

Spectral statistics and localization properties of a C_3 -symmetric billiardMatic Orel  and Marko Robnik *Center for Applied Mathematics and Theoretical Physics (CAMTP), University of Maribor, Mladinska 3, SI-2000 Maribor, Slovenia, European Union*

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We revisit the spectral statistics of the C_3 -symmetric billiard introduced by F. Leyvraz *et al.* [*J. Phys. A: Math. Gen.* **29**, L575 (1996)], which exhibits both Gaussian orthogonal ensemble (GOE) and Gaussian unitary ensemble (GUE) statistics depending on the symmetry block. Using high-precision Beyn's contour-integral method for the nonlinear Fredholm eigenvalue problem with built-in separation of irreducible subspaces, we compute 2.8×10^5 eigenvalues in each symmetry subspace, enabling statistically meaningful comparisons with random matrix theory. The improved spectra reveal clear GOE-GUE correspondence and resolve previously observed deviations in long-range spectral correlations. Furthermore, we analyze phase-space eigenstate localization through the distribution of entropy localization measures, which, for chaotic states follow a β distribution whose standard deviation decays as a power law with energy, consistent with the onset of quantum ergodicity as described by Schnirelman's theorem.

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I. INTRODUCTION

Quantum chaos (or more generally wave chaos) addresses the relationship between quantum or wave dynamics and the structure of the underlying classical phase space, particularly in the classical (short-wavelength) limit [1,2]. A central observation is how classical integrability, chaos, or mixed dynamics manifest themselves in quantum observables such as energy spectra, eigenfunctions, along with their phase-space representations. In this context, Wigner and Husimi functions [3–6] provide a natural framework for visualizing quantum states in phase space and identifying their correspondence with classical invariant structures. More generally, the term quantum chaos also encompasses quantum systems (e.g., many body) exhibiting universal spectral and eigenstate statistics even in the absence of a well-defined classical limit [7]. In this work, quantum billiards have a clear classical limit. A fundamental result of quantum chaos is the universality of spectral statistics, where classically integrable systems exhibit uncorrelated energy levels with Poissonian statistics [8,9], whereas fully chaotic systems are described by random matrix theory (RMT). Importantly, depending on the presence of antiunitary symmetries, the universality classes are described by either Gaussian orthogonal ensemble (GOE) or Gaussian unitary ensemble (GUE) [10,11]. This correspondence is known as the Bohigas-Giannoni-Schmit conjecture, which has its roots in the works of Casati, Guarneri, and Valz-Gris [12,13].

Billiard systems play a central role in studies of quantum chaos, as they provide conceptually simple yet dynamically

rich models in which the correspondence between classical and quantum mechanics can be explored in detail. The classical dynamics of a billiard is fully determined by its geometry, ranging from integrable shapes (such as rectangles and circles) to fully chaotic ones (e.g., Sinai or Bunimovich billiards), as well as mixed-type geometries exhibiting both regular and chaotic motion [9,14–18]. Their quantum counterparts, defined by the Helmholtz or Schrödinger equation with hard-wall boundary conditions, are experimentally accessible in microwave resonators, optical cavities, and acoustic systems, making billiards a natural testing ground for universal spectral statistics and phase-space localization phenomena [1,19–22]. For fully chaotic billiards, universality is not limited to spectral statistics but is also reflected in the structure of eigenfunctions. In the semiclassical limit, chaotic eigenstates are expected to become uniformly distributed in phase space, in accordance with the random-wave model [23], whereas finite-energy effects give rise to deviations in the form of localization of eigenfunctions [24,25], scars [26], and phenomenological-level spacing distributions [27–29]. A quantitative study of eigenfunction localization is thus essential for determining how rapidly this universal behavior is attained.

Discrete symmetries of a billiard play a crucial role in shaping its quantum spectral statistics. The presence of a symmetry group leads to a decomposition of the Hilbert space into independent subspaces associated with the irreducible representations (irreps) of the group [30], and spectral statistics must therefore be analyzed separately within each irrep, as superpositions of levels from different symmetry sectors trivially drive the statistics toward Poissonian, therefore obscuring the physical origin of the statistical properties of the individual level sequences. In most billiard systems studied, the relevant symmetries are reflection symmetries, giving rise to one-dimensional irreps. In chaotic systems,

each such symmetry-reduced spectrum follows the statistics of the Gaussian orthogonal ensemble, reflecting the presence of an effective time-reversal symmetry within each sector. In contrast, symmetry groups possessing higher-dimensional irreps can lead to a fundamentally different universality class. First studied by Leyvraz *et al.* [31], subspectra associated with complex irreducible representations exhibit GUE statistics despite the full system being time-reversal invariant [32]. A complete semiclassical theory for the coexistence of GOE and GUE symmetry classes in such systems was later developed by Joyner *et al.* [33]. Billiards with discrete rotational symmetries thus provide a natural but sparsely studied setting in which different irreps exhibit distinct random-matrix universality classes, offering a sensitive probe of the interplay between chaos, symmetry, and quantum statistics [34–36]. Interestingly, GUE statistics can also arise in certain other time-reversal-invariant systems such as the mesoscopic nanoelectromechanical quantum-dot systems [37].

The paper is organized as follows. In Sec. II, we introduce the geometry of the billiard and discuss its symmetry properties. In Sec. III, we outline the boundary integral method used to solve the Helmholtz equation and to compute the eigenvalues and eigenfunctions. Section IV is devoted to the analysis of spectral statistics, including nearest-neighbor-level spacings and the spectral rigidity Δ_3 . In Sec. V, we study the localization properties of eigenfunctions. In particular, Sec. VA introduces the Poincaré-Husimi (PH) function, Sec. VB its associated entropy localization measure, while Sec. VC analyzes the statistical behavior of the entropy localization measures by calculating the standard deviation of the fitted β distribution as a function of the wave number k . Section VD analyzes and interprets the final results of Sec. VC. Finally, Sec. VI summarizes the results and presents our conclusions.

II. GEOMETRY

In this study we have constructed the boundary from the following parametrization Fig. 1:

$$r(\phi) = \frac{1}{2}(1 + a(\cos 3\phi - \sin 6\phi)) \quad \phi \in [0, 2\pi) \\ a \in [0, 0.55], \quad (1)$$

where we primarily use $a = 0.2$ [32,36,38] for the study of level spacings and entropy localization measures because it provides a practically fully ergodic phase space without large variations of curvature on the boundary (allowing for a simple trapezoidal Nöystrom discretization scheme) or any traces of dynamical localization at small k . Generally this billiard is of the mixed type while being almost fully ergodic for $a \in [0.04, 0.31]$ Fig. 2.

The C_3 billiard is invariant under the cyclic rotation group C_3 generated by $R \equiv R_{2\pi/3}$ (rotation by $2\pi/3$). For $a \neq 0$ there are no reflections, so the discrete symmetry is purely rotational. The Hilbert space decomposes into three one-dimensional irreps labeled by the rotation eigenvalue

$$R \psi_m = e^{2\pi i m/3} \psi_m, \quad m = 0, 1, 2. \quad (2)$$

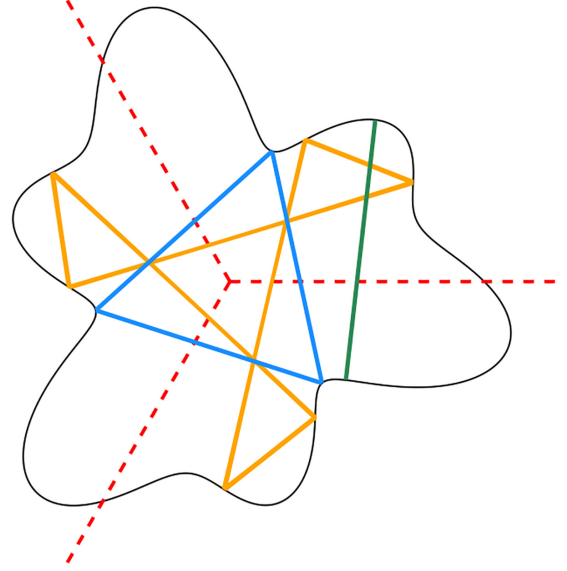


FIG. 1. Primitive two- and three-bounce periodic orbits of the C_3 billiard at $a = 0.2$. The boundary is shown in black, the three-bounce orbit in blue, the two-bounce orbit in green, and a six-bounce orbit in orange. Red dashed lines mark the symmetry axes defining the fundamental domain.

The $m = 0$ sector is real (self-conjugate), while $m = 1$ and $m = 2$ are complex conjugates and form a time-reversal pair (Table I).

The finite-time maximal Lyapunov exponent is computed using the standard Benettin algorithm adapted to billiard systems. Between collisions, the infinitesimal offset vector $\delta\Gamma_m = (\delta q_m, \delta v_m)$ evolves under the linearized free-flight dynamics,

$$\mathbf{q}_{m+1} = \mathbf{q}_m + \Delta t_m \cdot \mathbf{v}_m, \quad \delta \mathbf{v}_{m+1} = \delta \mathbf{v}_m, \\ \delta \mathbf{q}_{m+1} = \delta \mathbf{q}_m + \Delta t_m \delta \mathbf{v}_m, \quad (3)$$

where Δt_m denotes the flight time between the m th and $(m+1)$ th collisions. At the $(m+1)$ th collision with the boundary, the $\delta\Gamma_{m+1}$ transforms according to the linearized reflection map (for details see Ref. [40])

$$\delta \mathbf{q}_{m+1} = \delta \mathbf{q}_m - 2(\delta \mathbf{q}_m \cdot \mathbf{n})\mathbf{n}, \\ \delta \mathbf{v}_{m+1} = \delta \mathbf{v}_m - 2(\delta \mathbf{v}_m \cdot \mathbf{n})\mathbf{n} - \frac{2\kappa}{\cos \alpha}(\delta \mathbf{q}_m \cdot \mathbf{t}_i)\mathbf{t}_f, \quad (4)$$

where $\mathbf{n}$ is the inward unit normal at the collision point, κ the local boundary curvature at the collision point, α the incidence angle at the collision point, and $\mathbf{t}_i, \mathbf{t}_f$ denote the incoming and outgoing orthogonal unit vectors at the collision point (see

TABLE I. Irreducible representations of C_3 (rotation by $2\pi/3$), labeled as $R \equiv R_{2\pi/3}$. The nontrivial eigenvalues are the cubic roots of unity $\omega = e^{2\pi i/3}$ and $\omega^2 = e^{-2\pi i/3}$.

Irrep	m	Eigenvalue of R	Time-reversal
A	0	1	Self-conjugate
E_+	1	$\omega = e^{+2\pi i/3}$	E_-
E_-	2	$\omega^2 = e^{-2\pi i/3}$	E_+

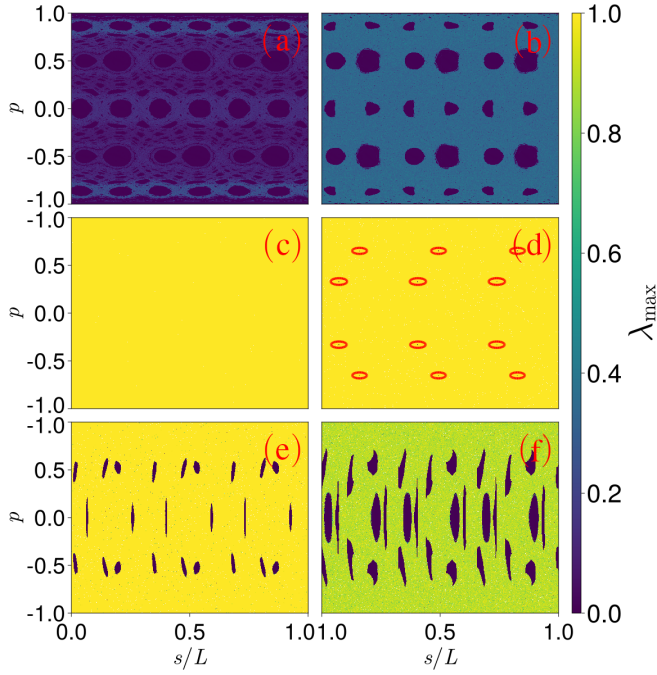


FIG. 2. Finite-time maximal Lyapunov exponent heatmaps $\lambda_{\max}$ [39] for the C_3 billiard on the Poincaré section in the fundamental domain $s/L \in [0, 1/3]$, where s is the arclength of a boundary point on the curve (1) and L being the total arclength of that curve. For visual clarity the color scale is capped at $\lambda_{\max} = 1$ (values above the cap are shown with the same maximal color). Dark regions ($\lambda_{\max} \approx 0$) indicate regular motion, whereas brighter regions correspond to chaotic dynamics with positive $\lambda_{\max}$. Panels show increasing deformation amplitude: (a) $a = 0.005$, (b) $a = 0.01$, (c) $a = 0.05$, (d) $a = 0.2$, (e) $a = 0.35$, (f) $a = 0.5$. For $0.05 \lesssim a \lesssim 0.31$, the phase space is predominantly chaotic; for larger a , it becomes mixed. Our case $a = 0.2$ (d) has very small stability islands [barely visible, encircled in panel (d)] corresponding to the orange PO in Fig. 1. The heatmaps were constructed on a 1000×1000 grid with 5000 collisions for each initial condition.

