

Designing topological cluster synchronization patterns with the Dirac operator

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Designing stable cluster synchronization patterns is a fundamental challenge in nonlinear dynamics of networks with great relevance to understanding neuronal and brain dynamics. So far, cluster synchronization has been studied exclusively in a node-based dynamical approach, according to which oscillators are associated only with the nodes of the network. Here we propose a topological synchronization dynamics model based on the use of the topological Dirac operator, which allows us to design cluster synchronization patterns for topological oscillators associated with both nodes and edges of a network. In particular, by modulating the ground state of the free energy associated with the dynamical model, we construct topological cluster synchronization patterns. These are aligned with the eigenstates of the topological Dirac equation that provide a very useful decomposition of the dynamical state of node and edge signals associated with the network. We use linear stability analysis to predict the stability of the topological cluster synchronization patterns and provide numerical evidence of the ability to design several stable topological cluster synchronization states on real connectome data, random graphs, and on stochastic block models.

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I. INTRODUCTION

Designing [1], controlling [2,3], and engineering [4–8] stable cluster synchronization patterns in networks are topics of crucial importance for a wide variety of systems, from chemical oscillators [9] to neural and brain networks [10,11]. Cluster synchronization on simple [12–14] and generalized network structures [15,16] defines a state in which different clusters of nodes of a network oscillate at different frequencies, often reflecting the underlying symmetries of the network. The current approaches to achieve [1,17] or destabilize [18] cluster synchronization rely on symmetries and network fibrations [19–21], network topology [22], time delays, edge weights [23], and phase lags [6,7].

All the approaches proposed so far exclusively address node-based cluster synchronization, i.e., associate the oscillators exclusively with the nodes of the network. In this work, we propose a topological dynamical model, called Dirac-Equation Synchronization Dynamics (DESD) to design patterns of cluster synchronization of topological signals given by oscillators associated with both nodes and edges of the network. We investigate such patterns on real connectomes, random graphs, and block models, and we provide a theoretical framework for predicting their stability.

The emerging field of higher-order topological dynamics [24,25] investigates the collective phenomena involving topological signals in networks and simplicial complexes with

drastically different properties with respect to the properties of the corresponding node dynamics. This field, combining algebraic topology [25] and nonlinear dynamics, is transforming our understanding of synchronized dynamical states [26–31] and higher-order diffusion [32–34] and has great potential for brain research [35,36] and AI [37–40].

Broadly speaking, topological signals are dynamical variables defined on nodes, edges, and even higher-dimensional simplices of networks and simplicial complexes. Examples of edge signals are ubiquitous and range from synaptic signals and edge signals among brain regions [41] to biological fluxes [42] or currents at different locations of the ocean [43].

Topological signals are naturally coupled to each other through the topological Dirac operator [44]. This operator has its roots in the Kogut and Susskind staggered fermions [45] defined on lattices and is receiving increasing attention in the study of nonlinear dynamics [43,46–51] and the development of AI algorithms [37–40,52–54].

In particular, the topological Dirac operator leads to the topological Dirac equation [44] that acts on the topological spinor given by the direct sum of node and edge signals. The eigenstates offer a very powerful spectral decomposition of node and edge signals that can be exploited by very advanced AI algorithms [38,39].

In higher-order nonlinear topological dynamics, the topological Kuramoto model [26,27] and the topological Dirac synchronization [43,46] are fundamental Kuramoto-like dynamical processes that lead to synchronization of topological signals. The topological Kuramoto model on a network captures the synchronization of edge signals and it displays a synchronized state aligned along the harmonic eigenvector of the 1-Hodge Laplacian. The topological Dirac synchronization [43,46] nontrivially couples node and edge signals associated with the network under study. Here we propose the DESD that is formulated starting from a free energy whose fundamental state is designed in order to achieve cluster

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synchronization aligned with given eigenstates of the topological Dirac equation [44]. These eigenstates are defined on both nodes and edges and lead to the topological cluster synchronization state of DESD in which the node and edge frequencies are determined by the eigenstates of the topological Dirac equation. Therefore, the DESD allows the design of the topological cluster synchronization patterns by modifying the coupling terms among the phase oscillators [55] without changing the network topology. This procedure is common in physical approaches to optimization. In these approaches, the ground state of the Hamiltonian represents the solution of the optimization problem that one desires to solve, and by minimizing the Hamiltonian one obtains the optimized solution. For instance, this procedure is common to both simulated annealing [56] and quantum annealing [57]. Similarly, here the coupling term between the phases is derived from the free energy whose fundamental state encodes the topological cluster synchronization patterns that we wish to design. In this way, the proposed approach is able to induce different topological cluster synchronization patterns on the same network by simply changing the parameters of the dynamical model.

Here, using linear stability analysis [58–61] and adapting it for the study of topological signals, we establish the conditions under which the topological cluster synchronization patterns are stable. Moreover, we demonstrate the ability of DESD to induce several stable topological cluster synchronization patterns on the same network without changing its topology and symmetries.

We found that DESD leads to stable patterns of the topological cluster synchronization if the eigenstates of the topological Dirac operator are isolated. These states can be found, for instance, in random graphs and in stochastic block models (SBM). In particular, on the SBM we show that we can exploit the important relation between the spectral properties of these networks and their community structure [62–65] to induce different patterns of cluster synchronization of nodes and edges that correlate with the modular structure of the network. In general, these states can also be related to network symmetries [66] through the spectral properties of the networks. As an illustration of the wide applicability of this approach, we studied the DESD on a real connectome derived from diffusion-weighted magnetic resonance imaging (MRI) [67], demonstrating that this approach can be used to reveal the left-right hemispheric separation as well as anterior-posterior differentiation.

Note that this research is based on higher-order topological dynamics [24,25], a research direction that is distinct from the field that aims to detect edge currents with mechanisms similar to those in condensed matter [68–70]. However, it is not excluded that these two research directions could be related to each other in the future.

This paper is structured as follows: In Sec. II we provide a motivation of our work and discuss the relevance of our approach to design topological cluster synchronization in real data, discussing in particular the application of the DESD dynamics to study a brain connectome dataset. In Sec. III we provide the necessary background on the topological Dirac operator and on the Kuramoto-like Dirac topological synchronization defined on nodes and edges of a network. In Sec. IV we discuss the major properties of the topological Dirac

equation and its eigenstates. In Sec. V we propose DESD which is able to design the patterns of topological cluster synchronization. We provide a linear stability analysis to assess the stability of these patterns and provide numerical evidence of topological cluster synchronization on random graphs and on SBM. Finally, in Sec. VI we provide the concluding remarks. The paper is accompanied by a set of Appendixes that provide the details of the linear stability analysis.

II. OVERVIEW OF THE DIRAC EQUATION SYNCHRONIZATION DYNAMICS

In this work we embrace a topological approach to network dynamics [24] that associates dynamical variables, also called topological signals, to both nodes and edges of a network (see Fig. 1). Anticipating the mathematical background that will be explained in detail in Sec. III we encode the topological signals of a network in a topological spinor $\Psi = (\theta, \phi)^T$, where the node signals and the edge signals are captured by the vectors $\theta \in \mathbb{R}^{N_0}$ and $\phi \in \mathbb{R}^{N_1}$, respectively, where N_0 is the number of the nodes of the network and N_1 is the number of the edges.

Examples of topological signals are ubiquitous as the edge signals can generally be used to treat currents and flows along the edges, as exemplified for instance by the classical example of an electric circuit in which one associates a voltage to each node and a current to each edge of the network. More recently, however, edge signals have been shown to undergo collective phenomena revealing important topological aspects of higher-order network dynamics [24,26] and they have been investigated in many different interdisciplinary contexts, including brain research [41,71], and topological AI algorithms [42,72].

Note that the topological approach to network dynamics that we will adopt in this manuscript cannot be directly reduced to dynamics on the line graph of the network [73,74], where nodes represent the edges of the original network. The reasons for this are twofold. First, in the topological approach the dynamics is defined at the same time on both nodes and edges, while the use of the line graph would require to define the dynamics exclusively on the edges of the original network. Second, the adoption of the topological approach ensures that the topology of the networks is fully taken into account and that the approach can be extended safely to higher-order networks (simplicial complexes). In particular, the topological approach preserves the notion of harmonic modes and their connection to the Betti numbers of the network, which would otherwise not be accounted for correctly, as extensively discussed in Ref. [72].

In this work we will define the DESD that will allow us to design topological cluster synchronization on both nodes and edges of a network (for codes see Repository [75]). This topological cluster synchronization state is designed in such a way to be aligned to isolated eigenstates of the topological Dirac equation [44] which are defined on both nodes and edges of the network and reveal its modular structure and symmetries. Before we discuss in detail the theoretical framework that will allow us to draw different topological cluster synchronization states without modifying the network structure, let us motivate

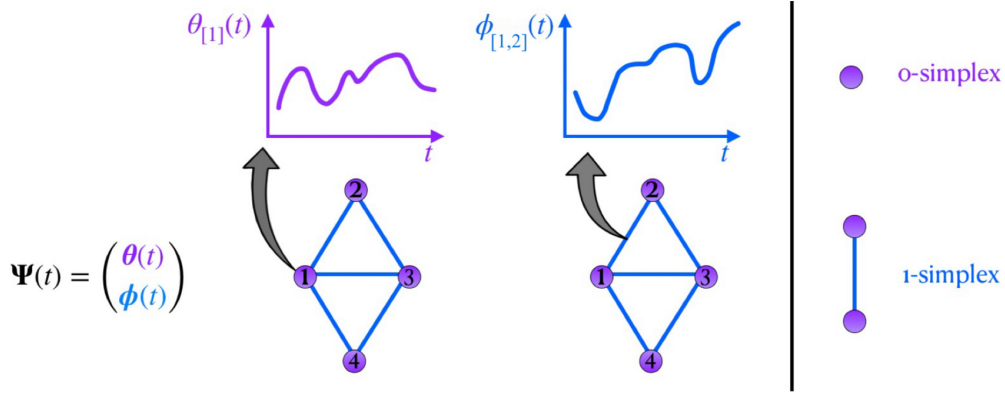


FIG. 1. Schematic representation of a network and its dynamical state encoded in the topological spinor Ψ comprising the node signals $\theta \in \mathbb{R}^{N_0}$ and the edge signals $\phi \in \mathbb{R}^{N_1}$. Here the node signal associated with node i (0-simplex) of the network is indicated with $\theta_{[i]}(t)$, while the edge signal associated with each edge $\ell = [i, j]$ (1-simplex) of the network is indicated with $\phi_\ell(t)$.

our research question by illustrating a direct application of our theoretical framework to a real-world scenario.

Here we demonstrate the applicability of the DESD to brain research by investigating this dynamical process on the structural connectivity of a human brain network derived from diffusion-weighted MRI [67] (Fig. 2). The isolated eigenstates of the topological Dirac equation, represented by the color code of node and edges in the network visualizations shown in Figs. 2(a) and 2(b) and Figs. 2(e) and 2(f), naturally reveal macroscale partitions that correspond to well-

established functional subdivisions: most prominently, a left-right hemispheric separation and an anterior-posterior differentiation. When the DESD is designed in such a way as to recover a topological cluster synchronization state aligned along these eigenstates, the frequencies of the node and edge signals, given by $\dot{\theta}$ and $\dot{\phi}$ respectively, induce topological patterns of cluster synchronization aligned with these anatomical modules, illustrating how structural topology constrains coherent functional activity. This is evident from the clear proportionality between the frequencies of nodes $\dot{\theta}_i$

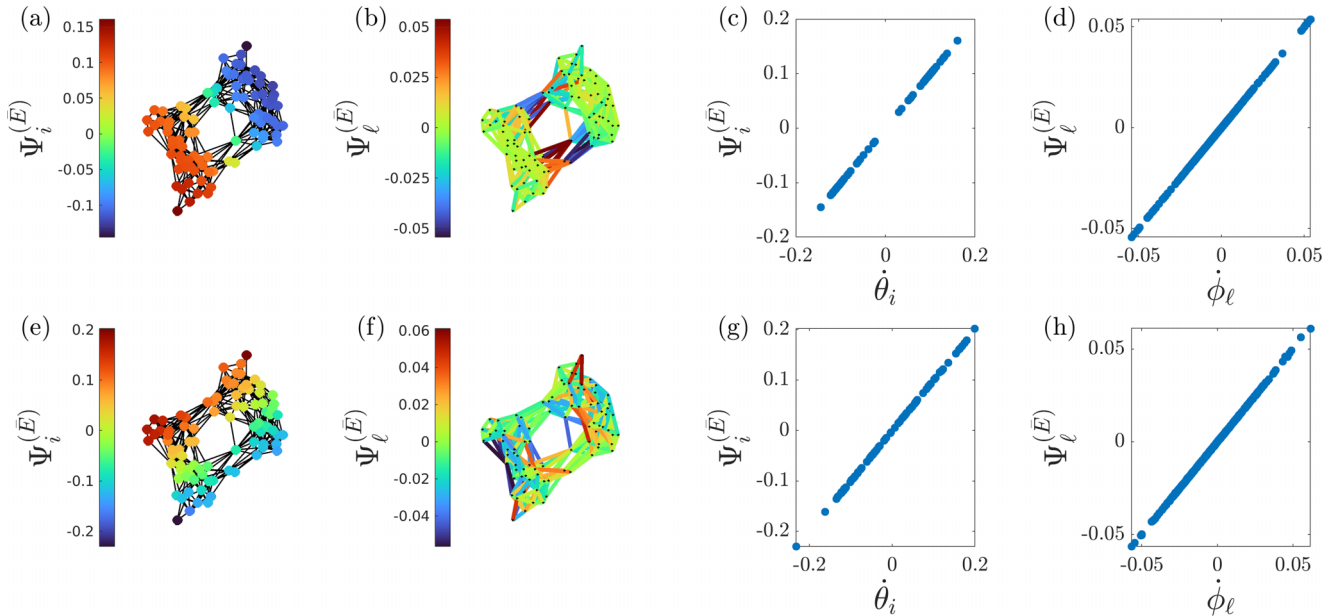


FIG. 2. Illustration of the application of DESD to the brain structural connectome obtained by diffusion-weighted MRI in Ref. [67]. On this network, DESD allows us to design two stable topological cluster synchronization patterns aligned along the first two isolated eigenstates of the topological Dirac equation $\bar{E} = 1.2392 \dots$ [(a)–(d)] and $\bar{E} = 1.5053 \dots$ [(e)–(h)] of the network. These two topological cluster synchronization patterns partition nodes [(a) and (e)] and edges [(b) and (f)] of the connectome revealing its symmetries and modular structure. In panels (a) and (e), the nodes are colored according to the associated component $\Psi_i^{(E)}$ of the selected eigenstate of the topological Dirac equation revealing in panel (a), the left (red) and right (blue) hemispheres, and in panel (e) the anterior (red) and posterior (blue) parts. In panels (b) and (f), the edges are colored according to the component $\Psi_\ell^{(E)}$ of the selected eigenstate of the topological Dirac equation revealing distinct fiber bundles mediating interactions between these functional modules. Panels (c), (d), (g), and (h) show that the node frequencies $\dot{\theta}_i$ and edge frequencies $\dot{\phi}_\ell$ associated with the DESD are proportional to the components $\Psi_i^{(E)}$ and $\Psi_\ell^{(E)}$ of the selected eigenstate of the topological Dirac equation. The simulation of DESD is performed for a mass $m = 1$ and a coupling constant $\sigma = 15$.

and edges ϕ_ℓ and, respectively, the component of the eigenstate of the topological Dirac equation defined on nodes $\Psi_i^{(E)}$ and edges $\Psi_\ell^{(E)}$ of the network obtained for a sufficiently large coupling constant of the DESD [Figs. 2(c) and 2(d) and Figs. 2(g) and 2(h)]. Thus, when the DESD designs the topological cluster synchronization along the respective eigenstate of the topological Dirac equation, the dynamics on both nodes and edges reveal the left and right hemispheres and the segmentation between the anterior and posterior regions of the brain. These results show that the DESD provides a natural bridge between anatomical connectivity and emergent functional organization, with cluster synchronization patterns arising coherently in both nodes and edges—something not captured by classical node-based dynamical models. Specifically, DESD identifies physically interpretable synchronization modes linking structural and functional brain modularity, thus addressing the key challenge of bridging topology and dynamics in real neural systems.

