

Screening and localization in the nonlinear Anderson problem

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We study the spreading dynamics of an initially localized wave packet of finite norm in one-dimensional nonlinear Schrödinger lattices with random potential. The problem has gained considerable interest in the literature, and it continues to attract attention due to its connection with the general properties of behavior of systems with competition between nonlinearity, nonlocality, and randomness. It is shown that adding small dielectric coupling to the ambient random medium leads to asymptotic localization of the nonlinear field regardless of the Kerr nonlinearity strength. If the electric susceptibility is zero, then the nonlinear field undergoes sharp localization-delocalization transition above a certain critical value of the nonlinearity parameter. The nonlinear localization length is found to be $\Lambda_{\text{loc}} \simeq \exp[(\pi/\varepsilon_r \tan \delta) \ln \beta]$, where $\tan \delta$ is the dielectric loss tangent, ε_r is the relative permeability of the medium, and β characterizes the Kerr nonlinearity. The model predicts a possibility of self-induced localization when the “medium” to which the wave field is dielectrically coupled is the wave function itself. The mathematical methods, stipulated here, pave the way towards understanding the wave processes in complex media with competition between randomness, dispersiveness, and nonlinearity, such as Anderson localization of a wave packet interacting with the environment.

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I. INTRODUCTION

Anderson localization [1] is a peculiar form of dynamic localization, which characterizes the transport of waves in disordered systems. The physical picture is that of a falling wave scattered by impurities of the medium. For strong disorder, this scattering process occurs in multiple scattering paths along which the linear field interferes with itself, leading to a spatially localized wave function akin to a standing wave. The phenomenon was initially discussed [1–3] for the dynamics of noninteracting electrons in a semiconductor and at present is fairly well understood [4].

Can nonlinearity or interactions destroy Anderson localization in random lattices? The question—to which no unambiguous answer has been provided—challenges the classic theories of localization [5], which assume that the lattice is static and the absence of interactions. Over the years, two nonequivalent approaches to the problem and respectively two different lines of inquiry have emerged.

One—dating back to Fleishman and Anderson [6]—pursued the interaction path, with a major milestone set by Altshuler *et al.* [7]. This line of inquiry has evolved over time into a vast area of research commonly referred to as many-body localization (MBL) (see Refs. [8–12] for reviews and references therein). The result of interest here is that

electron-electron interactions alone are unable to cause the relaxation and establish the thermal equilibrium, that is, one needs a thermal bath (coupling to phonons) to destroy the localized state [8].

The other line of inquiry—initiated by Pikovsky and Shepelyansky [13] and Flach *et al.* [14], and which is the main interest of the present study—followed the nonlinearity path [15,16], with most attention paid to the spreading dynamics of an initially localized wave packet of finite norm in one-dimensional (1D) nonlinear Schrödinger lattices with random potential. The process is described by the discrete Anderson nonlinear Schrödinger equation (DANSE):

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \beta |\psi_n|^2 \psi_n, \quad (1)$$

where $\psi_n = \psi(n, t)$ is the complex wave function which depends on time t and the position coordinate n ,

$$\hat{H}_L \psi_n = \varepsilon_n \psi_n + V(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \quad (2)$$

is the Anderson Hamiltonian in tight-binding approximation [1], V is the hopping matrix element, $\beta > 0$ characterizes nonlinearity, on-site energies ε_n are randomly distributed with zero mean across a finite energy range, and the probability distribution is normalized to $\sum_n |\psi_n|^2 = 1$.

The various aspects of DANSE (1) are discussed in Refs. [17–28]. The results can be summarized by a critical value $\beta = \beta_c$, above which the nonlinear field can spread indefinitely along the lattice, and below which it is Anderson localized similar to the linear field.

The correspondence between DANSE (1) and MBL is not at all trivial. Indeed, DANSE (1) is a single-particle problem, which mimics the behavior of an expanding

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Bose-Einstein condensate in 1D disorder potentials (see, e.g., Refs. [13,19,29,30]). As such, it focuses on the interplay between randomness and nonlinearity, with physical underpinnings inherited from the (mathematically related) problem of quantum localization of dynamical chaos [15,16]. In contrast, MBL shifts focus to many-particle systems of interacting fermions and the idea that such systems can behave fundamentally different from their bosonic counterparts, with major attention paid to the metal-insulator transition at finite temperatures, suggesting the possibility of a stable localized phase in the presence of interactions [7,8,31,32].

