

Finite-temperature topological transitions in the presence of quenched uncorrelated disorder

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We address issues related to the presence of defects at finite-temperature topological transitions, in particular when defects are modeled in terms of further variables associated with a quenched disorder, corresponding to the limit in which the defect dynamics is very slow. As a paradigmatic model, we consider the classical three-dimensional lattice $\mathbb{Z}_2$ gauge model in the presence of quenched uncorrelated disorder associated with the plaquettes of the lattice, whose topological transitions are characterized by the absence of a local order parameter. We study the critical behaviors in the presence of weak disorder. We show that they belong to a topological universality class that differs from those of known statistical models, and in particular different from that of the lattice $\mathbb{Z}_2$ gauge models without disorder, in agreement with the Harris criterium for the relevance of uncorrelated quenched disorder when the pure system undergoes a continuous transition with positive specific-heat critical exponent.

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I. INTRODUCTION

Statistical systems with quenched disorder, modeling the presence of defects subject to a slow dynamics, generally develops peculiar critical phenomena associated with universality classes that are different from those of pure systems, such as transitions to glassy phases. Finite-temperature phase transitions and critical behaviors in disordered systems have been mostly studied within lattice spin models with various realizations of local disorder, in particular with types of disorder preserving the symmetry of the pure system, such as random-bond and random-site disorder (random-field disorder instead generally breaks the global symmetry), and spatially uncorrelated probability distributions of the quenched variables, see, e.g., Refs. [1–39].

While a large amount of disorder may lead to glassy phases and glassy critical behaviors in classical three-dimensional (3D) spin systems, ferromagnetic transitions may still occur for sufficiently weak disorder. However, even though the quenched disorder does not change the symmetry-breaking pattern, the universality class of the critical behavior may differ from that of the ferromagnetic transition of the pure system. According to the Harris criterium [4] [see, e.g., [1,3] for the renormalization-group (RG) interpretation/generalization of the original argument], which applies to a spatially uncorrelated quenched disorder that is effectively coupled to the energy density, transitions in the presence of weak disorder may develop a new universality class when the specific-heat exponent α of the pure transition is positive, as it happens in

3D random-bond and random-site Ising systems [1,2,7,8]. On the other hand, if the pure-system continuous transition has a negative specific-heat exponent, then its universality class turns out to be stable with respect to weak disorder, as it happens in 3D random-bond O(N) vector models for any $N \geq 2$, such as the XY and Heisenberg spin models [40], corresponding to $N = 2$ and $N = 3$, respectively. Moreover, with increasing disorder, spin systems may develop peculiar multicritical behaviors along the so-called Nishimori line [2,6,9,28,33,34], and then transitions to glassy phases, see, e.g., Refs. [14,16–29,31].

The effects of quenched disorder have been much less investigated at topological transitions, such as those occurring in systems with gauge symmetries, without a local order parameter driving the transition [41–44]. A paradigmatic model for systems undergoing finite-temperature topological transitions is provided by the 3D lattice $\mathbb{Z}_2$ gauge model [41,42], whose confinement-deconfinement transition is not driven by a local order parameter, but by the nonlocal order parameter associated with the area/perimeter law of the Wilson loops [41,42]. 3D lattice $\mathbb{Z}_2$ gauge models have been also considered in the presence of local spatially uncorrelated quenched disorder associated with each plaquette of the lattice [45,46] (denoted by RPGM, for random plaquette gauge model, in the following), i.e., with randomly chosen plaquettes having couplings with the wrong (negative instead of positive) sign. The RPGM was originally introduced in relation to the theory of the quantum error correction. However, it also provides a paradigmatic example of systems undergoing topological transitions in the presence of disorder. Some results on its temperature-disorder phase diagram were reported in Refs. [46,47], but the critical behavior at the transition lines has never thoroughly been analyzed. Therefore the characterization of the universality classes of its topological transitions in the presence of quenched disorder is still an open issue.

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In this paper we investigate the effects of disorder at finite-temperature topological transitions, focusing on the paradigmatic 3D RPGM. We recall that the pure 3D lattice $\mathbb{Z}_2$ gauge model is related to the standard 3D Ising model by a duality relation [41,48], thus they share the same length-scaling critical exponent $\nu_{\mathcal{I}} \approx 0.630$ [40], which implies that the specific-heat exponent is positive, $\alpha_{\mathcal{I}} \approx 0.110$. We present a numerical analysis of the critical behaviors of the 3D lattice $\mathbb{Z}_2$ gauge model for weak disorder, which is made difficult by the fact that its topological transitions do not have gauge-invariant local order parameters. We thus study the critical behaviors of this system by focusing on the scaling behavior of the gauge-invariant energy cumulants. The numerical results show that the quenched disorder destabilizes the pure-system topological transition. Indeed, we observe critical behaviors belonging to another universality class with $\nu = 0.82(2)$, which is significantly larger than the Ising length-scale exponent $\nu_{\mathcal{I}} \approx 0.630$ in the absence of disorder (and also different from any universality class of Ising systems with quenched disorder). In the final part of the paper we also briefly discuss generalizations of the RPGM, considering generic 3D lattice $\mathbb{Z}_N$ gauge models with weak disorder.

