

Hydrodynamic fluctuations of stochastic charged cellular automata

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We study charge fluctuations of a family of stochastic charged cellular automata away from the deterministic single-file limit and obtain the exact typical charge probability distributions, known to be anomalous, using hydrodynamics. The cellular automata considered are examples of linearly degenerate systems where two distinct mechanisms of diffusion, namely, normal and convective diffusion, coexist. Our formalism, based on macroscopic fluctuation theory, allows us to describe current fluctuations stemming from these two diffusive processes, and we expect it to be applicable to generic linearly degenerate systems. The derived probability distributions match the exact microscopic result and numerical simulations.

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I. INTRODUCTION

Hydrodynamics is a powerful tool that allows us to describe the large-scale dynamics of systems in a universal way. Although hydrodynamics traditionally deals with the dynamics of mean values of observables, recently its scope has been extended to hydrodynamic fluctuations prescribed by correlation functions at hydrodynamic scales, culminating in the inception of macroscopic fluctuation theory (MFT) [1] and its ballistic generalization, ballistic MFT (BMFT) [2,3]. Armed with experimental advances in probing the dynamics of cold atoms (e.g., quantum gas microscopy [4–6]), there has also been growing interest in experimental observations of hydrodynamic fluctuations, in part also fueled by the discovery of anomalous spin current fluctuations in the isotropic Heisenberg chain [7,8], tantalizingly hinting at their surprising connection [9–14] with the Kardar-Parisi-Zhang universality class [15–17].

While hydrodynamic fluctuations in generic nonintegrable and integrable systems have been rather well understood so far, there is a class of systems whose hydrodynamic fluctuations are underexplored: linearly degenerate systems [18–22]. Loosely speaking, these are nonintegrable systems that are as close to integrability as possible in the sense that, in linearly degenerate systems, fluid modes do not self-interact. Such self-interactions are indeed also absent in integrable systems as trajectories of quasiparticles are only influenced by scatterings with other quasiparticles [23,24]. The lack of self-interactions precludes the standard mechanism of superdiffusion [25,26], whereby leading corrections to ballistic

transport become diffusive. Importantly, there are two mechanisms of diffusion, namely, normal and convective [27,28] diffusion. The former is the standard source of diffusion in nonintegrable systems whereas the latter, which often goes by the name of diffusion from convection, is an unconventional mechanism of diffusion and unique to linearly degenerate systems. While generically these two diffusion mechanisms coexist and affect typical fluctuations (hydrodynamic fluctuations at the diffusive scale) in linearly degenerate systems, it has remained unclear how to describe them consistently using hydrodynamic ideas. In this work, we address this issue by providing a framework based on MFT that enables us to systematically characterize typical fluctuations in generic linearly degenerate systems. It should also be emphasized that these systems are in fact not uncommon; recently, some models with particle-hole symmetry such as Dirac fluids [29] and quasi-one-dimensional Hamiltonian systems [30] have been shown to be linearly degenerate.

As a concrete example to which we apply our framework, we focus on a family of stochastic charged cellular automata (SCCA). Their dynamics have been solved microscopically [31–33], including dynamical current fluctuations [34–36]. Strikingly, the SCCA exhibit anomalous fluctuations [7,29,34,37], manifested in a non-Gaussian typical charge current probability distribution. Recently, in Ref. [38] we have developed a MFT-based hydrodynamic approach to evaluate hydrodynamic fluctuations at the deterministic point of the SCCA where normal diffusion is absent, successfully reproducing earlier microscopic results [34]. Our hydrodynamic framework, building on the approach developed in Ref. [38], recovers the typical charge fluctuations of the SCCA obtained recently in Ref. [36] using methods of integrable combinatorics.

II. LINEARLY DEGENERATE SYSTEMS

Consider a continuous one-dimensional quantum many-body system (generalization to higher dimensions, classical

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systems, etc., is straightforward) with N conserved charges $\hat{Q}_i = \int dx \hat{q}_i(x)$, where the density $\hat{q}_i(x)$ satisfies the continuity equation $\partial_t \hat{q}_i(x, t) + \partial_x \hat{j}_i(x, t) = 0$ with the associated current $\hat{j}_i(x)$. Since linear degeneracy is a concept associated to hyperbolic systems of conservation laws, we assume ballistic transport in the system. Namely, we assume that the flux jacobian $\mathbf{A}_i^j[\langle \hat{q} \rangle] = \partial \langle \hat{j}_i \rangle / \partial \langle \hat{q}_j \rangle$ (here the underline indicates that the quantity is a vector), where the average $\langle \dots \rangle$ is with respect to a generic generalized Gibbs ensemble (GGE), is a nonzero matrix. The corresponding Euler hydrodynamic equation for the Euler scale average densities $\mathbf{q}_i(x, t) = \lim_{\tau \rightarrow \infty} \langle \hat{q}_i(\tau x, \tau t) \rangle$ and currents $\mathbf{j}_i(x, t) = \lim_{\tau \rightarrow \infty} \langle \hat{j}_i(\tau x, \tau t) \rangle$ reads $\partial_t \mathbf{q}_i(x, t) + \partial_x \mathbf{j}_i(x, t) = 0$, which can also be expressed as

$$\partial_t \mathbf{q}_i(x, t) + \mathbf{A}_i^j[\mathbf{q}(x, t)] \partial_x \mathbf{q}_j(x, t) = 0. \quad (1)$$

