solve the Schrödinger equation is a transformation into a rotating coordinate system to remove the time dependency. With the rotated wave function,

$$\psi_R = e^{-i\frac{\omega_R}{2}t\sigma_z} \psi \quad (4)$$

the Schrödinger equation is transformed into

$$\dot{\psi}_R = \frac{i}{2}[(\omega_0 - \omega_R)\sigma_z + 2\omega_1 \cos(\omega_R t)\sigma_{x,R}]\psi_R \quad (5)$$

with

$$\begin{aligned} 2\cos(\omega_R t)\sigma_{x,R} &= 2\cos(\omega_R t)e^{i\frac{\omega_R}{2}t\sigma_z}\sigma_x e^{-i\frac{\omega_R}{2}t\sigma_z} \\ &= \sigma_x + \cos(2\omega_R t)\sigma_x - \sin(2\omega_R t)\sigma_y. \end{aligned} \quad (6)$$

Due to the transformation into the rotating system the field component rotating synchronously with the coordinate system has become static, whereas the other component now rotates with the double frequency in the opposite direction. Ignoring this component, a standard practice in NMR, the Hamiltonian has become static in the rotating system,

$$H_{\text{NMR}}^R = -\frac{1}{2}[(\omega_0 - \omega_R)\sigma_z + \omega_1\sigma_x]. \quad (7)$$

The eigenvalues of H_{NMR}^R are given by

$$\omega_{\pm} = \pm\frac{1}{2}\sqrt{(\omega_0 - \omega_R)^2 + \omega_1^2}, \quad (8)$$

see Fig. 1(b). Relative errors resulting from the neglect of the rotating fields are of the order of $(\omega_1/\omega_0)^2 = (B_1/B_0)^2$, typically about 10^{-6} .

In a standard NMR experiment the magnetization M , proportional to the spin polarization, is studied as a function of ω_0 or ω_R . The avoided crossing exhibited by the eigenvalues in the rotating frame at $\omega_0 = \omega_R$ implies a Lorentzian resonance curve,

$$M/M_0 = 1 - \frac{\omega_1^2}{(\omega_0 - \omega_R)^2 + \omega_1^2}, \quad (9)$$

see Fig. 1(c), where M_0 is the equilibrium polarization, usually resulting from a Boltzmann polarization. Details can be found, e.g., in chap. 2 of the monograph of Slichter [20].

III. EXPERIMENTAL SETUP

Joyner *et al.* [4] proposed a graph with a specific symmetry to mimic a spin $\frac{1}{2}$, which was realized by us in a microwave network [5] though not in a one-to-one adoption. The graph consists of two subgraphs, one the complex conjugate of the other, realized in the experiment by microwave circulators with opposite sense of rotation. The second ingredient is a pair of bonds coupling, e.g., vertices n and m in one subgraph with the symmetry equivalent ones $\bar{m}$ and $\bar{n}$, in the other one. For a symplectic symmetry to hold there must be a phase difference of π (or any odd integer multiple of π) for the waves propagating through the two bonds. In the spirit of the spin analogy one subgraph corresponds to the spin-up and the other subgraph to the spin-down component, and the phase difference of π between the two bonds mimics the fact that a spinor rotation by 2π changes the sign of the spinor. Symplectic symmetry implies Kramers degeneracy, i.e., each eigenvalue appears as a doublet, and each lifting of the symplectic symmetry may be interpreted in terms of a Zeeman splitting in an effective magnetic field.

The realization of an NMR equivalent needs the emulation of two magnetic fields, one of them time dependent. Our previous studies [6] suggested to do this by means of two pairs of bonds coupling the two subgraphs, but there is a simpler way: Figure 2(a) shows a sketch of the idealized graph. The unperturbed graph consists of four nodes 1, 2, $\bar{1}$, $\bar{2}$, obeying Neumann boundary conditions, coupled by four bonds of equal length l to form a square. At each of the nodes a dangling bond is attached, again of length l , and terminated by a short end corresponding to a Dirichlet boundary condition. Along the bond $\bar{1}2$ the wanted phase shift of π is applied. The static field is realized by applying static relative length changes $\pm\alpha$ of opposite sign at bonds $\bar{1}2$ and $\bar{2}1$. Along bonds 12 and $\bar{1}\bar{2}$ time-dependent relative length changes $\pm\beta(t)$ are applied to realize the radio-frequency field. A system with the same symmetry had already been used by us for a realization of the chiral symplectic ensemble [21], not in graphs, however, but with coupled dielectric resonators.

The system shown in Fig. 2(a) means an idealization, which cannot be realized experimentally one-to-one, but it allows for an analytic treatment, essential for an understanding of what is going on. Figure 2(b) shows a photograph of the actual experimental setup. The graph has the same geometry as the idealized one, but there are some differences in detail. The geometrical length of the individual bonds realized by means of standard coaxial cables amounted to $l_{\text{geo}} = 30.38$ cm, corresponding to an optical length of $l_{\text{opt}} = n_0 l_{\text{geo}} = 43.75$ cm, where $n_0 = 1.44$ is the index of refraction of the teflon dielectric in the cables. All lengths given in the following refer to the optical ones.

The experimental realization relies on the fact that the only quantity of relevance is the phase $\varphi = kl$ a wave acquires when traveling through a bond of length l . For a fixed frequency, corresponding to a constant wave number k , length and phase shifts are thus equivalent. To take advantage of this feature we had to restrict ourselves first of all to just one Kramers doublet. For the realization of the phase shift of π an extra length corresponding to half of the wavelength of the selected doublet was added to bond $\bar{1}2$. The static length shifts

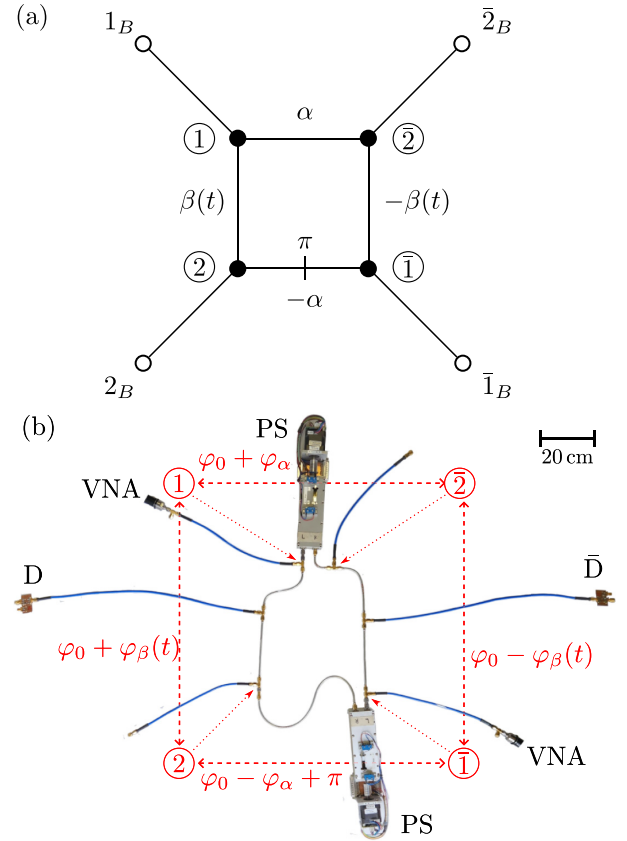


FIG. 2. (a) Sketch of the idealized graph, (b) Photo of the experimental setup. The graph is connected to a vector network analyzer (VNA) via two cables attached at nodes 1_B and $\bar{1}_B$. Two phase shifters (PS) allow for a change of the lengths of bonds $\bar{1}2$ and $\bar{2}1$. The π jump is realized by an appropriate additional length of bond $\bar{1}2$. By means of diodes (D) and ($\bar{D}$), attached at the ends of dangling bonds, the effective lengths of bonds 12 and $\bar{1}\bar{2}$ can be switched between two states with frequencies up to 125 MHz. In addition the phase shifts are given the waves acquire when traveling through the respective bonds.

have been performed by so-called phase shifters, microwave standard devices. Notwithstanding the notation they do not change the phase directly but indirectly by varying the length, just like a trombone. By means of diodes, attached at the end of dangling bonds, see Fig. 2(b), the effective lengths of bonds 12 and $\bar{1}\bar{2}$ could be switched between two states with frequencies up to 125 MHz. The lengths of the bonds thereby are not varied explicitly, in contrast to the other pair of bonds, but the phase shifts waves experience when traveling through the bonds. Details can be found in Appendix A.

Note that there are no circulators breaking time-reversal symmetry in the two subgraphs. Circulators always introduce considerable absorption, which showed up to be intolerable for the present studies. For a symplectic symmetry a break of time-reversal symmetry is not really needed, it is sufficient that the two subgraphs are complex conjugates of each other, which is trivially the case if both of them are real.