While in this work we focus on the effect of weak disorder at finite-temperature topological transitions, it is important to note that the presence of disorder can lead to very interesting phenomenological consequences also in many quantum (i.e., zero temperature) contexts, in particular for what regards quantum critical phenomena, see, e.g., Refs. [39,49], and topological phases of matter, see, e.g., Refs. [50–61].

The paper is organized as follows: in Sec. II we introduce the RPGM and discuss some properties of its phase diagram, then in Sec. III we investigate the critical behavior of this model in the regime of weak disorder, providing a quite accurate estimate of the corresponding critical exponent ν . Finally, in Sec. IV we report our conclusions and discuss some generalizations of the RPGM.

II. THE MODEL

In the statistical RPGM, i.e., a 3D lattice $\mathbb{Z}_2$ gauge model with uncorrelated quenched disorder associated with the plaquettes, spin variables $\sigma_{x,\mu} = \pm 1$ are associated with the links (starting from site x in the positive μ direction, $\mu = 1, 2, 3$) of a cubic lattice of size L , and quenched disorder variables $w_{x,\mu\nu} = \pm 1$ are associated with the plaquettes identified by the site x and the directions $\mu > \nu$. Its Hamiltonian reads [41,45,46]

$$H = -K \mathcal{H}, \quad \mathcal{H} = \sum_{x,\mu>\nu} w_{x,\mu\nu} \Pi_{x,\mu\nu}, \quad (1)$$

where K is the gauge coupling, $\Pi_{x,\mu\nu}$ with $\mu > \nu$ is the plaquette operator

$$\Pi_{x,\mu\nu} = \sigma_{x,\mu} \sigma_{x+\hat{\mu},\nu} \sigma_{x+\hat{\mu}+\hat{\nu},\mu} \sigma_{x,\nu}, \quad (2)$$

$\hat{\mu}$ indicates the unit vector along the direction μ and periodic boundary conditions are used on the lattice. The quenched disorder variables $w_{x,\mu\nu}$ ($\mu > \nu$) are spatially uncorrelated, and each disorder configuration is chosen according to the

probability distribution

$$P_w(q) = \prod_{x,\mu>\nu} [(1-q)\delta(w_{x,\mu\nu} - 1) + q\delta(w_{x,\mu\nu} + 1)], \quad (3)$$

where $0 \leq q \leq 1$ is a global parameter. Therefore, q is the probability of getting an extra minus sign in the Hamiltonian weight of the plaquette. The free-energy density of the RPGM is obtained by averaging over disorder configurations:

$$F(K) = \frac{1}{V} \sum_{\{w\}} P_w(q) \ln Z_w(K),$$

$$Z_w(K) = \sum_{\{\sigma\}} e^{-H/T}, \quad V = L^3, \quad (4)$$

where $\{w\}$ and $\{\sigma\}$ indicate disorder and spin configurations, respectively. Without loss of generality, in the following we fix $T = 1$. Therefore, the relevant phase diagram of the lattice $\mathbb{Z}_2$ gauge model with uniform spatially uncorrelated quenched disorder should be studied in the q - K plane.

In the absence of disorder, which corresponds to the model in Eq. (1) for $q = 0$, the 3D lattice $\mathbb{Z}_2$ -gauge model undergoes a continuous topological transition at a finite value of K [41–44], $K_c(q = 0) = 0.761413292(11)$, separating the high- K deconfined phase from the low- K confined phase. This transition is topological, as it is not driven by a local order parameter. The behavior of the nonlocal Wilson loop W_C , defined as the product of the link variables along a closed contour C within a plane, provides a nonlocal order parameter for the transition [41,42]. Indeed, its size dependence for large contours changes at K_c , from an area law $W_C \sim \exp(-c_a A_C)$ for small values of K , where A_C is the area enclosed by the contour C and $c_a > 0$ is the so called string tension, to a perimeter law $W_C \sim \exp(-c_p P_C)$ for large values of K , where P_C is the perimeter of the contour C and $c_p > 0$ is a constant. The topological transition of the 3D lattice $\mathbb{Z}_2$ gauge model can be related to the continuous transition of the standard 3D Ising model, because a nonlocal duality mapping relates the partition functions of the two models [41,48]. Duality implies that thermal observables have the same critical behavior in the two models and, in particular, the correlation-length exponent ν is the same. The most accurate estimate for the 3D Ising universality class is $\nu_{\mathcal{I}} = 0.629971(4)$ [62], see also Refs. [40,63–68] for other results.¹

