

Irreversibility of the pendulum revisited from Husimi's adiabaticity parameter

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(Received 7 September 2025; accepted 9 December 2025; published 29 December 2025)

We revisit the irreversibility of the pendulum with time-dependent angular frequency, considered in a classical paper by K. Husimi. He introduced a parameter that measures the adiabaticity of a process utilizing an adiabatic invariant for the equation of motion of the pendulum. With this adiabaticity parameter, Husimi showed the irreversibility of the pendulum for a cyclic process, which is reminiscent of the Planck principle in thermodynamics, based on the microscopic mechanics. In this study, we generalize the argument by Husimi to a damped pendulum with friction, and highlight the role of conservation of a phase-space area on the Husimi's adiabaticity parameter. Moreover, we also investigate the second law of thermodynamics and its generalization for a general noncyclic process as well as a cyclic process, and elucidate how the Husimi's adiabaticity parameter impacts this law. In particular, for a general noncyclic process, we show the law of entropy nondecrease for the pendulum without friction by using the property of the Husimi's adiabaticity parameter.

DOI: [10.1103/r7cg-vxz1](https://doi.org/10.1103/r7cg-vxz1)

I. INTRODUCTION

Deriving irreversibility from an underlying microscopic mechanics is one of the goals in nonequilibrium statistical mechanics since Boltzmann. Such attempts have been made for both classical and quantum mechanical systems, and under some reasonable assumptions, the second law of thermodynamics for time-dependent operations was derived in general setups [1–9].

One of the key points to connect thermodynamics and mechanics is to focus on the adiabatic theorem in mechanics. In classical mechanics, when a parameter of a system is adiabatically (sufficiently slowly) changed, the phase-space volume enclosed by a constant energy surface is proportional to an adiabatic invariant and is conserved in time. Because the thermodynamic entropy in macroscopic systems is given by the logarithm of the phase-space volume enclosed by a constant energy surface, its conservation may correspond to the conservation of the entropy of a thermally isolated system under a quasistatic operation [10].

In a classical paper [11], Husimi established a connection between thermodynamics and mechanics in terms of the adiabatic theorem. In an attempt to calculate a transition probability of a quantum parametric oscillator, Husimi found that the generating function of the transition probability is characterized by an adiabaticity parameter. It measures the adiabaticity of a process in terms of an adiabatic invariant of

the corresponding *classical* parametric oscillator, which we refer to as the pendulum hereafter. The parameter takes unity under an adiabatic change of a parameter reflecting the existence of adiabatic invariants, while it is larger than unity under a nonadiabatic change, thus serving as a degree of adiabaticity. By using this property of the adiabaticity parameter, Husimi succeeded in showing the thermodynamic irreversibility of the pendulum in a cyclic operation. To be specific, he proved the following relation:

$$E_{t_1} = E_{t_0} Q_{t_1}^*. \quad (1)$$

Here, E_{t_0} and E_{t_1} are the average energies of the pendulum before and after the cyclic operation starting from an initial equilibrium distribution, and $Q_{t_1}^*$ is Husimi's adiabaticity parameter [see Eq. (17) for its definition]. Because of the property $Q_{t_1}^* \geq 1$, we obtain

$$E_{t_1} \geq E_{t_0}. \quad (2)$$

This is reminiscent of the Planck principle in thermodynamics, and Husimi showed this principle based on mechanics for the pendulum. For an interpretation of this inequality, let us quote from Husimi's paper (p. 391 of [11]):

“Let the string of a pendulum, while in oscillation, be shortened or lengthened in an arbitrary manner and let it return to its initial length after a time. Then we expect, *on the average*, to find the energy of the pendulum increased.” Husimi's adiabaticity parameter received renewed interest in the thermodynamics of small systems and found many applications, such as in work distributions [12–14], finite-time heat engines [15–17], and shortcuts to adiabaticity [18–22].

In this paper, we revisit the irreversibility of the pendulum Husimi studied in [11] and generalize it. Our generalizations are twofold: First, we generalize the setup so that it includes a damped pendulum with friction. This generalization highlights the role of non-Hamiltonian dynamics as under-

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lying mechanics of Husimi's adiabaticity parameter. Such a generalization would also be of interest in the context of thermodynamics for non-Hamiltonian systems [23]. Second, we investigate the second law of thermodynamics and its generalization not only for a cyclic process but also for a noncyclic process in terms of the Husimi's adiabaticity parameter, where these two correspond to the Planck's principle and the law of entropy nondecrease in thermodynamics, respectively.

The remainder of the paper is organized as follows. In Sec. II, we introduce our pendulum model and specify the setup of the model. In Sec. III, we present our main results. In Sec. IV, concluding remarks are given.

II. MODEL

Let us consider a one-dimensional pendulum obeying the following equation of motion:

$$\ddot{z}_t = -\omega_t^2 z_t - \Gamma_t \dot{z}_t, \quad (3)$$

where the dot denotes time derivative, and $\omega_t > 0$ and Γ_t denote a time-dependent angular frequency and a time-dependent friction coefficient of the pendulum, respectively. Here, we allow Γ_t to take any value, including a negative value, so that the energy can be injected into the pendulum. The case $\Gamma_t = 0$ was considered in [11], and the present setup thus generalizes it by including the nonvanishing cases $\Gamma_t \neq 0$.

