

Convection patterns in nonequilibrium Kawasaki dynamics at low temperature

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We study a conservative stochastic lattice gas (Kawasaki dynamics) coupled in the bulk to a heat bath, which leads to standard phase separation at low uniform temperatures. Instead, a macroscopic temperature gradient drives the system into a nonequilibrium steady state. In this state, the usual long-range order is replaced by robust convection patterns, featuring regularly spaced stripe structures. We show that these nonequilibrium states differ markedly from equilibrium configurations with the same local temperature profiles. Finally, we develop a macroscopic description that captures these behaviors and provides a unified framework for understanding the observed patterns.

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I. INTRODUCTION

Nonequilibrium stationary states are central to many-body physics, yet their structure is far less understood than that of equilibrium states; see, e.g., Refs. [1–5]. Because they do not arise from free-energy minimization, even simple driven systems may exhibit behaviors with no equilibrium counterpart. This is especially true when a conserved quantity is present, so that a steady state must satisfy a continuity equation. This constraint allows for long-range correlations, self-organized criticality [6–10], and many of the collective phenomena observed in active matter [11–14]. Considerable progress has been made in systems that stay in local equilibrium, particularly in boundary-driven settings, with a very detailed description of the nonequilibrium fluctuations in some systems; see, e.g., Refs. [15–20].

In this work we investigate a conservative lattice gas in contact everywhere with a heat bath (Kawasaki dynamics), whose temperature varies smoothly across the system on a macroscopic scale. Particles interact through an Ising Hamiltonian, which at low and uniform temperature produces a symmetry-broken, phase-separated state. The imposed temperature gradient drives the system out of equilibrium by breaking detailed balance weakly, with a strength that scales inversely with the system size. This setting raises the fundamental question: how does a symmetry-broken phase respond to a weak nonequilibrium drive that preserves local equilibrium?

We find that the system develops convection-driven structures that fundamentally reorganize the macroscopic low-temperature phase. While the steady state is locally consistent with a Gibbs measure at the corresponding temperature, it

globally forms a regular array of convection cells manifesting as density stripes; see Fig. 1. These cells appear in the subcritical region where equilibrium would produce complete phase separation, and their number increases with system size, thereby destroying conventional long-range order. The resulting phenomenology is closely related to Rayleigh-Bénard convection [21–24], Turing-type pattern formation [25], and dissipative structures more broadly [26–28]. Its essential physical ingredients are local equilibrium, phase separation, and diffusive particle transport. At low temperature, this transport occurs within the bulk of the high and low-density phases (see the current lines in Fig. 1), where density fluctuations, or defects, propagate diffusively.

Previous studies of phase-separating systems driven out of equilibrium include models with a uniform macroscopic drive [29,30], where domains elongate along the drive, or the driven Widom-Rowlinson gas [31,32], which exhibits stripes perpendicular to it. Related behavior is found in systems under shear flow [33,34] or directed currents [35]; in these cases, tangential fluxes along the interfaces suppress fluctuations. Additionally, Kawasaki dynamics with antiferromagnetic interactions and a uniform drive was studied in Ref. [36], where distinct nonequilibrium patterns were observed.

Other approaches have focused on the competition between different local rules, such as the simultaneous presence of Glauber and Kawasaki dynamics [37–39], or the linear instabilities appearing in nonreciprocal Ising models [40]. Further, boundary-driven perturbations of Kawasaki dynamics have shown that coupling to magnetization reservoirs can induce uphill diffusion [41–44].

Notably, our results are consistent with Refs. [45,46], which examined a setup close to ours consisting of two subsystems separated by a sharp interface across which the temperature jumped from subcritical to infinite. In those studies, convection structures and density-modulated nonequilibrium stationary states were likewise observed. Similar stripe-like patterns were also found in thin films with a temperature gradient perpendicular to the film [47].