Fig. 2 for more information). To ensure numerical stability, the tangent-space vectors are periodically orthonormalized using the QR algorithm [41,42]. The finite-time maximal Lyapunov exponent [39] is then obtained as

$$\lambda_{\max}(T) = \frac{1}{T} \sum_{m=1}^{M(T)} \ln \frac{\|\delta \Gamma_m\|}{\|\delta \Gamma_{m-1}\|}, \quad (5)$$

where M is the total collision count and $T = \sum_{m=1}^{M(T)} \Delta t_m$ is the total flight time. Figure 2 shows a heatmap of maximal Lyapunov exponents for characteristic deformation amplitudes a in the boundary definition (1).

III. BOUNDARY INTEGRAL METHOD AND THE HELMHOLTZ EQUATION

We consider the stationary Helmholtz equation

$$(\Delta + k^2)\psi(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \quad (6)$$

in a two-dimensional billiard domain Ω with Dirichlet boundary conditions

$$\psi(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \quad (7)$$

where $\partial\Omega$ is the total boundary parametrized as in Eq. (1). For simply connected domains with Dirichlet boundary conditions, we employ a boundary integral formulation based on the double-layer potential [43], where

$$u(\mathbf{y}) = \partial_{\mathbf{n}_y}(\psi(\mathbf{y})) \quad (8)$$

is the boundary function, $\partial_{\mathbf{n}_y}$ denoting the outward normal derivative at $\mathbf{y}$,

$$G_k(\mathbf{x}, \mathbf{y}) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \quad (9)$$

is the free-space Green's function, and $H_m^{(1)}$ is the Hankel function of the first kind of order m [44]. The boundary integral equation (BIE) is

$$A(u(\mathbf{x})) = \frac{1}{2} u(\mathbf{x}) + \int_{\partial\Omega} \partial_{\mathbf{n}_y} G_k(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) ds(\mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega, \quad (10)$$

where the $\frac{1}{2}u(\mathbf{x})$ term follows from the jump relation of the double-layer operator [43,45], $ds(\mathbf{y})$ is the differential arclength at the point $\mathbf{y} \in \partial\Omega$, and $\partial_{\mathbf{n}_y} G_k(\mathbf{x}, \mathbf{y})$ is given as

$$\partial_{\mathbf{n}_y} G_k(\mathbf{x}, \mathbf{y}) = -\frac{ik}{4} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{d(\mathbf{x}, \mathbf{y})} H_1^{(1)}(kd(\mathbf{x}, \mathbf{y})), \quad (11)$$

where $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2}$ is the Euclidian distance between the points and $\mathbf{n}(\mathbf{y})$ the outward pointing normal vector at point $\mathbf{y}$. Due to the present C_3 rotational symmetry it is numerically easier to work with the fundamental domain Γ_{fund} instead of the full domain Γ_{full} . They are defined as

$$\Gamma_{\text{fund}} = \Gamma\left(\frac{2\pi}{3}\right), \quad \Gamma_{\text{full}} = \Gamma(2\pi) = \partial\Omega, \quad (12)$$

where

$$\Gamma(\theta) = \{(r(\phi) \cos \phi, r(\phi) \sin \phi) | \phi \in [0, \theta]\}.$$

The BIE (10) can be reduced to a boundary integral over Γ_{fund} and decomposed into irreducible symmetry sectors labeled by $m = 0, 1, 2$. The resulting symmetry-projected kernel [36] is (for derivation see Appendix B 1)

$$A(u_m(\mathbf{x})) = \frac{1}{2} u_m(\mathbf{x}) + \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \times \int_{\Gamma_{\text{fund}}} \partial_{\mathbf{n}_y} G_k(\mathbf{x}, \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}), \quad (13)$$

where $u_m(\mathbf{x})$ is the boundary function in the m th irreducible representation, $\mathbf{R}$ the rotation operator given in Eq. (B15) and $\chi_m(\mathbf{R}^\ell)^*$ being the complex conjugate of the character of the rotation $\mathbf{R}^\ell$ in the m th irreducible representation [see Eq. (B20)]. Similarly the wave function constructed using Eq. (9),

$$\psi(\mathbf{x}) = - \int_{\Gamma} G_k(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \Omega, \quad (14)$$

can be constructed in a given irreducible representation m (for derivation see Appendix B 2),

$$\psi_m(\mathbf{x}) = \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \int_{\Gamma_{\text{fund}}} G_k(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}),$$

$$\mathbf{x} \in \Omega. \quad (15)$$

IV. SPECTRAL STATISTICS

A. Nearest-neighbor-level spacings

To characterize short-range spectral correlations, we analyze the nearest-neighbor-level spacing distribution (NNLS) of the unfolded spectrum. After unfolding, spacings are defined as

$$s_n = E_{n+1} - E_n, \quad (16)$$

where $\{E_n\}$ denotes the unfolded eigenvalues ordered by increasing energy. Owing to the exact C_3 rotational symmetry of the billiard, the spectrum decomposes into statistically independent symmetry sectors corresponding to the irreducible representations $m = 0, 1, 2$. Since the $m = 1$ and $m = 2$ sectors form a complex-conjugate pair, having the same energy (doublets), related by time-reversal symmetry, we restrict the analysis to the $m = 0$ and $m = 1$ sectors. The NNLS distributions for the desymmetrized spectra in the $m = 0$ and $m = 1$ symmetry sectors are shown in Fig. 3, each based on approximately 2.8×10^5 eigenvalues.

The numerical results are compared with the Wigner surmises

$$P_{\text{GOE}}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi s^2}{4}\right), \quad (17)$$

$$P_{\text{GUE}}(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4s^2}{\pi}\right), \quad (18)$$

corresponding to the GOE and GUE, respectively.

We observe excellent agreement with GOE statistics in the $m = 0$ sector, consistent with the presence of time-reversal symmetry in this real irrep. In contrast, the $m = 1$ sector follows GUE statistics, reflecting the effective breaking of time-reversal symmetry in the complex irreps. This behavior is expected for systems with discrete rotational symmetry, where complex irreps support unitary symmetry classes leading to GUE statistics even in the absence of an external magnetic field [38].

B. Spectral rigidity Δ_3

While nearest-neighbor spacing statistics probe short-range correlations, long-range spectral correlations are characterized by the Dyson-Mehta [46] spectral rigidity $\Delta_3(L)$. For an unfolded spectrum $\{E_n\}$ with unit mean-level density, we introduce the spectral staircase function

$$N(E) = \text{No.}\{n : E_n \leq E\}. \quad (19)$$

The rigidity measures the mean-square deviation of $N(E)$ from its best linear fit on an interval of length L ,

$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{E_0}^{E_0+L} [N(E) - (AE + B)]^2 dE \right\rangle, \quad (20)$$

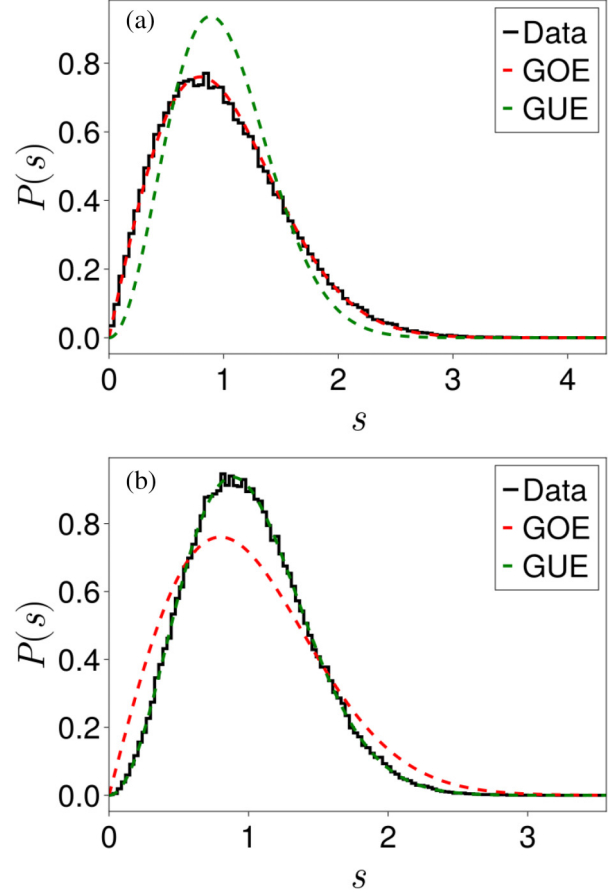


FIG. 3. Nearest-neighbor-level spacing distributions for the desymmetrized spectrum of the C_3 billiard. (a) $m = 0$ symmetry sector: the numerical histogram (approximately 2.8×10^5 unfolded eigenvalues) is compared with the Wigner surmise of the Gaussian orthogonal ensemble (GOE), reflecting the presence of time-reversal symmetry in this real irreducible representation. (b) $m = 1$ symmetry sector: comparison with the Gaussian unitary ensemble (GUE) Wigner surmise, indicating effective time-reversal symmetry breaking in the complex irreducible representations.

where $\langle \cdots \rangle_{E_0}$ denotes averaging over the energy interval $[E_0, E_0 + L]$. An equivalent characterization of long-range correlations is provided by the number variance

$$\Sigma^2(L) = \langle [N(E_0 + L) - N(E_0) - L]^2 \rangle_{E_0}, \quad (21)$$

which measures fluctuations of the number of levels in an interval of length L . The two quantities are related by the identity [1]

$$\Delta_3(L) = \frac{2}{L^4} \int_0^L (L^3 - 2L^2 r + r^3) \Sigma^2(r) dr. \quad (22)$$

For uncorrelated (Poissonian) spectra, the rigidity grows linearly,

$$\Delta_3^{\text{Poisson}}(L) = \frac{L}{15}, \quad (23)$$

whereas for chaotic systems random-matrix theory predicts a much slower, logarithmic growth. In the Gaussian ensembles,

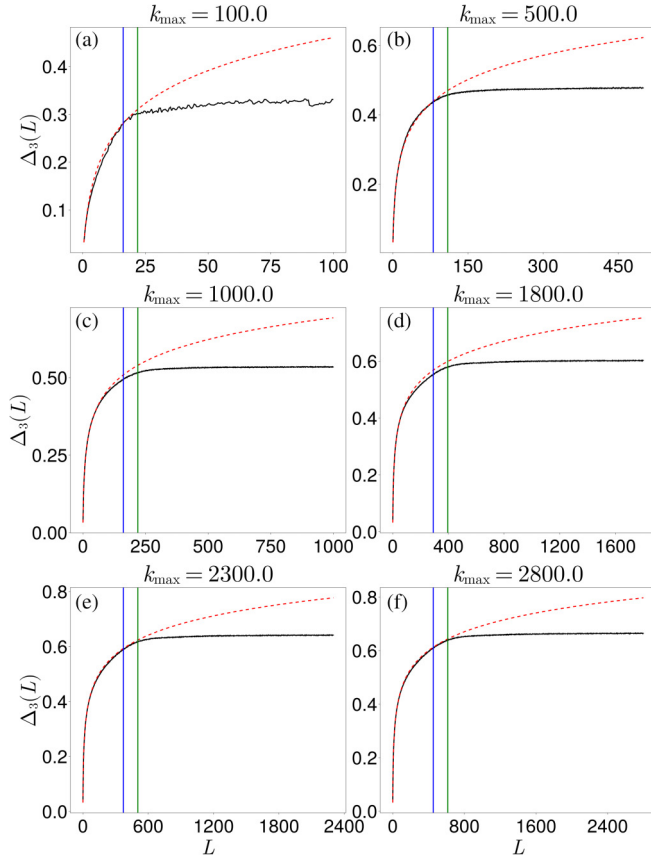


FIG. 4. $\Delta_3(L)$ for the $m = 0$ irreducible representation of the billiard with $a = 0.2$ at various wave numbers k . The dashed red line is the theoretical curve for the $\Delta_3^{\text{GOE}}(L)$, the blue vertical represents the length of the three-bounce periodic orbit and the green line represents the two-bounce periodic orbit, as shown in Fig. 1. It is observed that the spectrum starts to deviate from RMT around $L_{\text{max}} \approx \frac{A_{\text{fund}} \times k_{\text{max}}}{l_0}$, which matches the value of $l_0 = 1.2415$ corresponding to the two-bounce orbit (green vertical line, shown in green in Fig. 1), where A_{fund} is the area of the desymmetrized domain ($A_{\text{fund}} = \frac{A}{3}$) and k_{max} the largest wave number present in the spectrum (title in each subfigure) [47,48]. The blue vertical line corresponding to the three-bounce orbit with $l_0 = 1.6834$ (shown in blue in Fig. 1) which was used in Ref. [36] does not correctly capture the observed saturation trend.