In a preliminary study the spectrum of the graph was determined in dependence of the phase difference $\Delta\varphi$ for the $\bar{1}2/\bar{1}\bar{2}$ pair and no detuning of the $12/\bar{1}\bar{2}$ pair. The length of bonds $\bar{1}2$

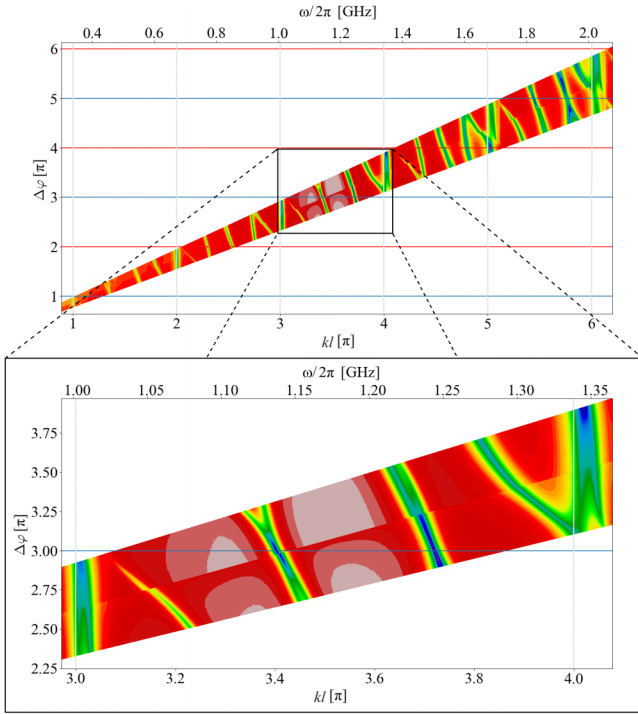


FIG. 3. Reflection spectra as a function of frequency $\nu = \omega/2\pi$ for the graph shown in Fig. 2(b) for different phase differences $\Delta\varphi = k\Delta l$ (Δl : difference of the lengths of bonds $\bar{1}\bar{2}$ and $\bar{1}\bar{2}$). Each resonance appears in a light color on a dark background. The abscissa label above the plot shows the frequency $\nu = \omega/2\pi$, the label below the plot the corresponding kl value in units of π . The Kramers doublet at 1.12 GHz had been used for the time-dependent studies.

was changed by the phase shifter stepwise to vary $\Delta\varphi$. Using the relation $\Delta\varphi = k\Delta l$ the length shift Δl was translated into a corresponding phase shift $\Delta\varphi$. A detailed description of the technique can be found in Sec. 2.1.3. of Ref. [3].

Figure 3 shows spectra of the reflection S_{11} in dependence of $\Delta\varphi$ in a color plot. The resonances appear in green on a red background. At the odd integer multiples of π , depicted by horizontal blue lines, the graph is symplectically symmetric, and all resonances appear as degenerate Kramers doublets. Any deviation from the π lines means a splitting of the Kramers doublets, which may be interpreted in terms of a Zeeman splitting in a magnetic field. A detailed interpretation of the resonances will be given in Sec. IV B.

The wedge shape of the experimentally explored region reflects the limits of the phase shifter. It allows for a limited length change by $\Delta l = 4$ cm only, corresponding to a $\Delta\varphi$ range of $0.1 \dots 0.3\pi$ for the shown frequency range. To increase the accessible range, additional pieces of fixed lengths had been introduced. The figure shows two successive wedges such obtained. The borderline between the two measurements is clearly seen in the enlarged region in the lower plot.

IV. THEORY

A. Wave equation

There are two constraints governing wave functions in graphs. First, energy conservation implies that all wave

functions meeting at a node have the same value at this node. And, second, for each node there is a continuity equation guaranteeing that the sum over all incoming and outgoing currents is zero. For details we refer to the report of Kottos and Smilansky [22], which also has been the basis for this presentation. An introduction into graphs from the viewpoint of a mathematician can be found in the article by Kuchment [23].

The bonds are coaxial microwave cables, i.e., dielectric circular waveguides with a teflon dielectric ($n_0 = 1.44$) separating the central core from the outer cylinder. For frequencies below 2 GHz as applied in the present work the wave guides support one traveling mode only. The wave function in the bond connecting nodes n and m may then be written as

$$\psi_{nm}(x, t) = \varphi_{nm}(x, t) + \varphi_{mn}(l_{nm} - x, t), \quad (10)$$

where l_{nm} is the length of the bond. The $\varphi_{nm}(x, t)$ satisfy the boundary conditions

$$\varphi_{nm}(0, t) = \varphi_n(t), \quad \varphi_{nm}(l_{nm}, t) = 0, \quad (11)$$

whence follow the boundary conditions $\psi_{nm}(0, t) = \varphi_n(t)$ and $\psi_{nm}(l_{nm}, t) = \varphi_m(t)$ for $\psi_{nm}(x, t)$ as demanded by energy conservation. The usual separation of the time dependence is not possible as long as the graph is time dependent.

The $\varphi_{nm}(x, t)$ obey the time-dependent wave equation. In the calculations variations of length l with time would mean time-dependent boundary conditions, which would be a technical challenge. But since the $\varphi_{nm}(x, t)$ depend on the optical length $l_{\text{opt}} = n_0 l_{\text{geo}}$ only, this problem can be overcome by shifting the time dependence from the boundary condition to the index of refraction. We hence assume the same constant length l for all bonds and introduce an effective index of refraction via $n_0 l_{\text{geo}} = n_{\text{eff}} l$, or

$$n_{\text{eff}} = n_0 l_{\text{geo}}/l = n_0(1 + \Delta l/l), \quad (12)$$

where $\Delta l/l$ is the relative length change of the respective bond. It is assumed throughout that these changes are small and can be treated in first order. In the experiment $\Delta l/l$ never exceeded ten percent. The $\varphi_{nm}(x, t)$ then are solutions of a correspondingly rescaled wave equation,

$$\left[\square + 2v_{nm} \frac{\partial^2}{\partial x^2} \right] \varphi_{nm}(x, t) = 0, \quad (13)$$

where

$$\square = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \quad (14)$$

is the d'Alembert operator in one dimension, $c = c_0/n_0$ is the velocity of light in the bonds, and the v_{nm} are given by $v_{nm} = 0$ for the dangling bonds, and $v_{1\bar{2}} = \alpha$, $v_{\bar{1}2} = -\alpha$, $v_{12} = \beta(t)$, $v_{\bar{1}\bar{2}} = -\beta(t)$ for the other bonds. Here α and $\beta(t) = 2\beta \cos(\omega_R t)$ are the relative length changes due to phase shifter and diode, respectively.

We arrange the φ_{nm} in terms of spinorlike column vectors,

$$\varphi_B = \begin{pmatrix} \varphi_{11_B} \\ \varphi_{\bar{1}\bar{1}_B} \\ \varphi_{2\bar{2}_B} \\ \varphi_{\bar{2}2_B} \end{pmatrix}, \quad \varphi_P = \begin{pmatrix} \varphi_{1\bar{2}} \\ \varphi_{\bar{1}2} \\ \varphi_{2\bar{1}} \\ \varphi_{\bar{2}1} \end{pmatrix}, \quad \varphi_D = \begin{pmatrix} \varphi_{12} \\ \varphi_{\bar{1}\bar{2}} \\ \varphi_{21} \\ \varphi_{\bar{2}\bar{1}} \end{pmatrix}, \quad (15)$$

where the subscripts refer to dangling bonds (B), phase shifters (P), and diodes (D), respectively. Throughout this paper rank-4 spinors and matrices are denoted in bold letters to distinguish them from the corresponding rank-2 quantities, which are indicated in ordinary letters. The arrangement of the wave functions in terms of components of rank-4 spinors will show up to be crucial for the treatment of the time-dependent case. For time-independent graphs this step would not have been needed.

For the column vectors (15) the wave equation (13) may be compactly written as

$$\left\{ \square \cdot \mathbb{1}_4 + 2V_A \frac{\partial^2}{\partial x^2} \right\} \varphi_A = 0 \quad (16)$$

($A = B, P, D$), where $\mathbb{1}_4$ is the four-dimensional unit matrix, and

$$V_B = 0 \cdot \mathbb{1}_4, \quad V_P = \alpha \sigma_\alpha, \quad V_D = \beta(t) \sigma_\beta \quad (17)$$

with

$$\sigma_\alpha = \begin{pmatrix} \sigma_z & \cdot \\ \cdot & -\sigma_z \end{pmatrix}, \quad \sigma_\beta = \begin{pmatrix} \sigma_z & \cdot \\ \cdot & \sigma_z \end{pmatrix}. \quad (18)$$

B. Continuity equation

The wave functions connecting the nodes may compactly be written in terms of the $\varphi_A(x, t)$ as

$$\begin{aligned} \Psi_B(x, t) &= \varphi_B(x, t), \\ \Psi_P(x, t) &= \begin{pmatrix} \varphi_{12} \\ \varphi_{1\bar{2}} \\ \varphi_{2\bar{1}} \\ \varphi_{\bar{2}1} \end{pmatrix} (x, t) + \begin{pmatrix} \varphi_{21} \\ -\varphi_{2\bar{1}} \\ -\varphi_{\bar{1}2} \\ \varphi_{1\bar{2}} \end{pmatrix} (l - x, t) \\ &= \varphi_P(x, t) + \begin{pmatrix} \cdot & i\sigma_y \\ -i\sigma_y & \cdot \end{pmatrix} \varphi_P(l - x, t), \\ \Psi_D(x, t) &= \begin{pmatrix} \varphi_{12} \\ \varphi_{1\bar{2}} \\ \varphi_{21} \\ \varphi_{2\bar{1}} \end{pmatrix} (x, t) + \begin{pmatrix} \varphi_{21} \\ \varphi_{2\bar{1}} \\ \varphi_{12} \\ \varphi_{1\bar{2}} \end{pmatrix} (l - x, t) \\ &= \varphi_D(x, t) + \begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix} \varphi_D(l - x, t), \end{aligned} \quad (19)$$

where Eq. (10) has been used. The minus signs in the second column vector contributing to $\Psi_P(x, t)$ are a consequence of the phase shift of π along the bond $\bar{1}2$.