While it is natural to write down hydrodynamic equations for physical charges, a more convenient basis of hydrodynamic modes is that of *normal modes*. Diagonalizing the flux jacobian as $\mathbf{R} \mathbf{A} \mathbf{R}^{-1} = \text{diag } v_i^{\text{eff}}$ where v_i^{eff} are the eigenvalues of $\mathbf{A}$, the normal modes n_i are defined via the transition matrix $\mathbf{R}_i^j$ by the relation $\mathbf{R}_i^j = \partial n_i / \partial \mathbf{q}_j$, in terms of which Eq. (1) becomes

$$\partial_t n_i(x, t) + v_i^{\text{eff}}(x, t) \partial_x n_i(x, t) = 0. \quad (2)$$

Normal modes also elucidate the physical meaning of linear degeneracy—a hydrodynamic system, Eq. (2), is linearly degenerate precisely when *every normal mode n_i is non-self-interacting*, $\partial v_i^{\text{eff}} / \partial n_i = 0$, that is, the dynamics of a normal mode is not influenced by the density of the mode itself. This condition is nongeneric and self-interaction is known to be the source of superdiffusion in one-dimensional continuum systems [25]. A notable class of linearly degenerate systems are integrable systems where normal modes are given by quasiparticles, whose trajectories are changed only by collisions with other quasiparticles, hence the absence of self-interaction.

Since superdiffusion is avoided due to the absence of self-interactions in linearly degenerate systems [39], leading corrections to ballistic transport are expected to be diffusive. In particular, there are generically two sources of diffusion: normal and convective diffusion. Normal diffusion is a ubiquitous source of diffusion where a slow mode constantly undergoes collisions with fast modes, giving rise to a jiggling movement of the slow mode. Such movement is characterized by Fick's law for the slow mode's current while the coarse-grained fast modes are modeled as an additive white noise to ensure the fluctuation-dissipation relation.

Convective diffusion, also known as diffusion from convection [28], has a completely different origin. In linearly degenerate systems, hydrodynamic modes interact in a way that propagates initial fluctuations ballistically, and as a result, initial fluctuations survive even at finite times, contributing to the diffusive scale. This can be best appreciated in integrable systems: in these systems quasiparticles scatter purely elastically, indicating that hydrodynamic fluctuations are transferred without being dissipated.

In what follows, we focus on a specific class of linearly degenerate systems called SCCA. While our choice facilitates

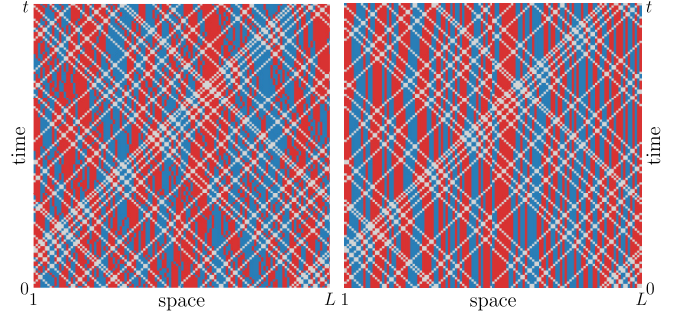


FIG. 1. Many-body trajectories of the cellular automata starting from the same initial state for stochastic (left, $0 < \Gamma < 1$) and deterministic (right, $\Gamma = 0$) single-file dynamics. Vacancies (light gray) propagate freely while neighboring positive (red) and negative (blue) particles scatter stochastically with crossing probability Γ .

analytic calculations, we expect the developed formalism and results are applicable to any linearly degenerate system.