the asymptotic behavior at large L reads

$$\Delta_3^{(\beta)}(L) \sim \frac{1}{\beta\pi^2} \ln L + C_\beta, \quad \beta = \begin{cases} 1, & \text{GOE,} \\ 2, & \text{GUE,} \end{cases} \quad (24)$$

with ensemble-dependent constants C_β . For the case of $a = 0.2$, $\Delta_3(L)$ is shown for symmetry classes $m = 0, 1$ in Figs. 4 and 5, respectively. It shows agreement with spectral rigidity RMT up until saturation due to the shortest periodic orbit. Another complementary result is the dependence of the saturation value of the Δ_3^∞ as a function of wave number k . The saturation value represents the maximal possible value of spectral rigidity achievable at a given maximal energy E_{max} or equivalent wave number k_{max} in the spectrum and signifies the contribution of the short classical periodic orbit. As described in detail in Ref. [47], the growth of Δ_3^∞ is logarithmic in $E = k^2$ and universal in each symmetry class, while the

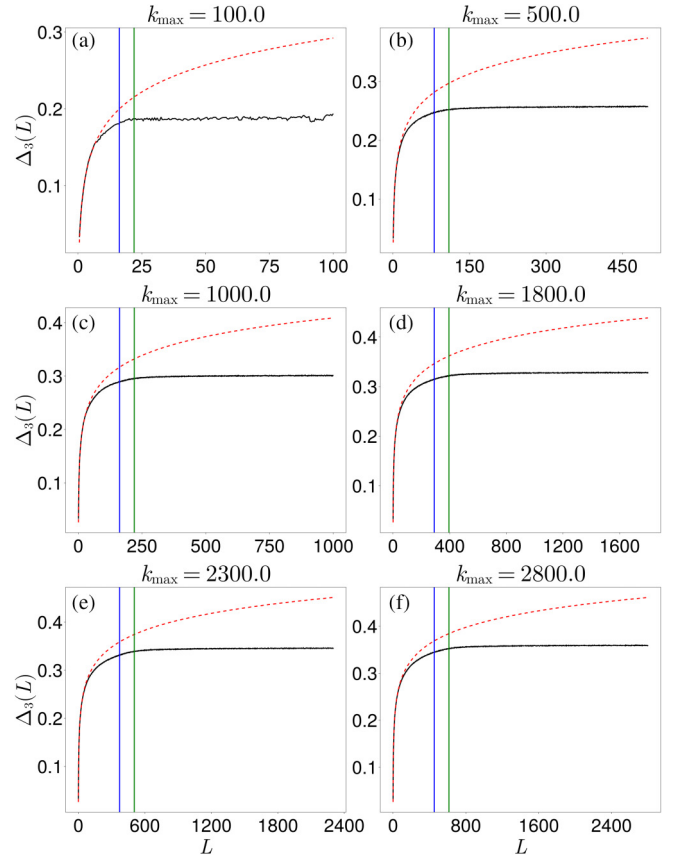


FIG. 5. $\Delta_3(L)$ for the $m = 1$ irreducible representation of the billiard with $a = 0.2$ at various wave numbers k . The plotted quantities are the same as in Fig. 4, where importantly the value of L_{max} does not depend on the irreducible symmetry class m , allowing for a direct comparison of the onset of saturation across all of them. In contrast to the $m = 0$ case the deviation from $\Delta_3^{\text{GUE}}(L)$ RMT is observed already before the prediction given by the length of the shortest periodic orbit (green vertical line), while the saturation is still observed to start at the calculated L_{max} .

additive constant C is system-dependent. The trend is clearly observed in Fig. 6, showing a clear agreement of the numerical data with the theoretical result.

V. PHASE-SPACE ENTROPY LOCALIZATION MEASURES

A. Poincaré-Husimi function

To analyze the phase-space structure of individual eigenstates we employ PH functions [5], which provide a reduced phase-space representation on the billiard boundary. This construction is particularly well suited for billiard systems, where the boundary acts as a natural global Poincaré section [49].

Let $\psi_n(\mathbf{x})$ be an eigenfunction satisfying Eq. (6). Then the eigenfunction is uniquely determined by its normal derivative on the boundary [49], which is the boundary function $u_n(s)$ defined in Eq. (8). The Poincaré section is parameterized by boundary position $q \in [0, L)$ and momentum

$$p = \sin(\alpha), \quad (25)$$

where α is the reflection angle of the specular reflection.

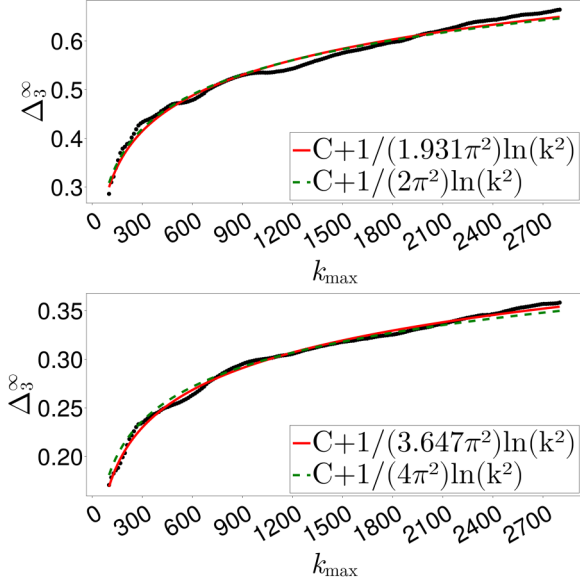


FIG. 6. Spectral rigidity Δ_3^∞ as a function of the maximal wave number $k_{\max}$ for the $m = 0$ (GOE, top panel) and $m = 1$ (GUE, bottom panel) symmetry classes. Black dots denote numerical results, while the solid red curves show fits of the form $\Delta_3^\infty = C + \frac{1}{\alpha\pi^2} \ln(k^2)$ with $\alpha = 1.931$ (GOE) and $\alpha = 3.647$ (GUE). The value of C is billiard dependent and was determined by fitting. Theoretical values are represented by the dashed green line showing agreement with the numerical results.

Following Ref. [4], coherent states on the boundary are defined as periodized Gaussian wave packets,

$$c_{(q,p),k}(s) = \left(\frac{k}{\pi}\right)^{1/4} \sum_{m \in \mathbb{Z}} \exp\left[-\frac{k}{2}(s - q + mL)^2 - ikp(s - q + mL)\right]. \quad (26)$$

and the unnormalized Poincaré-Husimi function associated with the eigenstate ψ_n is defined as

$$h_n(q, p) = \left| \int_0^L c_{(q,p),k_n}(s) u_n(s) ds \right|^2. \quad (27)$$

This quantity represents the projection of the boundary function $u_n(s)$ onto a minimum-uncertainty wave packet centered at (q, p) on the Poincaré section. Due to the fact that we are working in irreps the PH functions were constructed with symmetry reduction in mind, allowing for less numerical effort in their evaluation (for derivation see Appendix B 3). Examples are shown in Figs. 7(a)–7(d).

The symmetry of $h_n(q, p)$ reflects the underlying discrete symmetries of the billiard. In particular:

(1) For real irreducible representations [time-reversal invariant, GOE-like sectors, $m = 0$, Figs. 7(a) and 7(b)],

$$h_n(q, p) = h_n(q, -p). \quad (28)$$

(2) For complex irreducible representations (GUE-like sectors, $m = 1, 2$), time-reversal symmetry is broken and no such $p \rightarrow -p$ symmetry exists [Figs. 7(c) and 7(d)].

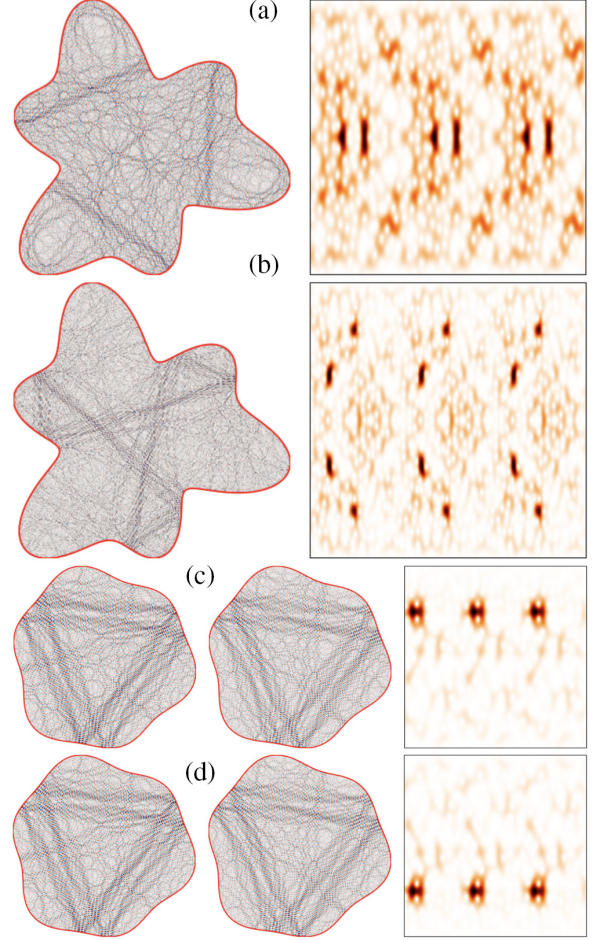


FIG. 7. (a) Probability density of a scar associated with the shortest periodic orbit (two-bounce orbit) together with the corresponding PH function (27) for $a = 0.2$ and $m = 0$. Darker regions correspond to larger PH function values and highlight the scar support on the underlying unstable periodic orbit. (b) Probability density with a scar like feature supported by the periodic orbit linked to the stability islands visible in the finite-time maximal Lyapunov exponent phase-space analysis, along with its PH representation for $a = 0.2$ and $m = 0$ [see Figs. 1 and 2(d)]. (c) Real and imaginary parts of a quantum eigenfunction with $a = 0.05$, $m = 1$, and $k \approx 331.25$. While the individual components do not exhibit explicit $2\pi/3$ rotational symmetry, their probability density preserves the symmetry. The corresponding PH function displays the breaking of the $p \rightarrow -p$ symmetry. Panel (d) shows the doublet partner of panel (c) with $a = 0.05$, $m = 2$, and $k \approx 331.25$. For the relevant periodic orbits seen in panels (a), (b) see Fig. 1.