The second constraint for the wave functions, in addition to energy conservation, is current conservation [22], resulting in a continuity equation for each node. It may be compactly expressed in terms of the $\Psi_A(x, t)$ as

$$\Psi'_B(0, t) + \Psi'_P(0, t) + \Psi'_D(0, t) = 0 \quad (20)$$

or, using Eq. (19),

$$\begin{aligned} \varphi'_B(0, t) + \varphi'_P(0, t) + \varphi'_D(0, t) \\ - \sigma_P \varphi'_P(l, t) - \sigma_D \varphi'_D(l, t) = 0, \end{aligned} \quad (21)$$

where

$$\sigma_P = \begin{pmatrix} \cdot & i\sigma_y \\ -i\sigma_y & \cdot \end{pmatrix}, \quad \sigma_D = \begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix}. \quad (22)$$

Equation (20) holds for Neumann boundary conditions, as realized, e.g., in standard T junctions.

V. TIME-INDEPENDENT GRAPH

A. Secular matrix

For a time-independent perturbation $\beta(t) = \beta = \text{const.}$ the time dependence can be separated,

$$\varphi_A(x, t) = e^{-i\omega t} \varphi_A(x), \quad (23)$$

turning the wave equation (16) into

$$\varphi_A''(x) = -K_A^2 \varphi_A(x), \quad (24)$$

where

$$K_A = k[\mathbb{1}_4 - V_A], \quad (25)$$

with $k = \omega/c$. All equations are correct in linear order of the perturbations. The solution of Eq. (24) is given by

$$\varphi_A(x) = \frac{\sin(K_A[l - x])}{\sin(K_A l)} \varphi_0, \quad \varphi_0 = \begin{pmatrix} \varphi_1 \\ \varphi_{\bar{1}} \\ \varphi_2 \\ \varphi_{\bar{2}} \end{pmatrix}, \quad (26)$$

where the boundary conditions (11) have been taken into account.

Plugging in this result into the continuity equation (21) one obtains a secular equation

$$\mathbf{h} \varphi_0 = 0, \quad (27)$$

for φ_0 . $\mathbf{h}$ is the secular matrix [5,6], in the present case given by

$$\mathbf{h} = \mathbf{h}_0 + \mathbf{h}_\alpha + \mathbf{h}_\beta, \quad (28)$$

where

$$\mathbf{h}_0 = \begin{pmatrix} -3f \mathbb{1} & g(\mathbb{1} + i\sigma_y) \\ g(\mathbb{1} - i\sigma_y) & -3f \mathbb{1} \end{pmatrix}, \quad (29)$$

is the secular matrix for the unperturbed graph, with $f = k \cot(kl)$, $g = k / \sin(kl)$. $\mathbf{h}_\alpha, \mathbf{h}_\beta$ are the contributions resulting from the perturbations. Explicit expressions for $\mathbf{h}_\alpha, \mathbf{h}_\beta$ are given in Appendix B.

Now we move to the basis where $\mathbf{h}_0$ is diagonal. This is achieved by means of the transformation

$$\tilde{\varphi}_A = \mathbf{R} \varphi_A, \quad (30)$$

where

$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon_y & i\sigma_y \varepsilon_y^* \\ i\sigma_y \varepsilon_y & \varepsilon_y^* \end{pmatrix}, \quad \varepsilon_y = e^{i\frac{\pi}{8} \sigma_y}. \quad (31)$$

The off-diagonal blocks of $\tilde{\mathbf{h}}_\alpha$ and $\tilde{\mathbf{h}}_\beta$ may be discarded, as long as they are small compared to the splitting of the eigenvalues of $\tilde{\mathbf{h}}_0$. $\tilde{\mathbf{h}}$ then adopts block diagonal form with diagonal blocks $\tilde{\mathbf{h}}_+$ and $\tilde{\mathbf{h}}_-$, given by

$$\tilde{\mathbf{h}}_\pm = (-3f \pm g\sqrt{2}) \mathbb{1} + k \left(\pm \frac{f'}{\sqrt{2}} - g' \right) (\alpha \sigma_z - \beta \sigma_x). \quad (32)$$

Here f' and g' denote the derivatives of f and g with respect to k . Details of the derivation can be found in Appendix B. In the absence of the perturbations there are, as expected,

two twofold degenerate zeros of $|\mathbf{h}|$ at $3f = \pm\sqrt{2}g$, whence follows

$$\tan(k_{\pm}l) = \pm\sqrt{\frac{7}{2}}, \quad (33)$$

or $k_{\pm}l \approx \pm 0, 344\pi + n\pi$, $n \in \mathbb{N}$.

For time-independent graphs the approach applied here is completely equivalent to the one adopted in Ref. [22], and the expressions for $\mathbf{h}$ could have been obtained much easier. There is, however, a difference in the definition of the $\mathbf{h}$ matrices: Usually the wave number k is the same in all bonds, and appears in the continuity equation (20) as a common factor, resulting from the spatial derivatives. It can thus be canceled, which had been done in Ref. [22] in the definition of the $\mathbf{h}$ matrices. In the present situation, with differing wave numbers in the bonds such a cancellation is not possible. Therefore the $\mathbf{h}$ matrices introduced here differ from the previous ones by a factor of k .

The diagonalization (30) of $\mathbf{h}_0$ is not unique. Every additional transformation $R_1 = \begin{pmatrix} R_{1+} & \\ & R_{1-} \end{pmatrix}$, not destroying the block diagonal structure of $\tilde{\mathbf{h}}_0$, is allowed. This freedom has been used to obtain expressions for $\tilde{\mathbf{h}}_{\alpha}$ and $\tilde{\mathbf{h}}_{\beta}$, which are particularly suited for the treatment of the time-dependent case.

There are thus two Kramers doublets within each k window ranging from $kl = (n-1)\pi$ to $kl = n\pi$, $n \in \mathbb{N}$, meaning altogether four eigenvalues. But there is a problem: the mean density of state is $\bar{\rho} = l_{\text{tot}}/\pi$, where $l_{\text{tot}} = 8l$ is the total length of the graph. Thus there should be eight eigenvalues within each k window. Where are the missing four ones? The answer comes from the spectral duality property of the secular matrix $h(k)$ [24]: The zeros of $h(k)$ constitute the spectrum of the graph, but there is another spectrum obtained from the poles of $h(k)$. It corresponds to the superposition of the spectra of the individual bonds. If all bond lengths are incommensurable, the two spectra are disjoint. But for commensurable bond lengths it may happen that eigenvalues are at the same time members of both spectra. In this case there is a cancellation of zeros in the nominator and denominator of $h(k)$, meaning that the eigenvalues in question are missing in the set of zeros of $h(k)$. This type of eigenvalue has been termed “topological” by Gnutzmann *et al.* [25]. In the present case there are four of them: one resonance living along the bonds $1_B 1-12-22_B$ and being zero on all the other bonds, as well as its three symmetry equivalents. In the experiment the topological resonances appear at the boundaries of the k windows, with a fourfold degeneracy at the odd integer π lines, see Fig. 3.

B. Scattering matrix

For the measurement the graph has been connected via vertices 1_B and $\bar{1}_B$ to a vector network analyzer measuring reflection amplitudes S_{11} , $S_{\bar{1}\bar{1}}$ and transmission amplitudes $S_{1\bar{1}}$, $S_{\bar{1}1}$. This has been achieved by replacing the short end terminators at 1_B and $\bar{1}_B$ by T junctions with the side junction terminated by a short end thus caring for a weak coupling.

The scattering matrix

$$S = \begin{pmatrix} S_{11} & S_{1\bar{1}} \\ S_{\bar{1}1} & S_{\bar{1}\bar{1}} \end{pmatrix} \quad (34)$$

relates vectors $a = (a_1, a_{\bar{1}})^T$, $b = (b_1, b_{\bar{1}})^T$ of incoming and outgoing amplitudes, respectively, via

$$b = Sa. \quad (35)$$

As long as the perturbations are static, we can apply time-independent scattering theory [26] establishing a relation between the scattering matrix S and the Green function G

$$S = \mathbb{1} - \frac{2i\gamma G}{\mathbb{1} + i\gamma G}, \quad (36)$$

where γ contains the information on the coupling of the ports. In the present case with only one pair of open ports at symmetry equivalent points S and G are 2×2 matrices. G is related to the secular matrix $\mathbf{h}$ via [22]

$$G = \begin{pmatrix} (\mathbf{h}^{-1})_{11} & (\mathbf{h}^{-1})_{1\bar{1}} \\ (\mathbf{h}^{-1})_{\bar{1}1} & (\mathbf{h}^{-1})_{\bar{1}\bar{1}} \end{pmatrix}, \quad (37)$$

see Appendix C for details. Changing into the basis, where $\mathbf{h}_0$ is diagonal, the scattering equation (35) is turned into

$$\tilde{b} = \tilde{S}\tilde{a}, \quad (38)$$

with

$$\tilde{a} = \varepsilon_y a, \quad \tilde{b} = \varepsilon_y b. \quad (39)$$

The transformed scattering matrix is given by

$$\begin{aligned} \tilde{S} &= \varepsilon_y S \varepsilon_y^\dagger \\ &= \mathbb{1} - 2i\gamma \tilde{G} \frac{\mathbb{1}}{\mathbb{1} + i\gamma \tilde{G}} \end{aligned} \quad (40)$$

with the transformed Green's function $\tilde{G}$. Details of the calculation are presented in Appendix D.