We note that DESD allows the design of multiple topological cluster synchronization states without changing the topology or the symmetries of the network. Interestingly, this phenomenology is also achieved without changing the geometry of the network which is naturally captured by the edge weights. Recently, weights have been identified as a powerful tool to modify the dynamical properties of a network or a simplicial complex in the context of node-based cluster synchronization [23], global (topological) synchronization [29,76], and diffusion [77] and random walks [78]. Here we embrace a purely topological approach and we adopt trivial edge weights equal to one on every edge. The rich phenomenology that we observe can be further extended including in the definition of the DESD nontrivial edge weights according to the algebraic topology approach to network dynamics [77], thereby exploring the effect of network geometry in DESD. However, this extension is beyond the scope of this work and will be addressed in future investigations.

Having now provided an overview of DESD and having demonstrated its applicability to real brain networks, in the rest of the manuscript we will adopt a comprehensive approach to DESD providing the algebraic topology and dynamical system background to this theoretical framework and investigating the stability of its associated topological cluster synchronization dynamics.

III. BACKGROUND ON DIRAC TOPOLOGICAL SYNCHRONIZATION

A. The topological Dirac operator of a network

We consider a network $G = (E, V)$ consisting of a set of N_0 nodes $V = \{1, 2, \dots, i, \dots, N_0\}$ and a set E of N_1 edges $E = \{1, 2, \dots, \ell, \dots, N_1\}$. Here and in the following, we will indicate with $\mathcal{N}$ the total number of simplices in the network (nodes and edges) and with $r \in \{1, 2, \dots, \mathcal{N}\}$ the generic simplex. In the field of higher-order topological dynamics [24,25] the dynamical state of the network (see Fig. 1) is encoded by the *topological spinor* $\Psi \in \mathbb{R}^{\mathcal{N}}$ given by the direct sum of the

node signal $\theta \in \mathbb{R}^{N_0}$ and the edge signal $\phi \in \mathbb{R}^{N_1}$, i.e.,

$$\Psi = \begin{pmatrix} \theta \\ \phi \end{pmatrix}. \quad (1)$$

The topological Dirac operator $\mathbf{D}$ [44] is a fundamental algebraic topology operator for performing discrete exterior calculus on the topological spinor. It is defined as

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{0} \end{pmatrix},$$

where the incidence matrix $\mathbf{B}$ is an $N_0 \times N_1$ matrix of elements

$$B_{i\ell} = \begin{cases} 1 & \text{if } \ell = [j, i], j < i \\ -1 & \text{if } \ell = [i, j], i < j \\ 0 & \text{otherwise.} \end{cases}$$

Note that the incidence matrix $\mathbf{B}$ is a matrix representation of the boundary operator, mapping each edge to its end nodes. The Dirac acts on the topological spinor as

$$\mathbf{D}\Psi = \begin{pmatrix} \mathbf{B}\phi \\ \mathbf{B}^\top\theta \end{pmatrix}. \quad (2)$$

Thus, it projects the node signals onto the edge signals and the edge signals onto the node signals thereby allowing crosstalk between them. Let us now discuss in more detail the nature of this transformation, which is deeply rooted in discrete exterior calculus [24,25,79]. The projection of the node signals onto edge signals $\mathbf{B}^\top\theta \in \mathbb{R}^{N_1}$ is nothing other than the gradient of the node signals and has elements

$$[\mathbf{B}^\top\theta]_\ell = \theta_j - \theta_i, \quad (3)$$

given by the difference in the value of θ at the two end nodes of the edge $\ell = [j, i]$. Moreover, the projection of the edge signals onto the node signals $\mathbf{B}\phi \in \mathbb{R}^{N_0}$ is given by the divergence of the edge signals and has elements

$$[\mathbf{B}\phi]_i = \sum_{\ell=[j,i]} \phi_\ell - \sum_{\ell=[i,j]} \phi_\ell \quad (4)$$

expressing the difference between the flux in and the flux out of node i .

A key property of the Dirac operator [44] is that it can be treated as the "square root" of the Laplacian. Indeed, we have $\mathbf{D}^2 = \mathcal{L}$, where $\mathcal{L}$ is the Gauss-Bonnet Laplacian given by

$$\mathcal{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix}, \quad (5)$$

where

$$\mathbf{L}_{[0]} = \mathbf{B}\mathbf{B}^\top \quad \mathbf{L}_{[1]} = \mathbf{B}^\top\mathbf{B}. \quad (6)$$

Here $\mathbf{L}_{[0]}$ is the famous graph Laplacian of the network, describing diffusion from node to node passing through edges, while $\mathbf{L}_{[1]}$ is the 1-Hodge Laplacian describing the diffusion from edge to edge passing through nodes. Note that here and in the following we indicate with $\lambda > 0$ the singular value of $\mathbf{B}$ and with $\mathbf{u}_\lambda, \mathbf{v}_\lambda$ its corresponding left and right singular vectors. Given the definition [Eq. (6)] of the graph Laplacian $\mathbf{L}_{[0]}$ and the 1-Hodge Laplacian $\mathbf{L}_{[1]}$, it follows that $\mathbf{L}_{[0]}$ and $\mathbf{L}_{[1]}$ are isospectral, i.e., they have the same nonzero eigenvalues $\mu = \lambda^2$, but generally have different degeneracies

for the harmonic eigenvalues, given by the Betti numbers β_0 and β_1 , respectively. The n th Betti number is a topological invariant equal to the number of n -dimensional holes in the network. In particular, β_0 is equal to the number of connected components and β_1 is equal to one-dimensional holes (i.e., cycles). In a connected network, we thus have $\beta_0 = 1$ and $\beta_1 = N_1 - (N_0 - 1)$.

Given that $\mathbf{D}^2 = \mathcal{L}$, it can be shown [37,44] that the eigenvalues Λ of $\mathbf{D}$ are given by

$$\Lambda = \pm\lambda = \pm\sqrt{\mu}, \quad (7)$$

where μ is the generic eigenvalue of $\mathbf{L}_{[0]}$. Additionally, the eigenvectors of $\mathbf{D}$ can be encoded in the matrix:

$$(\mathbf{W}^- \quad \mathbf{W}_{\text{harm}}^- \quad \mathbf{W}_{\text{harm}}^+ \quad \mathbf{W}^+). \quad (8)$$

Here $\mathbf{W}^\pm$ are the matrices associated with the eigenvectors $\Psi^{(\Lambda)}$ with eigenvalues $\Lambda > 0$ and eigenvalues $\Lambda < 0$, respectively. These eigenvectors can be explicitly expressed as

$$\begin{aligned} \Psi^{(\Lambda)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{u}_\lambda \\ \mathbf{v}_\lambda \end{pmatrix} \quad \text{for } \Lambda > 0, \\ \Psi^{(\Lambda)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{u}_\lambda \\ -\mathbf{v}_\lambda \end{pmatrix} \quad \text{for } \Lambda < 0, \end{aligned} \quad (9)$$

where $\mathbf{u}_\lambda$ and $\mathbf{v}_\lambda$ are the left and right singular vectors of the boundary operator $\mathbf{B}$ associated with the singular value λ , and $\mathcal{C}^\pm$ are normalization constants.

The matrices $\mathbf{W}_{\text{harm}}^\pm$ encode the harmonic eigenvectors associated with $\Lambda = 0$. These eigenvectors are $\beta_0 + \beta_1$ degenerate and there is a basis in which they have the following structure:

$$\begin{aligned} \Psi_N^{(0)} &= \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{0} \end{pmatrix}, \quad \text{with degeneracy } \beta_0, \\ \Psi_L^{(0)} &= \begin{pmatrix} \mathbf{0} \\ \mathbf{v}_0 \end{pmatrix}, \quad \text{with degeneracy } \beta_1. \end{aligned} \quad (10)$$

As previously mentioned, in a connected network $\beta_0 = 1$ and its corresponding eigenvector $\mathbf{u}_0 \propto \mathbf{1}_{N_0}$ is constant over each node of the network, while $\mathbf{v}_0$ is the generic harmonic eigenvector on the edges (harmonic eigenvectors of $\mathbf{L}_{[1]}$). The eigenvectors $\mathbf{v}_0$ are as many as the independent cycles [i.e., $\beta_1 = N_1 - (N_0 - 1)$ degenerate], and there is a basis in which each of them localizes around a single independent cycle of the network.

B. Dirac topological synchronization

Topological synchronization is a collective synchronization phenomenon that involves topological signals. Thus, on a simple network it involves both node and edge signals. Here we consider topological synchronization of Kuramoto-like [80–84] models involving nonidentical oscillators of phases θ and ϕ associated with the nodes and edges of the network, respectively. Thus, the dynamical state of the network is encoded in the topological spinor defined in Eq. (1), and the node and edge topological oscillators are coupled via the Dirac operator. In the absence of interactions ($\sigma = 0$), the dynamical equations describe nonidentical oscillators associated with nodes and edges, each oscillating with a heterogeneous

intrinsic frequency. Specifically, we have

$$\frac{d\Psi}{dt} = \Omega = \begin{pmatrix} \omega \\ \hat{\omega} \end{pmatrix}, \quad (11)$$

where $\omega \in \mathbb{R}^{N_0}$ and $\hat{\omega} \in \mathbb{R}^{N_1}$ are the vectors of intrinsic frequencies associated with the nodes and edges, respectively. When interactions are turned on ($\sigma > 0$), the equations for the Dirac topological synchronization (DTS) are

$$\frac{d\Psi}{dt} = \Omega - \frac{\delta\mathcal{F}}{\delta\Psi}, \quad (12)$$

where $\mathcal{F}$, here called the *free energy*, encodes for the interactions among phases associated with nodes and edges. In its simplest form $\mathcal{F}$ is given by [24,30]

$$\mathcal{F} = -\sigma \mathbf{1}^\top \cos(\mathbf{D}\Psi), \quad (13)$$

where the cosine function is taken element-wise and $\mathbf{1}$ is the $\mathcal{N}$ -column vector whose elements are $1_r = 1$. With this choice of the free energy $\mathcal{F}$, we obtain the DTS equation of motion given by

$$\frac{d\Psi}{dt} = \Omega - \sigma \mathbf{D} \sin(\mathbf{D}\Psi), \quad (14)$$

where the sine function is taken element-wise. This system of equations can also be expressed as

$$\frac{d\theta}{dt} = \omega - \sigma \mathbf{B} \sin(\mathbf{B}^\top \theta), \quad (15)$$

$$\frac{d\phi}{dt} = \hat{\omega} - \sigma \mathbf{B}^\top \sin(\mathbf{B}\phi). \quad (16)$$

The first thing we notice from these equations is that, with this choice of free energy, the node signal θ and the edge signal ϕ are decoupled. Equation (15) for the node signals θ can be shown (see Ref. [26]) to be the standard Kuramoto model defined on the network G , while Eq. (16) for the edge signals ϕ is the topological synchronization proposed and studied in Refs. [26,27].

The fully synchronized state of DTS corresponds to the ground state of the free energy $\mathcal{F}$ given by Eq. (13) and satisfies

$$\mathbf{D}\Psi = \mathbf{0}. \quad (17)$$

This implies that $\Psi \in \ker(\mathbf{D}) = \ker(\mathcal{L})$ and thus, in a fully connected network, implies that Ψ is aligned along a linear combination of $\Psi_N^{(0)}$ and $\Psi_L^{(0)}$, which are given by Eq. (10). Assuming that neither θ nor ϕ are zeros, this implies that θ is the same on each node, i.e., $\theta = \mathbf{1}$ and ϕ is aligned along the harmonic eigenvectors of the network, i.e., it is localized along the cycles of the network.

Interestingly, the linearized dynamics of DTS Eq. (14), valid when $\mathbf{D}\Psi$ is small, is given by

$$\frac{d\Psi}{dt} = \Omega - \sigma \mathcal{L}\Psi, \quad (18)$$

which can also be written explicitly as

$$\frac{d\theta}{dt} = \omega - \sigma \mathbf{L}_{[0]}\theta, \quad (19)$$

$$\frac{d\phi}{dt} = \hat{\omega} - \sigma \mathbf{L}_{[1]}\phi. \quad (20)$$

The linearized Kuramoto model Eq. (19) has been studied on networks [58,59] and lattices [60,61] to investigate the stability of the fully synchronized state, revealing the role of a finite spectral dimension in determining whether the fully synchronized state of the nodes can be observed. Note, however, that the linear stability of the topological synchronization state of edge signals starting from Eq. (20) in the presence of a finite spectral dimension has not been investigated so far.

Going beyond the linear approximation regime, and embracing the full nonlinear nature of the DTS, we can investigate the behavior of the dynamics by separately studying the harmonic and nonharmonic component of Ψ . To this end, we observe that Ψ can be expressed in a unique way as

$$\Psi = \Psi^{\text{harm}} + \Psi^{(D)}, \quad (21)$$

where $\Psi^{\text{harm}} \in \ker(\mathbf{D})$ and $\Psi^{(D)} \in \text{im}(\mathbf{D})$ and that similarly the vector of intrinsic frequencies Ω can be decomposed into

$$\Omega = \Omega^{\text{harm}} + \Omega^{(D)}, \quad (22)$$

where $\Omega^{\text{harm}} \in \ker(\mathbf{D})$ and $\Omega^{(D)} \in \text{im}(\mathbf{D})$. By using this decomposition, it can be readily shown that the harmonic component Ψ^{harm} in the full nonlinear dynamics of DTS is decoupled and noninteracting and obeys

$$\frac{d\Psi^{\text{harm}}}{dt} = \Omega^{\text{harm}}. \quad (23)$$

Thus this harmonic component continues to oscillate unaffected by the nonlinear dynamics. In order to investigate the dynamics of the nonharmonic component of the signal, i.e., $\Psi^{(D)}$ we consider

$$\Theta = \mathbf{D}\Psi = \mathbf{D}\Psi^{(D)}. \quad (24)$$

By expressing the topological Dirac operator $\mathbf{D}$ explicitly, we can express the node and edge components of Θ as

$$\Theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{0} \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix}. \quad (25)$$

Thus, Θ is unaffected by the harmonic component of Ψ and comprises the projection of the edge signal onto the nodes, α , and the projection of the node signal onto the edges, β . Using Eq. (14) and the definition of Θ [Eq. (24)], we can easily obtain the dynamical equation of motion for Θ that are given by

$$\frac{d\Theta}{dt} = \mathbf{D}\Omega - \sigma \mathcal{L} \sin(\Theta), \quad (26)$$

or equivalently by

$$\begin{aligned} \frac{d\alpha}{dt} &= \mathbf{B}\hat{\omega} - \sigma \mathbf{L}_{[0]} \sin(\alpha), \\ \frac{d\beta}{dt} &= \mathbf{B}^\top \omega - \sigma \mathbf{L}_{[1]} \sin(\beta). \end{aligned} \quad (27)$$

From the study of the dynamics of Ψ^{harm} and $\Theta = \mathbf{D}\Psi^{(D)}$ the following physical understanding of the DTS dynamics emerges. The fully synchronized state of DTS is the state in which Θ freezes and for which only the dynamics of Ψ^{harm} remains described by Eq. (23). This occurs in general for high values of σ . Under these conditions, the node phases oscillate in unison as their corresponding harmonic eigenvector is $\mathbf{u}_0 \propto$

$\mathbf{1}$, while the edge phases oscillate by remaining aligned with the harmonic eigenvectors $\mathbf{v}_0$ and thus have dynamics that localize around one-dimensional holes (i.e., cycles). Thus, the localization of edge signals around holes can be interpreted as the dynamics learning an aspect of the network topology.