III. THE MODEL

We consider a family of SCCA [32] on a one-dimensional lattice of $L \in 2\mathbb{N}$ sites. The lattice configuration at fixed time $t \in \mathbb{Z}$ is given by a string $\mathbf{s}^t \equiv s_1^t s_2^t \dots s_L^t$ of symbols $s_x^t \in \{\emptyset, +, -\}$, which correspond to vacancies and positively or negatively charged particles, respectively. Symbol dynamics are locally given by a one-parameter stochastic two-body map $(s_L^t, s_R^t) = \Phi(s_L, s_R)$ that encodes the following dynamical rules $(\emptyset, \emptyset) \rightarrow (\emptyset, \emptyset)$, $(\emptyset, c) \leftrightarrow (c, \emptyset)$ while $(c, c') \rightarrow (c', c)$ with crossing probability $0 \leq \Gamma \leq 1$ and $(c, c') \rightarrow (c, c')$ with reflection probability $\bar{\Gamma} \equiv 1 - \Gamma$ where $c, c' \in \{-, +\}$. The many-body dynamics is realized as a "brickwork" circuit obtained by imposing periodic boundary conditions, $s_x^t = s_{L+x}^t$, and coupling alternating sites as $\mathbf{s}^{2t+2} = \Phi^{\text{odd}}(\mathbf{s}^{2t+1})$ and $\mathbf{s}^{2t+1} = \Phi^{\text{even}}(\mathbf{s}^{2t})$, where

$$\Phi^{\text{odd}} = \prod_{x=1}^{L/2} \Phi_{2x-1, 2x}, \quad \Phi^{\text{even}} = \prod_{x=1}^{L/2} \Phi_{2x, 2x+1} \quad (3)$$

and $\Phi_{x, x+1}$ acts nontrivially only on sites x and $x+1$ (see Fig. 1 for representative many-body trajectories). For $\Gamma = 0$ the dynamics is deterministic and corresponds to *single-file* dynamics [31, 33–35, 40] where the only source of diffusive scale dynamics is convective diffusion [38], while for $\Gamma = 1$ the dynamics is free with no diffusive dynamics.

We restrict our study to a minimal subset of conserved charges, comprising the number of right- and left-moving particles $\hat{Q}_{\pm}$ and the total charge $\hat{Q}_c$, all of which are given as sums of local densities $\hat{Q}_i(\mathbf{s}) = \sum_{x=1}^L \hat{q}_{x,t}^i(\mathbf{s})$ where $i \in \mathcal{I} \equiv \{+, -, c\}$ and a caret indicates a microscopic quantity [see Supplemental Material (SM) [41] for details on charge densities and discrete space-time continuity relations]. The SCCA are superintegrable and support an exponential (in L) number of local conserved quantities [42] that are invariant under charge conjugation due to the underlying free particle dynamics. Note, however, that, for $\Gamma \neq 1$, only the total charge is odd under charge conjugation, while for $\Gamma = 1$ there exist exponentially many conserved quantities odd under charge conjugation.

We consider GGEs of the form $\mathbb{P}(\mathbf{s}) = e^{\beta^i \hat{Q}_i(\mathbf{s})} / \sum_{\mathbf{s}'} e^{\beta^i \hat{Q}_i(\mathbf{s}')}$, where repeated upper and lower indices indicate summation. The average of an observable $o(\mathbf{s})$ is accordingly $\langle o \rangle = \sum_{\mathbf{s}} \mathbb{P}(\mathbf{s}) o(\mathbf{s})$. Identifying the ensemble parameters as $\rho_{\pm} = [1 + e^{-\beta_{\pm}} / (2 \cosh \beta_c)]^{-1}$, $b = \tanh \beta_c$, the ensemble factorizes $\mathbb{P}(\mathbf{s}) = \prod_{x=1}^{L/2} p_+(s_{2x}) p_-(s_{2x+1})$ in terms of normalized one-site measures $p_{\pm}(\pm) = \rho_{\pm} \frac{1 \pm b}{2}$, $p_{\pm}(\emptyset) = \bar{\rho}_{\pm}$ with $0 \leq \rho_{\pm} \leq 1$ and $\bar{\rho}_{\pm} \equiv 1 - \rho_{\pm}$ the densities of right (left) movers and vacancies on right (left) running diagonals, respectively.

The dynamics of a GGE comprising the three charges $\hat{Q}_{\pm}$ and $\hat{Q}_c$ is closed, i.e., at the Euler scale their densities evolve without reference to the dynamics of other local conserved charges. This is reflected in the structure of the three-mode Euler hydrodynamics obtained by evaluating the flux jacobian [38]

$$[\mathbf{A}[\underline{\rho}]]_i^j = \frac{\partial j_i[\underline{\rho}]}{\partial \rho_j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ b\rho_-/\rho & -b\rho_+/\rho & v \end{pmatrix}, \quad (4)$$

where the three modes are labeled as $\rho_1 = \rho_+/2$, $\rho_2 = \rho_-/2$, and $\rho_3 = \rho_c$, and $v = p/\rho$ is the charge velocity with $p = \rho_1 - \rho_2$ and $\rho = \rho_1 + \rho_2$. Here $j_i[\underline{\rho}]$ is the average current defined via the continuity equation $\partial_t \varrho_i + \partial_x j_i[\underline{\rho}] = 0$ where ϱ_i is a space-time dependent density. While the dynamics of the left and right movers are trivial $\partial_t \varrho_{\pm} \pm \partial_x \varrho_{\pm} = 0$, charge transport is nontrivially affected by particle dynamics; the Euler equation accordingly becomes