$\tilde{G}$ exhibits poles at the positions of the Kramers doublets. Expanding G as a sum over the poles, and restricting the discussion to the neighborhood of just one Kramers doublet, Eq. (40) simplifies to

$$\tilde{S} = \mathbb{1} - \frac{2i\gamma'}{(\omega + i\gamma')\mathbb{1} - H}, \quad (41)$$

where

$$H = -\frac{\omega_1}{2}\sigma_z - \frac{\omega_2}{2}\sigma_x, \quad (42)$$

with $\gamma' = \gamma g_n$, where ω_n and g_n are position and residuum of the selected Kramers doublet, and $\omega_1 \sim \alpha$, $\omega_2 \sim \beta$. The position of the unsplit Kramers doublet has been shifted to zero. The notation has been chosen to be in accordance with NMR practice, see Eq. (2). We have thus established the equivalence of the splitting of a Kramers doublet by the two perturbations with the Zeeman splitting of a spin $\frac{1}{2}$ by two magnetic fields in z and x directions.

The eigenvalues of the Hamiltonian (42) are given by

$$\omega_{\pm} = \pm \frac{1}{2} \sqrt{\omega_1^2 + \omega_2^2}. \quad (43)$$

Figure 4 shows an experimental example. The data have been taken from an earlier study, where the two fields had been realized by two pairs of coupling bonds [6]. The eigenvalues are plotted in dependence of $\Delta\varphi_1 = k\Delta l_1$ and $\Delta\varphi_2 = k\Delta l_2$, the phase differences the two waves acquire when traveling

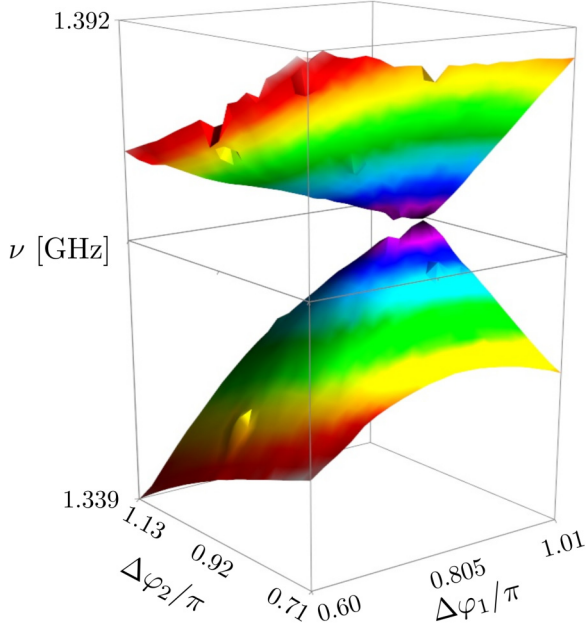


FIG. 4. Eigenfrequencies of a split Kramers doublet at $\omega/2\pi = \nu = 1.391$ GHz in a graph, composed two subgraphs, complex-conjugates of each other, with two pairs of coupling bonds to simulate the magnetic fields, in dependence of the phase differences $\Delta\varphi_1 = k\Delta l_1$ and $\Delta\varphi_2 = k\Delta l_2$.

through the pairs of bonds. In the three-dimensional space with the ω axis perpendicular to the $\Delta\varphi_1$ - $\Delta\varphi_2$ plane the eigenvalues (43) form a double cone, a diabolo, see Fig. 4. At the diabolo point $\Delta\varphi_1 = \Delta\varphi_2 = \pi$, corresponding to $\omega_1 = \omega_2 = 0$ the graph is symplectically symmetric as has been discussed in Sec. I.

The appearance of the Pauli matrices in Eq. (42) should not surprise: symplectic symmetry means that all matrix elements of the Hamiltonian must be quaternionic real [1,2]. Hence for any perturbation of the symplectic symmetry there must be quaternionic imaginary contributions. But because of the relations $\tau_k = \frac{1}{i}\sigma_k$ ($k = x, y, z$) between the spin matrices and the quaternions these perturbations are just linear combinations of the spin matrices. It had been the motivation of this section to illustrate the equivalence of a spin $\frac{1}{2}$ in an external magnetic field with a Kramers doublet in a graph with perturbed symplectic symmetry. For this purpose spin matrices are better suited than quaternions, which are usually used in the context of symplectic symmetries.

VI. TIME-DEPENDENT CASE

A. From the laboratory to the rotating frame

Now we turn to the situation, where β is time dependent

$$\beta(t) = 2\beta \cos(\omega_R t). \quad (44)$$

Guided by the approach applied in NMR we look for a spin rotation removing the time dependence in the wave equation (16) for φ_D , without introducing new time dependencies in the wave equation for φ_P , and in the continuity

equation (21). This is achieved by means of the rotation

$$\varphi_{AR}(x, t) = \mathbf{R}_t \tilde{\varphi}_A(x, t) \quad (45)$$

with

$$\mathbf{R}_t = \begin{pmatrix} e^{-i\frac{\omega_R t}{2}\sigma_z} & \cdot \\ \cdot & e^{-i\frac{\omega_R t}{2}\sigma_z} \end{pmatrix}. \quad (46)$$

$\tilde{\sigma}_P$ and $\tilde{\sigma}_D$ are not changed by the transformation. The continuity equation (21) thus holds also for the wave functions in the rotating frame. But the transformation removes the time dependencies in the wave equation, up to terms rotating with $2\omega_R t$ which are neglected. Details can be found in Appendix E. This is exactly the same situation one meets when removing the time dependence of the NMR Hamiltonian (2), see Sec. II.

We now can repeat all procedures for the static case, starting with the separation of the time dependence. We finally arrive at a scattering equation in the rotating frame,

$$b_R = S_R a_R, \quad (47)$$

where a_R, b_R are the amplitude vectors of the incoming and outgoing waves in the rotating frame, and

$$S_R = \mathbb{1} - \frac{2i\gamma'}{(\omega + i\gamma')\mathbb{1} - H_R} \quad (48)$$

is the scattering matrix in the rotating frame with

$$H_R = -\frac{\omega_0 - \omega_R}{2}\sigma_z - \frac{\omega_1}{2}\sigma_x. \quad (49)$$

We have obtained exactly the NMR Hamiltonian in the rotating frame, see Eq. (7). Details can be found in Appendix E. But how can we get access to the scattering matrix $S_R(\omega)$ in the rotating frame from a measurement, which obviously has to be performed in the laboratory frame? Here comes the recipe, for the details we refer again to Appendix E:

In order to determine the scattering matrix in the rotating frame,

(i) the spin-up component has to be excited at the frequency $\omega_- = \omega - \frac{\omega_R}{2}$, and the spin-down component at the frequency $\omega_+ = \omega + \frac{\omega_R}{2}$,

(ii) the detection, too, has to be taken at the shifted frequencies ω_- and ω_+ for spin-up and spin-down component, respectively.

We need, so to speak, twisted excitations and detections, a really strange situation. Excitation of the spin-up and spin-down components here essentially means excitation via ports 1_B or $\bar{1}_B$, but there is an admixture also from the excitation via the other port, resulting from the transformation (30) into the diagonal basis of $\mathbf{h}$.

The upper left and lower right matrix elements of $S_R(\omega)$ may thus be interpreted as ordinary scattering matrix elements in the laboratory frame,

$$(S_R)_{11}(\omega) = \tilde{S}_{11}(\omega_-), \quad (S_R)_{\bar{1}\bar{1}}(\omega) = \tilde{S}_{\bar{1}\bar{1}}(\omega_+), \quad (50)$$

but taken at shifted frequencies ω_- and ω_+ , respectively. The off-diagonal elements of S_R cannot such be determined, this would mean an excitation of the graph at the frequency ω_+ , and a detection at ω_- , and vice versa, possible in principle,