For the statistical mechanics description, we introduce the momentum $p_t \equiv \dot{z}_t$ and write the equation of motion (3) as the following equation for (z_t, p_t) in the phase plane:

$$\dot{z}_t = p_t, \quad (4)$$

$$\dot{p}_t = -\omega_t^2 z_t - \Gamma_t p_t, \quad (5)$$

which, for $\Gamma_t = 0$, recovers the Hamiltonian dynamics with the Hamiltonian $H_t(z_t, p_t)$:

$$H_t(z_t, p_t) \equiv \frac{p_t^2}{2} + \frac{\omega_t^2 z_t^2}{2}. \quad (6)$$

Under an adiabatic change of the angular frequency ω_t , the following quantity is conserved as an adiabatic invariant of the pendulum:

$$\frac{H_t(z_t, p_t)}{\omega_t} = \frac{p_t^2 + \omega_t^2 z_t^2}{2\omega_t} \approx \text{const.} \quad (7)$$

We introduce the average energy $E_t \equiv \langle H_t \rangle$ that serves as the internal energy of the pendulum:

$$E_t = \langle H_t \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_t dp_t f_t(z_t, p_t) H_t(z_t, p_t), \quad (8)$$

where $f_t(z_t, p_t)$ is the probability distribution at time t , and $\langle \cdot \rangle$ denotes the average with respect to the probability distribution. At an initial time $t = t_0$, the system is assumed to obey the microcanonical distribution of energy E_{t_0} as the same initial distribution as the original consideration by Husimi [11]:

$$f_{t_0}(z_{t_0}, p_{t_0}) = \frac{\delta(E_{t_0} - H_{t_0}(z_{t_0}, p_{t_0}))}{A_{t_0}}, \quad (9)$$

where

$$\begin{aligned} A_{t_0} &\equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} \delta(E_{t_0} - H_{t_0}(z_{t_0}, p_{t_0})) \\ &= \frac{2\pi}{\omega_{t_0}}, \end{aligned} \quad (10)$$

is a normalization factor. The realization of this microcanonical ensemble is given as follows [24]. Let us make contact between the pendulum and a heat bath, and after a sufficient time, we detach the heat bath from the pendulum. The pendulum may obey the canonical distribution, and we collect the states with energy E_{t_0} sampled from this distribution, which constitutes the desired microcanonical ensemble. We assume that no work cost is taken for this process. For $t > t_0$, the dynamics obeys Eq. (3) and is deterministic. The pendulum is thermally isolated during the time evolution for $\Gamma_t = 0$.

III. MAIN RESULTS

A. Internal energy in terms of Husimi's adiabaticity parameter

We calculate the internal energy in Eq. (8) and express it in terms of Husimi's adiabaticity parameter. The general solution to Eq. (3) can be written in terms of the fundamental solutions X_t and Y_t :

$$z_t = z_{t_0} Y_t + p_{t_0} X_t, \quad (11)$$

where X_t and Y_t satisfy

$$\ddot{X}_t = -\Gamma_t \dot{X}_t - \omega_t^2 X_t; \quad X_{t_0} = 0, \quad \dot{X}_{t_0} = 1, \quad (12)$$

and

$$\ddot{Y}_t = -\Gamma_t \dot{Y}_t - \omega_t^2 Y_t; \quad Y_{t_0} = 1, \quad \dot{Y}_{t_0} = 0, \quad (13)$$

respectively. Note that X_t has dimension of time and Y_t is dimensionless as seen from Eq. (11). By differentiating Eq. (11) with respect to time t , we also have

$$p_t = z_{t_0} \dot{Y}_t + p_{t_0} \dot{X}_t. \quad (14)$$

Equations (11) and (14) relate (z_t, p_t) and (z_{t_0}, p_{t_0}) in terms of the fundamental solutions X_t and Y_t and their time derivatives.

We can rewrite the internal energy in Eq. (8) as

$$\begin{aligned} E_t &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} \frac{\delta(E_{t_0} - H_{t_0}(z_{t_0}, p_{t_0}))}{A_{t_0}} \\ &\quad \times H_t(z_t(z_{t_0}, p_{t_0}), p_t(z_{t_0}, p_{t_0})), \end{aligned} \quad (15)$$

in terms of the integral with respect to z_{t_0} and p_{t_0} , where we have used $f_t(z_t, p_t) dz_t dp_t = f_{t_0}(z_{t_0}, p_{t_0}) dz_{t_0} dp_{t_0}$. By putting Eqs. (11) and (14) into Eq. (15), and performing the integration with respect to z_{t_0} and p_{t_0} , we obtain (see Appendix A for the derivation)

$$E_t = E_{t_0} \frac{\omega_t}{\omega_{t_0}} Q_t^*. \quad (16)$$

Here, Q_t^* is Husimi's adiabaticity parameter defined as [11]

$$Q_t^* \equiv \omega_{t_0} \frac{\dot{X}_t^2 + \omega_t^2 X_t^2}{2\omega_t} + \omega_{t_0}^{-1} \frac{\dot{Y}_t^2 + \omega_t^2 Y_t^2}{2\omega_t}. \quad (17)$$

By considering the initial conditions in Eqs. (12) and (13), we have $Q_{t_0}^* = 1$.