The phase diagram for $q > 0$ has been discussed in Refs. [46,47]. A sketch of the q - K^{-1} phase diagram is shown in Fig. 1. It shows notable analogies with the phase diagrams of the random-bond Ising and random-phase XY models, see, e.g., Refs. [6,9,16,28,30–35]. The deconfined phase at sufficiently large values of K is expected to extend to nonzero values of q , and it is separated from the confined phase by a transition line, which starts from the critical point along the $q = 0$ line. The so-called Nishimori line [2,6], where K and q satisfy the relation [46]

$$e^{-2K} = \frac{q}{1-q}, \quad (5)$$

¹The universal features of the topological confinement-deconfinement transitions of 3D lattice $\mathbb{Z}_N$ models have been investigated in several works, see, e.g., Refs. [41,44,69–78].

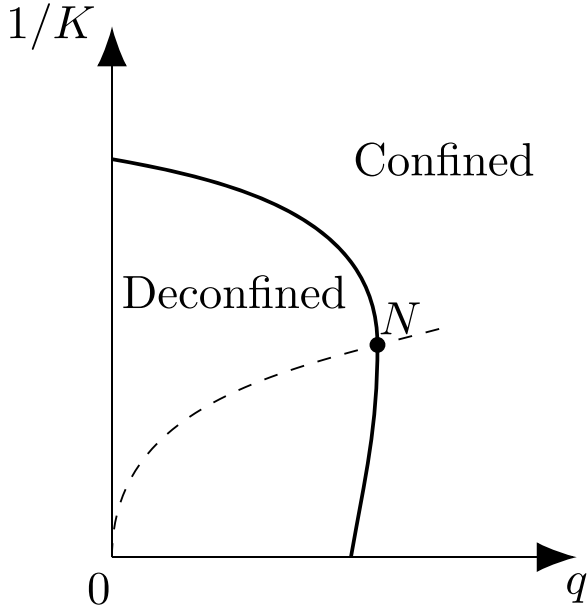


FIG. 1. The expected phase diagram of the 3D RPGM, see Refs. [46,47]. The continuous line separates the deconfined phase, at small K^{-1} and q , from the confined phase, while the dashed line denotes the Nishimori line. The black dot N represents the point where the deconfinement transition line intersects the Nishimori line, and the value of q at this point is denoted by q_N in the main text.

show an enhanced symmetry, which allows us to derive a number of exact relations [46]. In particular, the maximum value of q showing a deconfined phase corresponds to the location q_N of the topological confining-deconfining transition along the Nishimori line. Reference [47] reported the estimate $q_N \approx 0.033$. The transition line below the Nishimori critical point is likely slightly reentrant, as it also occurs in random-bond Ising models, see, e.g., Refs. [31,35]. There is no evidence of the existence of further gauge glassy phase for $q > q_N$, as characterized in Ref. [46], although its presence cannot be excluded yet. Therefore, the phase diagram should be similar to that of the 2D random-bond Ising model [6,32,35,36,46] with the ferromagnetic and paramagnetic phases replaced by the deconfined and confined phases, as sketched in Fig. 1. In this work we focus on the critical properties at the confined-deconfined topological transition for small disorder parameter q , up to the critical point N along the Nishimori line.

According to the Harris criterion [4] (see also, e.g., [1,3]), spatially uncorrelated quenched disorder coupled to the energy density of the system provides a relevant perturbation at the fixed points corresponding to the critical behavior of pure systems with positive specific-heat exponent. This is, for example, the case of the 3D random-bond Ising model where the specific-heat exponent of the pure Ising transition is $\alpha_{\mathcal{I}} = 2 - 3\nu_{\mathcal{I}} \approx 0.110$. In this case the critical behavior at the ferromagnetic transitions in the presence of spatially uncorrelated disorder belong to a different universality class, the so called randomly dilute Ising (RDI) universality class, whose length-scale critical exponent is $\nu_{\text{rdi}} = 0.683(2)$ [7,8,79], and the corresponding specific-heat exponent is negative, $\alpha_{\text{rdi}} = 2 - 3\nu_{\text{rdi}} = -0.049(6)$. An analogous change of universality

class occurs also in the 3D RPGM for $q > 0$. Indeed, also in this case we have spatially uncorrelated quenched variables coupled to the energy density, which corresponds to the plaquette term in lattice $\mathbb{Z}_2$ gauge models. Therefore, the universality class of the disordered topological transitions is expected to differ from that of the pure model at $q = 0$. Note also that there is no reason to expect that such transition belongs to the RDI universality class of disordered spin systems, since the duality mapping with the Ising model is lost in the presence of disorder.

We finally mention that the results reported in Refs. [46,47] along the confining-deconfining transition do not give hints of the universality class of the critical behavior for weak disorder, and in particular of the corresponding critical exponent ν .