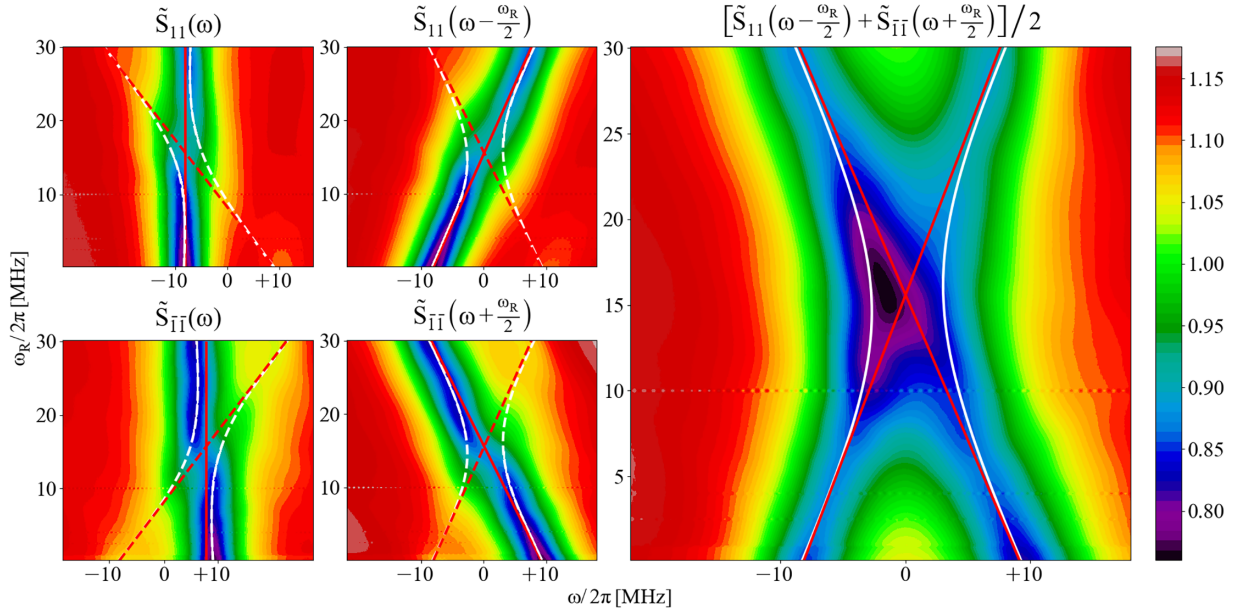


FIG. 5. From the laboratory to the rotating frame, illustrated for the Kramers doublet at $\nu_0 = \omega_0/2\pi = 1.112$ GHz, Zeeman split by $\nu_{\pm} = \pm 7.5$ MHz, see Fig. 3. Left column: Reflection spectra $\tilde{S}_{11}(\omega)$ (top) and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ (bottom) as a function of frequency $\nu = \omega/2\pi$ versus radio-frequency $\nu_R = \omega_R/2\pi$. The zeros of the abscissas have been shifted to ν_0 , the frequency of the nonsplit doublet. Middle column: $(S_R)_{11}(\omega) = \tilde{S}_{11}(\omega_-)$ (top) and $(S_R)_{\bar{1}\bar{1}}(\omega) = \tilde{S}_{\bar{1}\bar{1}}(\omega_+)$ (bottom), obtained by shear operations from the graphs shown in the left column, see text. Right column: $\frac{1}{2}\text{Tr} S_R = \tilde{S}_{11}(\omega_-) + \tilde{S}_{\bar{1}\bar{1}}(\omega_+)$, obtained from the sum of the two graphs shown in the middle column. The white lines correspond to the theoretically expected eigenvalues, see Eq. (49), the red ones to their limit for $\omega_1 \rightarrow 0$. The part of the eigenvalues, visible in $\tilde{S}_{11}(\omega)$ or $\tilde{S}_{\bar{1}\bar{1}}(\omega)$, is plotted by solid lines, the invisible part by dashed lines.

but not with the available equipment. But the trace of $S_R(\omega)$,

$$\text{Tr} S_R(\omega) = \tilde{S}_{11}(\omega_-) + \tilde{S}_{\bar{1}\bar{1}}(\omega_+), \quad (51)$$

is accessible.

Equation (51) constitutes the main theoretical result of this paper. It provides a tool to obtain the trace of the scattering matrix in the rotating frame from ordinary scattering matrix measurements in the laboratory frame taken at two different frequencies. With a more versatile vector network analyzer allowing for exciting and detecting two different frequencies at the same time the complete scattering matrix in the rotating frame would have been accessible.

The poles of the scattering matrix $S_R(\omega)$ (48) in the rotating system are given, up to the absorption term, by the eigenvalues of H_R , see Eq. (8). For $\omega_1 = 0$ the scattering matrix $S_R(\omega)$ is diagonal, see Eqs. (48) and (49), with diagonal elements

$$(S_R)_{11}(\omega) = 1 - \frac{2i\gamma'}{\omega_- + i\gamma' + \frac{\omega_0}{2}} = \tilde{S}_{11}(\omega_-),$$

$$(S_R)_{\bar{1}\bar{1}}(\omega) = 1 - \frac{2i\gamma'}{\omega_+ + i\gamma' - \frac{\omega_0}{2}} = \tilde{S}_{\bar{1}\bar{1}}(\omega_+), \quad (52)$$

illustrating relations (50) explicitly for the special case of $\omega_1 = 0$. $\tilde{S}_{11}(\omega)$ thus sees only the lower-frequency component of the Kramers doublet for $\omega_1 = 0$ and is blind for the other one. For $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ it is vice versa.

As an illustration we take the Kramers doublet at $\nu_0 = \omega_0/2\pi = 1.113$ GHz, see Fig. 3, Zeeman split into two components by $\nu_{\pm} = \omega_{\pm}/2\pi = \pm 7.5$ MHz. The left column of Fig. 5 shows $\tilde{S}_{11}(\omega)$ (top) and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ (bottom), with

$\nu_R = \omega_R/2\pi$ versus $\nu = \omega/2\pi$, in a color plot. The resonances appear as blue valleys in a red plateau. Both $\tilde{S}_{11}(\omega)$ and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ show up to be nearly independent of ω_R . Only at $\nu_R = \omega_R/2\pi = 15$ MHz, corresponding to the splitting $\Delta\omega/2\pi$ of the two Kramers doublets, there is an indication that something is happening. In addition the theoretically expected poles of $\tilde{S}_{11}(\omega)$ and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ for $\omega_1 = 0$, see Eq. (52), are plotted, the visible ones with solid red lines, the invisible ones dashed. Furthermore the eigenvalues of $\tilde{S}_{11}(\omega)$ and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ for a small $\omega_1 \neq 0$ are plotted in white. Now the straight lines found for $\omega_1 = 0$ are converted into hyperbolas. Close to the avoided crossings both eigenvalues become visible both for $\tilde{S}_{11}(\omega)$ and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$, but off the avoided crossings still only the vertical branches of the hyperbolas are seen. The white lines follow nicely the valley of the experimental eigenvalues. To this end ω_1 had been adjusted. In agreement with the prediction $\tilde{S}_{11}(\omega)$ sees only the lower Kramers component at $\nu_- = -7.5$ MHz, and $\tilde{S}_{\bar{1}\bar{1}}(\omega)$ only the higher one at $\nu_+ = +7.5$ MHz, apart from the neighborhood of the avoided crossing.

The central column of Fig. 5 shows the same data, but with $\tilde{S}_{11}$ and $\tilde{S}_{\bar{1}\bar{1}}$ plotted in dependence of the twisted frequencies $\omega_-/2\pi$ and $\omega_+/2\pi$, respectively. The figure on the right finally shows the sum of the two latter quantities corresponding to $\frac{1}{2}\text{Tr} S_R(\omega)$, see Eq. (51). Now both hyperbolic branches are clearly visible, corresponding to the eigenvalues in the rotating frame, and exhibiting an anticrossing at the resonance position $\omega_R = \Delta\omega$.

By means of this somewhat tricky operation we have been able to get the trace of the scattering matrix $S_R(\omega)$ in the rotating frame from the measured S -matrix components $\tilde{S}_{11}(\omega_-)$

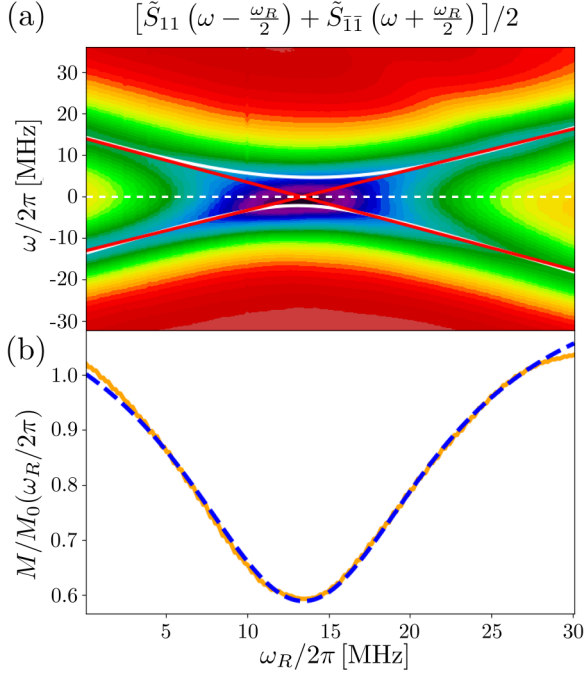


FIG. 6. (a) Same as right column of Fig. 5, but with interchanged axes. The hyperbolas plotted in white correspond to the eigenvalues of the Kramers doublet at 1.112 GHz in the rotating frame. (b) Corresponding NMR resonance curve, see Eq. (53). The orange line has been extracted from the value of $\frac{1}{2}\text{Tr}S_R(\omega)$ along the dotted white symmetry line, the dashed blue line reflects the best fit with a Lorentzian.

and $\tilde{S}_{11}(\omega_+)$ in the laboratory frame. As a result we have obtained the spectrum of the NMR Hamiltonian in the rotating frame, see Eq. (8), a quantity not even available in a standard NMR experiment.

B. Microwave analog of spin resonance

Figure 6(a) shows the right column of Fig. 5 again, but with interchanged axes to adjust the presentation to the one customary in NMR, see Fig. 1(b). $S_R(\omega)$ depends both on ω and ω_R , but the latter dependence had not been noted explicitly to simplify the notation. This poses a small problem in notation here: in the right part of Fig. 5 ω plays the role of the variable and ω_R the one of a parameter. Due to the interchange of axes in Fig. 6(a) the roles of variable and parameter are exchanged.