B. Entropy of localization measures

To quantify the degree of phase-space localization, we discretize $h_n(q, p)$ on a uniform (q, p) grid and normalize it such that $\sum_{q,p} h_n(q, p) = 1$ [52]. The localization entropy is then defined as

$$S_n = - \sum_{q,p} h_n(q, p) \ln h_n(q, p), \quad (29)$$

and the corresponding normalized localization measure,

$$A_n = \frac{e^{S_n}}{N_{\text{eff}}}, \quad (30)$$

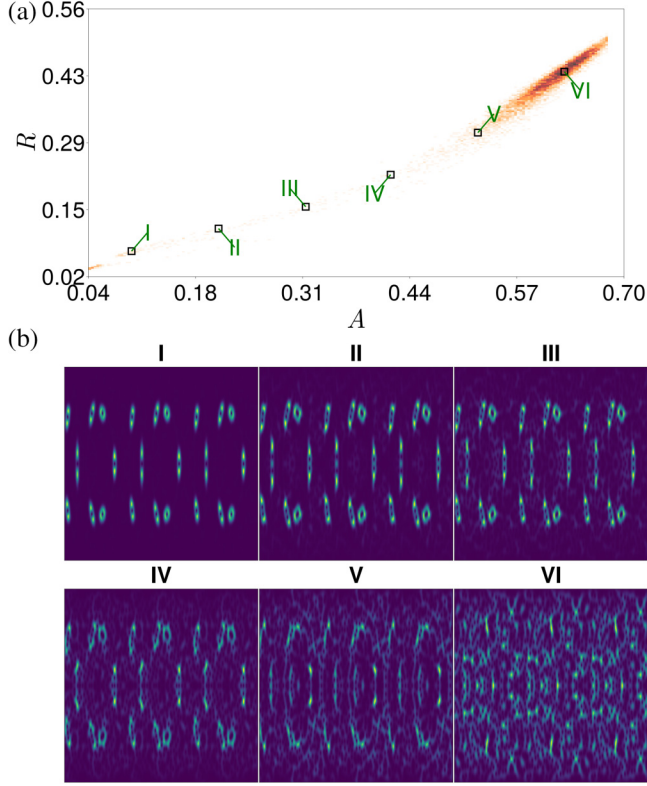


FIG. 8. Localization measures for the C_3 billiard with $a = 0.35$ in the $m = 0$ irreducible subspace from $k = 800$ to $k = 1000$ (approx. 8500 eigenstates). (a) Joint distribution of the localization entropy A and the normalized inverse participation ratio R , defined as $R = \frac{1}{N_c} \frac{1}{\sum_{i,j} H_{i,j}^2}$, where N_c is the number of chaotic grid cells. Highlighted states (I–VI) are representative examples. (b) Corresponding PH functions (27) in the full domain [see Fig. 2(e) for the classical phase space]. Brighter regions correspond to larger PH function values and identify the phase-space regions most strongly associated with the quantum state, allowing direct comparison with the underlying classical phase-space structure. The distribution in panel (a) clearly separates into a dominant bulk and sparse tails. The bulk corresponds to chaotic eigenstates, centered around $A_{\max} \approx 0.66$ and well described by a fitted β distribution [17,24,50]. In contrast, the tail at low A ($0 < A < 0.15$) originates from regular or residual mixed-type states, characterized by strong phase-space localization. For the purpose of statistical modeling of chaotic eigenstates, these tail states are excluded from the fitting procedure [51]. This filtering isolates the chaotic ensemble and yields stable β distribution parameters, while the removed states represent physically distinct dynamical classes.

where N_{eff} denotes the number of accessible chaotic phase-space cells. By construction, $A_n \in (0, 1)$: small values of A_n indicate strong localization (e.g., scarring or regular islands), while values close to 0.66 correspond to fully extended states in phase-space [53]. The distribution $P(A)$ of localization measures is generally asymmetric and broad. Empirically it is found that $P(A)$ for localized chaotic states is well described by a β distribution [24],

$$P(A; \alpha, \beta) = CA^{\alpha-1}(A_0 - A)^{\beta-1}, \quad 0 < A < A_0, \quad (31)$$

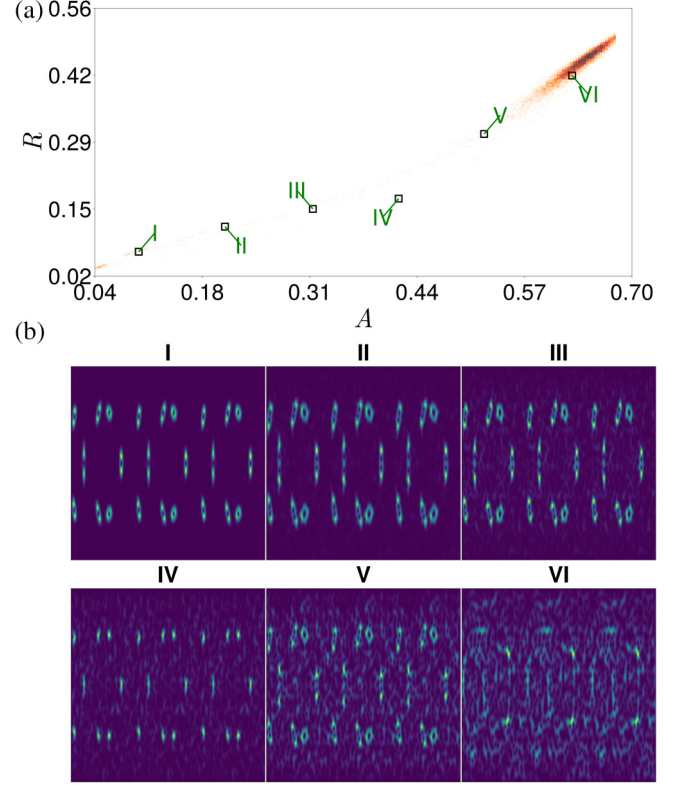


FIG. 9. Localization measures for the C_3 billiard with $a = 0.35$ in the $m = 1$ irreducible subspace from $k = 800$ to $k = 1000$. The detailed description mirrors Fig. 8. It is observed that comparing the width of the fitted β distribution between $m = 0$ and $m = 1$ irreducible subspaces shows consistent narrowing in the $m = 1$ case.

with normalization $C^{-1} = A_0^{\alpha+\beta-1} B(\alpha, \beta)$ and $B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$ is the Euler β function. The shape parameters (α, β) encode the relative weight of localized versus extended eigenstates and vary systematically with energy and classical phase-space structure. Provided are also Fig. 8 and Fig. 9 as cases of a mixed-type phase space with localization of various degrees. Examples for the case of $a = 0.2$ with $m = 0, 1$ symmetry sectors are given in Fig. 10 and Fig. 11.

C. Width of the β distribution versus k

To quantify the approach to the semiclassical limit, we analyze the statistical fluctuations of the localization-measure distribution obtained from the PH representations of the eigenstates. For each chaotic eigenfunction, the localization properties are characterized by fitting the $P(A)$ distribution with a β distribution. To this end, we examine the standard deviation (σ) which provides a direct measure of the width of the localization-measure distribution. The standard deviation of the β function is given by $\sigma(\text{Beta}(\alpha, \beta)) = \sqrt{\text{Var}(P(A))}$, where

$$\text{Var}(P(A)) = A_0^2 \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \quad (32)$$

is the variance of the fitted β distribution. For classically ergodic systems, Schnirelman's theorem [54] (quantum ergodicity) states that there exists a subsequence of

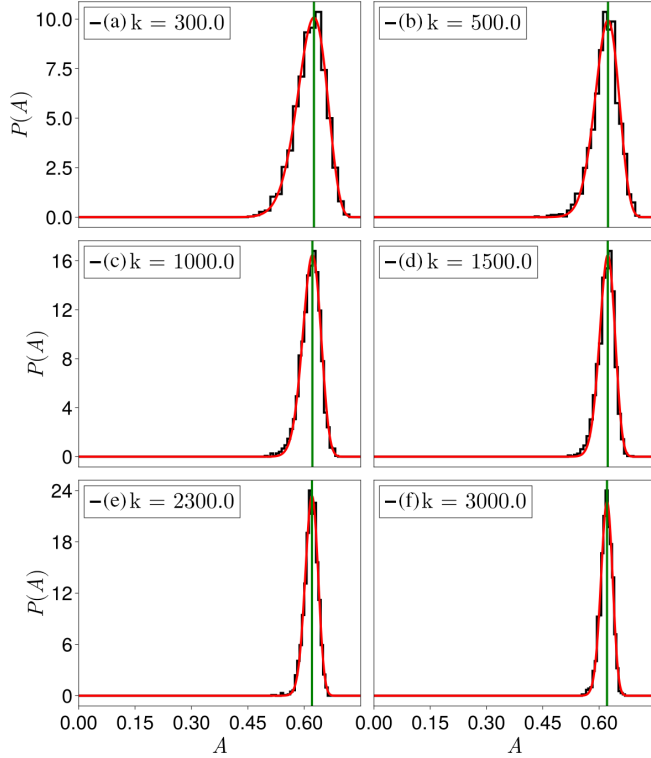


FIG. 10. Distribution $P(A)$ of the localization measure A derived from PH functions for the C_3 billiard at deformation parameter $a = 0.2$ in the $m = 0$ (GOE) symmetry sector. Panels (a)–(f) correspond to increasing wave numbers $k = 300, 500, 1000, 1500, 2300$, and 3000 , respectively. Black histograms show numerical data obtained from PH functions, while solid red curves represent fits by the β distribution (31). The vertical green line marks the maximum value of $P(A)$. With increasing k , the distribution becomes progressively narrower and more sharply peaked, indicating increasing phase-space delocalization and convergence toward the semiclassical ergodic limit.

eigenfunctions of density one whose semiclassical measures converge to the Liouville measure on the energy shell. As a consequence, in billiards projections of these measures into configuration space yield spatial equidistribution of the corresponding eigenfunctions. While the theorem itself does not provide quantitative rates of convergence, semiclassical arguments and random-wave models suggest that fluctuations of localization measures should systematically decrease with increasing energy. This is a special, limiting case of the principle of semiclassical condensation of Wigner and Husimi functions [55].

Across all studied billiard geometries, we observe a clear algebraic decay of the $\sigma(\text{Beta}(\alpha, \beta)) \propto k^{-\gamma}$ with exponents clustered around $\gamma = 0.4$. This behavior is robust across symmetry sectors ($m = 0, 1$) and deformation parameters a . The result indicates a systematic self-averaging trend consistent with semiclassical expectations associated with quantum ergodicity [56]. In physical terms, increasing energy progressively suppresses phase-space inhomogeneities. Importantly, the observed decay is algebraic rather than exponential, indicating that semiclassical convergence is governed by power-law scaling, as seen in Fig. 12.

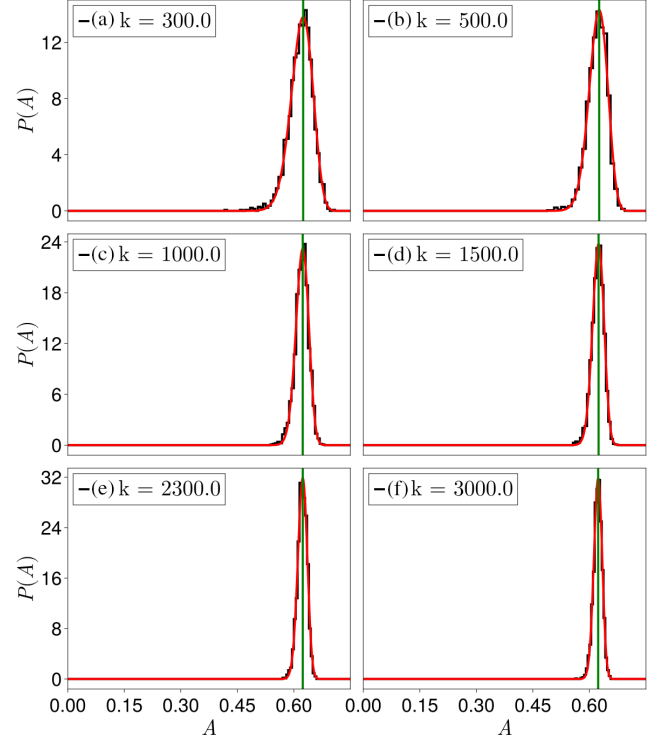


FIG. 11. Same as Fig. 10, but for the $m = 1$ (GUE) symmetry sector of the C_3 billiard at $a = 0.2$. The absence of time-reversal symmetry leads to systematically narrower distributions $P(A)$, consistent with enhanced phase-space delocalization compared to the GOE case.