III. NUMERICAL METHODS AND RESULTS

Topological transitions are notoriously difficult to investigate, due to the absence of local order parameters. It might seem that the natural strategy for investigating the topological transitions of lattice $\mathbb{Z}_2$ gauge models is to study the area or perimeter law behavior of large Wilson loops, as done in [47]. Such a strategy is, however, effective only if we want to investigate the phase structure of the model, and it becomes more and more inefficient as we approach a continuous phase transition. It indeed requires the use of very large lattices to estimate the (vanishing at the transition) string tension from the large distance behavior of Wilson loops. In our numerical study we follow a more convenient numerical approach, which lends itself for a finite-size scaling (FSS) analysis at the critical point.

We study the FSS behavior of the gauge-invariant energy cumulants, normalized by the lattice volume and averaged over disorder. These quantities, which will be denoted by B_k in the following, are the intensive quantities defined by

$$B_k = \left(\frac{\partial}{\partial K} \right)^k F(K) = \frac{1}{V} \sum_{\{w\}} P_w(q) \left(\frac{\partial}{\partial K} \right)^k \ln Z_w(K). \quad (6)$$

The cumulants can be related (for $k > 1$) to the central moments of the energy defined for each disorder realization by

$$m_k = \langle (\mathcal{H} - \langle \mathcal{H} \rangle)^k \rangle, \quad (7)$$

where $\mathcal{H}$ is defined in Eq. (1), and the statistical average $\langle \rangle$ is taken over the spin variables at fixed disorder configuration. One can easily see that B_1 is proportional to energy density, i.e.,

$$B_1 = \frac{1}{V} [\langle \mathcal{H} \rangle]_w, \quad (8)$$

where the square brackets $[\]_w$ indicate the average over the quenched disorder, while the second and third cumulants are given by

$$B_k = \frac{1}{V} [m_k]_w \quad \text{for } k = 2, 3. \quad (9)$$

In particular, B_2 is proportional to the specific heat. The relation between cumulants and central moments becomes more

complicated for higher cumulants, for example,

$$B_4 = \frac{1}{V} [m_4 - 3m_2^2]_w. \quad (10)$$

The energy cumulants are very useful to characterize topological transitions in which no local gauge-invariant order parameter is present (see, e.g., Refs. [80–84]), since for fixed q they are expected to show the FSS behavior [80,83]

$$B_k(K, L) \approx L^{k/\nu-3} \mathcal{B}_k(X) [1 + L^{-\omega} \mathcal{B}_{k,\omega}(X)] + b_k, \quad (11)$$

$$X = (K - K_c) L^{1/\nu},$$

where the constants b_k represent the analytic background [40,83], and the $O(L^{-\omega})$ suppressed term with $\omega > 0$ represents the leading scaling correction which is generally associated with the leading irrelevant RG perturbation at the fixed point [40] (for example $\omega \approx 0.8$ at the standard Ising transition, i.e., for $q = 0$). The scaling functions $\mathcal{B}_k(X)$ are universal apart from a multiplicative factor and a normalization of the argument, however, they generally depend on the boundary conditions adopted.

It is important to stress that the background term b_k in Eq. (11) turns out to be subleading with respect to the scaling term only when $k - 3\nu > 0$, thus, for $k \geq 2$ when the specific-heat exponent is positive, i.e., $\alpha = 2 - 3\nu > 0$, which is the case for $q = 0$. If instead $\alpha < 0$, which is the case expected along the confining-deconfining transition line for $q > 0$ due to Harris criterium, higher cumulants are needed to identify the scaling behavior, with minimum value of k depending on the critical exponent ν . Note however that higher cumulants also become more noisy, so it is typically convenient to use the cumulant corresponding to the smallest value of k such that $k - 3\nu > 0$.

To characterize the critical behavior of the RPGM for weak disorder, we present a numerical FSS analysis for $q = 0.015$, which according to the results of Ref. [47] should be sufficiently small for the system to develop a finite-temperature confining-deconfining transition. We thus perform simulations for $q = 0.015$ and several values of K , for lattice sizes up to $L = 32$. We use $O(10^3)$ disorder samples and for each disorder realization, we carry out 2×10^5 Metropolis updates of the whole lattice, always starting from the ordered configuration (since the average plaquette is quite close to one at the critical point) and discarding one quarter (half, for the largest lattices) of them as thermalization. Moreover, three independent simulations have been carried out for each disorder realization, in order to obtain unbiased estimates of the first three cumulants of the energy (see [8] for a discussion of this point).