For evaluating how well the synchronized dynamics of the projected phases freeze, two order parameters have been proposed, R_α and R_β , given by

$$R_\alpha = \left| \frac{1}{N_0} \sum_{i=1}^{N_0} e^{i\alpha_i} \right|, \quad R_\beta = \left| \frac{1}{N_1} \sum_{\ell=1}^{N_1} e^{i\beta_\ell} \right|. \quad (28)$$

These order parameters display on random graphs a smooth behavior, with a continuous onset of the nonzero values occurring at $\sigma^* = 0$.

The study of the DTS opens very interesting perspectives on the dynamics of topological oscillators. In particular, here, we have shown that DTS leaves node and edge signals decoupled. Thus, an important problem is to define dynamics that couples oscillators placed on nodes and edges. In the framework of Kuramoto-like models, this has been the subject of growing activity and includes approaches leading to discontinuous phase transitions demonstrated by the behavior of R_α and R_β . These models directly modify the dynamical Eq. (14) for DTS and cannot be written in terms of Eq. (12), i.e., they are not associated with a free energy $\mathcal{F}$. The first approach proposes to modulate the coupling constant σ with the order parameters [26,27]. The second approach, also called topological Dirac synchronization [43,46,50], instead introduces a phase lag into the equation of motion which then reads

$$\frac{d\Psi}{dt} = \Omega - \sigma \hat{\mathbf{D}} \sin((\hat{\mathbf{D}} - z\gamma \hat{\mathbf{D}}^2)\Psi), \quad (29)$$

where $z \in (0, 1)$, where $\hat{\mathbf{D}}$ is a suitably defined normalized Dirac operator and the matrix γ is given by

$$\gamma = \begin{pmatrix} \mathbf{I}_{N_0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{N_1} \end{pmatrix}. \quad (30)$$

Until now, we have shown that in DTS the synchronization dynamics occurs along the harmonic eigenvectors of the topological Dirac operator $\mathbf{D}$ that are constant on the nodes and localize on edges around cycles. Thus, an important question emerges: Can we design a dynamical system that synchronizes along other network patterns?

To this end, we will introduce the DESD which will affect the coupled dynamics of node and edge phases and will allow us to design spatial patterns on the network along which we can localize the synchronized dynamics. This synchronization model will exploit the properties of the topological Dirac equation. Hence, before discussing DESD, we first introduce the basic properties of the topological Dirac equation.

IV. THE TOPOLOGICAL DIRAC EQUATION

The topological Dirac equation [44] is a dynamical equation for the topological spinor that gives rise to a relativistic dispersion relation of its energy states, much like the Dirac equations defined on the continuum. This equation extends the traditional approach for staggered fermions on the lattice [45] to any arbitrary network structure and has been recently shown

to have wide applications in AI algorithms, including signal processing [37,38] and graph neural network algorithms [39]. The topological Dirac equation is defined as

$$i \frac{d\Psi}{dt} = \mathcal{H}\Psi, \quad (31)$$

where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \mathbf{D} + m\boldsymbol{\gamma}. \quad (32)$$

Here $m \geq 0$ is a parameter called *the mass* and the gamma matrix $\boldsymbol{\gamma}$ is given by Eq. (30).

Note that according to this definition, the gamma matrix $\boldsymbol{\gamma}$ anticommutes with the Dirac operator, i.e., $\{\mathbf{D}, \boldsymbol{\gamma}\} = \mathbf{D}\boldsymbol{\gamma} + \boldsymbol{\gamma}\mathbf{D} = \mathbf{0}$. The solution of the topological Dirac equations is

$$\Psi = e^{-iEt} \Psi^{(E)}, \quad (33)$$

where $\Psi^{(E)}$ is the eigenstate of the topological Dirac equations associated with the energy E . Any eigenstate $\Psi^{(E)}$ is independent of time and satisfies

$$E\Psi^{(E)} = \mathcal{H}\Psi^{(E)} \quad (34)$$

or, equivalently,

$$(\mathcal{H} - E\mathbf{I})\Psi^{(E)} = \mathbf{0}. \quad (35)$$

Iterating this eigenstate equation twice and using $\{\mathbf{D}, \boldsymbol{\gamma}\} = \mathbf{0}$ we find that

$$E^2\Psi^{(E)} = \mathbf{D}^2\Psi^{(E)} + m^2\Psi^{(E)}. \quad (36)$$

This relation reveals that the eigenstates of the Dirac operators are eigenstates of the Gauss-Bonnet Laplacian $\mathcal{L} = \mathbf{D}^2$ associated with the eigenvalue $\mu = \lambda^2$ and that the energy E of the topological Dirac equations obeys the relativistic dispersion relation

$$E^2 = \lambda^2 + m^2. \quad (37)$$

From this relation we conclude that the role of the mass is to introduce a gap in the energy spectrum, as the energy values must have an absolute value greater than or equal to the mass, i.e., $|E| \geq m$. A typical energy spectrum of the topological Dirac equation is plotted in Fig. 3, from which it is apparent that the spectrum includes both positive and negative energy values and every positive energy state with $E > m$ admits a conjugated negative energy state with the same absolute value $|E|$. This symmetry corresponds to the charge conjugation symmetry of the continuous Dirac equation. However, in the discrete case of the topological Dirac equation, this symmetry is broken for the energy states of energy $|E| = m$ which correspond to the harmonic eigenstates of the network and thus have a degeneracy given by the Betti number β_0 (number of connected components) for $E = m$ and a degeneracy given by the Betti number β_1 (number of independent cycles) for $E = -m$.

The mass not only changes the energy spectrum of the topological Dirac equation but also significantly changes the properties of the eigenstates associated with the energy E . In order to see this, let us discuss the structure of the eigenstates associated with the energy E and the singular value λ of the incidence matrix. To this end, let us characterize the matrix of the eigenstates

$$(\hat{\mathbf{W}}^- \quad \hat{\mathbf{W}}_{\text{harm}}^- \quad \hat{\mathbf{W}}_{\text{harm}}^+ \quad \hat{\mathbf{W}}^+). \quad (38)$$

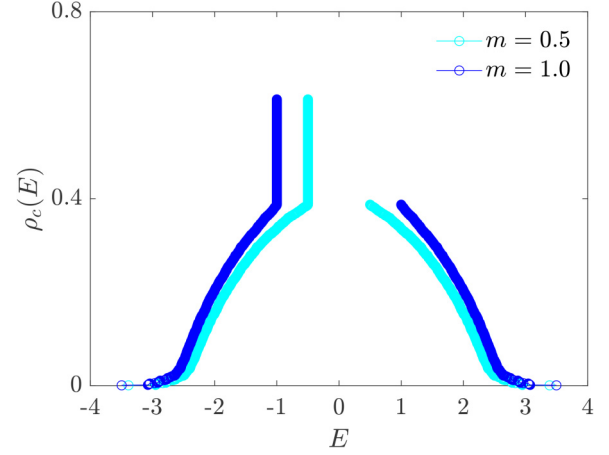


FIG. 3. The spectrum of the topological Dirac equations is described by the cumulative distribution of eigenstates $\rho_c(E)$ indicating for positive energy values E the number of eigenstates of energy E' with $E' \geq E$ and for negative energy values E the number of eigenstates of energy E' with $E' \leq E$. From this plot, two main observations can be made: (i) the nonzero mass m induces a spectral gap and (ii) the spectrum is symmetric with the only exception of the eigenstates of energy E with $|E| = m$. The data shown here is for a random network with $N_0 = 492$ nodes and average degree $c = 3$.

Here $\hat{\mathbf{W}}^\pm$ are the matrices associated with the eigenstate $\Psi^{(E)}$ with $E > m$ or $E < -m$, respectively (and $\lambda \neq 0$), which are given by

$$\begin{aligned} \Psi^{(E)} &= \mathcal{C}^+ \begin{pmatrix} \mathbf{u}_\lambda \\ \frac{\lambda}{|E|+m} \mathbf{v}_\lambda \end{pmatrix}, \quad \text{for } E > m \\ \Psi^{(E)} &= \mathcal{C}^- \begin{pmatrix} \frac{\lambda}{|E|+m} \mathbf{u}_\lambda \\ -\mathbf{v}_\lambda \end{pmatrix}, \quad \text{for } E < -m, \end{aligned} \quad (39)$$

where $\mathbf{u}_\lambda$ and $\mathbf{v}_\lambda$ are the left and right singular vectors of the boundary operator $\mathbf{B}$ associated with the singular value λ and $\mathcal{C}^\pm$ are normalization constants. Note that $\mathbf{u}_\lambda$ and $\mathbf{v}_\lambda$ are also eigenvectors of the graph Laplacian and of the 1-Hodge Laplacian respectively both associated with the eigenvalue λ^2 . Thus, these vectors reflect the symmetries of the network [66] and its community structure [62,64]. We note that the mass now enables the tuning of the relative normalization of the node and the edge signals, which can now have very different scales. Only for $m = 0$ do these eigenvectors reduce to the eigenvectors of the Dirac operator. The matrices $\hat{\mathbf{W}}_{\text{harm}}^\pm$ encode the harmonic eigenstates associated with $\lambda = 0$ and energies $E = \pm m$. These eigenvectors are independent of the value of the mass and are given by

$$\begin{aligned} \Psi^{(E)} &= \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{0} \end{pmatrix}, \quad \text{for } E = m \\ \Psi^{(E)} &= \begin{pmatrix} \mathbf{0} \\ \mathbf{v}_0 \end{pmatrix}, \quad \text{for } E = -m. \end{aligned}$$

Note that the degeneracy of the eigenvalue $E = m$ is equal to the degeneracy of the harmonic eigenvalue of $\mathbf{L}_{[0]}$ and is thus given by the zeroth Betti number β_0 , while the degeneracy of the eigenvalue $E = -m$ is equal to the degeneracy of the

harmonic eigenvalue of $\mathbf{L}_{[1]}$ and is thus given by the first Betti number β_1 .

V. DIRAC-EQUATION SYNCHRONIZATION DYNAMICS AND ITS STABILITY

A. Dirac-equation synchronization dynamics

The DESD allows us to design the synchronized state by aligning it to an eigenstate associated with the energy $\bar{E}$ of the topological Dirac equation. To this end, we consider the dynamics dictated by the equation of motion

$$\frac{d\Psi}{dt} = \Omega - \frac{\delta\mathcal{F}}{\delta\Psi}, \quad (40)$$

where the free energy $\mathcal{F}$ is now chosen to be

$$\mathcal{F} = -\sigma \mathbf{1}^\top \cos((\mathcal{H} - \bar{E}\mathbf{I})\Psi). \quad (41)$$

Notice that our choice of the free energy corresponds to the wish to design the synchronization pattern by imposing that the ground state of this free energy is given by the eigenstate associated with the energy $\bar{E}$ of the topological Dirac equation. Indeed, the free energy $\mathcal{F}$ is minimized for

$$(\mathcal{H} - \bar{E}\mathbf{I})\Psi = \mathbf{0}, \quad (42)$$

where here and in the following $\mathbf{I}$ is the identity matrix of linear size $\mathcal{N}$. Thus, the ground state of the free energy has topological spinors Ψ satisfying the eigenstate equation

$$\mathcal{H}\Psi = \bar{E}\Psi. \quad (43)$$

It follows that such a ground state only exists if $\bar{E}$ is an allowed energy state of the topological Dirac equation.

It is instructive at this point to provide the explicit expression for $\mathcal{H} - \bar{E}\mathbf{I}$

$$(\mathcal{H} - \bar{E}\mathbf{I}) = \begin{pmatrix} -(\bar{E} - m)\mathbf{I}_{N_0} & \mathbf{B} \\ \mathbf{B}^\top & -(\bar{E} + m)\mathbf{I}_{N_1} \end{pmatrix}, \quad (44)$$

revealing that this operator allows nontrivial cross-talk between nodes and edge signals. Inserting Eq. (41) into Eq. (40) we can thus derive the dynamical equations of motion of DESD which read

$$\frac{d\Psi}{dt} = \Omega - \sigma(\mathcal{H} - \bar{E}\mathbf{I})\sin((\mathcal{H} - \bar{E}\mathbf{I})\Psi). \quad (45)$$

Therefore, the DESD in general describes coupled dynamics of node and edge signals, although it still obeys Eq. (40). Note, however, that for $m = \bar{E} = 0$ we recover the DTS as Eq. (45) reduces to

$$\dot{\Psi} = \Omega - \sigma\mathbf{D}\sin(\mathbf{D}\Psi), \quad (46)$$

for which node and edge signals are decoupled. Here and in the following, we consider exclusively the case in which $\bar{E}$ is a nondegenerate energy state of the topological Dirac equation. Moreover the intrinsic frequencies Ω are given by

$$\Omega = \sum_E \Omega_E(t)\Psi^{(E)} \quad (47)$$

with Ω_E drawn from a standard Gaussian distribution for $E \neq \bar{E}$ while for $E = \bar{E}$ we put $\Omega_{\bar{E}} = 1$.