It is the purpose of the present study to demonstrate that DANSE (1) leads to asymptotic localization of the nonlinear field for *any* $\beta > 0$ as soon as a small dielectric coupling to the ambient medium is added. This coupling of the system and the environment introduces a nonlinear localization scale, which depends on dielectric loss of the material at low temperatures and the β value.

The paper is organized as follows. We introduce our approach first (Sec. II), followed by a derivation of the effective β value in the medium (Sec. III). The nonlinear Schrödinger model with dispersive nonlinearity and the associated mean-field reduction thereof are considered in Sec. IV. The nonlinear localization length is derived in Sec. V. We then augment our analysis with a demonstration that the nonlinear field can screen itself, leading to the possibility of self-localization (Sec. VI). We summarize our findings in Sec. VII.

II. PHYSICAL UNDERPINNINGS

Our idea is as follows: By its formulation, DANSE (1) refers to a wave process in a medium—not in vacuum. The effect of the medium is included via the random potential in Eq. (2), yet there might be a nonzero dielectric response from the same random medium that scatters the wave field. This dielectric response is missing in (1). We argue and confirm through the direct calculation that the dielectric response is such as to diminish (“screen”) the actual strength of nonlinear interaction compared to its strength in vacuum. The result is that the nonlinear field with screened interactions is asymptotically localized *regardless* of the β value, provided the lattice is infinite and the medium is in some sense random.

More explicitly, we start with DANSE (1), where we interpret β as the intrinsic value of the nonlinear field. We refer to this value as the bare value. Next we add dielectric coupling to the environment and wonder about the strength of nonlinear interaction under this coupling. We show that in the presence of randomness the nonlinearity becomes dispersive, that is, the actual β value starts to depend on the position coordinate n . It is the dispersive character of dielectric response from the surrounding random medium that localizes the nonlinear field in one dimension independently of the intrinsic β .

The theory approach designed above is inspired [33] by quantum electrodynamics, where one distinguishes between the bare (predicted by fundamental theory) electric charge and the actually measured charge [34,35]. The latter is somewhat smaller than the former as a result of screening by short-living electron-positron pairs surrounding the electric

charge in vacuum. Our analysis suggests the situation behind DANSE (1) is similar.

Assume for concreteness that the wave function ψ_n refers to an electromagnetic wave. It is understood that the electric field of the wave perturbs the surrounding dielectric medium causing its polarization. The action of the polarization field is that it partially screens the bare electric field of the wave, with the consequence that the actual strength of nonlinear interaction is decreased by a factor of

$$\Delta\beta/\beta \simeq -P_\omega/D_\omega, \quad (3)$$

where β is the bare value, and $\Delta\beta$ is the variation imposed by polarization. In the above, P_ω is the amplitude of the polarization field, and D_ω is the electric displacement, both frequency dependent in general. In a basic theory of dielectrics one writes the polarization and displacement fields respectively as $P_\omega = \varepsilon_0 \chi E_\omega$ and $D_\omega \equiv \varepsilon_0 E_\omega + P_\omega$, where ε_0 is the permittivity of free space, and χ is the dielectric susceptibility. Upon substitution in Eq. (3) this yields $\Delta\beta/\beta \simeq -\chi/(1+\chi)$, from which the effective control parameter is $\beta_{\text{eff}} = \beta + \Delta\beta \simeq \beta/(1+\chi)$.

If the wave function ψ_n refers to an electrically charged particle instead, then it is more convenient to calculate the effective β value by looking into charges, not fields. One has $\Delta\beta/\beta \simeq q_{\text{pol}}/q_0$, where q_{pol} is the induced polarization charge, and q_0 is the bare charge. Using for polarization charge $q_{\text{pol}} = -q_0 \chi/(1+\chi)$, one gets $\Delta\beta/\beta \simeq -\chi/(1+\chi)$, leading to $\beta_{\text{eff}} = \beta + \Delta\beta \simeq \beta/(1+\chi)$.