$$\partial_t \varrho_c + \partial_x (v \varrho_c) = 0. \quad (5)$$

Since the Euler hydrodynamic equations of the SCCA are manifestly linearly degenerate, corrections to the ballistic transport in the SCCA are generically diffusive as noted previously. Both normal and convective diffusion are present in the stochastic dynamics (for $0 < \Gamma < 1$), but the former disappears in the single-file limit $\Gamma = 0$, which is the case treated in our previous work, Ref. [38]. In what follows we characterize the typical charge current probability distribution when both diffusion mechanisms coexist using an idea based on MFT.

IV. TYPICAL CURRENT FLUCTUATIONS FROM MFT

A standard object used to probe the statistics of the time-integrated charge current $\hat{J}(t) = \int_0^t dt' \hat{j}_c(0, t')$ is the generating function $\langle e^{i\lambda \hat{J}(t)} \rangle = \int d\lambda e^{i\lambda \hat{J}} \mathcal{P}(J|t)$ of the probability distribution $\mathcal{P}(J|t)$. We are presently interested in the typical probability distribution for currents of magnitude $\hat{J}(t) \sim t^{1/2z}$, where the dynamical exponent z is set by the variance $\langle \hat{J}^2 \rangle^c(t) \sim t^{1/z}$. We accordingly define the normalized typical probability distribution $\mathcal{P}_{\text{typ}}(j) = \lim_{t \rightarrow \infty} t^{1/2z} \mathcal{P}(jt^{1/2z}|t)$. On the other hand, large current fluctuations of order t , i.e., $\hat{J}(t) \sim t$, satisfy a large deviation principle [43] of the form $\mathcal{P}(jt|t) \asymp e^{-tI(j)}$, where $I(j)$ is the rate function.

The exact microscopic probability distribution of the charge current in the single-file limit $\Gamma = 0$ of the SCCA was first obtained analytically using combinatorial techniques in Ref. [34] and later reproduced in Ref. [38] based on a

hydrodynamic approach we also employ here. While these studies focused on the deterministic case, more recently the full probability distribution for generic $\Gamma > 0$ was microscopically obtained by mapping the charge dynamics to the stochastic six vertex model in Ref. [36].

The fundamental tenet of MFT (and its ballistic generalization BMFT) is that for large fluctuations $\hat{J}(t) \sim t$, the generating function $\langle e^{i\lambda \hat{J}(t)} \rangle$ is fully determined by the saddle point of a path integral over fluctuating hydrodynamic fields. This means that MFT allows us to evaluate the generating function and in general any physical quantities controlled by large fluctuations using solely hydrodynamic data. In our recent work, Ref. [38], we have applied a similar idea to typical fluctuations and successfully reproduced the exact typical charge probability distribution for the SCCA at the deterministic point (see also a similar idea introduced in Ref. [29]). As we demonstrate below, we extend this approach in a way that applies to any linearly degenerate system, in which case diffusion has two distinct origins. We first outline the formalism for generic linearly degenerate systems and then apply it to the SCCA.

Let $q_i(x, t) = \hat{q}_i(\tau^{1/z} x, \tau t)$ be the fluctuating hydrodynamic fields at the typical scale, where $\tau \gg 1$ is the time over which we evaluate the current statistics. Note that the dynamical exponent can vary depending on the chosen charge e.g., $z = 2$ for charge transport while $z = 1$ for left (right) movers in the SCCA. At the typical scale the fields are initially Gaussian distributed by the central limit theorem and can be decomposed as

$$q_i(x, 0) \simeq \mathbf{q}_i + \tau^{-1/2z} \delta q_i(x, 0), \quad (6)$$

where $\mathbf{q}_i$ is the initial average density and $\delta q_i(x, 0)$ is a white noise whose variance is set by the susceptibility matrix $\mathbf{C}_{ij} = \partial \mathbf{q}_i / \partial \beta^j$. For simplicity we assume that the initial GGE is chosen such that $\mathbf{j}_i = 0$ for the charge whose statistics we study. With this initial condition, we claim that the fluctuating fields $q_i(x, t)$ evolve according to the following fluctuating hydrodynamic equation:

$$\partial_t q_i + \tau^{1/z} \mathbf{A}_i^j[\underline{q}] \partial_x q_j + \partial_x \left(-\frac{1}{2} \mathbf{D}_c^i \partial_x q_i + \eta_i \right) = 0, \quad (7)$$

where $\mathbf{D}_i^j$ is the *projected* diffusion matrix and η_i is a white noise $\langle \eta_i(x, t) \eta_j(x', t') \rangle = \tau^{-1/2} \sigma_{ij} \delta(x - x') \delta(t - t')$ with the noise strength set by the projected Onsager matrix $\sigma_{ij} = \int dt dx \langle \hat{j}_i(x, t) \hat{j}_j(0, 0) \rangle^c - \sigma_{ij}^{\text{conv}}$. Here $\sigma_{ij}^{\text{conv}}$ refers to the convective contribution to the Onsager matrix obtained in Ref. [28] (see Appendix B for its explicit expression), which stems convective diffusion, and the projected diffusion matrix is accordingly related to the projected Onsager matrix via the fluctuation-dissipation relation $\mathbf{D}\mathbf{C} = \sigma$, both of which are evaluated with respect to the initial GGE.