There is a one-to-one correspondence between Fig. 1(b) from NMR and Fig. 6(a) from the microwave analog. But there is also a microwave equivalent for a NMR resonance showing up as a dip in the magnetization of the sample, see Fig. 1(c). Of course there is no magnetization in a microwave graph. But there is another quantity playing this part, namely $\frac{1}{2}\text{Tr}S_R(\omega)$ taken at $\omega = 0$, corresponding to the horizontal symmetry line, dotted in white in Fig. 6(a). For this quantity one obtains from Eq. (48)

$$\frac{1}{2}\text{Tr}S_R(0) = 1 - \frac{4\gamma'^2}{(\omega_0 - \omega_R)^2 + \omega_1^2 + 4\gamma'^2}. \quad (53)$$

But this is exactly the formula for a typical Lorentzian-shaped magnetic resonance curve, see Eq. (9), with the only

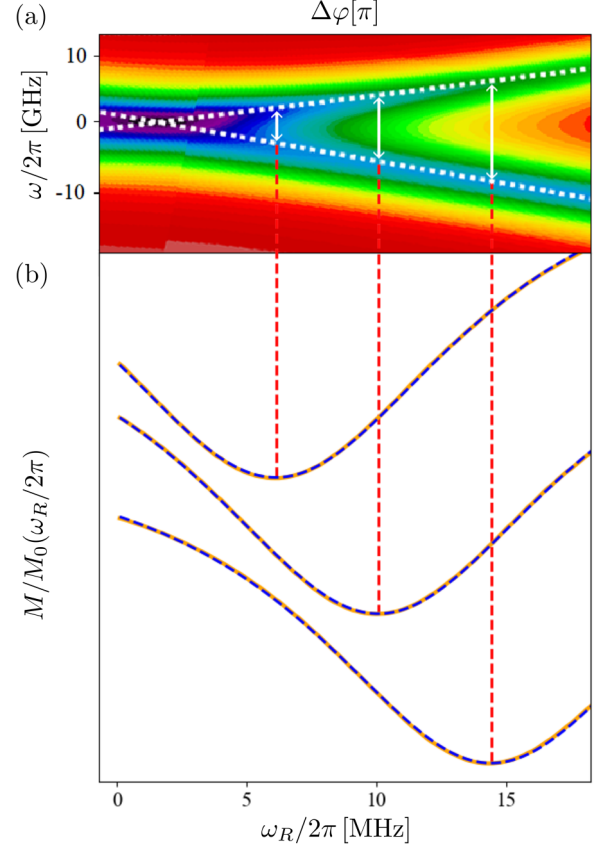


FIG. 7. (a) Splitting of the Kramers doublet at 1.113 GHz in dependence of $\Delta\varphi \sim \omega_0$. The vertical white arrows mark the positions, where resonance experiments have been performed. (b) Corresponding resonance curves (orange lines) and Lorentzian fits (dashed blue lines), see Fig. 6.

difference that there is an additional contribution to the resonance width from the coupling. Figure 6(b) shows the such obtained $\frac{1}{2}\text{Tr}S_R(0)$ in dependence of $\omega_R/2\pi$. Comparing Fig. 1(c) from NMR and Fig. 6(b) from the microwave analog there is again a complete equivalence.

Two dependencies can be studied. First, Eq. (53) predicts a resonance for $\omega_R = \omega_0$. This is illustrated in Fig. 7. The upper part of the figure shows the splitting of the Kramers doublet at 1.113 GHz in dependence of $\Delta\varphi \sim \omega_0$. The axes have been interchanged as compared to Fig. 3. For the ω_0 values marked by vertical lines a NMR equivalent has been performed. The lower part of the figure shows the obtained resonance curves. All resonance curves meet the resonance condition, and all of them are perfectly Lorentzian shaped.

Second, Eq. (53) predicts a resonance width of $\Delta\omega = \sqrt{\omega_1^2 + 4\gamma'^2}$ and a splitting of $2\Delta\omega$ of the two branches of the hyperbola in the rotating frame at the resonance position $\omega_R = \omega_0$. This is illustrated in Fig. 8. The upper part of the figure shows the eigenvalues in the rotating field for the Kramers doublet at 1.12 GHz for three different ω_1 values achieved by varying the length hubs induced by the diode in the dangling bonds, see Appendix A. The observed crossing of the hyperbolas is due to experimental imperfections. The lower part of the figure shows the corresponding NMR curves.

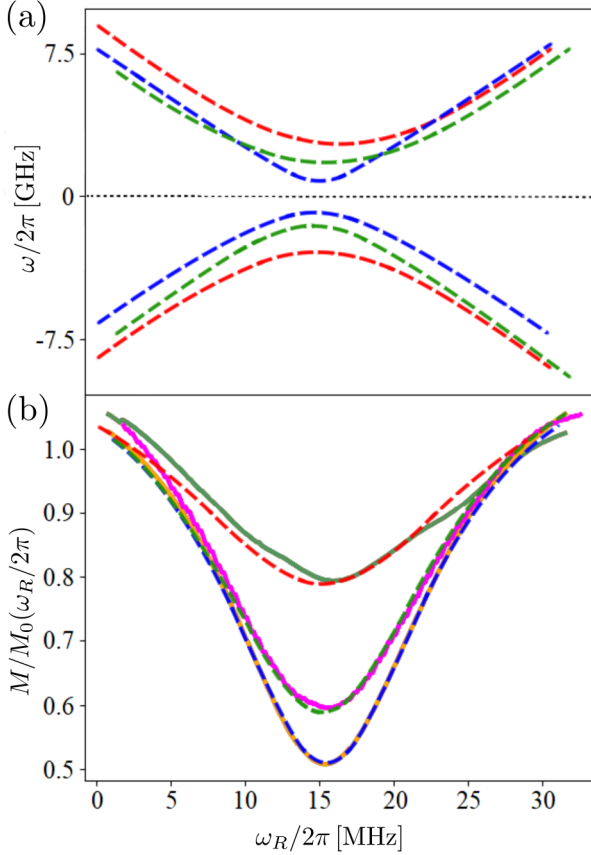


FIG. 8. (a) Eigenvalues of the Kramers doublet at 1.12 GHz in the rotation frame for three different radio-frequency strengths ω_1 , achieved by varying the length hubs induced by the diode. The dashed lines have been extracted from $\text{Tr}S_R(\omega_R)$ by a fit with a pair of hyperbolas, see Fig. 6. (b) Corresponding resonance curves. The solid lines have been extracted from $\text{Tr}S_R(\omega)$ along the dotted white line, see Fig. 6, the dashed lines reflect the best fits with a Lorentzian.

Again the line shapes are perfectly Lorentzian, and the line widths increase with ω_1 as expected. Further contributions to the line width are caused by absorption of the microwaves in the bonds, which in principle could be accounted for in terms of an imaginary contribution to ω . For a quantitative analysis of the resonance widths the data set was not sufficient.

VII. CONCLUSIONS

We presented an approach towards spin resonance where no spins are involved. It is based on the fact that a spin $\frac{1}{2}$ is not really needed to study spin physics, a system exhibiting symplectic symmetry is sufficient. A prerequisite for this achievement was the development of a technique allowing for rapid changes of the transmission properties of a network with frequencies up to 125 MHz. Among other things this opens an approach to spin relaxation studies. In standard NMR relaxation measurements are performed to get information on the origin of the fluctuating interactions. Here it is vice versa: it is easy to generate well-controlled fluctuating interactions, again by means of diodes, and to study their implications for the resonance line shape. In particular the old Bloembergen-Purcell-Pound model [27] could be directly addressed. It

assumes stochastic flips between two states, and this is exactly what can be simulated by microwave techniques, and in fact is realized in various systems found in nature. Furthermore, the here-developed technique to change effective bond lengths by means of diodes enables the experimental realization of time-dependent graphs, a class of systems inaccessible hitherto, in particular Floquet systems.

The graph studied in the present work, in the absence of perturbations just a square with dangling bonds at the corners, does exhibit both symplectic and chiral symmetry. All eigenvalues thus appear not only as Kramers doublets but also in pairs ω_e and $\omega_p = -\omega_e$, see Eq. (33). This feature had been used by us previously for the study of the chiral ensembles in a square arrangement of coupled dielectric resonators [21]. This allows for an interpretation of the two Kramers doublets, corresponding to ω_e and ω_p , in terms of electron and positron states. In fact the graph secular matrix $\mathbf{h}$ (28) may alternatively be written completely in terms of the γ matrices, and the column vectors φ_A appearing as solutions of the wave equation (16) may be interpreted as Dirac rank-4 spinors, see, e. g., the work by Bolte and Harrison [28] on the Dirac operator on graphs. This aspect has not been treated in this paper, but it shows that the square graph is also an excellent candidate for the study of the complete four-dimensional Dirac equation, not only of its reduced two-dimensional version.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DIODE

Figure 9 shows sketch and photograph of the diode construction attached at the end of the dangling bond denoted by D in Fig. 2(b). The high-speed double diode BAV 70 allows for bipolar flips with frequencies up to 125 MHz. At the other dangling bond, denoted in Fig. 2(b) by D', a double diode BAV 56 with opposite polarity is attached. Thus the lengths l_B of the dangling bonds can be periodically switched anticyclically between l_0 and $l_0 + \Delta\ell$, and $l_0 + \Delta\ell$ and l_0 , respectively, allowing for a switching without changing the total graph length.