The DESD can be studied using a similar approach previously outlined for DTS. In particular, the phases described by the topological spinor Ψ can be uniquely decomposed into

$$\Psi = \Psi^{(G)} + \Psi^{(H)}, \quad (48)$$

where $\Psi^{(G)} \in \ker(\mathcal{H} - \bar{E}\mathbf{I})$ and $\Psi^{(H)} \in \text{im}(\mathcal{H} - \bar{E}\mathbf{I})$. Similarly, the intrinsic frequencies Ω can be decomposed into

$$\Omega = \Omega^{(G)} + \Omega^{(H)}, \quad (49)$$

where $\Omega^{(G)} \in \ker(\mathcal{H} - \bar{E}\mathbf{I})$ and $\Omega^{(H)} \in \text{im}(\mathcal{H} - \bar{E}\mathbf{I})$. From the dynamical equations of DESD given by Eq. (45), it follows immediately that the component $\Psi^{(G)}$, aligned with the ground state of $\mathcal{F}$, obeys a free, uncoupled dynamics

$$\frac{d\Psi^{(G)}}{dt} = \Omega^{(G)}. \quad (50)$$

Let us consider the expansion of Ψ over the eigenstates of the topological Dirac equation thus getting

$$\Psi = \sum_E c_E(t)\Psi^{(E)}, \quad (51)$$

with $c_E(t) = \Psi^\top \Psi^{(E)}$. Equation (50) can be expressed now as

$$\frac{dc_{\bar{E}}}{dt} = \Omega_{\bar{E}}. \quad (52)$$

Therefore, the component aligned with the ground state of $\mathcal{F}$ is uncoupled and describes the fact that the phases of the networks, aligned with the pattern extending on nodes and edges along the eigenstate $\Psi^{(\bar{E})}$, are free to oscillate at constant frequency $\Omega_{\bar{E}}$ for any value of the coupling constant σ . The dynamics aligned with the excited states of $\mathcal{F}$ is captured by $\Psi^{(H)}$ given by

$$\Psi^{(H)} = \sum_{E \neq \bar{E}} c_E(t)\Psi^{(E)}. \quad (53)$$

In order to study these dynamics we can consider the spinor

$$\Theta = (\mathcal{H} - \bar{E}\mathbf{I})\Psi = (\mathcal{H} - \bar{E}\mathbf{I})\Psi^{(H)} \quad (54)$$

with the node component α and the edge component β given by

$$\Theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -(\bar{E} - m)\mathbf{I}_{N_0} & \mathbf{B} \\ \mathbf{B}^\top & -(\bar{E} + m)\mathbf{I}_{N_1} \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix},$$

and eigenstate decomposition,

$$\Theta = \sum_{E \neq \bar{E}} (E - \bar{E})c_E(t)\Psi^{(E)}. \quad (55)$$

The equation for Θ reads

$$\frac{d\Theta}{dt} = (\mathcal{H} - \bar{E}\mathbf{I})\Omega - \sigma(\mathcal{H} - \bar{E}\mathbf{I})^2 \sin(\Theta), \quad (56)$$

which is consistent with a freezing of the dynamics for $c_E(t)$ with $E \neq \bar{E}$ for sufficiently large coupling constant σ and a sufficiently long time on any finite network. We therefore conclude that for DESD the fully synchronized dynamics is encoded in the free dynamics of the component $\Psi^{(G)}$ of the topological spinor aligned along the ground state of $\mathcal{F}$ and is reached when the rest of the signal leads to a frozen dynamics of Θ .

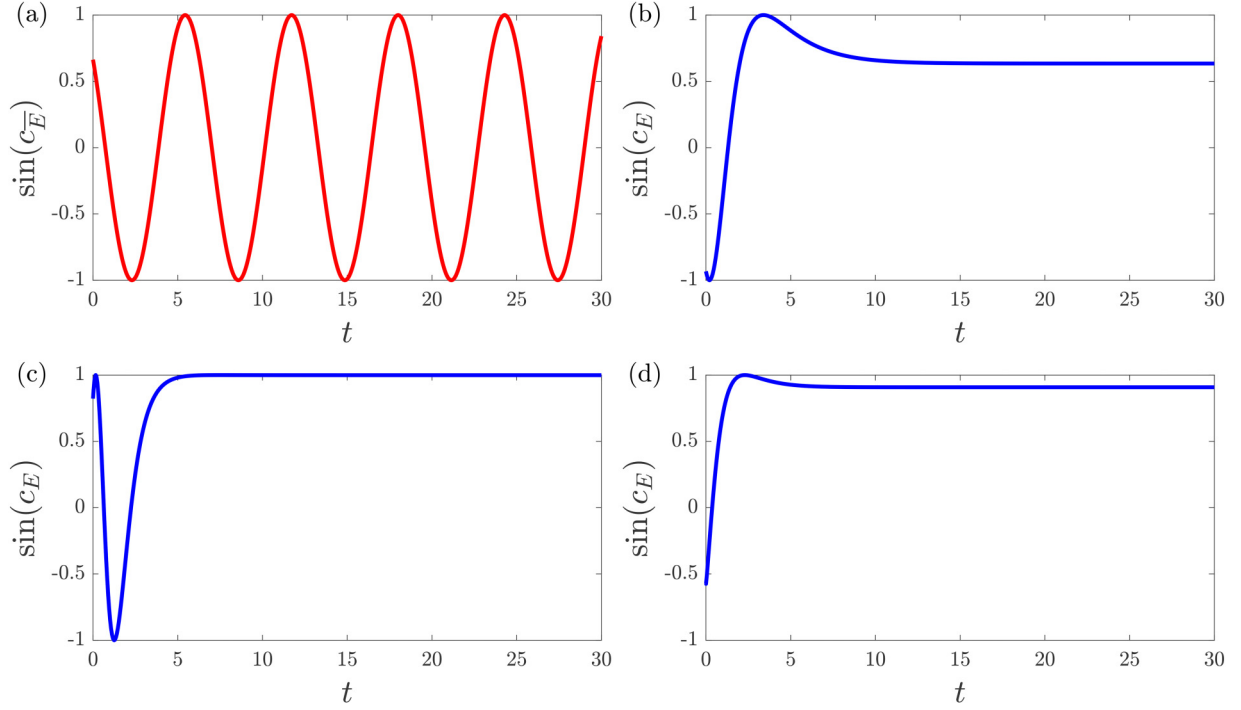


FIG. 4. The DESD along the isolated (gapped) eigenstate associated with energy state $\bar{E}$ is characterized by the eigenmode that oscillates freely, while the eigenmodes corresponding to the other energy states freeze asymptotically in time. Panel (a) shows the time series for $\sin(c_{\bar{E}}(t))$ for the eigenmode at energy $\bar{E} = 1.6642\dots$ corresponding to the isolated Fiedler singular value λ of the boundary operator of a Poisson network with $N_0 = 1000$ nodes and average degree $c = 12$. Panels (b)–(d) show the time series for $\sin(c_E(t))$ corresponding to the three eigenmodes associated with the energies E nearest to $\bar{E}$, namely (b) $E = 1.8484\dots$, (c) $E = 1.8797\dots$, and (d) $E = 1.8954\dots$ on the same network. The simulation of DESD is performed for a mass $m = 1$ and a coupling constant $\sigma = 15$.

In order to assess the extent to which the dynamics of Θ is frozen, we can monitor the average value of the free energy $\mathcal{F}$ together with the two order parameters R_α and R_β defined similarly to the DTS, i.e.,

$$R_\alpha = \frac{1}{N_0} \left| \sum_{i=1}^{N_0} e^{i\alpha_i} \right|, \quad R_\beta = \frac{1}{N_1} \left| \sum_{\ell=1}^{N_1} e^{i\beta_\ell} \right|. \quad (57)$$

To show numerical evidence for this behavior, in Fig. 4 we display the time-series of $\sin(c_E(t))$ for $E = \bar{E}$ and for $E \neq \bar{E}$ on a random graph for a large value of the coupling constant σ . We observe that $\sin(c_{\bar{E}}(t))$ displays a sustained oscillation with frequency $\Omega_{\bar{E}}$ while $\sin(c_E(t))$ with $E \neq \bar{E}$ saturate to a constant value after a transient time that is longer the closer the energy E is to $\bar{E}$. If we explore the DESD as a function of the coupling constant σ (see Fig. 5), by monitoring the order parameters R_α and R_β together with the free energy density $f = \mathcal{F}/(\sigma\mathcal{N})$, then we can appreciate that the synchronized state, for a large coupling constant, is increasingly aligned to the ground state of the free energy, i.e., to the eigenstate $\Psi^{(\bar{E})}$. This implies that the DESD does not occur uniformly over all nodes and edges of the networks but reflects the network patterns encoded by the eigenstate $\Psi^{(\bar{E})}$ leading to a topological cluster synchronization. Indeed, in the synchronized state we have

$$\Psi = \Psi^{(G)} + \Psi^{(H)} \simeq \Psi^{(G)}. \quad (58)$$

Thus, recalling that $\Psi^{(G)}$ can be expressed as

$$\Psi^{(G)} = c_E(t) \Psi^{(\bar{E})} \quad (59)$$

with

$$c_E(t) = \Omega_E t + c_E(0), \quad (60)$$

and that $\Psi^{(\bar{E})}$ is independent of time, then, by indicating with $\Psi_i^{(\bar{E})}$ and $\Psi_\ell^{(\bar{E})}$ the node and edge components of $\Psi^{(\bar{E})}$ respectively, we obtain

$$\frac{d\theta_i}{dt} = \Omega_{\bar{E}} \Psi_i^{(\bar{E})}, \quad \frac{d\phi_\ell}{dt} = \Omega_{\bar{E}} \Psi_\ell^{(\bar{E})}, \quad (61)$$

i.e., the frequencies of the node and edge signals are proportional to the associated components of the eigenstate $\Psi^{(\bar{E})}$. It follows that, while in the spectral domain only the eigenmode $\bar{E}$ oscillates, in the real space domain the oscillators placed on the nodes and edges might have different frequencies, with their value proportional to the components of the eigenstate $\Psi^{(\bar{E})}$. When the eigenstate $\Psi^{(\bar{E})}$ reveals a partition of the network into clusters, DESD leads to topological cluster synchronization defined on the nodes and edges of the network.

The formulation of DESD implies the ability to design the topological cluster synchronization pattern that extends to the nodes and edges of the network by tuning the values of $\bar{E}$. However, we note that not every eigenstate of the topological Dirac equation displays such a clear DESD. In particular, we observe that the most stable DESD are observed for eigenstates corresponding to gapped (or isolated) energy

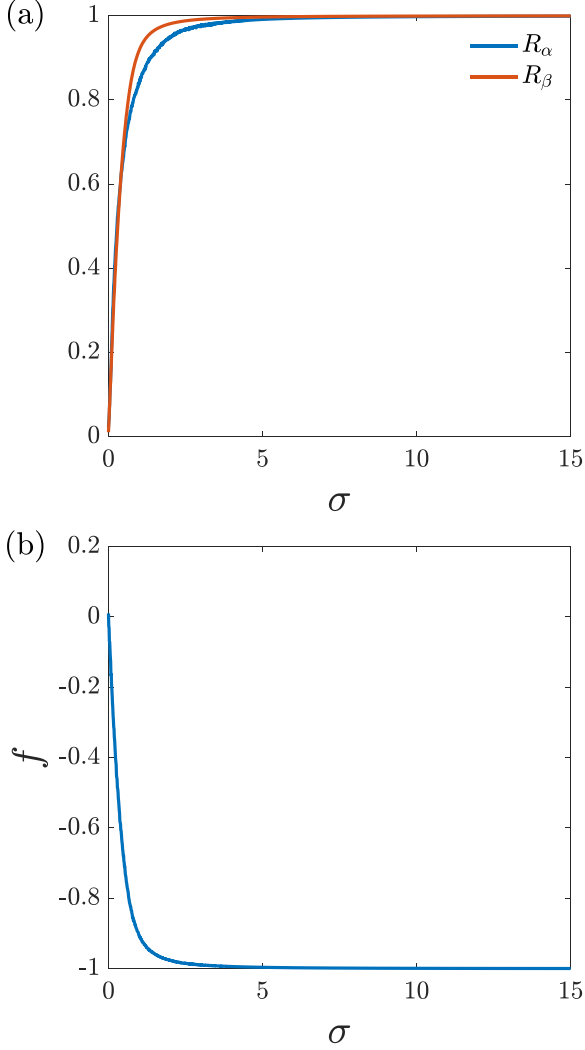


FIG. 5. The existence of a topological DESD is demonstrated by plotting in panel (a) the order parameters R_α and R_β and in panel (b) the free energy density f as functions of the coupling constant σ . The selected eigenstate is the Fiedler eigenstate, $\bar{E} = 1.6643$, of the same Poisson network studied in Fig. 4 and the mass is $m = 1$. The simulations are performed along the forward transition from random initial conditions. The equilibration time is $T_{\max} = 15$ with time steps of $dt = 0.001$. The values of R_α , R_β , and f were averaged over the final third of $T_{\max}$ for each σ .

states $\bar{E}$. In order to investigate this matter quantitatively, in the following we study the linear stability of the DESD.

B. Linear stability analysis and its numerical validation

In this section, we will investigate the stability of DESD along the generic eigenstate $\Psi^{(E)}$ associated with a nondegenerate energy E of the topological Dirac equation using an approach that generalizes previous approaches used for the standard (node-based) Kuramoto model on general networks [58,59] or on lattices [60,61]. By linearizing the DESD defined in Eq. (45) for $\|(\mathcal{H} - \bar{E}\mathbf{I})\Psi\| \ll 1$ we obtain

$$\frac{d\Psi}{dt} = \Omega - \sigma(\mathcal{H} - \bar{E}\mathbf{I})^2\Psi. \quad (62)$$

Let us expand Ψ and Ω over the eigenstates of the topological Dirac equation, thus getting

$$\begin{aligned} \Psi_r &= \sum_E c_E(t) \Psi_r^{(E)} \\ \Omega_r &= \sum_E \Omega_E(t) \Psi_r^{(E)}, \end{aligned} \quad (63)$$

where $c_E(t) = \Psi^\top \Psi^{(E)}$ and $\Omega_E = \Omega^\top \Psi^{(E)}$. Using Eq. (62) it can be readily shown that the amplitude $c_E(t)$ obeys the dynamical equation

$$\dot{c}_E(t) = \Omega_E - \sigma(E - \bar{E})^2 c_E(t). \quad (64)$$

It then follows that $c_E(t)$ is given by

$$\begin{aligned} c_E(t) &= \frac{\Omega_E}{\sigma(E - \bar{E})^2} (1 - e^{-\sigma(E - \bar{E})^2 t}) \\ &+ c_E(0) e^{-\sigma(E - \bar{E})^2 t} \end{aligned} \quad (65)$$

for $E \neq \bar{E}$, while for $E = \bar{E}$ the solution reads

$$c_{\bar{E}}(t) = \Omega_{\bar{E}} t + c_{\bar{E}}(0). \quad (66)$$

Thus, in this linear approximation, the eigenstate $E = \bar{E}$ is associated with a network pattern $\Psi^{(\bar{E})}$ that continues to drift unperturbed even at very large times, while the other modes corresponding to energies $E \neq \bar{E}$ are associated with an amplitude $c_E(t)$ that, for $t \rightarrow \infty$, saturate to a fixed value $\Omega_E / (\sigma(E - \bar{E})^2)$. Note, however, that this contribution diverges as $E \rightarrow \bar{E}$, thus if the spectral density of the eigenstates $\rho(E)$ is dense around $\bar{E}$, there is a possibility that the cumulative contributions to the phases due to the components associated with energies $E \neq \bar{E}$ diverge. When this happens, the dynamical synchronized state of DESD might not be stable in the thermodynamic limit, a feature that impedes its observability in real scenarios.

In order to study the contributions due to the component aligned with the eigenstates $E \neq \bar{E}$, we screen out the component aligned with the eigenstate associated with energy $\bar{E}$ and consider

$$\Phi = \Psi - (\Psi^\top \Psi^{(\bar{E})}) \Psi^{(\bar{E})}. \quad (67)$$

The dynamical properties of the linearized DESD can be investigated by evaluating: (i) the “roughness” W^2 associated with the phases Ψ and defined as

$$W^2 = \langle \|\Phi\|^2 \rangle = \frac{1}{\mathcal{N}} \left\langle \sum_{r=1}^{\mathcal{N}} \Phi_r^2 \right\rangle, \quad (68)$$

and (ii) the variance V^2 of the velocity of Φ

$$V^2 = \langle \|\dot{\Phi}\|^2 \rangle = \frac{1}{\mathcal{N}} \left\langle \sum_{r=1}^{\mathcal{N}} \dot{\Phi}_r^2 \right\rangle. \quad (69)$$

A diverging value of W^2 observed in the thermodynamic limit $\mathcal{N} \rightarrow \infty$ will indicate that the synchronized dynamics of DESD is unstable, while a value of V^2 converging to zero in the same limit will indicate that the phases are entrained.