The numerical results obtained for the second cumulant B_2 , corresponding to the specific heat, are shown in Fig. 2. They do not appear to diverge with increasing L , implying that $\alpha < 0$, consistently with the expectations coming from Harris criterium. In Fig. 3 we report data for the third cumulant B_3 , which is the first cumulant to display an apparently divergent behavior at the transition. To estimate K_c and ν we performed several fits using the FSS relation in Eq. (11), with a polynomial approximation of $\mathcal{B}_k(X)$ and taking also into account scaling corrections, i.e., using the functional form

$$B_3(K, L) = L^{3/\nu-3} \left(\sum_{i=0}^{n_l} a_i X^i + L^{-\kappa} \sum_{i=0}^{n_s} b_i X^i \right). \quad (12)$$

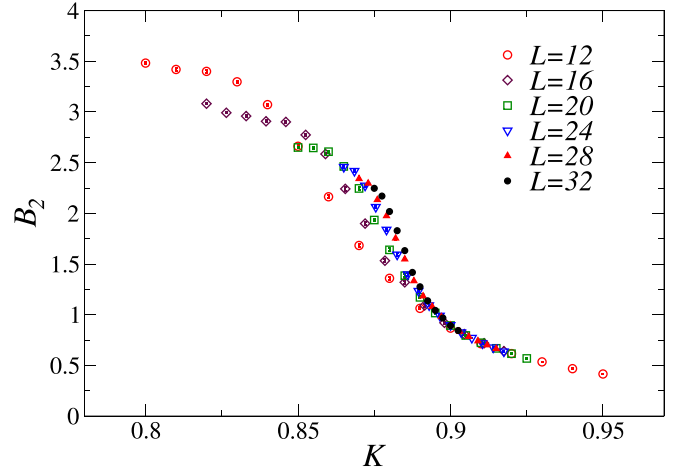


FIG. 2. Data for the second cumulants B_2 , proportional to the specific heat, across the topological transition for $q = 0.015$.

If $\omega < 3/\nu - 3$ then $\kappa = \omega$ and the $O(L^{-\kappa})$ term parametrizes the correction due to the leading irrelevant perturbation, see Eq. (11). If instead $\omega > 3/\nu - 3$ then $\kappa = 3/\nu - 3$, the $O(L^{-\kappa})$ term takes into account the analytic background in Eq. (11), and in this case $n_s = 0$ should be used.

The outcomes of some of these fits are reported in Table I, and our optimal estimates are

$$K_c = 0.8940(8), \quad \nu = 0.82(2). \quad (13)$$

The uncertainty on these estimates is dominated by the systematic errors, which have been estimated by varying the degree of the polynomial approximations (which has in fact a very mild effect), by varying the fit range, by systematically discarding the data coming from the smaller lattices in the fit to check the effects of scaling corrections, and by adding some scaling corrections, see Table I. In Fig. 4 we show the resulting FSS of B_3 when using our optimal estimates reported

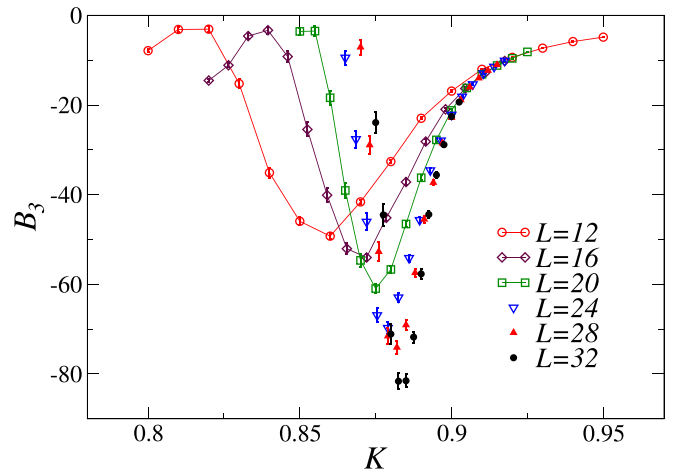


FIG. 3. Data for the third cumulant B_3 for $q = 0.015$, across the topological transition around $K \approx 0.89$. Continuous lines for $L = 12, 16, 20$ are drawn just to guide the eye.

TABLE I. Some of the fits performed to estimate K_c and ν for $q = 0.015$. $L_{\min}$ denotes the smallest value of the lattice size considered in the fit, and the fit range refers to the values of X , cf. Eq. (11), self-consistently used in the fit. Scaling corrections have been either totally neglected, or taken into account by fixing the reported value of κ (see text), or taken into account by leaving κ as a free parameter in the fit. In the last case, however, the value of κ could not be reliably estimated due to large statistical uncertainties. The errors on the various results are only statistical.