D. Approach to quantum ergodicity

Using Poincaré-Husimi functions, we analyzed entropy-based localization measures whose distributions are well described by β distributions over a broad range of energies and deformation parameters a . It is observed that the $\sigma(\text{Beta}(\alpha, \beta))$ for the $m = 1$ irrep is consistently smaller than for $m = 0$ at a given k . A characteristic feature observed across the studied cases is a power-law decay of the distribution width with increasing wave number,

$$\sigma(\text{Beta}(\alpha, \beta)) \propto k^{-\gamma}, \quad \gamma \approx 0.4. \quad (33)$$

This behavior indicates a gradual reduction of localization fluctuations toward the semiclassical limit. The extracted decay exponents remain comparable across symmetry sectors and deformation strengths within the predominantly ergodic regime. These findings are consistent with the expectations of quantum ergodicity, which predicts convergence of phase-space measures for a subsequence of eigenstates of density one. Although the theorem does not provide quantitative convergence rates, the observed scaling suggests that localization fluctuations decrease systematically with energy.

An interesting question that arises is whether a systematic relation exists between the average maximal Lyapunov exponent $\langle \lambda_{\max} \rangle$ within the chaotic region of phase space and the intercept D obtained from the linear fits in the loglog representation

$$\log_{10} \sigma(\text{Beta}(\alpha, \beta)) = D - \gamma \log_{10} k.$$

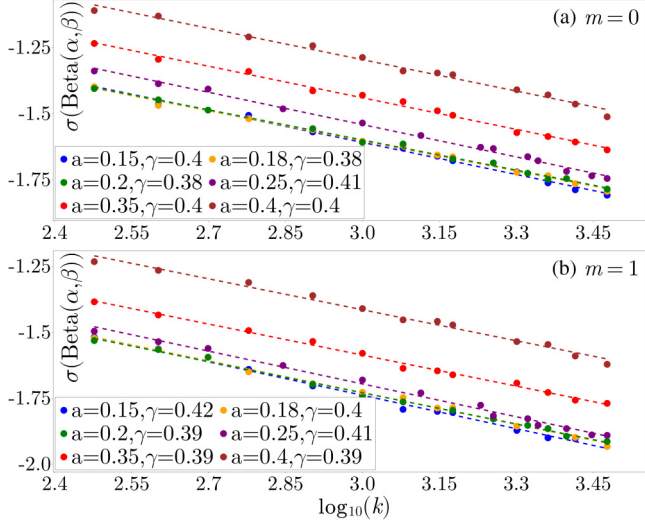


FIG. 12. Decay of the width $\sigma_\beta = \sigma(\text{Beta}(\alpha, \beta))$ of the fitted β distribution describing the chaotic eigenstate statistics of the localization measure A for the deformations parameters $a \in \{0.15, 0.18, 0.2, 0.25, 0.35, 0.4\}$. Shown is $\log_{10} \sigma_\beta$ versus $\log_{10} k$ (decadic logarithms). (a) $m = 0$ (GOE), (b) $m = 1$ (GUE). For all studied deformations $a = 0.15, 0.18, 0.20, 0.25, 0.35$, and 0.40 , the data follow an approximately linear trend on the log-log scale, consistent with $\sigma_\beta \propto k^{-\gamma}$, indicating a progressive suppression of localization fluctuations in the semiclassical limit. The power-law behavior holds throughout the predominantly ergodic regime ($0.05 \lesssim a \lesssim 0.31$), corresponding to $a = 0.15, 0.18, 0.20, 0.25$ and persisting in the mixed regime at $a = 0.35$ and 0.40 , where chaotic states exhibit only weak localization. Each dot represents the standard deviation of the β distribution fitted on an ensemble of 2000 PH functions.

As seen in Fig. 12, the intercept D increases with the deformation parameter a , while simultaneously $\gamma \approx 0.4$ remains unchanged as $\langle \lambda_{\max} \rangle$ decreases. This suggests a separation of roles between classical dynamical instability and the semiclassical scaling of fluctuations of the phase-space entropy localization measure: the Kolmogorov–Sinai entropy h_{KS} , which for the present billiard system coincides with the phase-space average of the unique positive Lyapunov exponent $h_{\text{KS}} = \langle \lambda_{\max} \rangle$, appears to set the prefactor (distribution width), while the decay exponent γ governs the approximately universal rate of convergence toward quantum ergodicity across all studied deformations. For a quantitative analysis of $D = f(h_{\text{KS}})$, the interval $a \in [0.2, 0.34]$ is particularly noteworthy, since h_{KS} depends approximately linearly on a in this regime Fig. 13. Overall, the results show that fluctuation properties of localization measures provide information complementary to conventional spectral diagnostics and offer a useful perspective on the trend toward semiclassical behavior.

VI. GENERAL CONCLUSIONS

We have presented a combined analysis of spectral correlations and eigenstate localization in the C_3 -symmetric billiard introduced in Ref. [32]. Using a contour-integral based approach and an explicit projection onto irreducible symmetry

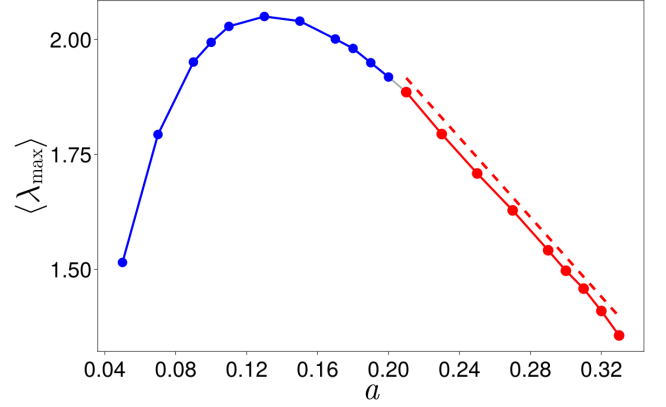


FIG. 13. Average maximal Lyapunov exponent $\langle \lambda_{\max} \rangle$ as a function of the deformation parameter a . In the interval $a \in [0.2, 0.34]$, the decrease of $\langle \lambda_{\max} \rangle$ is approximately linear. Comparing the $\langle \lambda_{\max} \rangle$ values for the given a values with their accompanying intercept values D from Fig. 12 it is seen that an increase of D is associated with a decrease of $\langle \lambda_{\max} \rangle$. For each initial condition in phase space, the trajectory was iterated until the finite-time maximal Lyapunov exponent stabilized within the prescribed numerical tolerance.

sectors, we obtained large spectra in the $m = 0$ and $m = 1$ subspaces and computed Poincaré–Husimi representations for large ensembles of eigenstates.

On the spectral side we analyzed the nearest-neighbor-spacing statistics showing excellent agreement with random-matrix universality when analyzed within each symmetry sector with eigenvalues in the interval $k \in [5, 3600]$: the real $m = 0$ irreducible representation follows GOE statistics, while the complex $m = 1$ irreducible representation follows GUE statistics, despite the time-reversal invariance of the underlying classical dynamics. Long-range correlations, quantified by the spectral rigidity $\Delta_3(L)$, agree with the corresponding GOE/GUE predictions up to the expected saturation scale controlled by the shortest relevant periodic orbit, consistent with semiclassical arguments. Our main focus has been on phase-space eigenfunction localization in the chaotic regime. We characterized the phase-space structure of eigenstates using Poincaré–Husimi functions and an entropy localization measure A , constructed from said functions. For predominantly chaotic ensembles, the empirical distribution $P(A)$ is well described by a β distribution across a broad range of energies and geometry deformations. To quantify the approach toward the semiclassical regime, we tracked the width of the fitted distribution via $\sigma(\text{Beta}(\alpha, \beta))$ and found a robust power-law decay $\sigma(\text{Beta}(\alpha, \beta)) \propto k^{-\gamma}$ with $\gamma \approx 0.4$ for all investigated deformations within the nearly fully ergodic interval. This trend indicates a systematic and quantifiable suppression of localization fluctuations with increasing energy, consistent with the qualitative expectations associated with quantum ergodicity [54]. Beyn’s method shows promise of being an extremely efficient method for calculating high-energy spectra of non-convex billiards with automatic verification of eigenvalue accuracy. To the best of our knowledge, it has not previously been applied to quantum billiard computations at this scale.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: NUMERICAL DETAILS:
BEYN CONTOUR METHOD

To compute high-lying eigenvalues of the Helmholtz operator in billiard domains, we employ a contour-integral method for nonlinear eigenvalue problems based on the framework of Beyn [57,58]. The primary motivation for adopting Beyn's contour-integral method is that within the deformation-parameter range of interest, the billiard boundary becomes nonconvex and exhibits strongly varying curvature. Under these conditions, the traditional Vergini-Saraceno method [59] is no longer directly applicable. Beyn's method, similar to the expanded boundary integral method [60], gives eigenvalues with a small imaginary component. This residual imaginary part provides a practical indicator of numerical accuracy, reflecting discretization errors associated with the finite-dimensional approximation of the Fredholm operator and highlighting spurious solutions.

Beyn's method can be used to find all solutions $(z, v(z))$ of operators satisfying the following equation $T(z)v(z) = 0$ where z is typically a complex parameter and $v(z)$ the solution vector. Since in the boundary integral method this is exactly the equation we are solving [43], Beyn's method is applicable. Let $T(k)$ denote the linear Fredholm operator arising from the boundary integral formulation of the Helmholtz equation. For k inside a contour Γ in the complex plane, the inverse operator admits the following expansion due to Keldysh's theorem [61]:

$$T(z)^{-1} = \sum_{n=1}^{N_\Gamma} \frac{1}{z - \lambda_n} v_n w_n^\dagger + R(z), \quad (\text{A1})$$

where $\{\lambda_n\}$ are N_Γ eigenvalues enclosed by Γ , v_n , and w_n are the corresponding right and left eigenvectors of $T(\lambda_n)$, with the normalization $w_n^\dagger T'(\lambda_n) v_n = 1$ [see Eqs. (6) and (7) in Ref. [57]], and $R(z)$ is analytic inside Γ . This representation forms the theoretical basis of contour-integration methods for nonlinear eigenvalue problems. The robustness of the method is guaranteed due to interior Helmholtz problems with closed boundaries having only real eigenvalues, implying that all the poles of Eq. (A1) are on the real axis and therefore we are never evaluating nearly singular Fredholm matrices on the contour (although care is taken when Fredholm matrices are being evaluated near the intersection of the contour with the real axis since an eigenvalue could be lying close). For

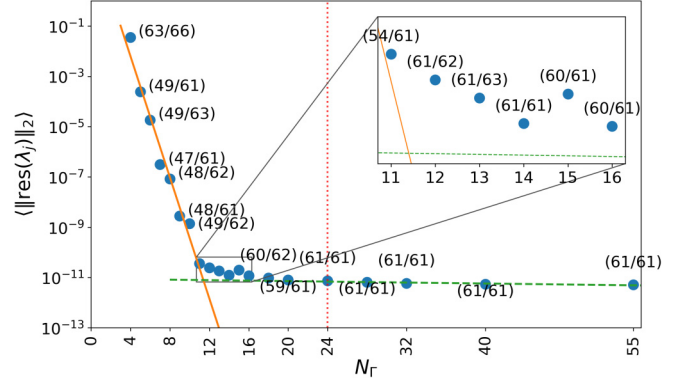


FIG. 14. Log-lin plot of the average residual $\langle \text{res}(\lambda_j) \rangle_2 = \langle \|T(\lambda_j)v(\lambda_j)\|_2 \rangle$ for $a = 0.2$ and $m = 0$ as a function of the contour discretization N_Γ . The contour ($R = 0.4$) encloses the interval $k \in (1404.0, 1404.8)$ containing 61 nonspurious eigenvalues, over which the average is taken. An exponential decrease of the residual is observed up to $N_\Gamma \approx 12$, followed by saturation at $\langle \text{res}(\lambda_j) \rangle_2 \approx 8 \times 10^{-12}$. The labels above each point show (keep/found), where “found” denotes the total number of eigenvalues detected inside the contour, while “keep” denotes the number of nonspurious eigenvalues that pass the post-processing filtering criteria and are retained for analysis. Although the residual magnitude appears stable for $N_\Gamma \in (12, 24)$, it does not find all the eigenvalues, as 1 or 2 eigenvalues close to the contour edges are lost. Therefore only at $N_\Gamma = 24$ (vertical dashed line) and beyond do we find all the eigenvalues inside the contour. For the full spectral range $k \in (5, 3600)$ a conservative choice $N_\Gamma = 50$ was adopted, yielding uniformly small residuals of order 10^{-10} .