The wave functions meeting at the T junction may be written as

$$\psi_l(x_l) = a_l e^{-ikx_l} - b_l e^{ikx_l}, \quad l = 0, 1, 2, \quad (\text{A1})$$

where the x_l are the distances from the T junction. The vectors $a = (a_0, a_1, a_2)^T$ and $b = (b_0, b_1, b_2)^T$ of the amplitudes of

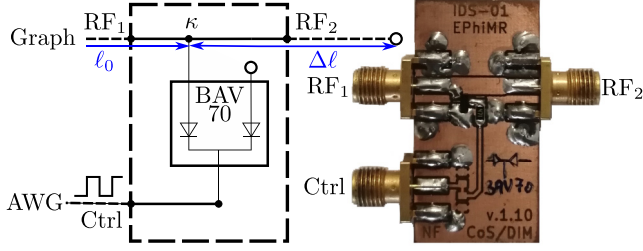


FIG. 9. Sketch and photograph of the diode construction attached at the end of the dangling bond denoted by D in Fig. 2(b). Depending on the state of the control signal (Ctrl), generated by a wave form generator (AWG), the microwaves entering via port RF₁ are either reflected at κ , the connection point to the diode BAV70 (high state), or after traveling through an additional bond of length of $\Delta\ell$ attached at port RF₂ (low state).

the incoming and outgoing waves obey the scattering equation

$$b = S_T a, \quad (\text{A2})$$

where

$$S_T = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \quad (\text{A3})$$

is the scattering matrix of the T junction. Assuming that the dangling bond is attached to port 0 and short terminated at the end, the amplitude of the incoming wave is obtained from the amplitude of the outgoing wave just by multiplication with a phase factor,

$$a_0 = e^{2ikl_B} b_0, \quad (\text{A4})$$

where l_B is the length of the bond. Eliminating a_0, b_0 from the scattering equation system (A2) using Eq. (A4), a 2×2 scattering matrix equation is obtained for the amplitudes of the incoming and outgoing waves in bonds 1 and 2,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S_V \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (\text{A5})$$

where

$$S_V = \frac{1}{\cos \alpha - 2i \sin \alpha} \begin{pmatrix} \cos \alpha & 2i \sin \alpha \\ 2i \sin \alpha & \cos \alpha \end{pmatrix}, \quad (\text{A6})$$

with $\alpha = kl_B$. Introducing another angle β via

$$\tan \beta = 2 \tan \alpha. \quad (\text{A7})$$

Equation (A6) may be alternatively written as

$$S_V = e^{i\beta} \begin{pmatrix} \cos \beta & i \sin \beta \\ i \sin \beta & \cos \beta \end{pmatrix}. \quad (\text{A8})$$

In its diagonal basis the scattering matrix is given by

$$(S_V)_D = \text{diag}(e^{2i\beta}, 1). \quad (\text{A9})$$

Expanding $l_B = l_0 + \Delta l$ in first order in Δl , one obtains using Eq. (A7),

$$(S_V)_D = (S_V)_{D0} \text{diag}(e^{ik(\Delta l)_{\text{eff}}}, 1), \quad (\text{A10})$$

where $(S_V)_{D0}$ is the value of $(S_V)_D$ for $l_B = l_0$, and

$$(\Delta l)_{\text{eff}} = \frac{4\Delta l}{1 + 3 \sin^2(kl_0)}. \quad (\text{A11})$$

Equation (A10) shows that part of the scattering matrix is not influenced by the dangling bond. But for the other part a change of the length of the dangling bond by Δl produces a phase factor equivalent to a change of the length of the sum of the two bonds meeting at the T junction by $(\Delta l)_{\text{eff}}$.

APPENDIX B: SECULAR MATRIX

Plugging in expression (26) for the wave equation into the continuity equation (20) one obtains for the secular matrix

$$\mathbf{h} = -\mathbf{F}_B - \mathbf{F}_P - \mathbf{F}_D + \sigma_P \mathbf{G}_P + \sigma_D \mathbf{G}_D, \quad (\text{B1})$$

where

$$\mathbf{F}_A = \mathbf{K}_A \cot(\mathbf{K}_A l), \quad \mathbf{G}_A = \mathbf{K}_A / \sin(\mathbf{K}_A l). \quad (\text{B2})$$

Expansion of the $\mathbf{F}_A, \mathbf{G}_A$ in linear order in the perturbations α, β yields

$$\mathbf{F}_A = f \mathbb{1}_4 - k f' \mathbf{V}_A, \quad \mathbf{G}_A = g \mathbb{1}_4 - k g' \mathbf{V}_A. \quad (\text{B3})$$

One thus obtains $\mathbf{h} = \mathbf{h}_0 + \mathbf{h}_\alpha + \mathbf{h}_\beta$ with $\mathbf{h}_0$ given by Eq. (29), and

$$\begin{aligned} \mathbf{h}_\alpha &= k f' \mathbf{V}_P - k g' \sigma_P \mathbf{V}_P \\ &= \alpha k [f' \sigma_\alpha - g' \sigma_P \sigma_\alpha] \\ \mathbf{h}_\beta &= k f' \mathbf{V}_D - k g' \sigma_D \mathbf{V}_D \\ &= \beta k [f' \sigma_\beta - g' \sigma_D \sigma_\beta]. \end{aligned} \quad (\text{B4})$$

Plugging in expressions (18) for σ_α and σ_β , we obtain

$$\mathbf{h}_\alpha = \alpha k \begin{pmatrix} f' \sigma_z & -g' \sigma_x \\ -g' \sigma_x & -f' \sigma_z \end{pmatrix}, \quad \mathbf{h}_\beta = \beta k \begin{pmatrix} f' \sigma_z & -g' \sigma_z \\ -g' \sigma_z & f' \sigma_z \end{pmatrix} \quad (\text{B5})$$

for the contributions resulting from the perturbations. Now we move to the basis where $\mathbf{h}_0$ is diagonal. By means of transformation (30) $\mathbf{h}_0$ is turned into

$$\tilde{\mathbf{h}}_0 = \begin{pmatrix} (-3f + g\sqrt{2}) \mathbb{1} & \\ & (-3f - g\sqrt{2}) \mathbb{1} \end{pmatrix}, \quad (\text{B6})$$

and the perturbations $\mathbf{h}_\alpha$ and $\mathbf{h}_\beta$ are turned into

$$\begin{aligned} \tilde{\mathbf{h}}_\alpha &= \alpha k \left[\frac{f'}{\sqrt{2}} \begin{pmatrix} \sigma_z & -\sigma_z \\ -\sigma_z & -\sigma_z \end{pmatrix} - g' \begin{pmatrix} \sigma_z & \\ & \sigma_z \end{pmatrix} \right], \\ \tilde{\mathbf{h}}_\beta &= \beta k \left[\frac{f'}{\sqrt{2}} \begin{pmatrix} -\sigma_x & -\sigma_x \\ -\sigma_x & \sigma_x \end{pmatrix} + g' \begin{pmatrix} \sigma_x & \\ & \sigma_x \end{pmatrix} \right]. \end{aligned} \quad (\text{B7})$$

All matrix transformations needed are compiled in Table I.

APPENDIX C: A RELATION BETWEEN THE GREEN'S FUNCTION AND THE SECULAR MATRIX

The secular equation $\mathbf{h}\phi = 0$ (27) holds for closed graphs. Let us now assume that there are L open channels attached at nodes $l, l = 1, \dots, L$. Again the wave functions within each channel may be written as

$$\psi_l(x_l) = a_l e^{-ikx_l} - b_l e^{ikx_l}, \quad (\text{C1})$$

TABLE I. Matrix transformations into the basis of $\mathbf{h}_0$.

	$\mathbf{A}$	$\tilde{\mathbf{A}}$
σ_P	$\begin{pmatrix} \cdot & i\sigma_y \\ -i\sigma_y & \cdot \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{pmatrix}$
σ_D	$\begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}$
σ_α	$\begin{pmatrix} \sigma_z & \cdot \\ \cdot & -\sigma_z \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_z & -\sigma_z \\ -\sigma_z & -\sigma_z \end{pmatrix}$
σ_β	$\begin{pmatrix} \sigma_z & \cdot \\ \cdot & \sigma_z \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} -\sigma_x & -\sigma_x \\ -\sigma_x & \sigma_x \end{pmatrix}$
$\sigma_P \sigma_\alpha$	$\begin{pmatrix} \cdot & \sigma_x \\ \sigma_x & \cdot \end{pmatrix}$	$\begin{pmatrix} \sigma_z & \cdot \\ \cdot & \sigma_z \end{pmatrix}$
$\sigma_D \sigma_\beta$	$\begin{pmatrix} \cdot & \sigma_z \\ \sigma_z & \cdot \end{pmatrix}$	$\begin{pmatrix} -\sigma_x & \cdot \\ \cdot & -\sigma_x \end{pmatrix}$

where the x_l are the distances from the respective nodes. Energy conservation fixes the value of the potential φ_l at node l by

$$\varphi_l = \psi_l(0) = a_l - b_l, \quad l = 1, \dots, L \quad (\text{C2})$$

Furthermore there are additional contributions to the current resulting from the open channels, given by

$$\lambda \frac{\partial \psi_l}{\partial x} \Big|_{x_l=0} = -ik\lambda(a_l + b_l), \quad l = 1, \dots, L. \quad (\text{C3})$$

Parameter λ depends on the boundary conditions for the attached channels, where $\lambda = 1$ means a perfect matching corresponding to Neumann boundary conditions. The homogeneous secular equation for the closed graph is thus turned by the open channels into the inhomogeneous equation

$$\mathbf{h} \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_L - b_L \\ \varphi_{L+1} \\ \vdots \end{pmatrix} = i\gamma \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_L + b_L \\ 0 \\ \vdots \end{pmatrix} \quad (\text{C4})$$

with $\gamma = k\lambda$. With the definition of Kottos and Smilansky [22] of the $\mathbf{h}$ matrices the factor k would be absent. It follows

$$\begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_L - b_L \end{pmatrix} = i\gamma G \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_L + b_L \end{pmatrix}, \quad (\text{C5})$$

where

$$G = (\mathbf{h}^{-1})_{\text{trunc}} \quad (\text{C6})$$

is the matrix obtained by taking only the first L rows and columns of $\mathbf{h}^{-1}$. Equation (37) is obtained as the special case for two ports only. In a trivial further step Eq. (C5) is transformed into $b = Sa$ with S given by Eq. (36).