Some comments are in order. First, the hydrodynamic equation, Eq. (7), reduces to the Euler equation in the absence of normal diffusion, which described typical charge fluctuations in the single-file limit of the SCCA [38], consistent with the appearance of the projected diffusion matrix. Second, Eq. (7) should not be mistaken for the standard hydrodynamic equation for the averages that takes a different form: in Ref. [44], the full hydrodynamic equation for linearly degenerate systems up to diffusive corrections is obtained by

generalizing the hydrodynamic equation reported in Ref. [26], which did not include normal diffusion.

V. TYPICAL CHARGE FLUCTUATIONS OF THE SCCA

We now apply Eq. (7) to charge fluctuations in the SCCA for ensembles with vanishing total charge where the charge dynamical exponent is $z = 2$. We first determine the projected diffusion and Onsager matrices, which are informed by microscopics. Since normal diffusion is unaffected by the underlying movement of particles, the relevant elements of the diffusion matrix vanish, $D_c^\pm = 0$, leaving only $D = D_c$ to be determined. Using the techniques developed in Ref. [36], in the SM [41] we explicitly evaluate the asymptotic behavior of the charge dynamical structure factor that, on the diffusive scale, turns out to be Gaussian with variance $(1 - \bar{\Gamma}\varrho)/\bar{\Gamma}\varrho$. The charge diffusion coefficient stemming from diffusion from convection is $(1 - \varrho)/\varrho$ and the projected charge diffusion coefficient thus becomes $D = (1 - \bar{\Gamma}\varrho)/\bar{\Gamma}\varrho - (1 - \varrho)/\varrho = \gamma/\rho$, where $\gamma = \Gamma/\bar{\Gamma}$, which also gives $\sigma_{cc} = \gamma$. The hydrodynamic equation for the fluctuating charge density thus reads

$$\partial_t \varrho_c + \tau^{1/2} \partial_x (v \varrho_c) + \partial_x \left(-\frac{1}{2} D \partial_x \varrho_c + \eta_c \right) = 0 \quad (8)$$

with $\overline{\eta_c(x, t) \eta_c(x', t')} = \tau^{-1/2} D \rho \delta(x - x') \delta(t - t')$. To solve Eq. (8) at large τ , we first note that stochasticity does not influence the dynamics of left (right) movers, and their dynamics are still given by $\partial_t \varrho_\pm \pm \tau^{1/2} \partial_x \varrho_\pm = 0$, which is solved by $\varrho_\pm(x, t) = \rho + \tau^{-1/4} \delta \varrho_\pm(x \mp \sqrt{\tau} t, 0)$. The charge density then satisfies $\varrho_c(x, t) \simeq \rho b(x, t)$ at large τ , allowing us to obtain the hydrodynamic equation for $b(x, t)$ from Eq. (8). The resulting equation is solved by a convolution (see Appendix A for the derivation), which in turn gives the full solution of $\varrho_c(x, t)$ at large τ :

$$\begin{aligned} \varrho_c(x, t) &\simeq \int dy G_t(x - X_t - y) \delta \varrho_c(y, 0) \\ &+ \int_0^t ds \int dy G_{t-s}(x - X_t - y) \partial_y \eta(y, s), \end{aligned} \quad (9)$$

where $G_t(x) = e^{-x^2/(2Dt)}/\sqrt{2\pi Dt}$ is the Gaussian kernel and X_t is the characteristics of $b(x, t)$ that satisfy $dX_t/dt = v(X_t, t)$ with $X_0 = 0$. Using the solution (9), the time-integrated current $\hat{J}(\tau) = \sqrt{\tau} \int_0^\infty dx [\delta \varrho_c(x, 1) - \delta \varrho_c(x, 0)]$ is also expressed in terms of the initial density fluctuations $\delta \varrho_c(x, 0)$ and the noise. We can now write down an MFT-like path integral for the generating function. In the Appendices, we carry out the path integral and show that it can be cast as a single-variable integral