an analysis of how N_Γ determines eigenvalue accuracy see Fig. 14. Following Beyn the following lowest contour moments are constructed:

$$A_0 = \frac{1}{2\pi i} \oint_\Gamma T(z)^{-1} V dz, \quad A_1 = \frac{1}{2\pi i} \oint_\Gamma z T(z)^{-1} V dz, \quad (\text{A2})$$

where V is a probing matrix with random columns. Inserting Eq. (A1) into the definition of A_0 gives

$$\begin{aligned} A_0 &= \frac{1}{2\pi i} \oint_\Gamma T(z)^{-1} V dz \\ &= \frac{1}{2\pi i} \oint_\Gamma \left(\sum_{n=1}^{N_\Gamma} \frac{1}{z - \lambda_n} v_n w_n^\dagger + R(z) \right) V dz \\ &= \sum_{n=1}^{N_\Gamma} \left[\frac{1}{2\pi i} \oint_\Gamma \frac{1}{z - \lambda_n} dz \right] v_n (w_n^\dagger V) + \frac{1}{2\pi i} \oint_\Gamma R(z) V dz. \end{aligned} \quad (\text{A3})$$

Since $R(z)V$ is analytic on and inside Γ , Cauchy's theorem gives $\oint_\Gamma R(z)V dz = 0$. Moreover, by the residue theorem, $\frac{1}{2\pi i} \oint_\Gamma \frac{1}{z - \lambda_n} dz = 1$ for each λ_n inside Γ . Therefore,

$$A_0 = \sum_{n=1}^{N_\Gamma} v_n (w_n^\dagger V). \quad (\text{A4})$$

Similarly, we have for the first moment

$$\begin{aligned}
 A_1 &= \frac{1}{2\pi i} \oint_{\Gamma} z T(z)^{-1} V dz \\
 &= \frac{1}{2\pi i} \oint_{\Gamma} z \left(\sum_{n=1}^{N_{\Gamma}} \frac{1}{z - \lambda_n} v_n w_n^{\dagger} + R(z) \right) V dz \\
 &= \sum_{n=1}^{N_{\Gamma}} \left[\frac{1}{2\pi i} \oint_{\Gamma} \frac{z}{z - \lambda_n} dz \right] v_n (w_n^{\dagger} V) \\
 &\quad + \frac{1}{2\pi i} \oint_{\Gamma} z R(z) V dz.
 \end{aligned} \tag{A5}$$

Again, $zR(z)V$ is analytic inside Γ , so its contour integral vanishes. For the pole terms we use the identity $\frac{z}{z - \lambda_n} = 1 + \frac{\lambda_n}{z - \lambda_n}$, hence $\frac{1}{2\pi i} \oint_{\Gamma} \frac{z}{z - \lambda_n} dz = \lambda_n$.

Therefore,

$$A_1 = \sum_{n=1}^{N_{\Gamma}} \lambda_n v_n (w_n^{\dagger} V). \tag{A6}$$

The common term gets relabeled as $\alpha_n^{\dagger} = w_n^{\dagger} V$,

$$A_0 = \sum_{n=1}^r v_n \alpha_n^{\dagger}, \quad A_1 = \sum_{n=1}^r \lambda_n v_n \alpha_n^{\dagger}, \tag{A7}$$

where $r = N_{\Gamma}$ is the number of eigenvalues inside Γ (all unique since we are in a given irreducible representation). Introducing the matrices

$$\begin{aligned}
 \Phi &= [v_1, \dots, v_r] \in \mathbb{C}^{N \times r}, \\
 \Lambda &= \text{diag}(\lambda_1, \dots, \lambda_r) \in \mathbb{C}^{r \times r}, \\
 C &= \begin{pmatrix} \alpha_1^{\dagger} \\ \vdots \\ \alpha_r^{\dagger} \end{pmatrix} \in \mathbb{C}^{r \times L},
 \end{aligned} \tag{A8}$$

where $r = \text{rank}(A_0)$ by construction, N is the dimension of $T(z)$, and L is the number of expected eigenvalues in the contour with additional padding (due to the fact that there are fluctuations around the smooth part of Weyl's law we reasonably increase the probing size of the random matrix V as to not lose potential eigenvalues.) Therefore, Eq. (A7) can be written compactly as

$$A_0 = \Phi C, \quad A_1 = \Phi \Lambda C. \tag{A9}$$

We compute the singular value decomposition

$$A_0 = U \Sigma W^{\dagger}, \quad U \in \mathbb{C}^{N \times r}, \quad \Sigma \in \mathbb{R}^{r \times r}, \quad W \in \mathbb{C}^{L \times r}, \tag{A10}$$

and since $A_0 = \Phi C$ has guaranteed rank r , the columns of U form an orthonormal basis of the eigenspace spanned by the vectors $\{v_n\}$, implying that the matrix

$$S = U^{\dagger} \Phi \in \mathbb{C}^{r \times r} \tag{A11}$$

is invertible. Constructing now the reduced matrix

$$B = U^{\dagger} A_1 W \Sigma^{-1}, \tag{A12}$$

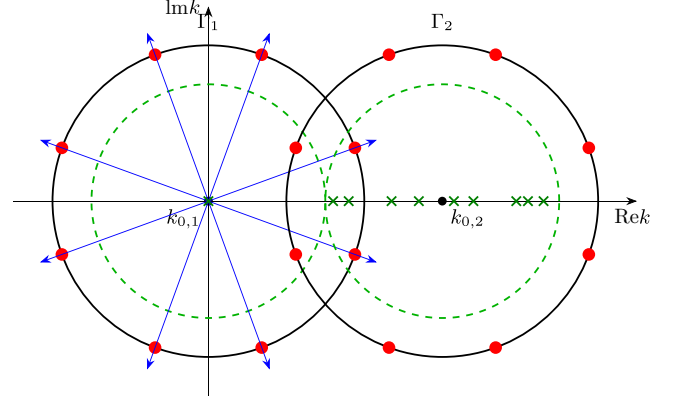


FIG. 15. Two successive complex contour centers $k_{0,1}$ and $k_{0,2}$ with their associated circular integration contours (black). Since the accuracy of the projected eigenvalues obtained from diagonalizing (A16) deteriorates near the contour boundary, adjacent contours are chosen to overlap slightly. Only eigenvalues lying within the inner, nonoverlapping regions (green dashed circles) are retained, thereby preserving numerical accuracy. The contour radius is estimated using Weyl's law, supplemented by a small safety margin to ensure that no eigenvalues are missed. In practice, we employ a trapezoidal discretization with $N = 50$ quadrature nodes and parametrize each contour as $\Gamma_i = \{k_{0,i} + R \exp(j\phi) | \phi = \frac{2\pi}{N}, j = 1, \dots, N-1\}$ where R is chosen such that at most 150 eigenvalues are expected within the contour. In the computation the contour radius was restricted to $R \leq 0.5$ (high k range), since larger values led to spurious solutions (manifested as unresolved small singular values, where the numerical threshold for zero becomes ambiguous) and reduced precision (see integral algorithm 1, step 4 in Ref. [57]). This discretization is sufficient for the moment matrix A_0 to become unambiguously numerically rank-deficient, indicating that all physical eigenvalues enclosed by the contour have been captured.

and using Eq. (A9), we obtain

$$\begin{aligned}
 B &= U^{\dagger} (\Phi \Lambda C) W \Sigma^{-1} \\
 &= (U^{\dagger} \Phi) \Lambda (C W \Sigma^{-1}) \\
 &= S \Lambda (C W \Sigma^{-1}).
 \end{aligned} \tag{A13}$$

Multiplying $A_0 = \Phi C$ on the left by $U^{\dagger}$ and using Eqs. (A9) and (A10) we get

$$U^{\dagger} A_0 = U^{\dagger} \Phi C \implies \Sigma W^{\dagger} = S C. \tag{A14}$$

Right multiplication by $W \Sigma^{-1}$ yields

$$I_r = S C W \Sigma^{-1}, \quad \text{hence} \quad C W \Sigma^{-1} = S^{-1}. \tag{A15}$$

Therefore,

$$B = S \Lambda S^{-1}, \tag{A16}$$

showing that the diagonal matrix Λ we are seeking is available by simply diagonalizing the reduced matrix B , whose construction is relatively straightforward. To accelerate the evaluation of $T(z)$ along the contour, matrix elements are interpolated using Chebyshev expansions constructed along radial “rays” in the complex plane Fig. 15 [62]. Acceptance intervals are introduced to maintain uniform interpolation accuracy in the presence of highly oscillatory Hankel kernels. In particular, for large arguments the outgoing Hankel function

admits the asymptotic form [44]

$$\begin{aligned} H_1^{(1)}(kr) &\sim \sqrt{\frac{2}{\pi kr}} \exp \left[i \left(kr - \frac{3\pi}{4} \right) \right] \\ &= \sqrt{\frac{2}{\pi kr}} \exp \left[i \left(\text{Re}(k)r - \frac{3\pi}{4} \right) \right] \exp(-\text{Im}(k)r), \end{aligned} \quad (\text{A17})$$

valid for $|kr| \rightarrow \infty$ where the complex wave number is written as $k = \text{Re}(k) + i\text{Im}(k)$. Equation (A17) shows that for complex wave numbers the kernel acquires an exponential factor $\exp(-\text{Im}(k)r)$. For $\text{Im}(k) < 0$, this term grows as $\exp(|\text{Im}(k)|r)$, imposing practical constraints on the contour shape (for instance, one may employ an elliptic contour whose minor axis is oriented perpendicular to the real axis and chosen significantly smaller than the major axis, the latter determining the expected number of enclosed eigenvalues) and on the maximum distance scale within the billiard for a prescribed numerical accuracy.

APPENDIX B: SYMMETRY REDUCED BIE FORMALISM

1. Desymmetrized Fredholm matrix

Here we derive the symmetric reduced BIE. We start by defining the action of an operator $\mathbf{R}$ (representing, e.g., rotation, reflection) on an arbitrary function of a vector variable $f(\mathbf{x})$ as the pullback $O_{\mathbf{R}}$,

$$O_{\mathbf{R}}(f(\mathbf{x})) = f(\mathbf{R}^{-1}\mathbf{x}). \quad (\text{B1})$$

For notational convenience we define the normal derivative of the Green function as $K(\mathbf{x}, \mathbf{y})$ [see Eq. (11)],

$$\partial_{n_y} G_k(\mathbf{x}, \mathbf{y}) = K(\mathbf{x}, \mathbf{y}) = -\frac{ik}{4} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{d(\mathbf{x}, \mathbf{y})} H_1^{(1)}(kd(\mathbf{x}, \mathbf{y})). \quad (\text{B2})$$

We would now like to show that it is left invariant under the action of $\mathbf{R}$, that is

$$O_{\mathbf{R}}(K(\mathbf{x}, \mathbf{y})) = K(\mathbf{R}^{-1}\mathbf{x}, \mathbf{R}^{-1}\mathbf{y}) = K(\mathbf{x}, \mathbf{y}), \quad (\text{B3})$$

which we now prove. Writing the action of $\mathbf{R}$ explicitly,

$$\begin{aligned} K(\mathbf{R}^{-1}\mathbf{x}, \mathbf{R}^{-1}\mathbf{y}) &= -\frac{ik}{4} \frac{\mathbf{n}(\mathbf{R}^{-1}\mathbf{y}) \cdot (\mathbf{R}^{-1}\mathbf{x} - \mathbf{R}^{-1}\mathbf{y})}{d(\mathbf{R}^{-1}\mathbf{x}, \mathbf{R}^{-1}\mathbf{y})} \\ &\quad \times H_1^{(1)}(kd(\mathbf{R}^{-1}\mathbf{x}, \mathbf{R}^{-1}\mathbf{y})), \end{aligned} \quad (\text{B4})$$

we can use the fact that the dot product between vectors is invariant to a simultaneous rotation of both

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{R}^{-1}\mathbf{x}) \cdot (\mathbf{R}^{-1}\mathbf{y}). \quad (\text{B5})$$

Both the numerator and the denominator prefactors to the Hankel function are dot products of vectors so they are left invariant

$$\begin{aligned} d(\mathbf{R}^{-1}\mathbf{x}, \mathbf{R}^{-1}\mathbf{y}) &= |\mathbf{R}^{-1}\mathbf{x} - \mathbf{R}^{-1}\mathbf{y}| \\ &= |\mathbf{R}^{-1}(\mathbf{x} - \mathbf{y})| \\ &= \sqrt{(\mathbf{R}^{-1}(\mathbf{x} - \mathbf{y})) \cdot (\mathbf{R}^{-1}(\mathbf{x} - \mathbf{y}))} \\ &= \sqrt{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})} \\ &= |\mathbf{x} - \mathbf{y}| \\ &= d(\mathbf{x}, \mathbf{y}). \end{aligned} \quad (\text{B6})$$

The same follows for the numerator

$$\begin{aligned} \mathbf{n}(\mathbf{R}^{-1}\mathbf{y}) \cdot (\mathbf{R}^{-1}\mathbf{x} - \mathbf{R}^{-1}\mathbf{y}) &= \mathbf{n}(\mathbf{R}^{-1}\mathbf{y}) \cdot \mathbf{R}^{-1}(\mathbf{x} - \mathbf{y}) \\ &= (\mathbf{R}^{-1}\mathbf{n}(\mathbf{y})) \cdot \mathbf{R}^{-1}(\mathbf{x} - \mathbf{y}) \\ &= (\mathbf{R}^{-1}\mathbf{n}(\mathbf{y})) \cdot (\mathbf{R}^{-1}(\mathbf{x} - \mathbf{y})) \\ &= \mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}), \end{aligned} \quad (\text{B7})$$

where in line 2 we have used the fact that normals transform as

$$\mathbf{n}(\mathbf{R}^{-1}\mathbf{y}) = \mathbf{R}^{-1}\mathbf{n}(\mathbf{y}). \quad (\text{B8})$$

Now using Eqs. (B8) and (B6) in Eq. (B4) it automatically follows that Eq. (B2) is invariant under $\mathbf{R}$ and Eq. (B3) is proven.