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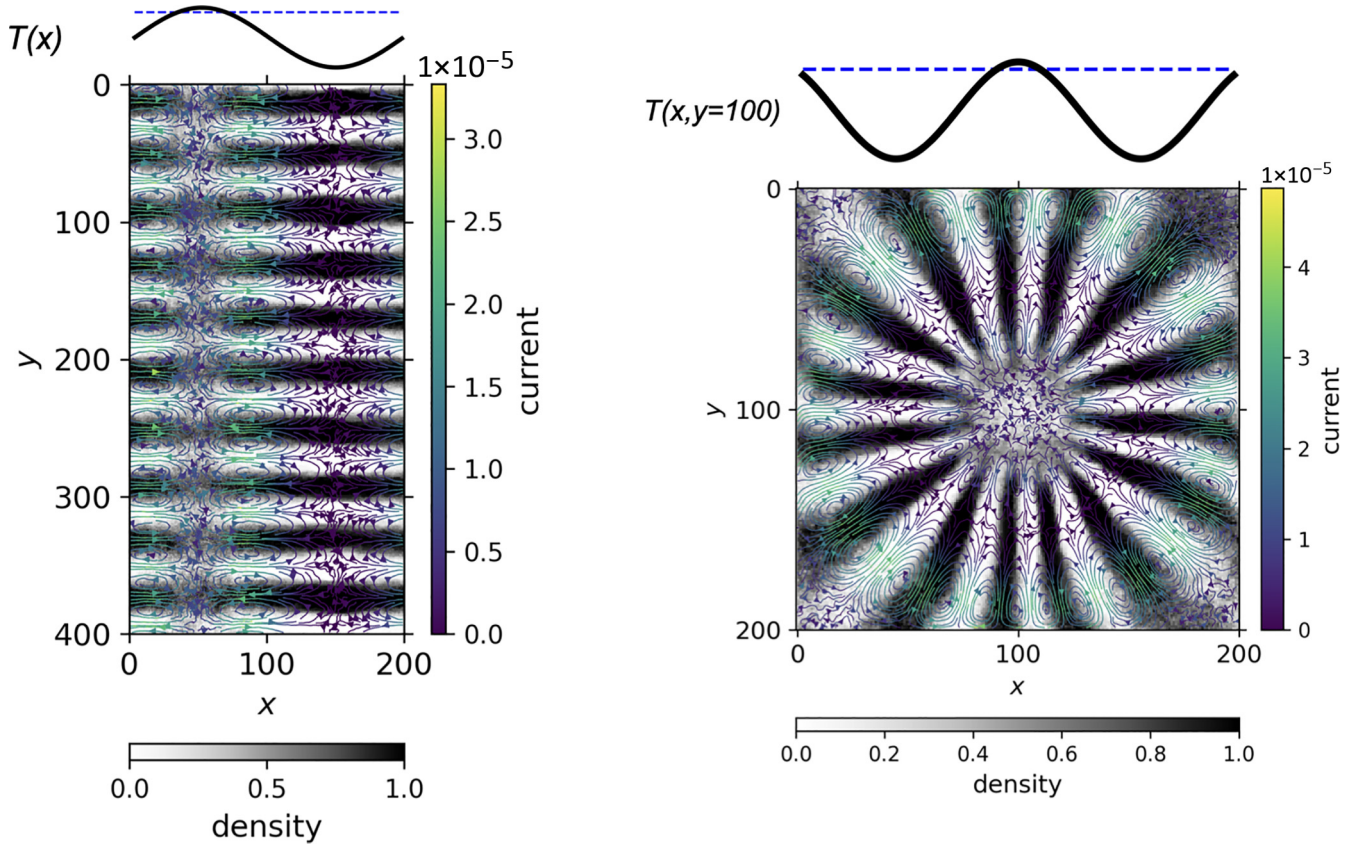


FIG. 1. Time-averaged densities (grayscale) and particle currents (colored lines) for a single realization. Averaging is performed over the final 25% of the total simulation time ($t = 5 \times 10^7$). Left: x -dependent temperature profile as in Eq. (2) with $L_y = 2L_x = 400$; the profile is shown in the upper inset, where dotted lines indicate the critical temperature $T_c \approx 0.57$. Right: Mexican-hat radial temperature profile as in Eqs. (3) and (13) with $L_x = L_y = 200$; $T(x, y = 100)$ is shown in the upper inset, with the dotted line again representing T_c . In both panels, $T_{\text{mean}} = 0.4$, $T_{\text{amp}} = 0.2$, and the filling factor is $\bar{\rho} = 1/2$.

Summary of main results

The model is described in detail in Sec. II. The core of this work is an extensive numerical investigation of the steady state of the model, the results of which are presented in Sec. III. Our primary findings are as follows:

1. *Robust convection cells.* The system develops convection patterns reminiscent of Rayleigh-Bénard convection; these serve as the channel for heat transfer from the hot reservoir to the cold one. This is a robust feature of the steady state, appearing across various initial conditions and geometries (though not all); see Sec. III B. The cells form an approximately periodic arrangement that may become fuzzy due to fluctuations as the system size grows; see Sec. III C. The number of cells increases with the linear system size L for a fixed aspect ratio. Our data are consistent with the number of cells scaling as $L^{1/2}$, though fluctuations on timescales beyond our simulation limits might modify this figure; see Sec. III D.