If one introduces the refraction index v_r , one can write $\beta_{\text{eff}} = \beta/v_r^2$, where the general expression $1+\chi = v_r^2$ has been applied. In the above we have assumed a nonmagnetic medium, for which the relative magnetic permeability can be set to 1. Note that β_{eff} behaves as the inverse of v_r *squared*. If the medium is close to its thermal equilibrium, then the refraction index v_r must be greater than 1 (for most dielectrics near the thermal equilibrium, v_r is *appreciably* greater than 1, i.e., $v_r \gg 1$) [36]. Thus it follows that the actual β resulting from renormalization is fundamentally smaller than the intrinsic β . One sees that there is an important possibility that the nonlinear field with coupling to the environment can be subcritical, with $\beta_{\text{eff}} = \beta/v_r^2 \ll \beta_c$, and hence be Anderson localized [13,23], while the corresponding vacuum field might be not.

III. EFFECTIVE β VALUE

In a more refined description, one writes χ as the real part of the complex frequency-dependent susceptibility of the medium, i.e., $\chi \equiv \text{Re } \chi(\omega)$. It is understood that $\chi(\omega)$ is a complex quantity, i.e., $\chi(\omega) = \chi_1(\omega) + i\chi_2(\omega)$, the components of which can depend on the frequency of the applied field. These frequency-dependent components are related to each other via the Kramers-Kronig (Hilbert) transform, i.e., $\chi_1(\omega) = (1/\pi)P \int \chi_2(\omega')d\omega'/(\omega' - \omega)$. In this context, one represents the effective control parameter as $\beta_{\text{eff}} = \beta/[1 + \chi_1(\omega)]$, where $\chi_1(\omega) \equiv \text{Re } \chi(\omega)$ is the real part of $\chi(\omega)$. The imaginary part $\chi_2(\omega) \equiv \text{Im } \chi(\omega)$ is expressed as $\chi_2(\omega) = \sigma_{\text{ac}}(\omega)/\varepsilon_0\omega$, where $\sigma_{\text{ac}}(\omega)$ is the real part of the complex frequency-dependent ac conductivity of the medium. We should stress that the frequency dependence of both $\chi(\omega)$ and $\sigma_{\text{ac}}(\omega)$ is particularly relevant for random systems where

spatial inhomogeneity dictates a distribution of time delays to a disordered medium's polarization response. Putting all the various pieces together, we have

$$\text{Re } \chi(\omega) = \frac{1}{\pi \varepsilon_0} P \int \frac{\sigma_{ac}(\omega')}{\omega'} \frac{d\omega'}{\omega' - \omega}. \quad (4)$$

At this point, one needs a model of ac conduction for the surrounding random medium. Of note here is that the charge carriers—the displacement of which is held responsible for the frequency-dependent P_ω —are subject to the same random potential in Eq. (2) which scatters the wave function ψ_n .

At low temperatures, the available charge conduction mechanisms fall into two main categories [37,38], i.e., transport of a carrier by quantum-mechanical tunneling through [39], or classical hopping over [40], the potential barrier separating two localized states. These both lead to linear frequency dependence of ac conductivity versus frequency—the former ensures that ac conductivity stays finite as temperature vanishes, and the latter accounts for some increase of ac conductivity as temperature rises.

More explicitly, in a basic theory of disordered dielectrics one introduces [37,39] the following general form of ac conductivity specialized to hopping conduction:

$$\sigma_{ac}(\omega) \propto \omega \int p(\tau) r^2 [\omega \tau / (1 + \omega^2 \tau^2)] d\tau, \quad (5)$$

where $p(\tau)$ is the probability distribution of relaxation times, r^2 derives from the square of the dipole moment, and the final term in square brackets is the Debye expression for dielectric loss. Because the random potential implies a spread of values (heights of the barriers), one has to integrate over τ . For given ω , the integral form above is dominated by the resonant contribution at $\omega \tau \approx 1$ (Tauberian theorem). Thus it follows that $\sigma_{ac}(\omega)$ is a linear function of frequency, i.e., $\sigma_{ac}(\omega) \propto \omega$ [37,38]. In this context, any departures from linearity arise from the possibility that the hopping distance r may be a function of ω [37].

We ought to note that temperature and frequency dependence of ac conductivity in disordered semiconductors has been the subject of very intense theoretical and experimental treatments (see, e.g., Refs. [37–43]). What is important for our purposes is that the linear scaling $\sigma_{ac}(\omega) \propto \omega$ applies universally for $T \rightarrow 0$ [43–46], where T is temperature, and that it is confirmed experimentally that in the absence of measurable dc conductivity the ac counterpart behaves linearly with ω [38,43].