$$\begin{aligned} \overline{\langle e^{\lambda \hat{J}_c(\tau)} \rangle} &\simeq \frac{1}{Z} \int \mathcal{D} \underline{\varrho} e^{-\mathcal{C}[\underline{\varrho}] \int dx \delta \varrho_i(x, 0) \delta \varrho_j(x, 0)/2} e^{\lambda \hat{J}_c(\tau)} \\ &= \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi \Delta^2}} e^{-x^2/2\Delta^2 + (\tau/2)^{1/2} \lambda^2 \omega g^2(x/\omega)/2}, \end{aligned} \quad (10)$$

where $g^2(x) = x \operatorname{erf} x + \pi^{-1/2} e^{-x^2}$ while $\Delta^2 = 2\rho\bar{\rho}$ (with $\bar{\rho} = 1 - \rho$) and $\omega^2 = 4\gamma\rho$. The typical probability distribution assumes a one-parameter scaling form $\mathcal{P}_{\text{typ}}(j) = \omega^{-1/2} f_r(j/\omega^{1/2})$ controlled by the ratio between convective

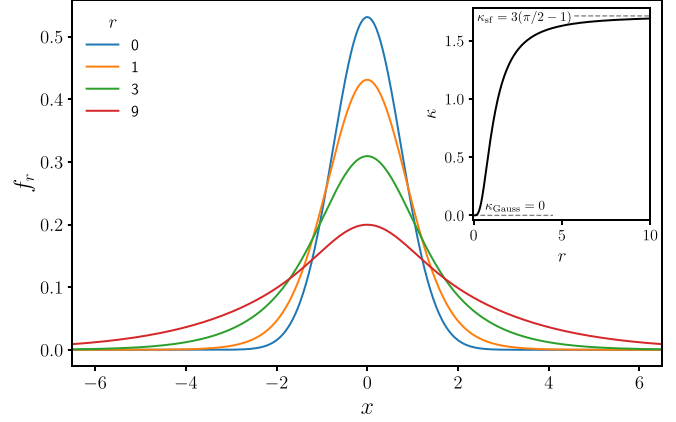


FIG. 2. Main panel: Scaling probability distribution (11) (colored curves) for different ratios r . The characteristic nonanalyticity of the single-file distribution for $r \rightarrow \infty$ is due to the divergent rescaling as $\omega \rightarrow 0$. Inset: Excess kurtosis κ (C6) of the scaling distribution (black curve) interpolates between the Gaussian value $\kappa_{\text{Gauss}} = 0$ and the single-file value $\kappa_{\text{sf}} = 3(\pi/2 - 1)$ (dashed gray lines) with increasing r .

and normal diffusion $r \equiv \Delta/\omega = \sqrt{\bar{\rho}/2\gamma}$:

$$f_r(x) = \int_{\mathbb{R}} \frac{dy}{2\pi g(ry)} e^{-y^2/2 - x^2/2g^2(ry)}. \quad (11)$$

The scaling distribution (11) (see Fig. 2) coincides with the microscopic result obtained in Ref. [36] using integrability (apart from a numerical factor due to different conventions used) and reduces to the single-file result [34,38] for $D \rightarrow 0$. We also note that in Ref. [29] a typical charge current probability distribution was obtained for Dirac fluids by numerically solving a hydrodynamic equation similar to Eq. (8).

The contributions of normal and convective diffusion to the dynamical structure factor are additive and therefore not readily distinguishable. On the other hand, knowledge of the distribution (11) allows one to quantify the relative strengths of the two sources of diffusion by determining the ratio r , e.g., by measuring the excess kurtosis $\kappa = \lim_{t \rightarrow \infty} \langle \hat{J}^4 \rangle^c(t) / [\langle \hat{J}^2 \rangle^c(t)]^2 - 3$ see inset of Fig. 2 and Eq. (C6) of Appendix C for its explicit form. We note that kurtosis measurements in quantum integrable systems are becoming feasible [45].

VI. CONCLUSIONS

We have developed a hydrodynamic framework based on MFT to describe fluctuations in linearly degenerate systems. Notably, two disparate mechanisms, namely, normal and convective diffusion, coexist in generic linearly degenerate systems, and our approach allows us to treat these diffusion processes consistently. To check the validity of our method, we studied a family of SCCA and evaluated the typical charge current probability distribution, which is known to be anomalous (non-Gaussian). The resulting distribution, Eq. (11), coincides with the microscopic result obtained recently, which in turn agrees with numerical simulations [36].

There are several directions that deserve further investigations. While linear degeneracy has a simple definition in terms

of fluid modes, it remains largely unclear how the hydrodynamic condition translates to microscopic models. In a similar vein, is it possible to systematically break integrability [46,47] in a way that linearly degeneracy is kept intact? Answering these questions would be of great interest and would give invaluable insights on the classification of nonintegrable and integrable systems. It would also be worthwhile to formulate linearly degenerate hydrodynamics up to diffusive corrections in a more mathematically rigorous way.