B. Inequality for Husimi's adiabaticity parameter

It can be shown that Q_t^* is bounded from the lower side as (see Appendix B for the derivation)

$$Q_t^* \geq W_t, \quad (18)$$

where W_t is the Wronskian for the fundamental solutions X_t and Y_t obeying Eqs. (12) and (13):

$$W_t \equiv \dot{X}_t Y_t - X_t \dot{Y}_t. \quad (19)$$

Note that

$$W_{t_0} = 1, \quad (20)$$

from the initial conditions in Eqs. (12) and (13). The Wronskian in Eq. (19) serves as an area expansion rate in the phase plane. In fact, the time evolution of an infinitesimal area element from an initial time t_0 to a time t in the phase plane is given by the Jacobian of the transformation from (z_{t_0}, p_{t_0}) to (z_t, p_t) :

$$dz_t dp_t = |J_t| dz_{t_0} dp_{t_0}. \quad (21)$$

The Jacobian matrix J_t associated with this transformation is given from Eqs. (11) and (14):

$$\begin{pmatrix} z_t \\ p_t \end{pmatrix} = \begin{pmatrix} Y_t & X_t \\ \dot{Y}_t & \dot{X}_t \end{pmatrix} \begin{pmatrix} z_{t_0} \\ p_{t_0} \end{pmatrix} = J_t \begin{pmatrix} z_{t_0} \\ p_{t_0} \end{pmatrix}, \quad (22)$$

which yields $|J_t| = \dot{X}_t Y_t - X_t \dot{Y}_t = W_t$.

We can obtain W_t in this model as follows. By differentiating Eq. (19) with respect to time and using Eqs. (12) and (13), we have

$$\dot{W}_t = -\Gamma_t W_t. \quad (23)$$

By solving this equation, we obtain

$$\begin{aligned} W_t &= W_{t_0} \exp\left(-\int_{t_0}^t \Gamma_{t'} dt'\right) \\ &= \exp\left(-\int_{t_0}^t \Gamma_{t'} dt'\right) > 0, \end{aligned} \quad (24)$$

where we used Eq. (20) in the second equality.

For the case $\Gamma_t = 0$, we recover $W_t = 1$ because the phase-plane area is conserved as seen from Eq. (24) (the Liouville's theorem). In this case, Eq. (18) reads [11]

$$Q_t^* \geq 1, \quad (25)$$

where the equality is achievable when the angular frequency ω_t is adiabatically changed so that $(\dot{X}_t^2 + \omega_t^2 X_t^2)/2\omega_t$ and $(\dot{Y}_t^2 + \omega_t^2 Y_t^2)/2\omega_t$ in Eq. (17) are conserved as the adiabatic invariants for the dynamics in Eqs. (12) and (13) with $\Gamma_t = 0$, respectively. Thus, Q_t^* measures the adiabaticity of the process.

For the case $\Gamma_t \neq 0$, it is more nontrivial to achieve the equality of Eq. (18), and its condition is given by (see Appendix B)

$$X_t = -\frac{\omega_{t_0}^{-1}}{\omega_t} \dot{Y}_t, \quad (26)$$

$$Y_t = \frac{\omega_{t_0}}{\omega_t} \dot{X}_t. \quad (27)$$

In the following, we show that this condition may be approximately met when a time-scale separation is assumed based on an elementary method [25].

We assume that the angular frequency ω_t is adiabatically (sufficiently slowly) changed in time, and the duration $t_1 - t_0$ taken for a finite change $\omega_{t_1} - \omega_{t_0}$ of ω_t is much longer than the oscillation period $T_t \equiv 2\pi/\omega_t$ ($T_t \ll t_1 - t_0$). Moreover, we also assume that the magnitude of Γ_t is sufficiently small ($|\Gamma_t| \ll \omega_t$). In other words, the time constant $1/\Gamma_t$ is much larger than the oscillation period T_t ($T_t \ll 1/\Gamma_t$).