Summarizing the above reasoning, we write the ac conductivity as $\sigma_{ac}(\omega) = A\omega$, where A is a coefficient, which can depend on temperature T , but not on the frequency ω . In material science [38,42,46], it is customary to express this coefficient as $A = \varepsilon_0 \varepsilon_r \tan \delta$, where ε_r is the relative permeability, and $\tan \delta$ is the dielectric loss of the material. With these definitions, Eq. (4) reduces to $\text{Re } \chi(\omega) = \mu P \int d\omega' / (\omega' - \omega)$, where we have introduced $\mu \equiv A / \pi \varepsilon_0 = (\varepsilon_r / \pi) \tan \delta$.

In the absence of meaningful dc conductivity, the integration in Eq. (4) can be performed from ω (which is assumed to be comparable to, yet somewhat greater than, the dielectric loss peak frequency, ω_m) and up to the actual frequency spread of the wave packet, which we shall denote by $\Delta\omega$.

Assuming a broad wave packet, with $\Delta\omega \gg \omega \gtrsim \omega_m$, from Eq. (4) one obtains, with logarithmic accuracy, $\text{Re } \chi(\omega) \simeq \mu \log |\Delta\omega/\omega|$. Translating frequencies into wave numbers using $\omega = v/n$, where v is the speed of the wave, one gets $\text{Re } \chi(n) \simeq \mu \log |\Delta n/n|$. If, however, the spread $\Delta\omega$ is large compared to ω , then the ratio $|\Delta n|/n$ is also large, i.e., $|\Delta n| \gg n$, permitting one to write the real part of χ as $\text{Re } \chi(\Delta n) \simeq \mu \log |\Delta n|$, without keeping an eye on n . To this end, the effective β value becomes $\beta_{\text{eff}} = \beta / [1 + \text{Re } \chi(\Delta n)] \simeq \beta / (1 + \mu \log |\Delta n|)$. Writing, under the logarithm sign, the spread Δn as $|n - n'|$, where $|n - n'| \gg 1$, one arrives at $\text{Re } \chi(n, n') \simeq \mu \log |n - n'|$, from which

$$\beta_{\text{eff}} = \beta_{\text{eff}}(n, n') \simeq \beta / (1 + \mu \log |n - n'|). \quad (6)$$

IV. FAILURE OF DANSE AND THE MODEL WITH DISPERSIVE NONLINEARITY

One sees, from Eq. (6), that the effective β value is dispersive (depends on the position coordinate where it is calculated). To this end, the DANSE model in Eq. (1)—which assigns one single value of β to all sites—becomes insufficient. That is, one needs to employ a more general dynamical equation that would directly take into account the dispersive character of interactions taking place. Here, we work with the nonlinear Schrödinger model with dispersive nonlinearity [28,47]:

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \sum_{n'} \beta_{n,n'} \frac{|\psi_{n'}|^2}{|n - n'|} \psi_n, \quad (7)$$

where $\beta_{n,n'} \equiv \beta(|n - n'|) > 0$ is the position-dependent nonlinearity parameter, the nonlocal kernel $\propto 1/|n - n'|$ represents the Poisson potential (here associated with electrostatic interaction), and $\hat{H}_L$ is the Anderson Hamiltonian of the type given by Eq. (2). Note that the nonlocal term in Eq. (7) is exact in the one-particle limit and does not require any further approximation such as, for instance, the dipolar approximation [48] or any other approximation alike [49]. Note, also, that Eq. (7) differs from the familiar Schrödinger-Newton equation (SNE) [50–54] in that the SNE invokes neither randomness nor dispersive interactions. Identifying, in dynamical Eq. (7), the position-dependent $\beta_{n,n'}$ with the effective value in Eq. (6), one obtains

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \sum_{n'} \frac{\beta}{1 + \mu \log |n - n'|} \frac{|\psi_{n'}|^2}{|n - n'|} \psi_n. \quad (8)$$

If $\mu \ll 1$, i.e., the medium is a poor conductor, then one can substitute the logarithmic form $1 + \mu \log |n - n'|$ with the algebraic power $|n - n'|^\mu$, i.e., $1 + \mu \log |n - n'| \approx |n - n'|^\mu$, leading to

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \beta \sum_{n'} \frac{|\psi_{n'}|^2}{|n - n'|^{1+\mu}} \psi_n, \quad (9)$$

which is equivalent to some modification of the Poisson potential in the medium. For a broad wave packet, characterized by $\Delta\omega \gg \omega$, the model in Eq. (9) simplifies to a superquadratic modification of DANSE (1), i.e.,