In this work, we derive some important results that allow us to predict whether we are able to observe a stable synchronization state of the DESD, or not, depending on the properties

of the density of states of the topological Dirac equation. These results are briefly summarized here. For details on the derivations, we refer the reader to the Appendixes. In order to characterize the properties of the density of states of the topological Dirac equation for $E \simeq \bar{E}$, we define the right spectral gap Δ_+ and the left spectral gap Δ_- as

$$\begin{aligned}\Delta_+ &= \min_{E > \bar{E}} (E - \bar{E}), \\ \Delta_- &= \min_{E < \bar{E}} (\bar{E} - E).\end{aligned}\quad (70)$$

Thus, if $\lim_{\mathcal{N} \rightarrow \infty} \Delta_{\pm} = 0$ for either choice of $\pm$, or both, then the eigenstate sits in the bulk of the density of states of the topological Dirac equation; otherwise, the eigenstate is isolated (or gapped). As we shall discuss in detail in the following, if the eigenstate is isolated, then the corresponding synchronization pattern of DESD is stable. Our criteria for assessing the stability of a synchronized state along the eigenstate $\bar{E}$ depend on the value of the exponents $\delta_{\pm}$. The exponent δ_+ is defined as follows:

(1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ = 0$, then we assume that the density of eigenstates scales as

$$\rho(E) \propto (E - \bar{E})^{\delta_+} \quad (71)$$

for $0 < E - \bar{E} \ll 1$.

(2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ > 0$, then we put by definition $\delta_+ = \infty$.

The exponent δ_- is defined according to the following similar classification:

(1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- = 0$, then we assume that the density of eigenstates scales as

$$\rho(E) \propto (\bar{E} - E)^{\delta_-} \quad (72)$$

for $0 < \bar{E} - E \ll 1$.

(2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- > 0$, then we put by definition $\delta_- = \infty$.

We are now in a position to summarize the results (see Appendixes) obtained by the linear stability approach for the synchronized state of DESD in terms of the values of $\delta_{\pm}$:

(1) If both $\delta_+ = \infty$ and $\delta_- = \infty$, then the synchronized state of DESD is stable. In this case, we will have $W^2 \rightarrow \text{cost}$ and $V^2 \rightarrow 0$ as $\mathcal{N} \rightarrow \infty$.

(2) If either $\delta_+ \leq 3$ or $\delta_- \leq 3$, and if they are both greater than 1, then the phases are entrained, but the synchronized state of DESD is not thermodynamically stable. In this case, we will have $W^2 \rightarrow \infty$ and $V^2 \rightarrow 0$ as $\mathcal{N} \rightarrow \infty$.

(3) If either $\delta_+ \leq 1$ or $\delta_- \leq 1$, then the entrained state might not even exist. In this case, we will have $W^2 \rightarrow \infty$ and V^2 might not converge to zero as $\mathcal{N} \rightarrow \infty$ since the linear approximation fails.

In order to numerically validate these theoretical predictions, we considered DESD defined on a random network of $N_0 = 1000$ and average degree $c = 12$. By setting $m = 1$, we first numerically evaluated the cumulative density of states $\rho_c(E)$ [see Fig. 6(a)]. We have focused on two specific eigenstates of positive energy E . The first eigenstate of energy $\bar{E} = 1.6642 \dots$ corresponds to the Fiedler singular state of the boundary operator and is isolated from the rest of the spectrum, thus $\delta_{\pm} = \infty$. The second eigenstate of energy $\bar{E} = 3.5261 \dots$ is deep in the bulk of the spectrum and the

density of states is characterized by $\delta_- < 1$ [see Fig. 6(b)]. We study the stability of the DESD by considering the full nonlinear DESD starting from an initial condition that is a small perturbation to the topological synchronized state. In particular, we take

$$\Psi(0) = \Psi(\bar{E}) + \epsilon \frac{\mathbf{x}}{|\mathbf{x}|}, \quad (73)$$

where $x_r \sim \mathcal{N}(0, 1)$ and $\epsilon = 0.1$. In Figs. 6(c) and 6(d) we plot V as a function of σ and W as a function of time t for $\sigma = 10$ for the DESD designed along the eigenstates $\bar{E} = 1.6642 \dots$ and $\bar{E} = 3.5261 \dots$. Our results show that, consistently with our theory, for the first considered eigenstate, the phases are entrained for large values of σ , i.e., V is very small in the large time limit, and the synchronized state admits small fluctuations, corresponding to a small roughness W . However, for the second eigenstate we observe large fluctuations W around the topological synchronized state, which increase over time, indicating that the synchronized state of DESD is not stable for this choice of the eigenstate $\bar{E}$.

In Fig. 7, we provide further numerical evidence of the stability, or lack thereof, of DESD aligned with the two mentioned eigenstates of the topological Dirac equation corresponding to the Fiedler eigenstate $\bar{E} = 1.6642 \dots$ (stable) and the eigenstate in the bulk of the spectrum $\bar{E} = 3.5261 \dots$ (unstable). Starting from random initial conditions, we plot $\dot{c}_E(t)$ for positive values of the energy [Figs. 7(a) and 7(b)] and negative values of the energy [Figs. 7(c) and 7(d)]. We show that for both cases the eigenmode at energy $E = \bar{E}$ oscillates unperturbed with frequency $\dot{c}_{\bar{E}} = \Omega_{\bar{E}} = 1$ while the negative energy eigenmodes freeze rapidly by increasing the coupling constant σ . However, we observe a sharp difference in the dynamical behavior of positive energies with $E \simeq \bar{E}$ and $E \neq \bar{E}$. Indeed we have that for the stable topological DESD [Fig. 7(a)] $\dot{c}_E$ freezes quickly with σ albeit eigenmodes closer to $\bar{E}$ are the last to freeze, while for the unstable topological DESD the freezing is much slower and is to be understood as a finite-size effect.

C. Designing stable topological cluster synchronization patterns on the stochastic block model

The SBM provides an ideal setting for demonstrating the design of topological cluster synchronization patterns via the DESD model. The Dirac spectrum of the SBM (see Fig. 8) exhibits isolated eigenstates at energies slightly above $E = m$ or slightly below $E = -m$. Their associated eigenstates have a node and an edge sectors that naturally partition the graph into node clusters and edge bundles. Indeed, these eigenstates have the structure given by Eq. (39) where $\mathbf{u}_{\lambda}$ and $\mathbf{v}_{\lambda}$ are singular vectors of the boundary operator, i.e., eigenvectors of the graph Laplacian $\mathbf{L}_{[0]}$ and the $\mathbf{L}_{[1]}$ Hodge Laplacian often used in spectral clustering [62,64]. The relative normalization of the node component and the edge component of the Dirac eigenstate is set by the nonzero value of the mass m . In particular, for $m > 0$ the positive eigenstates have a more significant contribution associated with the nodes while the negative eigenstates have a more significant contribution associated with the edges.

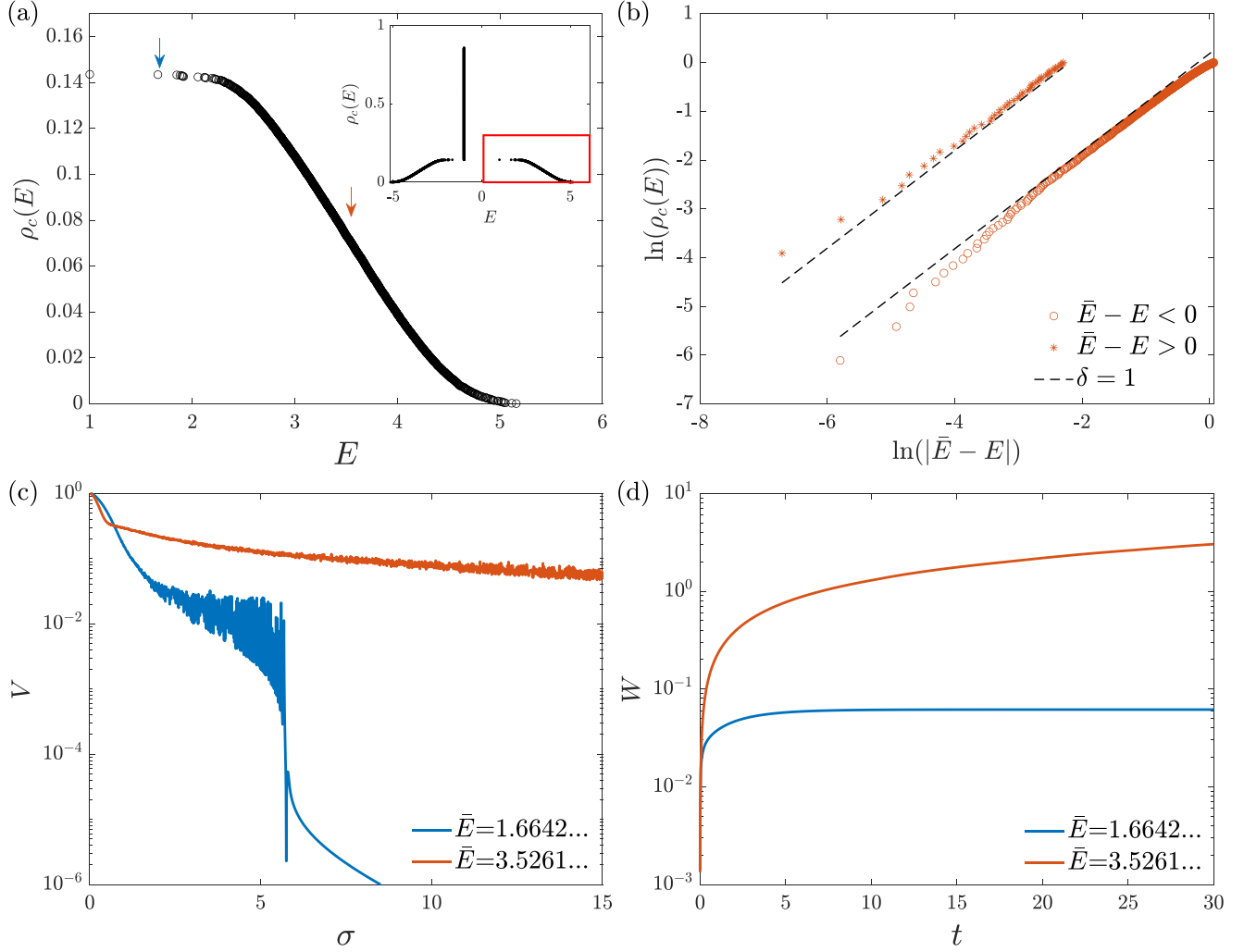


FIG. 6. The stability of DESD aligned with different eigenstates $\bar{E}$ of the topological Dirac equation is studied numerically and compared to the predictions of the linear stability analysis. In panel (a) the positive part of the cumulative density of eigenstates $\rho_c(E)$ of the topological Dirac equation (with $m = 1$), defined as in Fig. 3, is plotted for a Poisson network with $N_0 = 1000$ nodes and average degree $c = 12$ (the inset shows the entire spectrum). From the positive eigenstates, we select two positive eigenstates: the eigenstate at $\bar{E} = 1.6642 \dots$ corresponding to the gapped (isolated) Fiedler eigenstate and $\bar{E} = 3.5261 \dots$ located in the bulk of the spectrum. Panel (b) demonstrates that for the eigenstate in the bulk $\delta_- < 1$. In these conditions the linear stability analysis predicts that the Fiedler eigenstate $\bar{E} = 1.6642 \dots$ corresponds to a stable topological synchronized pattern while the eigenstate at energy $\bar{E} = 3.5261 \dots$ is not stable. Panels (c) and (d) display the standard deviation of the velocities of the phases V versus σ and the standard deviation of the roughness of the phases W versus t for $\sigma = 10$, showing that for the Fiedler eigenstate the DESD is stable while for the eigenstate at energy $\bar{E} = 3.5261 \dots$ DESD is not stable, in agreement with the linear stability analysis predictions.

In DESD, by setting $\bar{E}$ to the energy of any of the isolated eigenstates, the synchronization dynamics inherit the eigenstate's partitioning properties. Indeed, since the DESD occurs along the eigenstate $\Psi^{(\bar{E})}$, DESD leads to topological cluster synchronization. In this dynamical state, nodes belonging to different communities and edges belonging to different bundles oscillate in the real space at different frequencies with the frequencies proportional to the value of the eigenstate $\Psi^{(\bar{E})}$. This effect is illustrated in Fig. 9. Figures 9(a) and 9(d) highlight the node clusters and intercluster edge bundles, respectively. Figures 9(b) and 9(e) show the components of the eigenstate associated with the Fiedler energy ($\bar{E} = 1.4137 \dots$) on the nodes ($\Psi_i^{(\bar{E})}$) and on the intercluster edges [$\Psi_\ell^{(\bar{E})}$]. For direct comparison with Figs. 9(c) and 9(f), these

components of the eigenstate are plotted against the intrinsic frequencies of the nodes and edges, ω and $\hat{\omega}$, respectively. Figures 9(c) and 9(f) then show the resulting phase velocities of the node ($\dot{\theta}$) and edge ($\dot{\phi}$) signals, again as functions of ω and $\hat{\omega}$. A direct comparison reveals that the phase velocities synchronize with the Fiedler eigenstate, independently of the intrinsic frequencies.