This derivation relating the graph scattering matrix to the graph secular matrix is a streamlined version of the calculation presented in Sec. IIIC of Ref. [22]. In particular Eq. (23) of Ref. [22] for the scattering matrix, though expressing the

result in a somewhat different way, is equivalent to the expression obtained here. Only the sign of the scattering matrix differs, a consequence of a deviating sign convention of the outgoing amplitudes in Ref. [22].

APPENDIX D: A PARTIAL FRACTION EXPANSION OF THE GREEN'S FUNCTION

Using the transformation (30), $G = (\mathbf{h}^{-1})_{11}$, Eq. (37) may be written as

$$G = \tilde{\mathbf{r}}^\dagger \tilde{\mathbf{h}}^{-1} \tilde{\mathbf{r}}, \quad \tilde{\mathbf{r}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon_y \\ i\sigma_y \varepsilon_y \end{pmatrix}, \quad (\text{D1})$$

where $\tilde{\mathbf{r}}$ is the left column of $\mathbf{R}$, or

$$G = \frac{1}{2} \varepsilon_y^\dagger (\tilde{\mathbf{h}}_+^{-1} + \sigma_y \tilde{\mathbf{h}}_-^{-1} \sigma_y) \varepsilon_y = \varepsilon_y^\dagger \tilde{G} \varepsilon_y \quad (\text{D2})$$

with

$$\begin{aligned} \tilde{G} &= \frac{1}{2} (\tilde{\mathbf{h}}_+^{-1} + \sigma_y \tilde{\mathbf{h}}_-^{-1} \sigma_y) \\ &= \frac{1}{2} \left(\frac{\mathbb{1}}{h_{0+}\mathbb{1} + (\alpha\sigma_z - \beta\sigma_x)k_+} \right. \\ &\quad \left. + \sigma_y \frac{\mathbb{1}}{h_{0-}\mathbb{1} - (\alpha\sigma_z - \beta\sigma_x)k_-} \sigma_y \right) \\ &= \frac{1}{2} \left(\frac{\mathbb{1}}{h_{0+}\mathbb{1} + (\alpha\sigma_z - \beta\sigma_x)k_+} \right. \\ &\quad \left. + \frac{\mathbb{1}}{h_{0-}\mathbb{1} + (\alpha\sigma_z - \beta\sigma_x)k_-} \right), \end{aligned} \quad (\text{D3})$$

where

$$h_{0\pm} = -3f \pm g\sqrt{2}, \quad k_{\pm} = \left(\frac{f'}{\sqrt{2}} \mp g' \right) k. \quad (\text{D4})$$

In the last step of Eq. (D3) it has been used that σ_y anticommutes with σ_x and σ_z .

From Eq. (D3) an analytic expansion of $\tilde{G}$ in terms of partial fractions can be obtained. This is illustrated here for $\tilde{\mathbf{h}}_+^{-1}$ in the absence of the perturbation,

$$\begin{aligned} \tilde{h}_{0+}^{-1} &= \frac{1}{k(-3 \cot \varphi + \sqrt{2}/\sin \varphi)}, \quad \varphi = kl \\ &= \frac{1}{3k \cos \varphi_0 - \cos \varphi}, \end{aligned} \quad (\text{D5})$$

where $\cos \varphi_0 = \sqrt{2}/3$, see Eq. (33). The latter expression can be written as

$$\tilde{h}_{0+}^{-1} = \frac{1}{6k} (\cot \varphi_+ + \cot \varphi_-), \quad \varphi_{\pm} = \frac{\varphi \pm \varphi_0}{2} \quad (\text{D6})$$

Using the well-known partial fraction expansion for the cotangent, the familiar pole expansion of $\tilde{G}$ is obtained,

$$\tilde{G} = \sum_n \frac{g_n}{k - k_n}, \quad (\text{D7})$$

where the sum is over all Kramers doublets. The extension to the situation where there is a perturbation is straightforward. We do not proceed further in this direction, since the details are not of relevance in the present context.

APPENDIX E: SCATTERING MATRIX IN THE ROTATING FRAME

By means of the rotation (46) the wave equation (13) is transformed into

$$\left\{ \square_R \mathbb{1}_4 + 2\tilde{V}_{AR} \frac{\partial^2}{\partial x^2} \right\} \phi_{AR}(x, t) = 0, \quad (\text{E1})$$

and the time derivative in the d'Alembert operator is turned by the rotation into

$$\left(\frac{\partial}{\partial t} \right)_R \mathbb{1}_4 = \mathbf{R}_t \frac{\partial}{\partial t} \mathbf{R}_t^\dagger \mathbb{1}_4 = \frac{\partial}{\partial t} \mathbb{1}_4 + \frac{i\omega_R}{2} \begin{pmatrix} \sigma_z & \cdot \\ \cdot & \sigma_z \end{pmatrix}. \quad (\text{E2})$$

$\tilde{V}_P = \alpha \tilde{\sigma}_\alpha$ is not changed, $\tilde{V}_P = V_{PR}$, but

$$\begin{aligned} V_{DR} &= 2\beta \cos(\omega_R t) \mathbf{R}_t \tilde{\sigma}_\beta \mathbf{R}_t^\dagger \\ &= 2\beta \cos(\omega_R t) \mathbf{R}_t \frac{1}{\sqrt{2}} \begin{pmatrix} -\sigma_x & -\sigma_x \\ -\sigma_x & \sigma_x \end{pmatrix} \mathbf{R}_t^\dagger \\ &= 2\beta \cos(\omega_R t) \frac{1}{\sqrt{2}} \begin{pmatrix} -\sigma_{xR} & -\sigma_{xR} \\ -\sigma_{xR} & \sigma_{xR} \end{pmatrix}. \end{aligned}$$

With

$$\begin{aligned} 2 \cos(\omega_R t) \sigma_{xR} &= 2 \cos(\omega_R t) e^{-i\frac{\omega_R t}{2} \sigma_z} \sigma_x e^{+i\frac{\omega_R t}{2} \sigma_z} \\ &= 2 \cos(\omega_R t) e^{-i\omega_R t \sigma_z} \sigma_x \\ &= (1 + e^{-2i\omega_R t \sigma_z}) \sigma_x \end{aligned} \quad (\text{E3})$$

one arrives at

$$V_{DR} = \frac{\beta}{\sqrt{2}} \begin{pmatrix} -\sigma_x & -\sigma_x \\ -\sigma_x & \sigma_x \end{pmatrix} = \beta \tilde{\sigma}_D, \quad (\text{E4})$$

where the terms rotating with $2\omega_R t$ have been disregarded. Except for this neglect all time dependencies have disappeared in the rotation frame. The $\tilde{V}_A$ are exactly the same as for the static case, there is only an additional term resulting from the transformation of the d'Alembert operator.

By the transformation into the rotating frame the wave equation (E1) has become time independent. Hence, the time

dependence of the wave functions can be separated,

$$\phi_{AR}(x, t) = e^{-i\omega t} \phi_{AR}(x). \quad (\text{E5})$$

By this separation the time derivative of the d'Alembert operator in the rotating frame (E2) is turned into

$$\left(\frac{\partial}{\partial t} \right)_R \mathbb{1}_4 \rightarrow -i \left[\omega \mathbb{1}_4 - \frac{\omega_R}{2} \begin{pmatrix} \sigma_z & \cdot \\ \cdot & \sigma_z \end{pmatrix} \right]. \quad (\text{E6})$$

The equations obtained for the static case can thus be directly be adopted for the time-dependent case, if only $\omega \mathbb{1}$ is replaced by $\omega \mathbb{1} - \frac{\omega_R}{2} \sigma_z$.

The separation of the time dependence can be performed for the amplitude vectors of the incoming and outgoing waves in the rotating frame as well,

$$a_R(t) = e^{-i\omega t} a_R, \quad b_R(t) = e^{-i\omega t} b_R \quad (\text{E7})$$

resulting in the time-independent scattering equation (48) in the rotating frame, with a Hamiltonian H_R given by Eq. (49). Here we used that Eqs. (41) and (42) for the static case can be directly adopted for the time-dependent case by only replacing ω by $\omega - \frac{\omega_R}{2} \sigma_z$, see above.

The final step is a back transformation from the rotating into the laboratory frame. The vectors of incoming and outgoing amplitudes in the laboratory frame $\tilde{a}$ and $\tilde{b}$, see Eq. (39), are obtained from the corresponding quantities in the rotating frame, $a_R(t)$ and $b_R(t)$ via backward spinor rotations,

$$\tilde{a}(t) = e^{i\frac{\omega_R t}{2} \sigma_z} a_R(t) = e^{-i(\omega - \frac{\omega_R}{2} \sigma_z)t} a_R \quad (\text{E8})$$

or

$$\begin{pmatrix} \tilde{a}_1(t) \\ \tilde{a}_{\bar{1}}(t) \end{pmatrix} = \begin{pmatrix} e^{-i\omega_- t} (a_R)_1 \\ e^{-i\omega_+ t} (a_R)_{\bar{1}} \end{pmatrix}, \quad (\text{E9})$$

with $\omega_\pm = \omega \pm \frac{\omega_R}{2}$, where separation (E7) has been used. Equation (E9) and the corresponding expression for $\tilde{b}(t)$ illustrate the statement that for the determination of the scattering matrix in the rotation frame spin-up and spin-down components have to be excited and detected at shifted frequencies ω_- and ω_+ , respectively.

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