2. *Local equilibrium.* The steady state of our nonequilibrium dynamics satisfies local equilibrium throughout the system: in the large-volume limit, the expectations of local observables are compatible with those of a Gibbs measure at the local temperature and some local chemical potential. This property ultimately stems from the fact that detailed balance is only weakly broken, with violations in the transition rates being of order $1/L$; see Sec. III E.

3. *Diffusive transport.* The convection cells in the steady state feature persistent particle currents. In Sec. III F, we analyze how these currents scale with the system size L . Despite significant finite-size effects, the data are compatible with diffusive behavior, i.e., local currents scaling as $1/L$.

4. *Chemical potential profiles (numerical).* Most numerical results are obtained at half filling ($\bar{\rho} = 1/2$), where the data are compatible with a zero chemical potential across the system. The case $\bar{\rho} \neq 1/2$ is analyzed in Sec. III G, and the results are in sharp contrast with those expected from equilibrium. To highlight this, we compare the dynamics with an explicit equilibrium counterpart where the temperature is constant but the coupling strength varies spatially such that the ratio of coupling to temperature matches that of the nonequilibrium system. The resulting steady states are markedly different; see Fig. 2.

In the second part of this paper, Sec. IV, these findings are interpreted within the framework of a macroscopic theory that explicitly builds on the presence of local equilibrium and diffusive transport; see Sec. IV A for a precise statement of the underlying assumptions. The theory provides clarification on the following points:

1. *Chemical potential profiles (theoretical).* It yields a clear and testable prediction for the chemical potential profile across the entire system. This prediction fits the data,

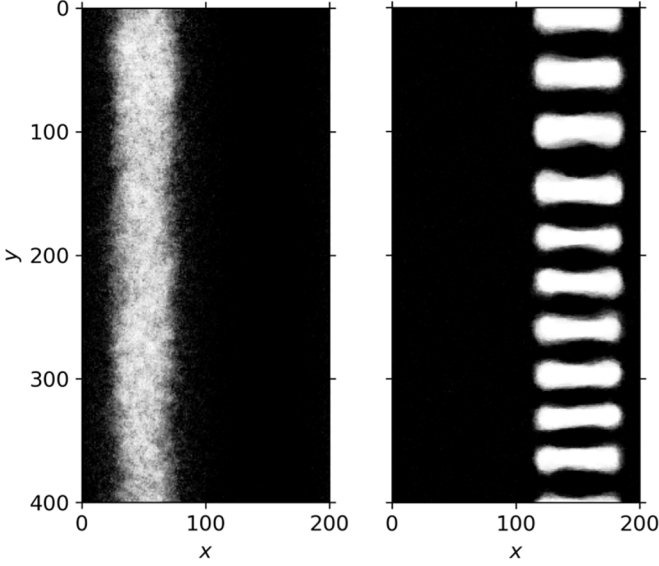


FIG. 2. Equilibrium (left) and nonequilibrium (right) steady states. Time-averaged particle density at filling factor $\bar{\rho} = 0.8$, obtained over the final 25% of a simulation of duration $t = 5 \times 10^7$ for a single realization (grayscale as in Fig. 1). The temperature varies along x as in Eq. (2), with $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$. The system size is $L_y = 2L_x = 400$. Equilibrium and nonequilibrium transition rates are given in Eqs. (10) and (5), respectively.

specifically in Sec. III G, with discrepancies of only a few percent. These results are detailed in Sec. IV C.

2. *Interface shape.* It explains why the interface shape is no longer determined by mean-curvature considerations, but rather by nonequilibrium persistent currents that become dominant in the large-volume limit; see Sec. IV B.

3. *Fragmentation into convection cells.* It shows that the number of convection cells must grow unbounded with the system size; see Sec. IV D.

Finally, we note that an understanding based on the linearization of the mean-field dynamics has been developed in Ref. [48]. There, it is shown that the convective phase emerges as a dynamical instability, in a manner analogous to a Turing instability.