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \beta |\psi_n|^{2s} \psi_n, \quad (10)$$

where $s = 1 + \mu$. In fact, if the nonlinear field is spread over a large number of states $\Delta n \gg 1$, and if the variation is so smooth that $|\psi_{n+1} - \psi_{n-1}|/2|\psi_n| \ll 1$, then one can replace the actual separation $|n - n'|$ in Eq. (9) with the characteristic spread Δn , take $1/|\Delta n|^{1+\mu}$ out of the summation sign, and sum over n' using the conservation of the total probability $\sum_{n'} |\psi_{n'}|^2 = 1$. On the other hand, if $\Delta n \gg 1$, then $\sum_{n'} |\psi_{n'}|^2 = 1$ dictates $|\psi_n|^2 \simeq 1/\Delta n$, from which $1/|\Delta n|^{1+\mu} \simeq |\psi_n|^{2(1+\mu)}$. Eliminating Δn from the nonlinear term with the aid of $\beta/|\Delta n|^{1+\mu} \simeq \beta|\psi_n|^{2(1+\mu)}$, one is led to Eq. (10), where $s = 1 + \mu > 1$.

V. NONLINEAR LOCALIZATION LENGTH

Equation (10) predicts that the nonlinear field can spread along the lattice up to a certain finite distance only, beyond which it is Anderson localized similar to the linear field. This is because the nonlinear term in Eq. (10) acts as a stochastic pump [13,55], the strength of which decays while spreading [24,56]. In fact, if the nonlinear field is spread over Δn states, then the nonlinear frequency shift is $\Delta\omega_{\text{NL}} \simeq \beta|\psi_n|^{2s} \simeq \beta/|\Delta n|^s$, while the distance between the eigenstates approaches zero as $\delta\omega \simeq 1/|\Delta n|$. Hence it follows that the Chirikov overlap parameter [57,58] decays with Δn as

$$K \equiv \Delta\omega_{\text{NL}}/\delta\omega \simeq \beta/|\Delta n|^\mu. \quad (11)$$

If $K \gg 1$, then the wave function in Eq. (10) behaves as a chaotic system, meaning the integrals of motion are destroyed, and thus the nonlinear field can expand in the n direction via a stochastic process. In contrast, if $K \ll 1$, then the behavior is regular, and the field is localized.

Assume the wave function is initially so narrow (Δn is so small at the time $t = 0$) that $K \gg 1$ (that is, the conditions for chaos are satisfied with a large margin). Then the wave field starts to spread along the lattice driven by the Kerr nonlinearity. On the other hand, because K decreases with Δn increasing, this spreading process naturally comes to a halt as soon as the K value in Eq. (11) drops down to unity, i.e., $K \simeq 1$. Further spreading is forbidden by the transition to regularity and the K values becoming less than 1.

One sees that the nonlinear field automatically (without tuning of parameters) evolves into a marginally regular state at the border of chaos and saturates exactly at that border. The phenomenon is similar to *self-organized criticality* [59–61], with that particularity that the driving noise term is incorporated directly in the dynamical equation. By the time regularity is reached the field has spread over $|\Delta n|_{\text{max}} \simeq \exp[(1/\mu) \ln \beta]$ states. In a 1D model, this is equivalent to a nonlinear localization scale:

$$\Lambda_{\text{loc}} \simeq \exp[(\pi/\varepsilon_r \tan \delta) \ln \beta], \quad (12)$$

where $\tan \delta$ is the dielectric loss tangent, and we have used $\mu = (\varepsilon_r/\pi) \tan \delta$. If losses are absent, then $\tan \delta$ vanishes, leading to $\Lambda_{\text{loc}} \rightarrow +\infty$. That means that the nonlinear field without dielectric coupling to the environment can spread to arbitrarily long spatial scales, provided $K \gg 1$. This recovers the type of dynamics considered previously [13,14,17,21–23] in the framework of DANSE (1). Note that the dependence of Λ_{loc} on $\mu \rightarrow +0$ is nonperturbative, as it should be [23,24].