$L_{\min}$	Fit range	Scal. corr.	K_c	ν	χ^2/dof
20	$[-1.5, 1]$	No	0.89436(33)	0.8162(56)	69/42
24	$[-1.5, 1]$	No	0.89442(53)	0.826(10)	32/28
20	$[-1, 1]$	No	0.89393(50)	0.8102(76)	65/35
24	$[-1, 1]$	No	0.89396(82)	0.821(14)	27/21
20	$[-0.5, 0.5]$	No	0.8933(17)	0.811(31)	12/11
24	$[-0.5, 0.5]$	No	0.8919(23)	0.782(46)	12/9
12	$[-1.5, 1]$	Fiit κ	0.89371(29)	0.8129(54)	67/58
16	$[-1.5, 1]$	Fiit κ	0.89414(63)	0.822(15)	46/46
16	$[-1.5, 1]$	$\kappa = 0.5$	0.89389(42)	0.802(10)	70/55
20	$[-1.5, 1]$	$\kappa = 0.5$	0.89406(67)	0.792(21)	36/36
16	$[-0.5, 0.5]$	$\kappa = 0.5$	0.89370(98)	0.827(14)	41/28
20	$[-0.5, 0.5]$	$\kappa = 0.5$	0.8927(21)	0.813(32)	18/17
16	$[-1.5, 1]$	$\kappa = 1.0$	0.89382(40)	0.804(10)	69/55
20	$[-1.5, 1]$	$\kappa = 1.0$	0.89301(69)	0.800(20)	46/38
16	$[-0.5, 0.5]$	$\kappa = 1.0$	0.89298(91)	0.817(13)	32/28
20	$[-0.5, 0.5]$	$\kappa = 1.0$	0.8932(20)	0.820(30)	16/16

in Eq. (13), showing the good quality of the collapse of the data.²

²The estimate of K_c that we report in Eq. (13) is significantly more accurate than the estimate reported in Ref. [47], which was $K_c \approx 0.866$, somehow obtained from the specific-heat data for $L = 24$ at $q = 0.015$.

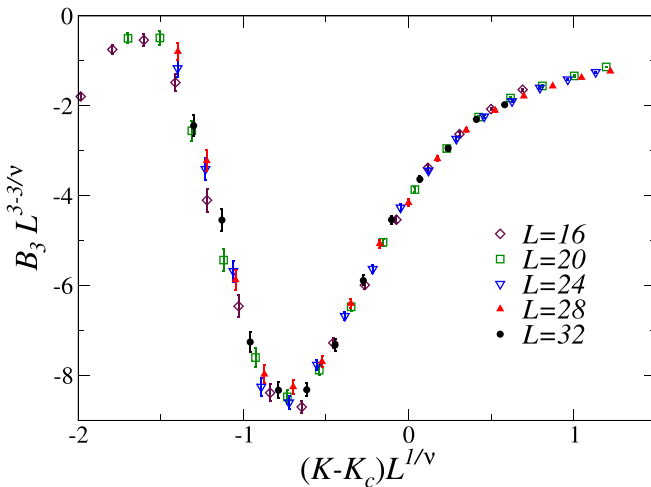


FIG. 4. Scaling of third cumulant B_3 for $q = 0.015$, obtained by plotting $L^{3-3/\nu} B_3$ versus $(K - K_c) L^{1/\nu}$ with our optimal estimates $K_c = 0.8940$ and $\nu = 0.82$.

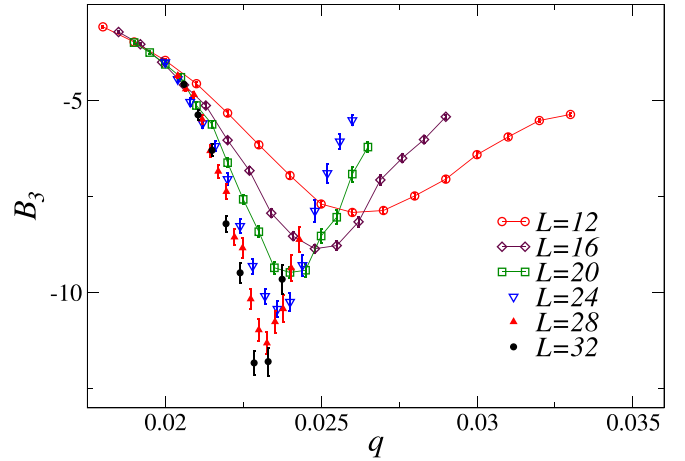


FIG. 5. Data for the third cumulant B_3 for $K = 1$, across the topological transition around $q \approx 0.022$. Continuous lines for $L = 12, 16, 20$ are drawn just to guide the eye.

A reasonable hypothesis is that the value $\nu = 0.82(2)$ of the length-scale critical exponent characterizes the critical behavior along the whole transition line up to the Nishimori point. Note that the value of ν that we have obtained is significantly larger than the Ising exponent $\nu_I \approx 0.630$ in the absence of disorder. Moreover, the corresponding value of the specific-heat exponent is negative, $\alpha = 2 - 3\nu = -0.46(6)$, as expected by the Harris criterium. We also note that the value ν obtained by our numerical analysis, cf. Eq. (13), implies that the background term in the FSS relation in Eq. (11) for the third cumulant is suppressed as $O(L^{-(3/\nu-3)})$ with $3/\nu - 3 = 0.66(7)$. Further $O(L^{-\omega})$ scaling corrections arising from the leading irrelevant RG perturbation are also expected, however, the available data are not sufficiently precise to reliably estimate the value of ω .