II. MODEL

We consider a stochastic lattice gas on a two-dimensional rectangular lattice Λ with $L_x \times L_y$ sites and periodic boundary conditions. Each site $\mathbf{x} \in \Lambda$ is either empty, $n_{\mathbf{x}} = 0$, or occupied, $n_{\mathbf{x}} = 1$. A configuration is denoted by $\eta = (n_{\mathbf{x}})_{\mathbf{x} \in \Lambda}$. The dynamics conserves the total particle number N , and we denote the filling factor by $\bar{\rho} = N/V$ with $V = L_x L_y$. We study large systems by increasing L_x up to 200 while keeping the aspect ratio L_x/L_y fixed.

A. Nonequilibrium Kawasaki dynamics

The energy of a configuration $\eta = (n_{\mathbf{x}})_{\mathbf{x} \in \Lambda}$ is given by the ferromagnetic Ising Hamiltonian

$$E(\eta) = -J \sum_{\mathbf{x} \sim \mathbf{y}} (2n_{\mathbf{x}} - 1)(2n_{\mathbf{y}} - 1), \quad (1)$$

with $J > 0$ ($J = 1/4$ in all simulations). Introducing spin variables $\sigma_{\mathbf{x}} = 2n_{\mathbf{x}} - 1 = \pm 1$ yields the standard Ising interaction.

The temperature varies on the macroscopic scale. Let $T(u_x, u_y)$ be a smooth function on $[0, 1]^2$, and define the microscopic temperature field as

$$T(x, y) = T(x/L_x, y/L_y).$$

Our main example is

$$T(u_x, u_y) = T_{\text{mean}} + T_{\text{amp}} \sin(2\pi u_x). \quad (2)$$

We also consider radial profiles,

$$T(u_x, u_y) = T_{\text{mean}} + T_{\text{amp}} \varphi(r), \quad (3)$$

with $r^2 = (u_x - 0.5)^2 + (u_y - 0.5)^2$ and $\int_{[0,1]^2} \varphi(r) du_x du_y = 0$. Periodic boundaries ensure continuity of T throughout.

The dynamics is a continuous-time Kawasaki exchange process. For a nearest-neighbor bond $(\mathbf{x}, \mathbf{y})$ and configuration η , let $\eta^{\mathbf{x}, \mathbf{y}}$ be the configuration obtained by exchanging $n_{\mathbf{x}}$ and $n_{\mathbf{y}}$. Define the bond temperature

$$T_{\mathbf{x}, \mathbf{y}} = T\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right),$$

as well as the energy difference

$$\Delta E_{\mathbf{x}, \mathbf{y}} = E(\eta^{\mathbf{x}, \mathbf{y}}) - E(\eta) = -4J(n_{\mathbf{y}} - n_{\mathbf{x}})\Delta_{\mathbf{x}, \mathbf{y}},$$

$$\text{with } \Delta_{\mathbf{x}, \mathbf{y}} = \sum_{\mathbf{z}: \mathbf{z} \sim \mathbf{x}, \mathbf{z} \neq \mathbf{y}} n_{\mathbf{z}} - \sum_{\mathbf{z}: \mathbf{z} \sim \mathbf{y}, \mathbf{z} \neq \mathbf{x}} n_{\mathbf{z}}. \quad (4)$$

The transition rate is

$$W(\eta \rightarrow \eta^{\mathbf{x}, \mathbf{y}}) = \frac{\gamma}{1 + e^{\Delta E_{\mathbf{x}, \mathbf{y}}/T_{\mathbf{x}, \mathbf{y}}}}. \quad (5)$$

Spatial variations in T break detailed balance. To demonstrate the robustness of the phenomenology, we explore alternative dynamics in Appendix A, including other temperature profiles and temperature-dependent rate γ .

B. Continuity equation

Conservation of particle number and the restriction of nonzero transition rates to nearest neighbors imply that the stochastic evolution of $n_{\mathbf{x}}$ can be written as

$$\frac{dn_{\mathbf{x}}}{dt} = -(\mathbf{j}_{\mathbf{x}, \mathbf{x}+\mathbf{e}_1} - \mathbf{j}_{\mathbf{x}-\mathbf{e}_1, \mathbf{x}} + \mathbf{j}_{\mathbf{x}, \mathbf{x}+\mathbf{e}_2} - \mathbf{j}_{\mathbf{x}-\mathbf{e}_2, \mathbf{x}}), \quad (6)$$

with $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Here, $\mathbf{j}_{\mathbf{x}, \mathbf{y}}$ denotes the stochastic particle current, that can be decomposed as

$$\mathbf{j}_{\mathbf{x}, \mathbf{y}} = j_{\mathbf{x}, \mathbf{y}} + \xi_{\mathbf{x}, \mathbf{y}},$$

where $j_{\mathbf{x}, \mathbf{y}}$ is an observable, i.e., a function of the configuration, and $\xi_{\mathbf{x}, \mathbf{y}}$ is a stochastic noise.