VI. SELF-SCREENING

Let us now explore a possibility that the “medium” to which the wave field is dielectrically coupled is the nonlinear field itself. The problem—which makes sense for broad wave packets—leads to a phenomenon that one may characterize as self-screening. The self-screening occurs because the Kerr nonlinearity $\sim \beta|\psi_n|^2$ causes the components of the wave field to overlap. This destabilizes the wave function at some locations, around which the wave field turns into a chaotic state. If such states appear sufficiently close to each other, then the nonlinear field can propagate between the different chaotic states by either tunneling through, or hopping over, the regular states in between. The process is similar to the transport mechanism considered by Elliott [37,38], with chaotic states thought of as the analog localized states.

More explicitly, in the basis of linearly localized modes, the evolution of the components a_k of the wave field is due to their nonlinear coupling, that is,

$$i\dot{a}_k = \beta \sum_{m_1, m_2, m_3} a_{m_1} a_{m_2}^* a_{m_3}, \quad (13)$$

where the star denotes the complex conjugate; the upper dot means time differentiation; $a_{-m} = a_m^*$ by way of temporal translation symmetry; the summation goes over all admissible wave numbers m_1 , m_2 , and m_3 , both positive and negative; and we have set $\hbar = 1$ for simplicity. The dynamical equations (13) can be obtained straightforwardly following the technique described in Refs. [23,24]. We do not repeat those derivations here.

If all couplings are next-neighbor-like (a situation of prime interest as such couplings have the largest cross section), then the evolution equation for a_k reduces to

$$i\dot{a}_k = 2\beta a_{k-1} a_k^* a_{k+1}, \quad (14)$$

which is obtained from Eq. (13) by letting $m_1 = k \mp 1$, $m_2 = k$, and $m_3 = k \pm 1$.

A. Mapping onto a Cayley graph

If $k \rightarrow \pm\infty$, then the set of dynamical equations (14) defines an infinite chain process with the topology of a Cayley tree [23]. This is shown by constructing a one-to-one correspondence between (14) and a Cayley graph as follows. Each cyclic term $i\dot{a}_k$ with the wave number k and the eigenfrequency $\omega_k = \omega(|k|)$ is represented as a node in the graph space labeled by the coordinate k . The coupling links between the different nodes are represented as bonds. From the structure of the nonlinear term in Eq. (14) one sees that there will be three and only three bonds at each node—one ingoing bond representing the wave process with amplitude a_k^* , and two outgoing bonds representing the processes with amplitudes a_{k-1} and a_{k+1} , respectively (see Fig. 1).

It is assumed, following Refs. [23,28], that each nonlinear oscillator with the equation of motion (14) can reside in either a chaotic (with probability p) or regular (with probability $1 - p$) state. The probability for an oscillator to be in the chaotic state is given by the Boltzmann factor $p = \exp(-\delta\omega/\Delta\omega_{\text{NL}})$, where $\delta\omega \simeq 1/\Delta n$ is the distance between Anderson

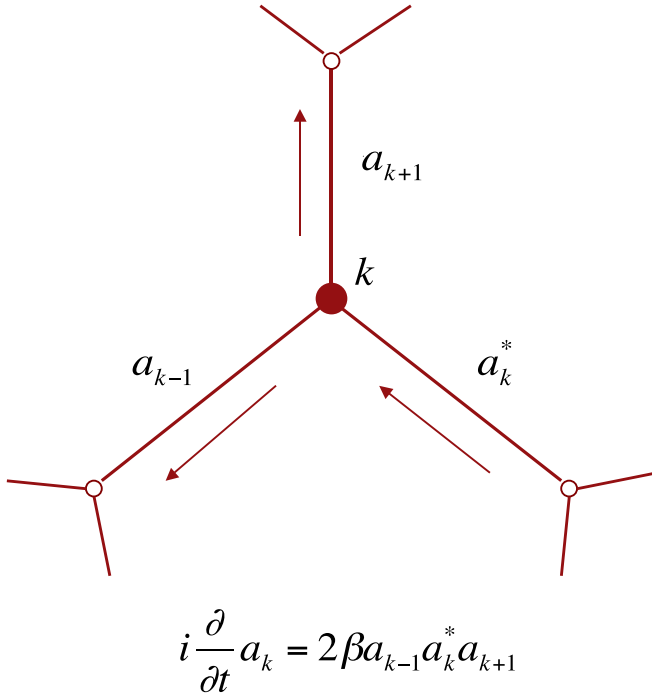


FIG. 1. Mapping dynamical equations (14) on a Cayley graph. Arrows indicate ingoing (a_k^*) and outgoing (a_{k-1} and a_{k+1}) wave processes at each node k . Thus the interaction term in Eq. (14) defines an infinite (for $k \rightarrow \infty$) Cayley graph with the coordination number equal to 3 exactly.

eigenstates in the frequency domain, and $\Delta\omega_{\text{NL}} = \beta|\psi_n|^2$ is the nonlinear frequency shift. Using here that $\sum_{n'} |\psi_{n'}|^2 = 1$ implies $|\psi_n|^2 \simeq 1/\Delta n$ for $\Delta n \gg 1$, one gets $p = \exp(-1/\beta)$ independently of Δn , that is, p is preserved through dynamics.