The eigenstates associated with the negative counterparts of the isolated positive energies exhibit identical structural partitions, differing only in the relative scaling of the node and edge components. Specifically, node signals are amplified relative to edge signals for positive energies and vice versa for negative ones. Thus, both positive and negative Fiedler eigenstates induce the same partitioning, though only the positive eigenstate is used in

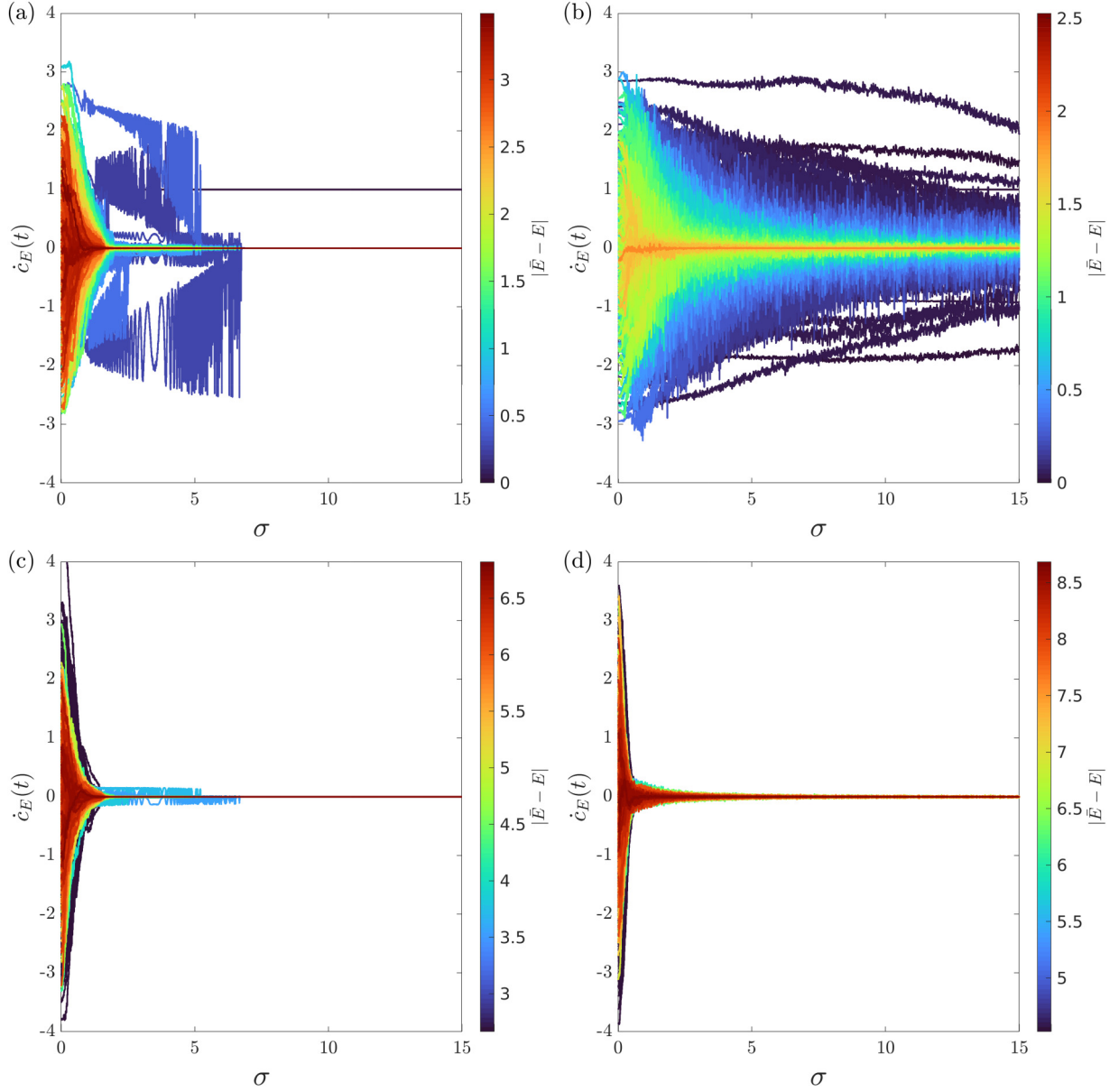


FIG. 7. The stability of the DESD designed along the two eigenstates of the Poisson network considered in Fig. 6 is studied starting from random initial conditions. Panels (a)–(d) show $\dot{c}_E(t)$ as a function of σ for the isolated Fiedler eigenstate at energy $\bar{E} = 1.6642 \dots$ [(a) and (c)] leading to stable topological cluster synchronization and for the eigenstate corresponding to the energy state $\bar{E} = 3.5261 \dots$ in the bulk of the spectrum [(b) and (d)]. The top row shows $\dot{c}_E(t)$ for $E > 0$ and the bottom row shows $\dot{c}_E(t)$ for $E < 0$. The color map represents the distance $|\bar{E} - E|$. It can be seen that as σ increases, the $\dot{c}_E(t)$ of more eigenstates $E \neq \bar{E}$ converge to zero, with those furthest away from $\bar{E}$ being quickest (including all with $E < 0$). In both cases, $\dot{c}_E$ for $E = \bar{E}$ remains constant and independent of σ , i.e., $\dot{c}_E = \Omega_{\bar{E}} = 1$. Simulations are performed across the forward transitions with σ increased adiabatically from 1 to 15 in steps of 0.01. At each value of σ , the dynamics were allowed to equilibrate for $T_{\max} = 15$ with time steps of $dt = 0.001$. The value of $\dot{\Psi}(t)$, from which $\dot{c}_E(t)$ was calculated, was averaged over the final third of $T_{\max}$ for each σ .

Figs. 9(b) and 9(c) and its negative counterpart in Figs. 9(e) and 9(f).

A notable property of DESD on the SBM is the possibility to design several topological cluster synchronization patterns on the same network thanks to the presence of several isolated eigenstates of the Dirac spectrum (see Fig. 10). In order to provide evidence for this remarkable property of DESD, we consider the three isolated Dirac eigenstates highlighted on the spectrum of the SBM shown in Fig. 8. The DESD

allows us to design different topological cluster synchronization patterns by selecting any of these different eigenstates. The choice of the eigenstate indeed produces a topological cluster synchronization on nodes and edges that partition both nodes and edges into node clusters and edge bundles oscillating at the same frequency independently of their intrinsic frequency. As we have shown in Sec. II, this phenomenology of SBM is also shared by real networks with modular structure, such as the connectome network shown in Fig. 2.

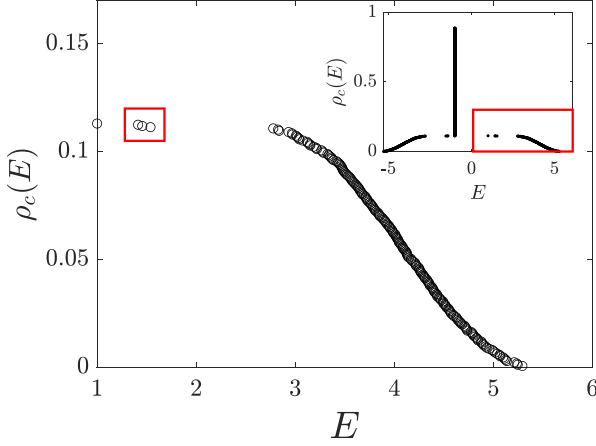


FIG. 8. The positive part of the cumulative density of eigenstates $\rho_c(E)$ of the topological Dirac equation (with $m = 1$), defined as in Fig. 3, is plotted for a SBM with four clusters, 50 nodes per cluster, an intracluster connection probability of $p_1 = 0.3$, yielding an average intracluster degree of $c_1 = 15$, and an intercluster connection probability of $p_2 = 0.007$, corresponding to an average intercluster degree of $c_2 = 1$ (the inset shows the entire spectrum). The red box in the main figure highlights the energies associated with eigenstates that partition the SBM. These are $E = 1.4137 \dots$, $1.4557 \dots$, and $1.5388 \dots$ and correspond to the lowest energies that satisfy $E \neq m$.

VI. CONCLUSION

In this work, we proposed DESD, which allows us to design topological cluster synchronization patterns extending to node and edge signals of a network encoded in the topological spinor. These topological cluster synchronization patterns are such that oscillators associated with different clusters of nodes and bundles of edges oscillate at distinct frequencies. These frequencies are dictated by the elements of the eigenstate corresponding to the fundamental state of the free energy associated with the DESD. The design principle adopted in DESD modulates this ground state with a procedure similar to the one adopted in optimization algorithms used in statistical mechanics approaches such as simulated or quantum annealing. Here the fundamental state of the free energy is taken to be one of the eigenstates of the topological Dirac equation which provides a very efficient way to decompose the dynamical state of topological spinors. We provide evidence that topological cluster synchronization states can be achieved on random graphs, stochastic block models, and real network datasets provided that their associated eigenstate is gapped (isolated). Moreover, if this condition is not met, then we provide conditions for the stability of the DESD defined along an eigenstate in the bulk of the Dirac spectrum of the network.

On random graphs, we can design stable topological cluster synchronization very reliably along the Fiedler eigenstate of the network. On the stochastic block model, DESD can induce different stable topological cluster synchronization states on nodes and edges which correlate with different partitions of the blocks and the edges connecting them. Similarly, on the considered connectome dataset, DESD can provide reliable partitions of the nodes and edges of the network revealing the left-right hemispheric separation and an anterior-posterior differentiation.

This work can be extended in different directions. From a theoretical perspective, an open question is to investigate topological cluster synchronization on higher-order networks where topological signals are not only defined on nodes and edges but also on triangles, tetrahedra, and so on. On the more applied side, further exploration of the design principle on real modular networks, such as connectomes, might shed light on the interplay between network structure and function.

DATA AVAILABILITY

The data that support the findings of this article are openly available [75], embargo periods may apply.

APPENDIX A: STABILITY OF DESD

In this Appendix, we will investigate the stability of DESD along the generic eigenstate $\Psi^{(E)}$ associated with an energy $\bar{E}$ of the topological Dirac equation. To this end, we follow the approach used for the standard Kuramoto model on simple networks and on lattices [58–61] and define Φ as the contribution to the phases Ψ that is orthogonal to the eigenstate $\Psi^{(E)}$, i.e.,

$$\Phi = \Psi - (\Psi^\top \Psi^{(E)}) \Psi^{(E)}. \quad (\text{A1})$$

In order to study the stability of the DESD we investigate the “roughness” W^2 associated with the phases Ψ calculated for the linearized DESD [Eq. (62)] and defined as

$$W^2 = \langle \|\Phi\|^2 \rangle = \frac{1}{N} \left\langle \sum_{r=1}^N \Phi_r^2 \right\rangle. \quad (\text{A2})$$

Using the orthogonality of the eigenstates of the topological Dirac equation we thus obtain

$$W^2 = \frac{1}{N} \sum_{E \neq \bar{E}} \langle c_E^2(t) \rangle. \quad (\text{A3})$$

The explicit expression of $c_E(t)$ is given by Eq. (65). Explicitly performing the average over the intrinsic frequencies Ω_E , we obtain

$$\begin{aligned} \langle c_E^2(t) \rangle &= \left\langle \left[\frac{\Omega_E}{\sigma(E - \bar{E})^2} (1 - e^{-\sigma(E - \bar{E})^2 t}) + c_E(0) e^{-\sigma(E - \bar{E})^2 t} \right]^2 \right\rangle \\ &= \frac{1}{\sigma^2(E - \bar{E})^4} (1 - e^{-\sigma(E - \bar{E})^2 t})^2 + c_E^2(0) e^{-2\sigma(E - \bar{E})^2 t}, \end{aligned} \quad (\text{A4})$$

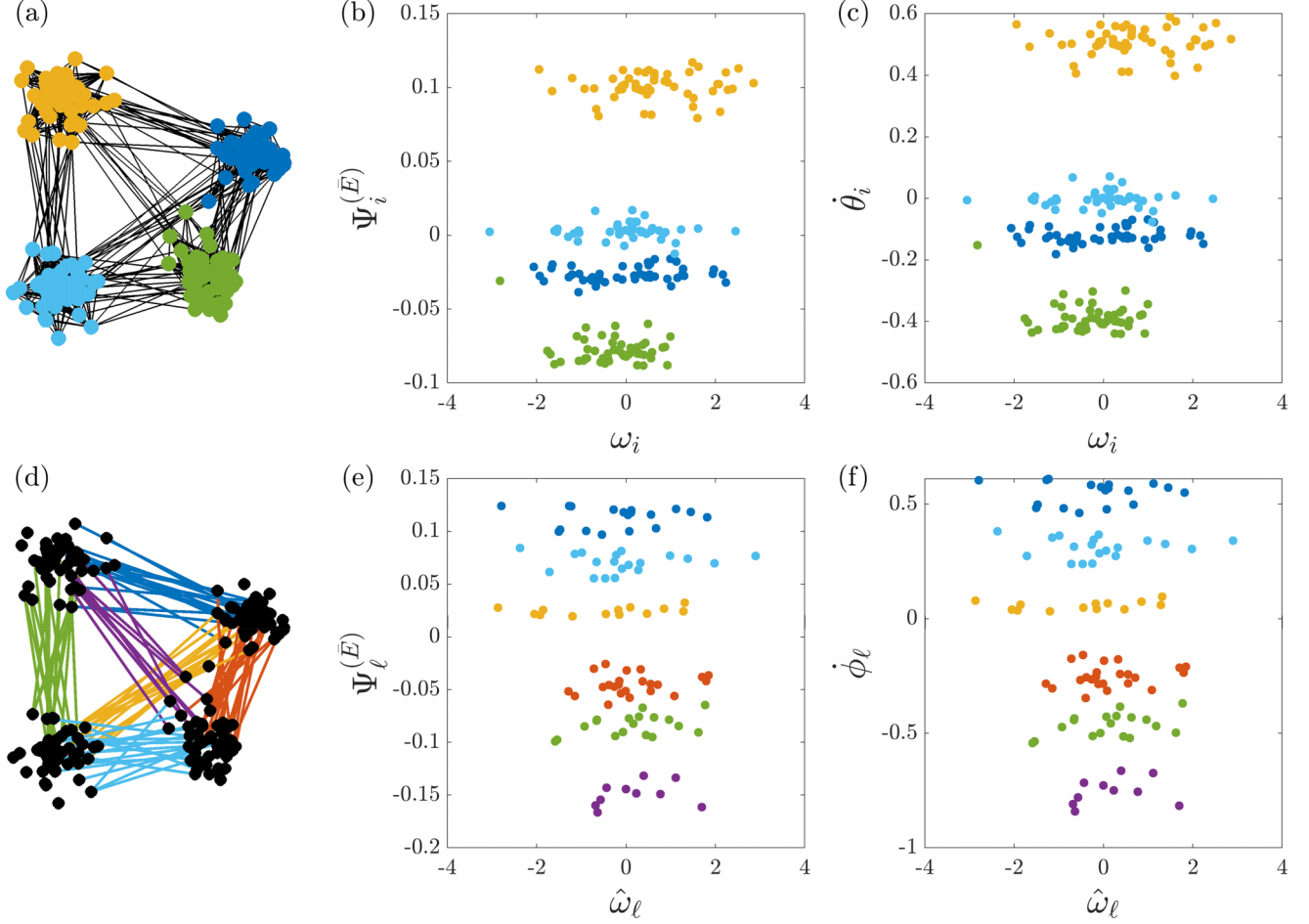


FIG. 9. The DESD leads to topological cluster synchronization defined on nodes and edges whose frequencies settle to values proportional to the elements of the eigenstate $\Psi^{(E)}$. The considered SBM is visualized in panel (a) with each cluster highlighted in a unique color. Panel (b) shows the element $\Psi_i^{(E)}$ of the eigenstate associated with the generic node i of the network which determines the frequency of $\dot{\theta}_i$ of the node (panel c) in the DESD regardless of the value of the node intrinsic frequency ω_i . Panel (d) provides a visualization of the different bundles of edges connecting the different communities in the same network indicated with different colors (the edges among nodes of the same community are omitted). Panel (e) shows the element $\Psi_\ell^{(E)}$ of the eigenstate associated with the generic link ℓ of the network which determines the frequency of $\dot{\phi}_\ell$ of the link [panel (f)] in the DESD regardless of the value of the link intrinsic frequency $\hat{\omega}_\ell$. The DESD is run up to a maximum time $T_{\max} = 30$ at $\sigma = 15$.

where we have assumed $\langle \Omega_E \rangle = 0$ and $\langle \Omega_E \Omega_{E'} \rangle = \delta_{E,E'}$. Thus, inserting this expression in Eq. (A3) we get, in the limit $t \rightarrow \infty$, the expression for W^2 given by

$$W^2 = W_+^2 + W_-^2 \quad (\text{A5})$$

with

$$W_+^2 = \frac{1}{\mathcal{N}} \sum_{E > \bar{E}} \frac{1}{\sigma^2(E - \bar{E})^4},$$

$$W_-^2 = \frac{1}{\mathcal{N}} \sum_{E < \bar{E}} \frac{1}{\sigma^2(E - \bar{E})^4}. \quad (\text{A6})$$

It is therefore sufficient that either W_+^2 or W_-^2 diverges in the thermodynamic limit, i.e., for $\mathcal{N} \rightarrow \infty$, for the synchronization state of DESD to lose its stability. Let us now determine the spectral properties of the topological Dirac equation that will determine whether W_+^2 diverges in the thermodynamic limit. A similar argument will also hold for W_-^2 . Let us de-

fine the right spectral gap Δ_+ and the left spectral gap Δ_- associated with the selected energy state $\bar{E}$ of the Dirac operator,

$$\Delta_+ = \min_{E > \bar{E}} (E - \bar{E}),$$

$$\Delta_- = \min_{E < \bar{E}} (\bar{E} - E). \quad (\text{A7})$$

In this scenario, for $\mathcal{N} \ll 1$ we can express W_+^2 and W_-^2 in the continuum approximation as

$$W_+^2 = \int_{E > \bar{E} + \Delta_+} dE \frac{\rho(E)}{\sigma^2(E - \bar{E})^4},$$

$$W_-^2 = \int_{E < \bar{E} - \Delta_-} dE \frac{\rho(E)}{\sigma^2(E - \bar{E})^4}. \quad (\text{A8})$$

Thus, we have different scenarios for W_+^2 in the limit $\mathcal{N} \rightarrow \infty$:

- (1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ > 0$, then W_+^2 is finite.