To verify that the universality class of the transitions does not change along the confinement-deconfinement transition line for weak disorder, we now present results obtained by fixing $K = 1.0$, performing a scan in q . Also in this case, the specific heat does not display a divergent behavior, and the first seemingly divergent cumulant is the third one, see Fig. 5. To extract the critical properties from these data we can follow the same strategy already adopted for the transition at $q = 0.015$, the only difference being that the scaling variable to be used in Eq. (11) is now

$$X = (q - q_c) L^{1/\nu} \quad (14)$$

instead of the definition reported in Eq. (11). The final result of this analysis are the estimates

$$q_c = 0.0219(3), \quad \nu = 0.84(8), \quad (15)$$

where the reported errors are largely dominated by systematic errors. The estimated value of the critical exponent ν is fully consistent with the one obtained previously, see Eq. (13), although significantly less accurate.

In Fig. 6 we show the scaling plot obtained by rescaling B_3 data using the optimal value q_c in Eq. (15) and the critical exponent $\nu = 0.82$ obtained at fixed $q = 0.015$. The results are substantially consistent, however we note that scaling

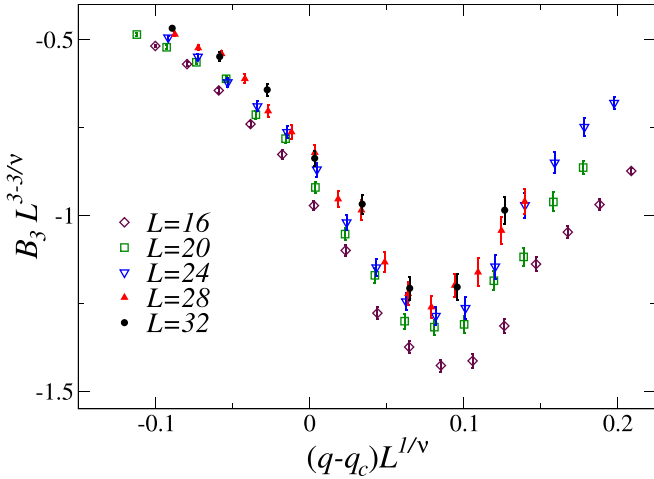


FIG. 6. Scaling of third cumulant B_3 for $K = 1$, obtained by plotting $L^{3-3/\nu} B_3$ versus $(q - q_c) L^{1/\nu}$ using our optimal estimates $q_c = 0.0219$ and the value $\nu = 0.82$ obtained in the $q = 0.015$ case.

corrections are much larger than those at fixed $q = 0.015$, see Fig. 4. This slower approach to the asymptotic critical behavior can be easily explained by the presence of larger scaling corrections, as generally expected when approaching the multicritical Nishimori point, where the critical behavior is expected to change. The qualitative behaviors of the B_3 curves obtained for $K = 1$ and $q = 0.015$ is the same, however, the presence of large scaling corrections for $K = 1$ makes it impossible to achieve a robust quantitative check that the asymptotic scaling curves associated with B_3 , cf. Eq. (11), are the same up to rescaling coefficients in the two cases.

In conclusion, we believe that our results provide a quite robust evidence that the topological transitions of the 3D lattice $\mathbb{Z}_2$ gauge model changes universality class in the presence of quenched uncorrelated disorder associated with the plaquette. Moreover, we note that the exponent ν of this new universality class significantly differs from that of the ferromagnetic transitions in 3D random-bond Ising models, for which [7] $\nu_{\text{rdi}} = 0.683(2)$.

IV. CONCLUSIONS

We have investigated the effects of quenched disorder at finite-temperature topological transitions, focusing on the 3D $\mathbb{Z}_2$ gauge model in the presence of an uncorrelated quenched disorder associated with the plaquettes of the lattice (RPGM). This is a paradigmatic model for systems undergoing finite-temperature topological transitions [41–44]; indeed, its confinement-deconfinement transitions are not driven by any local order parameter. We report a numerical analysis of their critical behaviors for weak disorder, and for this purpose we focus on the scaling behavior of the gauge-invariant energy cumulants defined in Eq. (6). The numerical investigation has been carried out using the binomial distribution for disorder variables, but it is natural to expect that the universality class of the results is independent of the disorder distribution, as it happens for the Ising model with quenched disorder, see, e.g., Ref. [8].