Let us derive an expression for the current $j_{\mathbf{x}, \mathbf{y}}$. The stochastic evolution of a generic observable $A(\eta)$ is given by

$$\frac{dA(\eta)}{dt} = \mathcal{L}A(\eta) + \xi_{A, \eta},$$

where $\xi_{A, \eta}$ is a noise (a martingale increment) and where $\mathcal{L}$ is the generator of the dynamics, given by

$$\mathcal{L}A(\eta) = \sum_{\eta'} (A(\eta') - A(\eta))W(\eta \rightarrow \eta').$$

This is Dykin's martingale formula; see, e.g., Ref. [49]. Taking $A = n_{\mathbf{x}}$ yields

$$\mathcal{L}n_{\mathbf{x}}(\eta) = \sum_{\mathbf{y}:\mathbf{y}\sim\mathbf{x}} (n_{\mathbf{x}}(\eta^{\mathbf{x},\mathbf{y}}) - n_{\mathbf{x}}(\eta))W(\eta \rightarrow \eta^{\mathbf{x},\mathbf{y}}).$$

Using

$$\frac{1}{1+e^x} = \frac{1}{2} \left(1 - \tanh \frac{x}{2}\right)$$

and the expression (5) for the rates, we obtain

$$\mathcal{L}n_{\mathbf{x}} = \frac{\gamma}{2} \sum_{\mathbf{y}:\mathbf{y}\sim\mathbf{x}} (n_{\mathbf{y}} - n_{\mathbf{x}}) \left(1 - \tanh \frac{\Delta E_{\mathbf{x},\mathbf{y}}}{2T_{\mathbf{x},\mathbf{y}}}\right), \quad (7)$$

where we omit the explicit dependence on η , knowing that $\eta = (n_{\mathbf{x}})_{\mathbf{x}}$. Because $\Delta E_{\mathbf{x},\mathbf{y}}$ depends on whether a particle moves from $\mathbf{x}$ to $\mathbf{y}$ or vice versa, we decompose

$$\begin{aligned} (n_{\mathbf{y}} - n_{\mathbf{x}}) \tanh \frac{\Delta E_{\mathbf{x},\mathbf{y}}}{2T_{\mathbf{x},\mathbf{y}}} &= n_{\mathbf{y}}(1 - n_{\mathbf{x}}) \tanh \frac{\Delta E_{\mathbf{x},\mathbf{y}}}{2T_{\mathbf{x},\mathbf{y}}} - n_{\mathbf{x}}(1 - n_{\mathbf{y}}) \tanh \frac{\Delta E_{\mathbf{x},\mathbf{y}}}{2T_{\mathbf{x},\mathbf{y}}} \\ &= (n_{\mathbf{y}}(1 - n_{\mathbf{x}}) + n_{\mathbf{x}}(1 - n_{\mathbf{y}})) \tanh \frac{2J\Delta_{\mathbf{x},\mathbf{y}}}{T_{\mathbf{x},\mathbf{y}}} \end{aligned}$$

using Eq. (4). Inserting this in Eq. (7) yields

$$\mathcal{L}n_{\mathbf{x}} = -(j_{\mathbf{x},\mathbf{x}+\mathbf{e}_1} - j_{\mathbf{x}-\mathbf{e}_1,\mathbf{x}}) - (j_{\mathbf{x},\mathbf{x}+\mathbf{e}_2} - j_{\mathbf{x}-\mathbf{e}_2,\mathbf{x}}),$$

with

$$\begin{aligned} j_{\mathbf{x},\mathbf{y}} &= -\frac{\gamma}{2}(n_{\mathbf{y}} - n_{\mathbf{x}}) + \frac{\gamma}{2}(n_{\mathbf{y}}(1 - n_{\mathbf{x}}) \\ &\quad + n_{\mathbf{x}}(1 - n_{\mathbf{y}})) \tanh \frac{2J\Delta_{\mathbf{x},\mathbf{y}}}{T_{\mathbf{x},\mathbf{y}}}. \end{aligned} \quad (8)$$