Using these prerequisites, the relation

$$A(u(\mathbf{x})) = \frac{1}{2}u(\mathbf{x}) + \int_{\Gamma_{\text{full}}} K(\mathbf{x}, \mathbf{y})u(\mathbf{y})ds(\mathbf{y}) = 0, \quad (\text{B9})$$

commutes with the action of $\mathbf{R}$, that is

$$[A, O_{\mathbf{R}}]u(\mathbf{x}) = A(O_{\mathbf{R}}u(\mathbf{x})) - O_{\mathbf{R}}A(u(\mathbf{x})) = 0. \quad (\text{B10})$$

To show this we will need the following relations, which follow from Eq. (B3) and basic geometry

$$\begin{aligned} K(\mathbf{x}, \mathbf{R}\mathbf{y}) &= K(\mathbf{R}^{-1}\mathbf{x}, \mathbf{R}^{-1}\mathbf{R}\mathbf{y}) = K(\mathbf{R}^{-1}\mathbf{x}, \mathbf{y}), \\ ds(\mathbf{R}^{-1}\mathbf{y}) &= ds(\mathbf{y}). \end{aligned} \quad (\text{B11})$$

Up to this point the symmetry operation $\mathbf{R}$ was arbitrary, but now we will restrict ourselves only to true symmetries of the boundary Γ_{full} , since we will require $O_{\mathbf{R}}\Gamma_{\text{full}} = \Gamma_{\text{full}}$. Explicitly for a rotation this statement says

$$\begin{aligned} O_{\mathbf{R}}\Gamma_{\text{full}} &= \{(r(\phi + \theta_R), \phi + \theta_R)\} \\ &= \{(r(\phi), \phi)\} = \Gamma_{\text{full}}, \end{aligned} \quad (\text{B12})$$

where $\theta_R = \text{angle}(\mathbf{R})$. Writing the action of $\mathbf{R}$ on $u(\mathbf{x})$ via Eq. (B1) and using Eqs. (B3) and (B11) we can write

$$\begin{aligned} A(O_{\mathbf{R}}u(\mathbf{x})) &= \frac{1}{2}u(\mathbf{R}^{-1}\mathbf{x}) + \int_{\Gamma_{\text{full}}} K(\mathbf{x}, \mathbf{y})u(\mathbf{R}^{-1}\mathbf{y})ds(\mathbf{y}) \\ &\stackrel{z=\mathbf{R}^{-1}\mathbf{y}}{=} \frac{1}{2}u(\mathbf{R}^{-1}\mathbf{x}) + \int_{\Gamma_{\text{full}}} K(\mathbf{x}, \mathbf{R}\mathbf{z})u(\mathbf{z})ds(\mathbf{z}) \\ &= \frac{1}{2}u(\mathbf{R}^{-1}\mathbf{x}) + \int_{\Gamma_{\text{full}}} K(\mathbf{R}^{-1}\mathbf{x}, \mathbf{z})u(\mathbf{z})ds(\mathbf{z}) \\ &= (O_{\mathbf{R}}Au)(\mathbf{x}), \end{aligned} \quad (\text{B13})$$

proving that Eq. (B10) and implying that if $u(\mathbf{x})$ is a solution of Eq. (B9), then $O_{\mathbf{R}}u(\mathbf{x}) = u(\mathbf{R}^{-1}\mathbf{x})$ is also a solution. This property has an important consequence. Since the solution space is closed under the action of the symmetry operator $O_{\mathbf{R}}$, solutions may be chosen to possess definite transformation behavior under symmetry operations.

For the rotational symmetry group $C_3 = \{I, \mathbf{R}, \mathbf{R}^2\}$ one has

$$O_{\mathbf{R}}^3 u(\mathbf{x}) = u(\mathbf{x}), \quad (\text{B14})$$

which follows directly from the cyclic nature of the symmetry and $\mathbf{R}$ is the rotation operator in $\mathbb{R}^2$,

$$\mathbf{R}\mathbf{x} = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \mathbf{x}. \quad (\text{B15})$$

The operator $O_{\mathbf{R}}$ therefore admits eigenfunctions. Denoting such eigenfunctions by $u_m(\mathbf{x})$, we define

$$O_{\mathbf{R}}(u_m(\mathbf{x})) = u_m(\mathbf{R}^{-1}\mathbf{x}) = \lambda_m u_m(\mathbf{x}). \quad (\text{B16})$$

Applying the symmetry operation three times yields

$$(O_{\mathbf{R}}^3 u_m)(\mathbf{x}) = \lambda_m^3 u_m(\mathbf{x}). \quad (\text{B17})$$

Consistency with the group property $O_{\mathbf{R}}^3 = I$ therefore requires

$$\lambda_m^3 = 1. \quad (\text{B18})$$

The eigenvalues are thus the cubic roots of unity,

$$\lambda_m = e^{-2\pi i m/3}, \quad m = 0, 1, 2, \quad (\text{B19})$$

and they can be identified exactly with characters of the irreducible representations,

$$\chi_m(\mathbf{R}^\ell) = e^{-2\pi i m \ell/3}. \quad (\text{B20})$$

Equation (B16) may therefore be written as the symmetry transformation law $u_m(\mathbf{R}^{-1}\mathbf{x}) = \chi_m(\mathbf{R}) u_m(\mathbf{x})$, or more generally,

$$u_m(\mathbf{R}^{-\ell}\mathbf{x}) = \chi_m(\mathbf{R}^\ell) u_m(\mathbf{x}). \quad (\text{B21})$$

Physically, this expresses that boundary functions can be classified according to their symmetry behavior. Each solution belongs to a definite symmetry sector. To extract solutions with definite symmetry, we introduce the projection operator onto the sector m ,

$$u_m(\mathbf{x}) = P_m u(\mathbf{x}) = \frac{1}{3} \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* O_{\mathbf{R}}^\ell u(\mathbf{x}). \quad (\text{B22})$$

Inserting $u_m(\mathbf{x})$ into Eq. (B9) we can see that it is indeed a solution whenever $u(\mathbf{x})$ is

$$A(u_m(\mathbf{x})) = A(P_m u(\mathbf{x})) = P_m(A(u(\mathbf{x}))) = 0, \quad (\text{B23})$$

where we have used the fact that P_m is a linear operator. Consequently, the boundary integral equation decomposes into independent symmetry sectors labelled by m . This allows the desymmetrization of Eq. (B9) into an integral over only the fundamental domains if we restrict ourselves to only solutions $u_m(\mathbf{x})$ in a given irreducible representation labeled with $m = 0, 1, 2$. Therefore we decompose the integration over the full boundary into an integral over the fundamental domain and the other parts of the domain will be constructed from symmetry

$$\begin{aligned} & \int_{\Gamma_{\text{full}}} K(\mathbf{x}, \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}) \\ &= \sum_{\ell=0}^2 \int_{\Gamma_{\text{fund}}} K(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{R}^\ell \mathbf{y}) ds(\mathbf{y}) \\ &= \sum_{\ell=0}^2 \chi_m(\mathbf{R}^{-\ell}) \int_{\Gamma_{\text{fund}}} K(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}) \\ &= \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \int_{\Gamma_{\text{fund}}} K(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}). \end{aligned} \quad (\text{B24})$$

Finally, inserting Eq. (B24) into Eq. (B9) we have the equation

$$A(u_m(\mathbf{x})) = \frac{1}{2} u_m(\mathbf{x}) + \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \int_{\Gamma_{\text{fund}}} K(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}). \quad (\text{B25})$$

Upon discretization of the fundamental boundary Γ_{fund} using quadrature nodes $\{\mathbf{x}_j\}_{j=1}^N$ and quadrature weights $\{\zeta_j\}_{j=1}^N$ (e.g., Gauss-Legendre), the BIE reduces to a matrix equation. The symmetry-reduced boundary integral equation becomes

$$\sum_{j=1}^N A_{ij}^{(m)} u_m(\mathbf{x}_j) = 0, \quad (\text{B26})$$

where the matrix elements are given by

$$A_{ij}^{(m)} = \frac{1}{2} \delta_{ij} + \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \zeta_j K(\mathbf{x}_i, \mathbf{R}^\ell \mathbf{x}_j), \quad (\text{B27})$$

The matrix (B27) is then used in Beyn's method [$T(z)$ matrix as used in Eqs. (A1) and (A2)] as described in Appendix A.

2. Wave function in a given irreducible representation

For Dirichlet boundary conditions $\psi|_{\Gamma_{\text{full}}} = 0$, Green's theorem reduces to a pure single-layer potential term. Hence the interior wave function in symmetry sector m is reconstructed as $[G_k(\mathbf{x}, \mathbf{y})]$ given in Eq. (9)]

$$\psi_m(\mathbf{x}) = - \int_{\Gamma_{\text{full}}} G_k(\mathbf{x}, \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \Omega. \quad (\text{B28})$$

Following the procedure above we reduce the boundary integral to the fundamental boundary segment Γ_{fund} using $\Gamma_{\text{full}} = \bigcup_{\ell=0}^2 \mathbf{R}^\ell \Gamma_{\text{fund}}$ and Eq. (B11), obtaining

$$\psi_m(\mathbf{x}) = - \sum_{\ell=0}^2 \int_{\Gamma_{\text{fund}}} G_k(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{R}^\ell \mathbf{y}) ds(\mathbf{y}). \quad (\text{B29})$$

Using the symmetry transformation law (B21), i.e., $u_m(\mathbf{R}^\ell \mathbf{y}) = \chi_m(\mathbf{R}^{-\ell}) u_m(\mathbf{y}) = \chi_m(\mathbf{R}^\ell)^* u_m(\mathbf{y})$, we arrive at the symmetry-reduced reconstruction formula

$$\psi_m(\mathbf{x}) = - \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \int_{\Gamma_{\text{fund}}} G_k(\mathbf{x}, \mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}), \quad (\text{B30})$$

where numerically the boundary integral over Γ_{fund} is discretized using the same procedure as going from Eqs. (B25) to (B27).