If $\beta = 0$, i.e., the problem is linear, then $p = 0$, so all oscillators are in the regular state, and the field is localized. If, however, $\beta \rightarrow \infty$, i.e., the nonlinear interaction is extremely strong, then $p \rightarrow 1$, and the space is filled with oscillators in the chaotic state.

The key step is to observe that there exists a critical p value, which is a fraction between 0 and 1, for which the oscillators in the chaotic state may form an infinite connected cluster for the first time ever, though without yet filling the ambient space.

This critical value is none other than the point of percolation transition in the graph space. As is well known [62], the percolation point on a Cayley graph lies at $p_c = 1/(z - 1)$ exactly, where z is the number of bonds at each node (the so-called coordination number [63]). For the Cayley graph in Fig. 1, this number is equal to 3.

Of note here is that the number $z = 3$ is exact and defined by the topology of nonlinear interaction in dynamical equations (14). The following lattice rule is key: One ingoing wave with the amplitude a_k^* and two outgoing waves with amplitudes a_{k-1} and a_{k+1} , provided the necessary conditions for the energy and momentum conservation are matched. This ever ramifying branching process dictates $z = 3$ in the graph space. Letting $z = 3$ in $p_c = 1/(z - 1)$ one gets $p_c = 1/2$ exactly.

B. Ac conductivity at percolation

If p is not too close to zero yet smaller than the percolation threshold probability $p_c = 1/2$, then the chaotic states are clustered and do *not* constitute a percolation path throughout the field (and hence do not contribute to the dc conduction). If, however, hops can occur between the adjacent chaotic states only, and if this process occurs near its percolation threshold, then the dependence of ac conductivity over the frequency is given by the power law [45,64]

$$\sigma_{\text{ac}}(\omega) \propto \omega^\eta, \quad (15)$$

where $\eta = (\theta + d - d_f)/(2 + \theta)$ is the power exponent, d_f is the Hausdorff dimension of chaotic clusters, and θ is the connectivity index [65]. Using known values $d = 6$, $d_f = 4$, and $\theta = 4$ for mean-field percolation on Cayley trees [66,67], one gets $\eta = 1$ exactly, hence $\sigma_{\text{ac}}(\omega) \propto \omega$. This linear scaling extends the classic result of Elliott [37,38] in Eq. (5) to DANSE (1).

C. Nonlinear localization length

Once the linear scaling $\sigma_{\text{ac}}(\omega) \propto \omega$ is established, one repeats the renormalization procedure described above starting from the Kramers-Kronig transform (4) to obtain, in place of DANSE (1), the nonlinear Schrödinger equation with superquadratic nonlinearity, Eq. (10), where the exponent of superquadratic power is written as $2s = 2(1 + \mu_0)$, $\mu_0 = A_0/\pi\epsilon_0 \ll 1$, and A_0 is the coefficient of ac conduction pertaining to the hopping motion. This type of nonlinearity steers automatically the nonlinear field to the margins of regularity via the dependence of the Chirikov overlap parameter $K \equiv \Delta\omega_{\text{NL}}/\delta\omega \simeq \beta/|\Delta n|^{\mu_0}$ on the number of states. As a consequence, the nonlinear field gets localized, with the nonlinear localization length

$$\Lambda_{\text{loc}} \simeq \exp[(1/\mu_0) \ln \beta]. \quad (16)$$

We define the range of validity of this result from the p values lying beneath the percolation point—provided just that the chaotic states can form connected clusters—to the onset of percolation at $p_c = 1/2$, the percolation point included (because the dc conductivity is still zero for $p = p_c$ exactly [66,67]).