Under the above assumptions, we may assume the following form for X_t and Y_t :

$$X_t = \frac{1}{\sqrt{\omega_{t_0}}} \rho_t \sin \theta_t, \quad (28)$$

$$Y_t = \sqrt{\omega_{t_0}} \rho_t \cos \theta_t, \quad (29)$$

where ρ_t denotes the amplitude

$$\rho_t \equiv \sqrt{\frac{\omega_{t_0}^2 X_t^2 + Y_t^2}{\omega_{t_0}}}, \quad (30)$$

which changes so slowly that it may be regarded as a constant during the oscillation period T_t , and θ_t denotes the phase. By using Eqs. (12) and (13) and Eqs. (28) and (29), we obtain the following equations for ρ_t and θ_t :

$$\ddot{\rho}_t - \rho_t \dot{\theta}_t^2 = -\Gamma_t \dot{\rho}_t - \omega_t^2 \rho_t, \quad (31)$$

$$\frac{d}{dt}(\rho_t^2 \dot{\theta}_t) = -\Gamma_t \rho_t^2 \dot{\theta}_t. \quad (32)$$

Equation (31) with $\Gamma_t = 0$ is referred to as the Ermakov equation [26–28]. By comparing Eqs. (23) and (32), we find

$$W_t = \rho_t^2 \dot{\theta}_t. \quad (33)$$

By putting $\dot{\theta}_t = W_t/\rho_t^2$ into Eq. (31), it is rewritten in terms of W_t as

$$\ddot{W}_t + \omega_t^2 \rho_t + \Gamma_t \dot{W}_t = \frac{W_t^2}{\rho_t^3}, \quad (34)$$

where W_t changes slowly, obeying Eq. (23) due to the assumption of sufficiently small $|\Gamma_t|$. Taking the time-average of Eq. (34) over the oscillation period, the terms with time derivative vanish while during this period ω_t and W_t may be regarded as time-independent constants. Therefore, we get the approximate solution to Eq. (34) as

$$\rho_t \simeq \sqrt{\frac{W_t}{\omega_t}}. \quad (35)$$

Further, by using this approximate solution, we can approximate θ_t as

$$\theta_t = \int_{t_0}^t \frac{W_{t'}}{\rho_{t'}^2} dt' \simeq \int_{t_0}^t \omega_{t'} dt'. \quad (36)$$

Finally, we obtain X_t and Y_t in Eqs. (28) and (29) as

$$X_t \simeq \sqrt{\frac{W_t}{\omega_{t_0} \omega_t}} \sin\left(\int_{t_0}^t \omega_{t'} dt'\right), \quad (37)$$

$$Y_t \simeq \sqrt{\frac{\omega_{t_0} W_t}{\omega_t}} \cos\left(\int_{t_0}^t \omega_{t'} dt'\right), \quad (38)$$

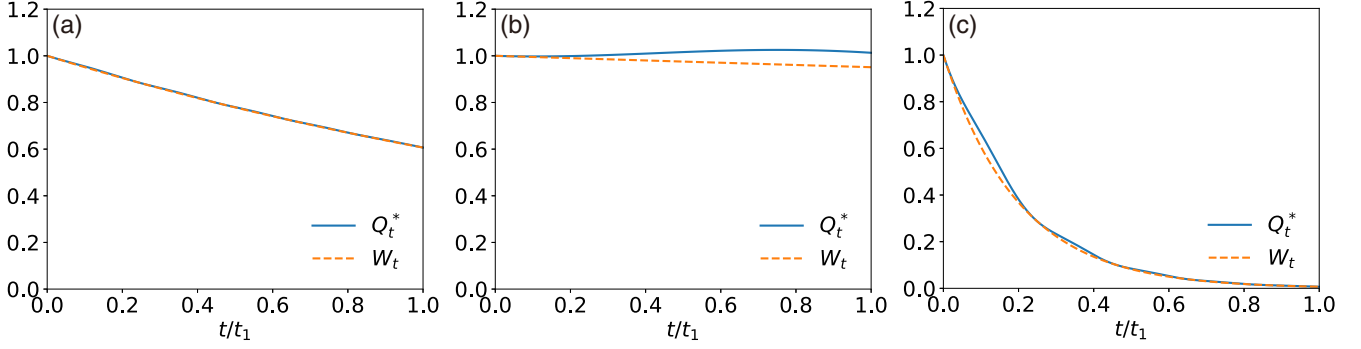


FIG. 1. The time evolutions of Husimi's adiabaticity parameter Q_t^* [Eq. (17)] and the Wronskian W_t [Eq. (19)] as the functions of X_t and Y_t obeying Eqs. (12) and (13), respectively, for three different (t_1, Γ) : (a) $(t_1, \Gamma) = (10, 0.05)$, (b) $(t_1, \Gamma) = (1, 0.05)$, and (c) $(t_1, \Gamma) = (10, 0.5)$. We used $t_0 = 0$, $\omega_{t_0} = 1.2$, and $\omega_{t_1} = 2$ as the other parameters.

respectively. Putting these into Eq. (17) and treating ω_t and W_t as if they are time-independent constants, we obtain $Q_t^* \simeq W_t$ as the equality condition of Eq. (18).

We demonstrate the attainability of the equality in Eq. (18) by numerically solving Eqs. (12) and (13). For simplicity, we adopt the following linear protocol for changing ω_t from ω_{t_0} at $t = t_0$ to ω_{t_1} at $t = t_1$:

$$\omega_t = \omega_{t_0} + (\omega_{t_1} - \omega_{t_0}) \frac{t - t_0}{t_1 - t_0} \quad (t_0 \leq t \leq t_1), \quad (39)$$

and a time-independent friction coefficient $\Gamma_t = \Gamma$.