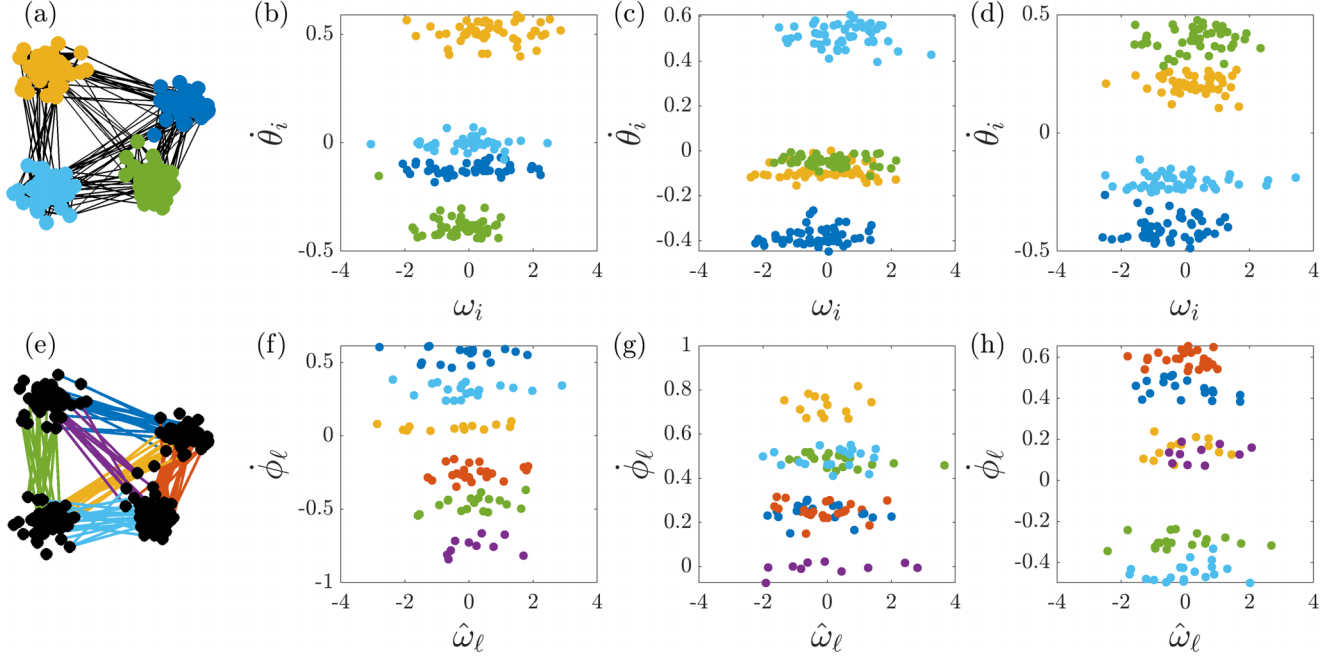


FIG. 10. The DESD allows us to design topological cluster synchronization pattern defined on nodes and edges on the SBM. By considering the DESD aligned with the isolated eigenstates of the topological Dirac equation of the SBM shown in Fig. 8 we show that DESD leads to cluster synchronization of nodes and edges where different communities of nodes and different bundles of links are partitioned into clusters that oscillate at the same frequency independently of their intrinsic frequencies. Here panels (a) and (e) define the color code of the node clusters and the edge clusters while panels (b)–(d) show the phase velocities of the nodes, $\dot{\theta}_i$, and the edges, $\dot{\phi}_\ell$, plotted against their intrinsic frequencies, ω and $\hat{\omega}$, for the DESD associated with the eigenstates $\bar{E} = \pm 1.4137 \dots$ [(b) and (f)], $\bar{E} = \pm 1.4557 \dots$ [(c) and (g)], and $\bar{E} = \pm 1.5388 \dots$ [(d) and (h)]. The DESD is run up to a maximum time $T_{\max} = 30$ at $\sigma = 15$.

(2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ = 0$, then, assuming that the density of eigenstates scales as $\rho(E) \propto (E - \bar{E})^{\delta_+}$ for $0 < E - \bar{E} \ll 1$, we obtain for any finite $\mathcal{N}$,

$$W_+^2 \sim \begin{cases} \Delta_+^{-(3-\delta_+)} & \text{if } \delta_+ < 3, \\ -\ln(\Delta_+) & \text{if } \delta_+ = 3, \\ \text{finite} & \text{if } \delta_+ > 3, \end{cases} \quad (\text{A9})$$

as long as $0 < E - \bar{E} \ll 1$. Thus, as $\mathcal{N} \rightarrow \infty$, $\Delta_+ \rightarrow 0$, W_+^2 diverges for $\delta_+ \leq 3$ and the synchronized patterns of the DESD cannot be thermodynamically stable.

Similar scenarios hold for W_-^2 :

(1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- > 0$, then W_-^2 is finite.

(2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- = 0$ then, assuming that the density of eigenstates scales as $\rho(E) \propto (\bar{E} - E)^{\delta_-}$ for $0 < \bar{E} - E \ll 1$, we obtain for any finite $\mathcal{N}$,

$$W_-^2 \sim \begin{cases} \Delta_-^{-(3-\delta_-)} & \text{if } \delta_- < 3, \\ -\ln(\Delta_-) & \text{if } \delta_- = 3, \\ \text{finite} & \text{if } \delta_- > 3, \end{cases} \quad (\text{A10})$$

as long as $0 < \bar{E} - E \ll 1$. Thus, as $\mathcal{N} \rightarrow \infty$, $\Delta_- \rightarrow 0$, W_-^2 diverges for $\delta_- \leq 3$ and the synchronized patterns of the DESD cannot be thermodynamically stable.

APPENDIX B: VALIDITY OF THE LINEAR APPROXIMATION

The linear approximation of the DESD given by Eq. (62) is valid when the elements of the argument of the sine function

are small, i.e., $(\mathcal{H} - \bar{E}\mathbf{I})\Psi$ is small. A useful global parameter for establishing the presence or absence of this condition is the correlation C defined as follows:

$$C = \langle \|(\mathcal{H} - \bar{E}\mathbf{I})\Psi\|^2 \rangle = \frac{1}{\mathcal{N}} \Psi^\top [\mathcal{H} - \bar{E}\mathbf{I}]^2 \Psi, \quad (\text{B1})$$

where we have used the fact that $\mathcal{H} - \bar{E}\mathbf{I}$ is a symmetric matrix. The divergence of C in the large network limit establishes the failure of the linear approximation. Using the orthogonality of the eigenstates, the correlation C can be expressed as

$$C = \frac{1}{\mathcal{N}} \left\langle \sum_{E \neq \bar{E}} c_E^2(t) (E - \bar{E})^2 \right\rangle. \quad (\text{B2})$$

By substituting Eq. (65) for $c_E(t)$ for $E \neq \bar{E}$ into Eq. (B2) we obtain

$$C = \frac{1}{\mathcal{N}} \sum_{E \neq \bar{E}} \left[\left(\frac{(1 - e^{-\sigma(E-\bar{E})^2 t})^2}{\sigma^2 (E - \bar{E})^4} + c_E^2(0) e^{-2\sigma(E-\bar{E})^2 t} \right) \times (E - \bar{E})^2 \right], \quad (\text{B3})$$

which gives in the asymptotic limit $t \rightarrow \infty$:

$$C = \frac{1}{\mathcal{N}} \sum_{E \neq \bar{E}} \frac{1}{\sigma^2 (E - \bar{E})^2}. \quad (\text{B4})$$

In the large network limit $\mathcal{N} \gg 1$ we have

$$C = C_+ + C_- \quad (\text{B5})$$

with

$$\begin{aligned} C_+ &= \int_{E > \bar{E} + \Delta_+} dE \frac{\rho(E)}{\sigma^2(E - \bar{E})^2}, \\ C_- &= \int_{E < \bar{E} - \Delta_-} dE \frac{\rho(E)}{\sigma^2(E - \bar{E})^2}, \end{aligned} \quad (\text{B6})$$

where $\Delta_{\pm}$ are defined in Eq. (A7). Thus, we have different scenarios for C_+ :

- (1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ > 0$, then C_+ is finite.
- (2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ = 0$ then, assuming that the density of eigenstates scales as $\rho(E) \propto (E - \bar{E})^{\delta_+}$ for $0 < E - \bar{E} \ll 1$, we obtain for any finite $\mathcal{N}$,

$$C_+ \sim \begin{cases} \Delta_+^{-(1-\delta_+)} & \text{if } \delta_+ < 1, \\ -\ln(\Delta_+) & \text{if } \delta_+ = 1, \\ \text{finite} & \text{if } \delta_+ > 1, \end{cases} \quad (\text{B7})$$

as long as $0 < E - \bar{E} \ll 1$. Thus, as $\mathcal{N} \rightarrow \infty$, $\Delta_+ \rightarrow 0$, C_+ diverges for $\delta_+ \leq 1$ and the linearization of the DESD loses its validity.

Similar scenarios hold for C_- :

- (1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- > 0$, then C_- is finite.
- (2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- = 0$ then, assuming that the density of eigenstates scales as $\rho(E) \propto (\bar{E} - E)^{\delta_-}$ for $0 < \bar{E} - E \ll 1$, we obtain

$$C_- \sim \begin{cases} \Delta_-^{-(1-\delta_-)} & \text{if } \delta_- < 1, \\ -\ln(\Delta_-) & \text{if } \delta_- = 1, \\ \text{finite} & \text{if } \delta_- > 1, \end{cases} \quad (\text{B8})$$

as long as $0 < \bar{E} - E \ll 1$. Thus, as $\mathcal{N} \rightarrow \infty$, $\Delta_- \rightarrow 0$, C_- diverges for $\delta_- \leq 1$ and the linearization of the DESD loses its validity.

APPENDIX C: ENTRAINED PHASES

Here we characterize the fluctuation in $\dot{\Phi}$ using the global parameter V^2 defined as follows:

$$V^2 = \langle \|\dot{\Phi}\|^2 \rangle = \frac{1}{\mathcal{N}} \left\langle \sum_{r=1}^{\mathcal{N}} \dot{\Phi}_r^2 \right\rangle \quad (\text{C1})$$

Using the orthogonality of the eigenstates we have

$$V^2 = \frac{1}{\mathcal{N}} \sum_{E \neq \bar{E}} \langle \dot{c}_E^2(t) \rangle. \quad (\text{C2})$$

Using the linearized solution (65), we obtain the following expression for $\dot{c}_E(t)$ for $E \neq \bar{E}$:

$$\dot{c}_E(t) = [\Omega_E - \sigma(E - \bar{E})^2 c_E(0)] e^{-\sigma(E - \bar{E})^2 t}. \quad (\text{C3})$$

Substituting Eq. (C3) into Eq. (C1), performing the average, and taking the limits $t \rightarrow \infty$ and $\mathcal{N} \rightarrow \infty$ we find that V^2 can be expressed as

$$V^2 \sim \frac{1}{\mathcal{N}} \sum_{E \neq \bar{E}} [1 + \sigma^2(E - \bar{E})^4 c_E^2(0)] e^{-2\sigma(E - \bar{E})^2 t}. \quad (\text{C4})$$

In the limit $\mathcal{N} \gg 1$ we can perform the continuous approximation

$$V^2 = V_+^2 + V_-^2 \quad (\text{C5})$$

with

$$\begin{aligned} V_+^2 &= \int_{E > \bar{E} + \Delta_+} dE \rho(E) [1 + \sigma^2(E - \bar{E})^4 c_E^2(0)] e^{-2\sigma(E - \bar{E})^2 t}, \\ V_-^2 &= \int_{E < \bar{E} - \Delta_-} dE \rho(E) [1 + \sigma^2(E - \bar{E})^4 c_E^2(0)] e^{-2\sigma(E - \bar{E})^2 t}. \end{aligned}$$

To the leading term we thus have

$$\begin{aligned} V_+^2 &\sim \int_{E > \bar{E} + \Delta_+} dE \rho(E) e^{-2\sigma(E - \bar{E})^2 t}, \\ V_-^2 &\sim \int_{E < \bar{E} - \Delta_-} dE \rho(E) e^{-2\sigma(E - \bar{E})^2 t}. \end{aligned} \quad (\text{C6})$$

We observe that under very general conditions, as long as the linear approximation of DESD is valid, the phases become entrained, meaning that $V^2 \rightarrow 0$ as $t \rightarrow \infty$ for $\mathcal{N} \gg 1$. In order to show this, let us consider the following scenarios for V_+^2 :

- (1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ > 0$ in the limit $\mathcal{N} \rightarrow \infty$, then we obtain

$$V_+^2 \sim e^{-2\sigma \Delta_+^2 t} \rightarrow 0 \text{ for } t \rightarrow \infty. \quad (\text{C7})$$

- (2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_+ = 0$, then, assuming that the density of eigenstates scales as $\rho(E) \propto (E - \bar{E})^{\delta_+}$ for $0 < E - \bar{E} \ll 1$, we obtain for any finite $\mathcal{N}$,

$$V_+^2 \sim t^{-2/\delta_+} \rightarrow 0 \text{ for } t \rightarrow \infty. \quad (\text{C8})$$

Similarly, for V_-^2 we obtain

- (1) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- > 0$ in the limit $\mathcal{N} \rightarrow \infty$, then we obtain

$$V_-^2 \sim e^{-2\sigma \Delta_-^2 t} \rightarrow 0 \text{ for } t \rightarrow \infty. \quad (\text{C9})$$

- (2) If $\lim_{\mathcal{N} \rightarrow \infty} \Delta_- = 0$, then, assuming that the density of eigenstates scales as $\rho(E) \propto (\bar{E} - E)^{\delta_-}$ for $0 < \bar{E} - E \ll 1$, we obtain for any finite $\mathcal{N}$,

$$V_-^2 \sim t^{-2/\delta_-} \rightarrow 0 \text{ for } t \rightarrow \infty. \quad (\text{C10})$$