To this end, our finding is as follows: The nonlinear field of the type given by the DANSE (1) is Anderson localized both at ($p = p_c$) and below ($p < p_c$) the percolation threshold. This is different from a conclusion drawn previously [23] that the nonlinear field in Eq. (1) is delocalized as soon as $p = p_c$. Here, we argue it is asymptotically localized through self-screening and the linear dependence $\sigma_{\text{ac}}(\omega) \propto \omega$ in proximity to the percolation transition. Note that $\Lambda_{\text{loc}} \simeq \exp[(1/\mu_0) \ln \beta]$ can be greater than the lattice size assumed in simulations, if $\mu_0 \ll 1$ is small enough.

D. Asymptotic scaling laws in proximity to percolation

With use of $p = \exp(-1/\beta)$ one can translate $p_c = 1/2$ into a critical β value above which the nonlinear field is delocalized by way of self-screening: $\beta_c = 1/\ln 2$. For the β values greater than this, asymptotic localization is only possible through coupling to an external medium.

If $p \rightarrow p_c$ from below, then one obtains the dispersion of field spreading as the mean-square displacement of a random walker on the incipient percolation cluster, leading to [64,67]

$$\langle (\Delta n)^2(t) \rangle \propto t^{2/(2+\theta)}, \quad (17)$$

where θ is the connectivity index. Using $\theta = 4$ for percolation on trees [66], one gets

$$\langle (\Delta n)^2(t) \rangle \propto t^{1/3}. \quad (18)$$

This scaling law has been observed numerically in Refs. [14,17,21] and associated with weak chaos. Here, we note that the linear scaling of the ac conductivity versus frequency, $\sigma_{ac}(\omega) \propto \omega$, and the subdiffusive scaling law in Eq. (18) derive universally and jointly from the same basic assumption that the transport of the wave field occurs through a next-neighbor random walk between chaotic states in the vicinity of percolation. Both scalings are equally important to understand the delocalization transition in the absence of coupling to environment, with the linear dependence $\sigma_{ac}(\omega) \propto \omega$ leading directly to self-screening.

VII. SUMMARY

The results of this work may be stated as follows. Augmenting DANSE (1) with a finite susceptibility response from the environment leads to asymptotic localization of the nonlinear field regardless of the initially assumed β value. The nonlinear localization length, Eq. (12), depends exclusively on dielectric loss of the surrounding random medium, its relative permeability ε_r , and the nonlinearity strength. If the dielectric susceptibility is set to zero exactly, then the DANSE (1) predicts sharp localization-delocalization transition at $\beta = \beta_c = 1/\ln 2$ consistently with previous analysis [23,28].

Of note here is that the dielectric loss takes a finite value at zero temperature (because the ac conductivity remains finite as temperature vanishes), meaning the nonlinear localization length in Eq. (12) is finite for $T \rightarrow 0$.

More generally and more importantly, we have seen that the occurrence of a nonlinear localization state is controlled by the ac, rather than dc, properties of the medium (because dielectric loss at zero temperature is due to the ac conductivity response, while the dc conduction becomes negligible). Also we have seen that the correct dynamical model with coupling

to the environment is dispersive, that is, the actual β value in the medium depends on the position coordinate.

The comprehension of dispersive interactions has led us to a nonlocal dynamical equation, Eq. (7), which includes a sum over n' in the Kerr term. The latter equation differs from the familiar SNE equation in that it (i) is dispersive and (ii) confronts the Poisson potential due to electrostatic interaction against the random potential due to underlying disorder.

For broad wave packets, the dispersive equation (7) reduces to a superquadratic generalization of the DANSE (1) and is represented in a final form by Eq. (10).

We have seen that the superquadratic equation (10) is fundamentally different from its DANSE counterpart in Eq. (1) in that the former does *not* preserve the property for the wave field to be chaotic or regular, while the latter equation does. More explicitly, if the wave field in Eq. (10) starts as chaotic, it becomes regular through nonlinear feedback dynamics, where the feedback is provided by the Chirikov overlap parameter depending on the number of states, Eq. (11). This transition to regularity is self-organized and explains the asymptotic localization of the nonlinear field regardless of the β value.

Finally, we have discussed the physics underpinnings of the superquadratic model (10), as well as its implications for the phenomena of self-screening and self-localization, the main theory predictions of the present study.

The mathematical methods, stipulated here, pave the way towards understanding the wave processes in complex media with competition between randomness, dispersiveness, and nonlinearity, such as Anderson localization of a broad wave packet interacting with the environment. Further applications can be found in the realm of wave turbulence [68,69] and will be reported in a forthcoming publication.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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