In Figs. 1(a)–1(c), we show the time evolutions of the Husimi's adiabaticity parameter Q_t^* and the Wronskian W_t for three different sets of (t_1, Γ) . In all cases, we find that the inequality in Eq. (18) is satisfied. In particular, in (a), where the values (t_1, Γ) are consistent with the assumptions of the time-scale separations ($T_t \ll t_1 - t_0$ and $T_t \ll 1/\Gamma_t$), we can see that Q_t^* almost attains the lower bound W_t . Meanwhile, in (b), where only $T_t \ll 1/\Gamma_t$ is satisfied, Q_t^* deviates from the bound W_t because ω_t is changed fast in time, thus violating the other condition $T_t \ll t_1 - t_0$. Conversely, in (c), where only $T_t \ll t_1 - t_0$ is satisfied, Q_t^* is relatively close to the bound W_t , but it is not perfect.

C. The Planck principle

Now that we have obtained Husimi's adiabaticity parameter [Eq. (17)] and the inequality for the parameter [Eq. (18)], we investigate how they are related to the second law of thermodynamics for the pendulum. First, we consider the second law of thermodynamics in terms of internal energy.

For a cyclic process that lasts for $t_1 - t_0$ with $\omega_{t_1} = \omega_{t_0}$ in Eq. (16), we have

$$E_{t_1} = E_{t_0} Q_{t_1}^*. \quad (40)$$

For $\Gamma_t = 0$, by noting Eq. (25), we obtain

$$E_{t_1} \geq E_{t_0}, \quad (41)$$

which is Eq. (2) and is already known in [11]. Though it is not explicitly mentioned so in [11], Eq. (41) may be considered as the Planck principle in thermodynamics, stating that internal energy (or, equivalently, temperature) of a thermally isolated system never decreases at the end of a cyclic process. It should be noted that the formulation of the Planck principle

in thermodynamics of small systems is not so obvious; see [24] for a counterexample to the Planck principle being true.

Meanwhile, in a general case with $\Gamma_t \neq 0$, Eq. (40) becomes

$$E_{t_1} \geq E_{t_0} W_{t_1}, \quad (42)$$

where we used Eq. (18). In this case, although the internal energy may not necessarily increase depending on the dynamical factor W_t , the existence of the inequality relation is the reminiscent of the second law of thermodynamics, which we may regard as a generalized version of the Planck principle.

D. The law of entropy nondecrease

We consider the second law of thermodynamics for the pendulum in terms of entropy, that is, the law of entropy nondecrease. The definition of entropy, which is valid for nonequilibrium processes, is by no means obvious. Here, we adopt the volume entropy following [2,5]. To this end, we introduce the phase-plane area enclosed by an elliptical trajectory of the pendulum with angular frequency ω and energy E :

$$\Omega \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz dp \Theta(E - H(z, p)) = 2\pi \frac{E}{\omega}, \quad (43)$$

where $\Theta(x)$ is a Heaviside step function satisfying $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$. We also define $\Omega_t(z_t, p_t)$ as the phase-plane area enclosed by an elliptical trajectory with ω_t and H_t at time t as

$$\Omega_t(z_t, p_t) \equiv 2\pi \frac{H_t(z_t, p_t)}{\omega_t}, \quad (44)$$

which is conditioned on (z_{t_0}, p_{t_0}) as $H_t(z_t, p_t) = H_t(z_t(z_{t_0}, p_{t_0}), p_t(z_{t_0}, p_{t_0}))$ in Eq. (44) is so. Note that $\Omega_t(z_t, p_t)$ is conserved under an adiabatic change of ω_t because $H_t(z_t, p_t)/\omega_t$ serves as the adiabatic invariant [Eq. (7)]. Then, we may define the volume entropy S_t [2,5] by

$$S_t \equiv \langle \ln \Omega_t(z_t, p_t) \rangle, \quad (45)$$

where we set the Boltzmann constant to unity and $S_{t_0} = \langle \ln \Omega_{t_0}(z_{t_0}, p_{t_0}) \rangle = \ln 2\pi (E_{t_0}/\omega_{t_0})$ for any initial value (z_{t_0}, p_{t_0}) . A related definition of entropy is a surface entropy given by the logarithm of the energy density of states. Though the volume and surface entropies for systems in equilibrium

coincide in the thermodynamic limit, the volume entropy is more appropriate here for its direct connection to an adiabatic invariant. Besides, a more appropriate choice for small systems is the volume entropy; see, e.g., [5] and references therein, including [29,30].

With the definition Eq. (45), the entropy change is calculated as (see Appendix C for the derivation)

$$\Delta S_t \equiv S_t - S_{t_0} = \left\langle \ln \frac{\Omega_t}{\Omega_{t_0}} \right\rangle = \ln \frac{Q_t^* + W_t}{2}. \quad (46)$$

For $\Gamma_t = 0$, because of $W_t = 1$, Eq. (46) recovers

$$\Delta S_t = \ln \frac{Q_t^* + 1}{2} \geq 0, \quad (47)$$

where the inequality is ensured from Eq. (25), and the equality is achieved when $Q_t^* = 1$.