3. Poincaré-Husimi function in a given irreducible representation

The Poincaré-Husimi function (unnormalized) associated with a boundary density is defined as [4,63]

$$H(q, p) = |\langle c_{q,p} | u \rangle|^2, \quad \langle c_{q,p} | u \rangle = \int_{\Gamma_{\text{full}}} c_{q,p}^*(\mathbf{y}) u(\mathbf{y}) ds(\mathbf{y}), \quad (\text{B31})$$

where $c_{q,p}(\mathbf{y})$ denotes a boundary coherent state localized at the phase-space point (q, p) . Restricting to a symmetry sector and using the boundary decomposition $\Gamma_{\text{full}} = \bigcup_{\ell=0}^2 \mathbf{R}^\ell \Gamma_{\text{fund}}$ together with the invariance $ds(\mathbf{R}^\ell \mathbf{y}) = ds(\mathbf{y})$, the projection amplitude becomes

$$\langle c_{q,p} | u_m \rangle = \sum_{\ell=0}^2 \int_{\Gamma_{\text{fund}}} c_{q,p}^*(\mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{R}^\ell \mathbf{y}) ds(\mathbf{y}) = \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \int_{\Gamma_{\text{fund}}} c_{q,p}^*(\mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}), \quad (\text{B32})$$

where Eq. (B21) has been used. The PH function in the m th irreducible representation therefore reads

$$H_m(q, p) = \left| \sum_{\ell=0}^2 \chi_m(\mathbf{R}^\ell)^* \int_{\Gamma_{\text{fund}}} c_{q,p}^*(\mathbf{R}^\ell \mathbf{y}) u_m(\mathbf{y}) ds(\mathbf{y}) \right|^2. \quad (\text{B33})$$

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- [1] H.-J. Stöckmann, *Quantum Chaos—An Introduction* (Cambridge University Press, Cambridge, UK, 1999).
 - [2] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 2010).
 - [3] K. Husimi, Some formal properties of the density matrix, *Proc. Phys. Math. Soc. Jpn.* **22**, 264 (1940).
 - [4] A. Bäcker, S. Fürstberger, and R. Schubert, Poincaré Husimi representation of eigenstates in quantum billiards, *Phys. Rev. E* **70**, 036204 (2004).
 - [5] A. Voros, Semi-classical ergodicity of quantum eigenstates in the Wigner representation, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems* (Springer, Berlin, 1979), pp. 326–333.
 - [6] E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40**, 749 (1932).
 - [7] A. Chan, S. Shivam, D. A. Huse, and A. D. Luca, Many-body quantum chaos and space-time translational invariance, *Nat. Commun.* **13**, 7484 (2021).
 - [8] M. V. Berry and M. Tabor, Level clustering in the regular spectrum, *Proc. Roy. Soc. Lond. A* **356**, 375 (1977).
 - [9] M. Robnik and G. Veble, On spectral statistics of classically integrable systems, *J. Phys. A: Math. Gen.* **31**, 4669 (1998).
 - [10] M. Robnik and M. V. Berry, False time-reversal violation and energy level statistics: The role of anti-unitary symmetry, *J. Phys. A: Math. Gen.* **19**, 669 (1986).
 - [11] M. Robnik, Antiunitary symmetries and energy level statistics, in *Quantum Chaos and Statistical Nuclear Physics* (Springer, Berlin, 1986), pp. 120–130.
 - [12] O. Bohigas, M. J. Giannoni, and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, *Phys. Rev. Lett.* **52**, 1 (1984).
 - [13] G. Casati, F. Valz-Gris, and I. Guarneri, On the connection between quantization of nonintegrable systems and statistical theory of spectra, *Lett. Nuovo Cimento* **28**, 279 (1980).
 - [14] M. Robnik, Classical dynamics of a family of billiards with analytic boundaries, *J. Phys. A: Math. Gen.* **16**, 3971 (1983).
 - [15] Y. G. Sinai, Dynamical systems with elastic reflections: Ergodic properties of dispersing billiards, *Russ. Math. Surv.* **25**, 137 (1970).
 - [16] L. A. Bunimovich, Mushrooms and other billiards with divided phase space, *Chaos* **11**, 802 (2001).
 - [17] Č. Lozej, D. Lukman, and M. Robnik, Effects of stickiness in the classical and quantum ergodic lemon billiard, *Phys. Rev. E* **103**, 012204 (2021).
 - [18] Č. Lozej, G. Casati, and T. Prosen, Quantum chaos in triangular billiards, *Phys. Rev. Res.* **4**, 013138 (2022).
 - [19] H. Alt, A. Backer, C. Dembowski, H.-D. Graf, R. Hofferbert, H. Rehfeld, and A. Richter, Mode fluctuation distribution for spectra of superconducting microwave billiards, *Phys. Rev. E* **58**, 1737 (1998).
 - [20] R. Hofferbert, H. Alt, C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, H. Rehfeld, and A. Richter, Experimental investigations of chaos-assisted tunneling in a microwave annular billiard, *Phys. Rev. E* **71**, 046201 (2005).
 - [21] X.-F. Jiang, C.-L. Zou, L. Wang, Q.-H. Gong, and Y.-F. Xiao, Whispering-gallery microcavities with unidirectional laser emission, *Laser Photon. Rev.* **10**, 40 (2016).
 - [22] H.-J. Stöckmann and U. Kuhl, Microwave studies of the spectral statistics in chaotic systems, *J. Phys. A: Math. Theor.* **55**, 383001 (2022).
 - [23] M. V. Berry, Regular and irregular semiclassical wavefunctions, *J. Phys. A: Math. Gen.* **10**, 2083 (1977).
 - [24] B. Batistić, Č. Lozej, and M. Robnik, The distribution of localization measures of chaotic eigenstates in the stadium billiard, *Nonlin. Phenom. Complex Syst. (Minsk)* **23**, 17 (2020).
 - [25] M. Robnik, Recent developments in quantum chaos of mixed-type systems: A mini review, *Nonlin. Phenom. Complex Syst.* **26**, 209 (2023).
 - [26] E. J. Heller, Bound-state eigenfunctions of classically chaotic Hamiltonian systems: Scars of periodic orbits, *Phys. Rev. Lett.* **53**, 1515 (1984).
 - [27] T. A. Brody, A statistical measure for the repulsion of energy levels, *Lett. Nuovo Cimento* **7**, 482 (1973).
 - [28] F. M. Izrailev, Intermediate statistics of the quasi-energy spectrum and quantum localisation of classical chaos, *J. Phys. A: Math. Gen.* **22**, 865 (1989).
 - [29] B. Batistić and M. Robnik, Quantum localization of chaotic eigenstates and the level spacing distribution, *Phys. Rev. E* **88**, 052913 (2013).
 - [30] M. Tinkham, *Group Theory and Quantum Mechanics* (McGraw-Hill, Columbus, OH, 1964).
 - [31] F. Leyvraz, C. Schmit, and T. H. Seligman, Anomalous spectral statistics in a symmetrical billiard, *J. Phys. A: Math. Gen.* **29**, L575 (1996).
 - [32] C. Dembowski, B. Dietz, H.-D. Gräf, A. Heine, F. Leyvraz, M. Miski-Oglu, A. Richter, and T. H. Seligman, Phase shift experiments identifying Kramers doublets in a chaotic

- superconducting microwave billiard of threefold symmetry, *Phys. Rev. Lett.* **90**, 014102 (2003).
- [33] C. H. Joyner, S. Müller, and M. Sieber, Semiclassical approach to discrete symmetries in quantum chaos, *J. Phys. A: Math. Theor.* **45**, 205102 (2012).
- [34] Z.-Y. Li and L. Huang, Quantization and interference of a quantum billiard with fourfold rotational symmetry, *Phys. Rev. E* **101**, 062201 (2020).
- [35] B. Dietz, Relativistic quantum billiards with threefold rotational symmetry: Exact, symmetry-projected solutions for equilateral neutrino billiard, *Acta Phys. Pol. A* **140**, 473 (2021).
- [36] W. Zhang and B. Dietz, Microwave photonic crystals, graphene, and honeycomb-kagome billiards with threefold symmetry: Comparison with nonrelativistic and relativistic quantum billiards, *Phys. Rev. B* **104**, 064310 (2021).
- [37] A. Gusso, M. G. E. da Luz, and L. G. C. Rego, Quantum chaos in nanoelectromechanical systems, *Phys. Rev. B* **73**, 035436 (2006).
- [38] B. Dietz, A. Heine, V. Heuveline, and A. Richter, Test of a numerical approach to the quantization of billiards, *Phys. Rev. E* **71**, 026703 (2005).
- [39] A. M. Lyapunov, The general problem of the stability of motion, *Int. J. Control* **55**, 531 (1992).
- [40] C. Dellago, H. A. Posch, and W. G. Hoover, Lyapunov instability in a system of hard disks in equilibrium and nonequilibrium steady states, *Phys. Rev. E* **53**, 1485 (1996).
- [41] G. Benettin, L. Galgani, A. Giorgilli, and J.-M. Strelcyn, Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems. Part I: Theory, *Meccanica* **15**, 9 (1980).
- [42] G. Benettin, L. Galgani, A. Giorgilli, and J.-M. Strelcyn, Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems: A method for computing all of them. Part 2: Numerical application, *Meccanica* **15**, 21 (1980).
- [43] A. Bäcker, Numerical aspects of eigenvalue and eigenfunction computations for chaotic quantum systems, in *The Mathematical Aspects of Quantum Maps*, edited by M. D. Esposti and S. Graffi (Springer, Berlin, Heidelberg, 2003), pp. 91–144.
- [44] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, Garden City, NY, 1964).
- [45] R. Kress, Boundary integral equations in time-harmonic acoustic scattering, *Math. Comput. Model.* **15**, 229 (1991).
- [46] M. L. Mehta, *Random Matrices* (Academic Press, Boston, 1991).
- [47] M. V. Berry, Semiclassical theory of spectral rigidity, *Proc. Roy. Soc. Lond. A* **400**, 229 (1985).
- [48] T. Prosen and M. Robnik, Energy level statistics in the transition region between integrability and chaos, *J. Phys. A: Math. Gen.* **26**, 2371 (1993).
- [49] A. Bäcker, S. Fürstberger, R. Schubert, and F. Steiner, Behaviour of boundary functions for quantum billiards, *J. Phys. A: Math. Gen.* **35**, 10293 (2002).
- [50] B. Batistić, Č. Lozej, and M. Robnik, Statistical properties of the localization measure of chaotic eigenstates and the spectral statistics in a mixed-type billiard, *Phys. Rev. E* **100**, 062208 (2019).
- [51] B. Batistić, T. Manos, and M. Robnik, The intermediate level statistics in dynamically localized chaotic eigenstates, *Europhys. Lett.* **102**, 50008 (2013).
- [52] B. Batistić, Č. Lozej, and M. Robnik, The level repulsion exponent of localized chaotic eigenstates as a function of the classical transport time scales in the stadium billiard, *Nonlin. Phenom. Complex Syst. (Minsk)* **21**, 225 (2018).
- [53] H. Yan and M. Robnik, Chaos and quantization of the three-particle generic Fermi-Pasta-Ulam-Tsingou model. II. Phenomenology of quantum eigenstates, *Phys. Rev. E* **109**, 054211 (2024).
- [54] A. I. Shnirel'man, Ergodic properties of eigenfunctions, *Usp. Mat. Nauk* **29**, 181 (1974).
- [55] M. Robnik, Topics in quantum chaos in generic systems, *Nonlin. Phenom. Complex Syst. (Minsk)* **1**, 1 (1998).
- [56] P. Leboeuf and A. Voros, Chaos-revealing multiplicative representation of quantum eigenstates, *J. Phys. A: Math. Gen.* **23**, 1765 (1990).
- [57] W.-J. Beyn, An integral method for solving nonlinear eigenvalue problems, *Linear Algebra Appl.* **436**, 3839 (2012).
- [58] A. Imakura, L. Du, and T. Sakurai, A map of contour integral-based eigensolvers for solving generalized eigenvalue problems, *Jpn. J. Ind. Appl. Math.* **33**, 721 (2016).
- [59] E. Vergini and M. Saraceno, Calculation by scaling of highly excited states of billiards, *Phys. Rev. E* **52**, 2204 (1995).
- [60] G. Veble, T. Prosen, and M. Robnik, Expanded boundary integral method and chaotic time-reversal doublets in quantum billiards, *New J. Phys.* **9**, 15 (2007).
- [61] R. Mennicken and M. Möller, *Non-self-adjoint Boundary Eigenvalue Problems*, North-Holland Mathematics Studies, Vol. 192 (Elsevier, Amsterdam, 2003).
- [62] L. Greengard, Mpspack: Hank106.f, GitHub source file (2026), <https://github.com/ahbarnett/mpspack/blob/master/@utils/hank106.f>.
- [63] B. Crespi, G. Perez, and S.-J. Chang, Quantum Poincaré sections for two-dimensional billiards, *Phys. Rev. E* **47**, 986 (1993).