We show that the topological transitions of the 3D RPGM belong to a topological universality class different from that of the pure system, in agreement with the Harris criterium for the relevance of uncorrelated quenched disorder when the pure system undergoes a continuous transition with positive specific-heat critical exponent. Indeed, the critical behavior in the presence of weak disorder turns out to be characterized by the length-scale critical exponent $\nu = 0.82(2)$, which is significantly larger than the Ising length-scale critical exponent $\nu_{\text{I}} \approx 0.630$ in the absence of disorder. Note that the same critical exponent is also expected to characterize the critical behavior of generic systems (e.g., with matter fields) which displays a $\mathbb{Z}_2$ topological transition, see Ref. [85] for the case of the plaquette-disordered $\mathbb{Z}_2$ gauge Higgs model.

The same universality class is expected to describe all the transitions happening for $q < q_N$ and $K < K_N$, where (q_N, K_N) is the intersection point of the deconfinement transition line and the Nishimori line, see Fig. 1. At this intersection point the RPGM is expected to develop a peculiar multicritical behavior, characterized by two relevant RG perturbations with RG dimensions $y_1 > y_2 > 0$. Analogously to the multicritical behavior at the Nishimori point of the 2D and 3D random-bond Ising models [6,9,33,34], we expect that the multicritical behavior of RPGM is characterized by two relevant RG perturbations associated with the scaling fields $u_1 = q - q_N$, whose corresponding line $u_1 = 0$ runs parallel to the transition line at the Nishimori point, and $u_2 = \tanh K + 2q - 1$, with corresponding RG dimensions y_1 and y_2 , respectively, so that $y_1 > y_2 > 0$. Unfortunately, this multicritical behavior cannot be investigated using the techniques adopted in the present study. This is due to the fact that the energy density, and all the energy cumulants, along the Nishimori line are analytic functions; indeed, the exact result [46]

$$\frac{1}{V} [\langle \mathcal{H} \rangle]_w = 3 - 6q \quad (16)$$

holds along the Nishimori line (5) (see also analogous behaviors [2,6] for the random-bond Ising model). Therefore, the scaling part of the cumulants vanishes along the Nishimori line, therefore to study the singular behavior of the free energy, one cannot move along the Nishimori line. This makes an investigation of the multicritical behavior at the Nishimori point extremely challenging from the computational point of view, since it requires the independent tuning of two different parameters (i.e., both K and q) to approach the unknown position of the multicritical point. We also note that the critical behaviors along the deconfinement-confinement transition line below the Nishimori point, see Fig. 1, is expected to belong to a different universality class, as it occurs for other systems with quenched disorder, see, e.g., Refs. [28,31,35].

We finally discuss possible extensions of this study to generic lattice $\mathbb{Z}_N$ gauge models with $N > 2$ in the presence of quenched disorder associated with the plaquette, which are simply obtained by replacing the $\mathbb{Z}_2$ link variable $\sigma_{x,v}$ in the plaquette operator (2) with $\lambda_{x,v} = \exp(i2\pi n_{x,v}/N)$ with $n_{x,v} = 0, 1, \dots, N - 1$, and using the real part of the plaquette operator in the Hamiltonian (for $N = 2$ we recover the $\mathbb{Z}_2$ gauge model). We recall that, in the absence of quenched disorder, therefore for $q = 0$ in Eq. (3), 3D lattice $\mathbb{Z}_N$ gauge models undergo finite-temperature topological transitions for

any N . For $N > 4$ the transitions are continuous and belong to the inverted XY (IXY) universality class [44,73,80,86]. For $N = 4$ the transition is continuous and belongs to the Ising universality class because its partition functions can be written as the square of the partition function of the lattice $\mathbb{Z}_2$ gauge model (note however that this holds only for the specific plaquette Hamiltonian that we consider [44]). Finally the transition is first order for $N = 3$. Therefore, since the specific-heat critical exponent of the IXY universality class is negative [40,44] (it is equal to that of the duality related XY universality class, $\alpha \approx -0.015$), according to the Harris criterium we expect that a weak quenched disorder is irrelevant for $N > 4$, and the topological transitions still belong to the IXY universality class. On the other hand, the quenched disorder must be relevant for $N = 4$ if the plaquette action is used, because its specific heat exponent is positive. Indeed, analogously to the pure case [83,87], one can easily prove that for each disorder configuration, the partition function of the $N = 4$ model satisfies the identity

$$Z_{N=4,w}(K) = Z_{N=2,w}(K/2)^2. \quad (17)$$

The corresponding free-energy density (4) in the presence of disorder is thus equivalent to that of the 3D lattice $\mathbb{Z}_2$ gauge model. Therefore, we expect to observe continuous transitions analogous to those found in the 3D lattice $\mathbb{Z}_2$ gauge model with quenched disorder, apart from a trivial change of the coupling K according to the identity (17). On the other hand, the situation is less clear for $N = 3$: the first-order transitions of pure systems are generically expected to be smoothed by the quenched disorder (see, e.g., [88]), and may give rise to continuous transitions belonging to a new universality class.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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