Meanwhile, as for general cases, including $\Gamma_t \neq 0$, because the area expansion rate in the phase plane W_t can take values larger or smaller than unity, the pendulum may not be regarded as being thermally isolated. Thus, we may not expect that we have the law of entropy nondecrease like Eq. (47). Nevertheless, by considering Eq. (18), we notice that we can bound the volume-entropy change from the lower side as

$$\Delta S_t = \ln \frac{Q_t^* + W_t}{2} \geq \ln W_t. \quad (48)$$

This inequality imposes a thermodynamic limitation on the allowed operation of the pendulum through the angular frequency, even in the case of $\Gamma_t \neq 0$, which is reminiscent of the second law of thermodynamics. When $\Gamma_t \neq 0$, the system is no longer thermally isolated and we may regard that there is an implicit coupling to a surrounding, such as a heat reservoir, through the friction term. However, the degrees of freedom of such a heat reservoir have not been taken into account explicitly here. Therefore, we are using the “second law” statement to mean that there is an operational restriction even on non-Liouvillian dynamics closed only by the system’s degree of freedom. This result also highlights the role of the volume entropy in a non-thermally isolated system beyond the role in a thermally isolated system.

Two more remarks are in order with respect to the second laws in Eqs. (47) and (48). First, can we obtain the similar inequalities if we use an alternative entropy, such as the Gibbs entropy, instead of the volume-entropy? The Gibbs entropy is defined by

$$\begin{aligned} S_t^G &\equiv \langle -\ln f_t(z_t, p_t) \rangle \\ &= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_t dp_t f_t(z_t, p_t) \ln f_t(z_t, p_t). \end{aligned} \quad (49)$$

As it can be easily shown (see Appendix D for the derivation), the Gibbs entropy change is given by

$$\Delta S_t^G \equiv S_t^G - S_{t_0}^G = \ln W_t, \quad (50)$$

in terms only of the Wronskian W_t . This does not include the Husimi’s adiabaticity parameter Q_t^* , and the second-law-like inequalities are not obtainable. In particular, when $\Gamma_t = 0$, the Gibbs entropy is conserved for an arbitrary dynamics due to the Liouville’s theorem. This indicates that the volume entropy may be a more relevant entropy to be adopted for the present model.

It is noteworthy that we may express Eq. (48) in a different way in terms of the Gibbs entropy in Eq. (50):

$$\Delta S_t \geq \Delta S_t^G. \quad (51)$$

Thus, the Gibbs entropy change may serve as a lower bound on the volume-entropy change. Again, when $\Gamma_t = 0$, the Gibbs entropy is conserved along a dynamics, and Eq. (51) consistently recovers Eq. (47).

Second, it is interesting to consider a relationship between the present result and the characterization of thermodynamic entropy as a Noether invariant, which is discussed in [31,32]. In fact, when $\Gamma_t = 0$, the following Ermakov-Lewis invariant is known as a Noether invariant of the pendulum obeying Eq. (3) [33–35]:

$$I_t = \frac{1}{2} \left\{ \frac{z_t^2}{\rho_t^2} + (p_t \rho_t - z_t \dot{\rho}_t)^2 \right\}, \quad (52)$$

where ρ_t satisfies the Ermakov equation Eq. (31) with $\Gamma_t = 0$. Because the Wronskian W_t is related to the Ermakov-Lewis invariant I_t by $I_t = W_t^2/2$ [36], we may express the entropy change in Eq. (46) in the quasistatic limit ($Q_t^* \approx W_t = \sqrt{2I_t}$) as

$$\Delta S_t \approx \frac{1}{2} \ln(2I_t), \quad (53)$$

in terms of the Noether invariant.

IV. CONCLUDING REMARKS

In this paper, stimulated by the classical paper by Husimi [11], we investigated the irreversibility of the damped pendulum that is initially distributed obeying the microcanonical distribution and then evolves deterministically under a time-dependent operation of the angular frequency. We investigated the generalized version of the second law of thermodynamics for the pendulum, in terms of internal energy in Eq. (42) and entropy in Eq. (48), respectively, generalizing the argument by Husimi [11]. In both cases, Husimi’s adiabaticity parameter, bounded by the Wronskian, played a key role in quantifying the irreversibility of the thermodynamic process.