Note added. Recently, we became aware of a related and independent work by Doyon [44]. In Ref. [44] the same projected diffusion matrix is obtained for linearly degenerate systems by evaluating the corrections to Euler-scale correlation functions, while in this work we directly take the diffusive scaling and evaluate current fluctuations and correlation functions.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SOLUTION OF THE HYDRODYNAMIC EQUATION FOR $\varrho_c(x, t)$

According to Eq. (8), the equation for the bias field $b(x, t)$ reads

$$\partial_t b + \sqrt{\tau} v \partial_x b = \frac{1}{2} D \partial_x^2 b + \rho^{-1} \partial_x \eta_c, \quad (\text{A1})$$

which is readily solved by a convolution. Let us first notice that the solution of the equation $\partial_t b + \sqrt{\tau} v \partial_x b = 0$, which we denote by $\tilde{b}(x, t)$, can be obtained in terms of characteristics. Defining $dX_{x,t}/dt = \sqrt{\tau} v(X_{x,t}, t)$ where $X_{x,0} = x$, it follows that $\tilde{b}(X_{x,t}, t)$ is constant in time, i.e., $\tilde{b}(X_{x,t}, t) = \tilde{b}(x, 0)$. In fact, when τ is large we can neglect the x dependence of $X_{x,t}$ because

$$\begin{aligned} \frac{dX_{x,t}}{dt} &\simeq \frac{\tau^{1/4} [\delta\varrho_+(X_{x,t} - \sqrt{\tau}t, 0) - \delta\varrho_-(X_{x,t} + \sqrt{\tau}t, 0)]}{2\rho} \\ &\simeq \tau^{1/4} \frac{\delta\varrho_+(-\sqrt{\tau}t, 0) - \delta\varrho_-(\sqrt{\tau}t, 0)}{2\rho}. \end{aligned} \quad (\text{A2})$$

We can thus approximate $X_{x,t}$ as $X_{x,t} \simeq x + X_t$ and have $\tilde{b}(x, t) = \tilde{b}(x - X_t, 0) \simeq \tau^{-1/4} \delta\varrho_c(x - X_t, 0)/\rho$. Then the solution of Eq. (A1) at large τ is given by

$$\begin{aligned} b(x, t) &\simeq \int dy G_t(x - y) \tilde{b}(y, t) \\ &+ \rho^{-1} \int_0^t ds \int dy G_{t-s}(x - X_t - y) \partial_y \eta(y, s) \end{aligned}$$

$$\begin{aligned} &= \int dy G_t(x - X_t - y) \tilde{b}(y, 0) \\ &+ \rho^{-1} \int_0^t ds \int dy G_{t-s}(x - X_t - y) \partial_y \eta(y, s), \end{aligned} \quad (\text{A3})$$

which yields Eq. (9).

APPENDIX B: PROJECTED DIFFUSION MATRIX

Here we shall show that the diffusion matrix used for the Fick's law in Eq. (8) has to be the projected one. In terms of the fluctuating charge field at the typical scale $\varrho_c(x, t)$, the total Onsager matrix is defined by

$$\sigma_{ij}^{\text{tot}} = \lim_{\tau \rightarrow \infty} \sqrt{\tau} \int dx \int_0^1 dt \int_0^1 dt' \langle j_i(x, t) j_j(0, t') \rangle^c. \quad (\text{B1})$$

To compute this, we first note that

$$\begin{aligned} \int_0^1 dt j_c(x, t) &= \int_x^\infty dy [\varrho_c(y, \tau) - \varrho_c(y, 0)] \\ &= \tau^{-1/4} \sqrt{\rho} \int dy \mu(x, y) \delta n_c(y, 0) \\ &+ \int_0^1 ds \int dy G_{1-s}(x - y - X_1) \eta(y, s), \end{aligned} \quad (\text{B2})$$

where $\mu(x, y) = \text{erfc}[(x - y - X_1)/\sqrt{2D}]/2 - \Theta(y - x)$. Using this expression, we then have

$$\begin{aligned} &\int dx \int_0^1 dt \int_0^1 dt' \langle j_c(x, t) j_c(0, t') \rangle^c \\ &= \tau^{-1/2} \rho \left(\int dx dy dz \langle \mu(x, y) \mu(0, z) \rangle_{\delta n_1, \delta n_2} + D \right), \end{aligned} \quad (\text{B3})$$

where $\langle \cdots \rangle_{\delta n_1, \delta n_2}$ indicates the (path-integral) averaging over δn_1 and δn_2 where we used that both $\delta n_c(x, 0)$ and $\eta(x, t)$ are Gaussian distributed with mean zero and $\int dx G_t(x) = 1$. A lengthy but straightforward calculation now shows that

$$\begin{aligned} &\tau^{-1/2} \rho \int dy \int dz \mu(x, y) \mu(0, z) \\ &= \tau^{-1/2} \rho |X_1|^2 = \tau^{-1} (1 - \rho) \left(\int_0^{\sqrt{\tau}} dx \delta \bar{n}(x, 0) \right)^2, \end{aligned} \quad (\text{B4})$$

where $\delta \bar{n}(x, 0) = [\delta n_1(x, 0) - \delta n_2(-x, 0)]/\sqrt{2}$. Performing the path integral over δn_1 and δn_2 , we obtain

$$\tau^{-1/2} \rho \int dx dy dz \langle \mu(x, y) \mu(0, z) \rangle_{\delta n_1, \delta n_2} = \tau^{-1/2} \bar{\rho}. \quad (\text{B5})$$