- [1] Y. S. Cho, T. Nishikawa, and A. E. Motter, Stable chimeras and independently synchronizable clusters, *Phys. Rev. Lett.* **119**, 084101 (2017).
- [2] J. Lehnert, P. Hövel, A. Selivanov, A. Fradkov, and E. Schöll, Controlling cluster synchronization by adapting the topology, *Phys. Rev. E* **90**, 042914 (2014).
- [3] C. Tomaselli, L. V. Gambuzza, G.-Q. Sun, S. Boccaletti, and M. Frasca, Taming cluster synchronization, *Chaos Solitons Fract.* **199**, 116576 (2025).
- [4] L. M. Pecora, F. Sorrentino, A. M. Hagerstrom, T. E. Murphy, and R. Roy, Cluster synchronization and isolated desynchronization in complex networks with symmetries, *Nat. Commun.* **5**, 4079 (2014).
- [5] F. Sorrentino, L. M. Pecora, A. M. Hagerstrom, T. E. Murphy, and R. Roy, Complete characterization of the stability of cluster synchronization in complex dynamical networks, *Sci. Adv.* **2**, e1501737 (2016).
- [6] I. Z. Kiss, C. G. Rusin, H. Kori, and J. L. Hudson, Engineering complex dynamical structures: Sequential patterns and desynchronization, *Science* **316**, 1886 (2007).
- [7] H. Kori, C. G. Rusin, I. Z. Kiss, and J. L. Hudson, Synchronization engineering: Theoretical framework and application to dynamical clustering, *Chaos* **18**, 026111 (2008).
- [8] T. Timofeyev and A. Patania, Cluster synchronization via graph Laplacian eigenvectors, *Chaos* **35**, 093109 (2025).
- [9] I. Z. Kiss, Y. Zhai, and J. L. Hudson, Emerging coherence in a population of chemical oscillators, *Science* **296**, 1676 (2002).
- [10] J. Shlens, F. Rieke, and E. J. Chichilnisky, Synchronized firing in the retina, *Curr. Opin. Neurobiol.* **18**, 396 (2008).
- [11] R. G. Erra, J. L. P. Velazquez, and M. Rosenblum, Neural synchronization from the perspective of non-linear dynamics, *Front. Comput. Neurosci.* **11**, 98 (2017).
- [12] F. Sorrentino and E. Ott, Network synchronization of groups, *Phys. Rev. E* **76**, 056114 (2007).
- [13] P. Ji, T. K. D. M. Peron, P. J. Menck, F. A. Rodrigues, and J. Kurths, Cluster explosive synchronization in complex networks, *Phys. Rev. Lett.* **110**, 218701 (2013).
- [14] C. H. Kim, J. Park, Y. J. Kim, S. Park, S. Boccaletti, and B. Kahng, Cluster-mediated synchronization dynamics in globally coupled oscillators with inertia, *Chaos, Solitons Fractals* **196**, 116281 (2025).
- [15] S. Jalan and A. Singh, Cluster synchronization in multiplex networks, *Europhys. Lett.* **113**, 30002 (2016).
- [16] A. Salova and R. M. D'Souza, Cluster synchronization on hypergraphs, *arXiv:2101.05464*.
- [17] V. Nicosia, M. Valencia, M. Chavez, A. Díaz-Guilera, and V. Latora, Remote synchronization reveals network symmetries and functional modules, *Phys. Rev. Lett.* **110**, 174102 (2013).
- [18] J. D. Hart, Y. Zhang, R. Roy, and A. E. Motter, Topological control of synchronization patterns: Trading symmetry for stability, *Phys. Rev. Lett.* **122**, 058301 (2019).
- [19] H. A. Makse, P. Boldi, F. Sorrentino, and I. Stewart, Symmetries of living systems: Symmetry fibrations and synchronization in biological networks, *arXiv:2502.18713*.
- [20] F. Morone, I. Leifer, and H. A. Makse, Fibration symmetries uncover the building blocks of biological networks, *Proc. Natl. Acad. Sci. USA* **117**, 8306 (2020).
- [21] I. Leifer, F. Morone, S. D. S. Reis, J. S. Andrade, Jr., M. Sigman, and H. A. Makse, Circuits with broken fibration symmetries perform core logic computations in biological networks, *PLoS Comput. Biol.* **16**, e1007776 (2020).
- [22] W. Lu, Bo Liu, and T. Chen, Cluster synchronization in networks of coupled nonidentical dynamical systems, *Chaos* **20**, 013120 (2010).
- [23] C. Duan, T. Nishikawa, and A. E. Motter, Prevalence and scalable control of localized networks, *Proc. Natl. Acad. Sci. USA* **119**, e2122566119 (2022).
- [24] A. P. Millán, H. Sun, L. Giambagli, R. Muolo, T. Carletti, J. J. Torres, F. Radicchi, J. Kurths, and G. Bianconi, Topology shapes dynamics of higher-order networks, *Nat. Phys.* **21**, 353 (2025).
- [25] G. Bianconi, *Higher-order Networks: An Introduction to Simplicial Complexes* (Cambridge University Press, Cambridge, UK, 2021).
- [26] A. P. Millán, J. J. Torres, and G. Bianconi, Explosive higher-order Kuramoto dynamics on simplicial complexes, *Phys. Rev. Lett.* **124**, 218301 (2020).
- [27] R. Ghorbanchian, J. G. Restrepo, J. J. Torres, and G. Bianconi, Higher-order simplicial synchronization of coupled topological signals, *Commun. Phys.* **4**, 120 (2021).
- [28] T. Carletti, L. Giambagli, and G. Bianconi, Global topological synchronization on simplicial and cell complexes, *Phys. Rev. Lett.* **130**, 187401 (2023).
- [29] R. Wang, R. Muolo, T. Carletti, and G. Bianconi, Global topological synchronization of weighted simplicial complexes, *Phys. Rev. E* **110**, 014307 (2024).
- [30] A. Arnaudon, R. L. Peach, G. Petri, and P. Expert, Connecting Hodge and Sakaguchi-Kuramoto through a mathematical framework for coupled oscillators on simplicial complexes, *Commun. Phys.* **5**, 211 (2022).
- [31] I. Bačić, M. T. Schaub, J. Kurths, and D. Witthaut, Phase locking and multistability in the topological Kuramoto model on cell complexes, *arXiv:2510.05831*.
- [32] J. J. Torres and G. Bianconi, Simplicial complexes: Higher-order spectral dimension and dynamics, *J. Phys.: Complexity* **1**, 015002 (2020).
- [33] C. Ziegler, P. S. Skardal, H. Dutta, and D. Taylor, Balanced Hodge Laplacians optimize consensus dynamics over simplicial complexes, *Chaos* **32**, 023128 (2022).
- [34] S. Krishnagopal and G. Bianconi, Topology and dynamics of higher-order multiplex networks, *Chaos Solitons Fract.* **177**, 114296 (2023).
- [35] F. A. N. Santos, E. P. Raposo, M. D. Coutinho-Filho, M. Copelli, C. J. Stam, and L. Douw, Topological phase transitions in functional brain networks, *Phys. Rev. E* **100**, 032414 (2019).
- [36] G. Petri, P. Expert, F. Turkheimer, R. Carhart-Harris, D. Nutt, P. J. Hellyer, and F. Vaccarino, Homological scaffolds of brain functional networks, *J. R. Soc. Interface* **11**, 20140873 (2014).
- [37] L. Calmon, M. T. Schaub, and G. Bianconi, Dirac signal processing of higher-order topological signals, *New J. Phys.* **25**, 093013 (2023).
- [38] R. Wang, Y. Tian, P. Liò, and G. Bianconi, Dirac-equation signal processing: Physics boosts topological machine learning, *PNAS Nexus* **4**, pgaf139 (2025).
- [39] C. Nauck, R. Gorantla, M. Lindner, K. Schürholt, A. S. J. S. Mey, and F. Hellmann, Dirac-Bianconi graph neural networks-enabling long-range graph predictions, in *ICML 2024 Workshop on Geometry-grounded Representation Learning and Generative Modeling* (2024).

- [40] C. Battiloro, L. Testa, L. Giusti, S. Sardellitti, P. Di Lorenzo, and S. Barbarossa, Generalized simplicial attention neural networks, in *IEEE Transactions on Signal and Information Processing over Networks* (2024).
- [41] J. Faskowitz, R. F. Betzel, and O. Sporns, Edges in brain networks: Contributions to models of structure and function, *Network Neurosci.* **6**, 1 (2022).
- [42] S. Barbarossa and S. Sardellitti, Topological signal processing over simplicial complexes, *IEEE Trans. Signal Process.* **68**, 2992 (2020).
- [43] L. Calmon, J. G. Restrepo, J. J. Torres, and G. Bianconi, Dirac synchronization is rhythmic and explosive, *Commun. Phys.* **5**, 253 (2022).
- [44] G. Bianconi, The topological Dirac equation of networks and simplicial complexes, *J. Phys.: Complexity* **2**, 035022 (2021).
- [45] J. Kogut and L. Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, *Phys. Rev. D* **11**, 395 (1975).
- [46] L. Calmon, S. Krishnagopal, and G. Bianconi, Local Dirac synchronization on networks, *Chaos* **33**, 033117 (2023).
- [47] L. Giambagli, L. Calmon, R. Muolo, T. Carletti, and G. Bianconi, Diffusion-driven instability of topological signals coupled by the Dirac operator, *Phys. Rev. E* **106**, 064314 (2022).
- [48] R. Muolo, T. Carletti, and G. Bianconi, The three way Dirac operator and dynamical turing and Dirac induced patterns on nodes and links, *Chaos, Solitons Fractals* **178**, 114312 (2024).
- [49] T. Carletti, L. Giambagli, R. Muolo, and G. Bianconi, Global topological Dirac synchronization, *J. Phys.: Complex.* **6**, 025009 (2025).
- [50] M. Nurisso, A. Arnaudon, M. Lucas, R. L. Peach, P. Expert, F. Vaccarino, and G. Petri, A unified framework for simplicial Kuramoto models, *Chaos* **34**, 053118 (2024).
- [51] R. Muolo, L. Giambagli, H. Nakao, D. Fanelli, and T. Carletti, Turing patterns on discrete topologies: From networks to higher-order structures, *Proc. Roy. Soc. A*, **480**, 20240235 (2024).
- [52] M. Alain, So Takao, B. Paige, and M. Peter Deisenroth, Gaussian processes on cellular complexes, *arXiv:2311.01198*.
- [53] Junjie Wee, G. Bianconi, and K. Xia, Persistent Dirac for molecular representation, *Sci. Rep.* **13**, 11183 (2023).
- [54] F. Suwayyid and G.-W. Wei, Persistent dirac of paths on digraphs and hypergraphs, *Found. Data Sci.* **6**, 124 (2024).
- [55] T. Stankovski, T. Pereira, P. V. E. McClintock, and A. Stefanovska, Coupling functions: Universal insights into dynamical interaction mechanisms, *Rev. Mod. Phys.* **89**, 045001 (2017).
- [56] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, Optimization by simulated annealing, *Science* **220**, 671 (1983).
- [57] A. Das and B. K. Chakrabarti, *Colloquium: Quantum annealing and analog quantum computation*, *Rev. Mod. Phys.* **80**, 1061 (2008).
- [58] A. P. Millán, J. J. Torres, and G. Bianconi, Synchronization in network geometries with finite spectral dimension, *Phys. Rev. E* **99**, 022307 (2019).
- [59] A. P. Millán, J. J. Torres, and G. Bianconi, Complex network geometry and frustrated synchronization, *Sci. Rep.* **8**, 9910 (2018).
- [60] H. Hong, H. Park, and M. Y. Choi, Collective synchronization in spatially extended systems of coupled oscillators with random frequencies, *Phys. Rev. E* **72**, 036217 (2005).
- [61] H. Hong, H. Chaté, H. Park, and L.-H. Tang, Entrainment transition in populations of random frequency oscillators, *Phys. Rev. Lett.* **99**, 184101 (2007).
- [62] A. Capocci, V. D. P. Servedio, G. Caldarelli, and F. Colaiori, Detecting communities in large networks, *Physica A* **352**, 669 (2005).
- [63] R. Guimera, M. Sales-Pardo, and L. A. Nunes Amaral, Modularity from fluctuations in random graphs and complex networks, *Phys. Rev. E* **70**, 025101 (2004).
- [64] U. Von Luxburg, A tutorial on spectral clustering, *Statist. Comput.* **17**, 395 (2007).
- [65] S. Hata and H. Nakao, Localization of Laplacian eigenvectors on random networks, *Sci. Rep.* **7**, 1121 (2017).
- [66] R. J. Sánchez-García, Exploiting symmetry in network analysis, *Commun. Phys.* **3**, 87 (2020).
- [67] A. Škoch, B. R. Bučková, J. Mareš, J. Tintěra, P. Sanda, L. Jajcay, J. Horáček, F. Španiel, and J. Hlinka, Human brain structural connectivity matrices—Ready for modelling, *Sci. Data* **9**, 486 (2022).
- [68] E. Tang, J. Agudo-Canalejo, and R. Golestanian, Topology protects chiral edge currents in stochastic systems, *Phys. Rev. X* **11**, 031015 (2021).
- [69] C. Zheng and E. Tang, A topological mechanism for robust and efficient global oscillations in biological networks, *Nat. Commun.* **15**, 6453 (2024).
- [70] S. N. Chowdhury and H. Meyer-Ortmanns, Topologically protected edge oscillations in nonlinear dynamical units, *arXiv:2506.18699*.
- [71] A. Santoro, F. Battiston, G. Petri, and E. Amico, Higher-order organization of multivariate time series, *Nat. Phys.* **19**, 221 (2023).
- [72] M. T. Schaub, Yu Zhu, J.-B. Seby, T. M. Roddenberry, and S. Segarra, Signal processing on higher-order networks: Livin' on the edge... and beyond, *Sign. Process.* **187**, 108149 (2021).
- [73] T. S. Evans and R. Lambiotte, Line graphs, link partitions, and overlapping communities, *Phys. Rev. E* **80**, 016105 (2009).
- [74] T. S. Evans and R. Lambiotte, Line graphs of weighted networks for overlapping communities, *Eur. Phys. J. B* **77**, 265 (2010).
- [75] A. A. A. Zaid and G. Bianconi, Dirac equation synchronization dynamics (2026), <https://github.com/qdxa7/Dirac-Equation-Synchronization-Dynamics>.
- [76] A. Gallo, Yu Tian, R. Lambiotte, and T. Carletti, Global synchronization in matrix-weighted networks, *arXiv:2507.12322*.
- [77] F. Baccini, F. Geraci, and G. Bianconi, Weighted simplicial complexes and their representation power of higher-order network data and topology, *Phys. Rev. E* **106**, 034319 (2022).
- [78] Yu Tian, S. Kojaku, H. Sayama, and R. Lambiotte, Matrix-weighted networks for modeling multidimensional dynamics: Theoretical foundations and applications to network coherence, *Phys. Rev. Lett.* **134**, 237401 (2025).
- [79] L. J. Grady and J. R. Polimeni, *Discrete Calculus: Applied Analysis on Graphs for Computational Science* (Springer, Berlin, 2010), Vol. 3.
- [80] Y. Kuramoto, Self-entrainment of a population of coupled nonlinear oscillators, in *International Symposium on Mathematical Problems in Theoretical Physics* (Springer, Berlin, 1975), pp. 420–422.

- [81] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge University Press, Cambridge, UK, 2001).
- [82] F. A. Rodrigues, T. K. D. M. Peron, P. Ji, and J. Kurths, The Kuramoto model in complex networks, *Phys. Rep.* **610**, 1 (2016).
- [83] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Synchronization in complex networks, *Phys. Rep.* **469**, 93 (2008).
- [84] S. Boccaletti, A. N. Pisarchik, C. I. Del Genio, and A. Amann, *Synchronization: From Coupled Systems to Complex Networks* (Cambridge University Press, Cambridge, UK, 2018).