Though the present results are limited to the one-dimensional damped pendulum, it would be straightforward to extend the present results to multidimensional pendulums and linearly coupled pendulums, which are analytically tractable owing to their linearity. We expect that essentially the same Husimi-Wronskian framework as the present results [Eq. (18)] could be applied for such dissipative systems. The generalization of the inequalities in Eqs. (42) and (48) to broader classes of dissipative systems, such as nonlinear pendulums beyond the linear pendulum, deserves further research. In addition, as we stated in Sec. III D, our model does not include the degrees of freedom of a heat bath. It is important to incorporate the degrees of freedom of a heat bath explicitly and perform the reduction of the model into the present model by focusing only on the system’s degrees of freedom or tracing out those of the heat bath. Deterministic thermostat models such as the Nosé-Hoover chain thermostat [37] may be appropriate models for such an investigation. In this context, the statistical-mechanics formulation of non-Hamiltonian systems [38] may be mathematically relevant and useful. Finally, an interesting task for open systems is to reveal the role of

the volume entropy in the formulation of the second law of thermodynamics for a total system consisting of a system and a heat bath.

We expect that the present results will contribute to elucidating the relationship between thermodynamics and mechanics.

ACKNOWLEDGMENTS

The author is grateful to anonymous referees for their valuable comments on the manuscript and to Kyosuke Watanabe for careful reading of the manuscript and useful comments on

it. This work was supported by JSPS KAKENHI Grant No. 25K07163.

DATA AVAILABILITY

The data that support the findings of this article are openly available [39].

APPENDIX A: DERIVATION OF EQ. (16)

The internal energy, as the average energy $E_t = \langle H_t \rangle$ in Eq. (16), is calculated as

$$E_t = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} \frac{\delta(E_{t_0} - H_{t_0}(z_{t_0}, p_{t_0}))}{A_{t_0}} H_t(z_t(z_{t_0}, p_{t_0}), p_t(z_{t_0}, p_{t_0}))$$

$$= \frac{1}{A_{t_0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} \delta\left(E_{t_0} - \frac{1}{2}(p_{t_0}^2 + \omega_{t_0}^2 z_{t_0}^2)\right) \frac{1}{2} \{p_{t_0}^2 (\dot{X}_t^2 + \omega_t^2 X_t^2) + z_{t_0}^2 (\dot{Y}_t^2 + \omega_t^2 Y_t^2) + 2z_{t_0} p_{t_0} (\dot{X}_t \dot{Y}_t + \omega_t^2 X_t Y_t)\}, \quad (\text{A1})$$

where we used Eqs. (6), (11), and (14) in the second equality. By performing the variable transformation ($0 \leq r < \infty, 0 \leq \vartheta < 2\pi$):

$$z_{t_0} = (r/\omega_{t_0}) \cos \vartheta, \quad (\text{A2})$$

$$p_{t_0} = r \sin \vartheta, \quad (\text{A3})$$

and using the following relation:

$$\delta\left(E_{t_0} - \frac{1}{2}(p_{t_0}^2 + \omega_{t_0}^2 z_{t_0}^2)\right) = \delta(E_{t_0} - r^2/2) = \frac{\delta(r - \sqrt{2E_{t_0}})}{\sqrt{2E_{t_0}}}, \quad (\text{A4})$$

we derive

$$\text{Eq. (A1)} = \frac{1}{A_{t_0} \omega_{t_0} \sqrt{2E_{t_0}}} \int_0^\infty \int_0^{2\pi} dr d\vartheta \delta(r - \sqrt{2E_{t_0}}) r$$

$$\times \frac{1}{2} \left\{ r^2 \sin^2 \vartheta (\dot{X}_t^2 + \omega_t^2 X_t^2) + \frac{r^2}{\omega_{t_0}^2} \cos^2 \vartheta (\dot{Y}_t^2 + \omega_t^2 Y_t^2) + \frac{2r^2}{\omega_{t_0}} \cos \vartheta \sin \vartheta (\dot{X}_t \dot{Y}_t + \omega_t^2 X_t Y_t) \right\}$$

$$= E_{t_0} \frac{\omega_t}{\omega_{t_0}} Q_t^*, \quad (\text{A5})$$

by performing the integrations, where we used Eqs. (10) and (17) in the second equality.

APPENDIX B: DERIVATION OF EQ. (18)

It is straightforward to derive Eq. (18) by rewriting the following non-negative quantity [11]:

$$0 \leq \omega_{t_0} \frac{(\omega_t X_t + \omega_{t_0}^{-1} \dot{Y}_t)^2}{2\omega_t} + \omega_{t_0}^{-1} \frac{(\omega_{t_0} \dot{X}_t - \omega_t Y_t)^2}{2\omega_t}, \quad (\text{B1})$$

$$= \omega_{t_0} \frac{(\omega_t^2 X_t^2 + 2\omega_t \omega_{t_0}^{-1} X_t \dot{Y}_t + \omega_{t_0}^{-2} \dot{Y}_t^2)}{2\omega_t} + \omega_{t_0}^{-1} \frac{(\omega_{t_0}^2 \dot{X}_t^2 - 2\omega_t \omega_{t_0} X_t \dot{Y}_t + \omega_t^2 Y_t^2)}{2\omega_t}$$

$$= \omega_{t_0} \frac{(\dot{X}_t^2 + \omega_t^2 X_t^2)}{2\omega_t} + \omega_{t_0}^{-1} \frac{(\dot{Y}_t^2 + \omega_t^2 Y_t^2)}{2\omega_t} + (X_t \dot{Y}_t - \dot{X}_t Y_t) = Q_t^* - W_t. \quad (\text{B2})$$

Therefore, we derived $Q_t^* \geq W_t$. The equality condition is satisfied when each term of the two terms in Eq. (B1) vanishes, which yields Eqs. (26) and (27).