Combining these, we thus end up with

$$\sigma_{cc}^{\text{tot}} = \bar{\rho} + D\rho. \quad (\text{B6})$$

Since $\sigma_{cc}^{\text{conv}} = \bar{\rho}$ and $\sigma_{cc} = D\rho$, we establish that σ_{cc} is indeed the projected Onsager matrix $\sigma_{cc} = \sigma_{cc}^{\text{tot}} - \sigma_{cc}^{\text{conv}}$. It should be stressed that the above derivation can be readily generalizable to any linearly degenerate systems. We thus expect that the

projected diffusion matrix controls the strength of normal diffusion via the Fick's law in generic linearly degenerate systems.

APPENDIX C: TYPICAL CHARGE PROBABILITY DISTRIBUTION

Here we compute the generating function $\overline{\langle e^{\lambda \hat{J}(\tau)} \rangle}$ using a path integral. We first carry out the path integral for δn_c and average over the noise η_c , which gives

$$\overline{\langle e^{\lambda \hat{J}_c(\tau)} \rangle} \simeq \langle e^{(1/2)\lambda^2 \rho \tau^{1/2} [|X_1| + \sqrt{2D/\pi} p(|X_1|/\sqrt{2D})]} \rangle_{\delta n_1, \delta n_2} \times \langle e^{(1/2)\lambda^2 \rho \tau^{1/2} [|X_1| + \sqrt{2D/\pi} p(|X_1|/\sqrt{2D})]} \rangle_{\delta \bar{n}}. \quad (C1)$$

Since $X_1 = \tau^{-1/4} \sqrt{\rho/\rho} \int_0^\tau dx \delta \bar{n}(x, 0)$ is still Gaussian distributed, this path integral can be reduced to a one-dimensional integral. Namely, we have that

$$\int \mathcal{D}\phi e^{-\int dx [\phi(x)^2/2] + f[\int dx g(x)\phi(x)]} = \int dX e^{-X^2/2 + f(\hat{g}X)} \quad (C2)$$

for generic functions $f(x)$ and $g(x)$ where $\hat{g} = \sqrt{\int dx g(x)^2}$, which follows from the following discrete version of the identity (C2):

$$\int \frac{d^n x}{(2\pi)^{n/2}} e^{-(1/2)x \cdot x} e^{f(\underline{g} \cdot x)} = \int \frac{dx}{\sqrt{2\pi}} e^{-(1/2)x^2} e^{f(\hat{g}x)}, \quad (C3)$$

where $\hat{g} = \sqrt{\underline{g} \cdot \underline{g}}$ and $\underline{g} \in \mathbb{R}^n$. The continuum limit of this then yields Eq. (C2), which in turn allows us to obtain Eq. (10). Inverting the Laplace transform, the typical probability distribution $\mathcal{P}_{\text{typ}}(j)$ is then given by Eq. (11) whose second moment reads

$$m_2 = \int_{\mathbb{R}} dx x^2 P_r(x) = \sqrt{(1 + 2r^2)/\pi}. \quad (C4)$$

We also compute the fourth moment

$$m_4 = \frac{3}{\pi} \frac{1 + 6r^2}{1 + 2r^2} \frac{1}{\sqrt{1 + 4r^2}} + 3r^2 \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}} y^2 \text{erf}^2(ry) e^{-y^2/2}, \quad (C5)$$

whence we extract the excess kurtosis (see inset of Fig. 2)

$$3 + \kappa(r) = m_4/m_2^2 = \frac{3}{1 + 2r^2} \left[\frac{1 + 6r^2}{1 + 2r^2} \frac{1}{\sqrt{1 + 4r^2}} + \pi r^2 \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}} y^2 \text{erf}^2(ry) e^{-y^2/2} \right], \quad (C6)$$

with the limits $\kappa(0) = 0$ and $\kappa(\infty) = 3(\pi/2 - 1)$, which agree with Gaussian and single-file values, respectively. Note that the kurtosis is unchanged by rescaling so that (C6) holds for the typical distribution even for $r \rightarrow \infty$ despite the involved divergent rescaling as $\omega \rightarrow 0$.

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