APPENDIX C: DERIVATION OF EQ. (46)

The entropy change $\Delta S_t = \langle \ln \Omega_t / \Omega_{t_0} \rangle$ is given as

$$\begin{aligned} \Delta S_t &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} \frac{\delta(E_{t_0} - H_{t_0}(z_{t_0}, p_{t_0}))}{A_{t_0}} \ln \left\{ \frac{H_t(z_t(z_{t_0}, p_{t_0}), p_t(z_{t_0}, p_{t_0}))}{\omega_t} \cdot \frac{\omega_{t_0}}{E_{t_0}} \right\} \\ &= \frac{1}{A_{t_0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} \delta \left(E_{t_0} - \frac{1}{2} (p_{t_0}^2 + \omega_{t_0}^2 z_{t_0}^2) \right) \\ &\quad \times \ln \left\{ \frac{\{ p_{t_0}^2 (\dot{X}_t^2 + \omega_t^2 X_t^2) + z_{t_0}^2 (\dot{Y}_t^2 + \omega_t^2 Y_t^2) + 2z_{t_0} p_{t_0} (\dot{X}_t \dot{Y}_t + \omega_t^2 X_t Y_t) \}}{2\omega_t} \cdot \frac{\omega_{t_0}}{E_{t_0}} \right\}. \end{aligned} \quad (C1)$$

By performing the variable transformation in Eqs. (A2) and (A3) and using Eq. (A4), we derive

$$\begin{aligned} \text{Eq. (C1)} &= \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \ln \left\{ \frac{\omega_{t_0} \sin^2 \vartheta (\dot{X}_t^2 + \omega_t^2 X_t^2) + \omega_{t_0}^{-1} \cos^2 \vartheta (\dot{Y}_t^2 + \omega_t^2 Y_t^2) + 2 \cos \vartheta \sin \vartheta (\dot{X}_t \dot{Y}_t + \omega_t^2 X_t Y_t)}{\omega_t} \right\} \\ &= \ln \frac{Q_t^* + W_t}{2}. \end{aligned} \quad (C2)$$

Here, we have calculated the integral by using the following ($a = \omega_{t_0} (\dot{X}_t^2 + \omega_t^2 X_t^2)/\omega_t$, $b = \omega_{t_0}^{-1} (\dot{Y}_t^2 + \omega_t^2 Y_t^2)/\omega_t$, and $c = 2(\dot{X}_t \dot{Y}_t + \omega_t^2 X_t Y_t)/\omega_t$):

$$\begin{aligned} \int_0^{2\pi} d\vartheta \ln(a \sin^2 \vartheta + b \cos^2 \vartheta + c \cos \vartheta \sin \vartheta) &= \int_0^{2\pi} d\vartheta \ln \left(\frac{a+b}{2} + \frac{1}{2} \sqrt{(a-b)^2 + c^2} \cos(2\vartheta - \phi) \right) \\ &= \int_0^{2\pi} d\vartheta' \ln \left(\frac{a+b}{2} + \frac{1}{2} \sqrt{(a-b)^2 + c^2} \cos \vartheta' \right) \\ &= 2\pi \ln \left(\frac{a+b + \sqrt{4ab - c^2}}{4} \right), \end{aligned} \quad (C3)$$

for $4ab > c^2$, where $\phi \equiv \tan^{-1}(c/(b-a))$. Also, we used Eq. (19) in the second equality in Eq. (C2).

APPENDIX D: DERIVATION OF EQ. (50)

From the definition of the Gibbs entropy in Eq. (49), we can show Eq. (50) as follows:

$$\begin{aligned} S_t^G &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_t dp_t f_t(z_t, p_t) \ln f_t(z_t, p_t) \\ &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_{t_0} dp_{t_0} f_{t_0}(z_{t_0}, p_{t_0}) \ln \{ W_t^{-1} f_{t_0}(z_{t_0}, p_{t_0}) \} \\ &= - \ln W_t^{-1} + S_{t_0}^G \\ &= \ln W_t + S_{t_0}^G. \end{aligned} \quad (D1)$$

Here, we used $f_t(z_t, p_t) = f_{t_0}(z_{t_0}, p_{t_0})|J_t|^{-1} = f_{t_0}(z_{t_0}, p_{t_0})W_t^{-1}$ and $f_t(z_t, p_t)dz_t dp_t = f_{t_0}(z_{t_0}, p_{t_0})dz_{t_0} dp_{t_0}$ in the second equality. It should be noted that this result is valid not only for a microcanonical distribution but for any initial distribution. Moreover, because the Wronskian W_t may be replaced with the Jacobian $|J_t|$ of the transformation in general cases, the result is not restricted to the pendulum.

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