C. Inhomogeneous equilibrium dynamics

For comparison, we introduce a related process that satisfies detailed balance; its rates differ from the nonequilibrium process described above only by corrections of order $1/L$. Define the modified energy

$$E_T(\eta) = -J \sum_{\mathbf{x}\sim\mathbf{y}} \frac{1}{T_{\mathbf{x},\mathbf{y}}} (2n_{\mathbf{x}} - 1)(2n_{\mathbf{y}} - 1), \quad (9)$$

where $-J/T_{\mathbf{x},\mathbf{y}}$ now plays the role of a spatially varying coupling strength. The Kawasaki rates are

$$W_{\text{eq}}(\eta \rightarrow \eta^{\mathbf{x},\mathbf{y}}) = \frac{\gamma}{1 + e^{\Delta E_{T,\mathbf{x},\mathbf{y}}}}, \quad (10)$$

with $\Delta E_{T,\mathbf{x},\mathbf{y}}$ given by

$$\begin{aligned} \Delta E_{\mathbf{x},\mathbf{y}} &= E_T(\eta^{\mathbf{x},\mathbf{y}}) - E_T(\eta) = -4J(n_{\mathbf{y}} - n_{\mathbf{x}})\Delta_{\mathbf{x},\mathbf{y}}^T, \\ \text{with } \Delta_{\mathbf{x},\mathbf{y}}^T &= \sum_{\mathbf{z}:\mathbf{z}\sim\mathbf{x},\mathbf{z}\neq\mathbf{y}} \frac{n_{\mathbf{z}}}{T_{\mathbf{x},\mathbf{z}}} - \sum_{\mathbf{z}:\mathbf{z}\sim\mathbf{y},\mathbf{z}\neq\mathbf{x}} \frac{n_{\mathbf{z}}}{T_{\mathbf{y},\mathbf{z}}}. \end{aligned} \quad (11)$$

This process is conservative and the continuity equation (6) still holds, provided the current $j_{\mathbf{x},\mathbf{y}}$ is now replaced by

$$\begin{aligned} j_{\mathbf{x},\mathbf{y}}^T &= -\frac{\gamma}{2}(n_{\mathbf{y}} - n_{\mathbf{x}}) \\ &\quad + \frac{\gamma}{2}(n_{\mathbf{y}}(1 - n_{\mathbf{x}}) + n_{\mathbf{x}}(1 - n_{\mathbf{y}})) \tanh(2J\Delta_{\mathbf{x},\mathbf{y}}^T). \end{aligned} \quad (12)$$

Crucially, because the rates in Eq. (10) define an equilibrium dynamics and because the process is conservative, the invariant states are Gibbs states of the type $e^{-(E_T - \mu N)}$, for some chemical potential μ . Since our focus is on the stationary states, the dynamics is here merely a way to access this Gibbs state, rather than the fundamental object of study.

III. NUMERICAL RESULTS

We present the main conclusions of the numerical analysis of the nonequilibrium dynamics with rates given in Eq. (5). We work at filling factor $\bar{\rho} = 1/2$, except in Sec. III G, where we also use the equilibrium dynamics with rates given in Eq. (10) as a comparison point. The initial state is either an infinite-temperature (white noise) configuration, obtained by placing $N = \bar{\rho}L_xL_y$ particles uniformly at random on the lattice, or a phase-separated configuration of prescribed geometry. We have examined cases in which the temperature profile crosses the critical point as well as cases that remain entirely subcritical. Although finite-size scaling is more difficult to analyze when the critical point is crossed, the qualitative phenomenology appears to be the same in both cases.

A. Simulation method and convergence to stationarity

We simulate the dynamics using a standard kinetic Monte Carlo scheme. One Monte Carlo cycle consists of the following steps:

- (1) Choose a particle at random.
- (2) Among the unoccupied nearest neighboring sites, select one at random with probability $W\Delta t$, where W is the transition rate given in Eqs. (5) and (10), and where $\Delta t = 1/(4\gamma)$. If a site is selected, move the particle; otherwise it remains in place. The choice $\Delta t = 1/(4\gamma)$ guarantees that the probability is independent of the value of γ and that the probability of moving at all never exceeds unity.
- (3) Repeat steps 1–2 a total of N times and then increment time by 1 unit.

Time is measured in units of $1/\gamma$, with γ as in Eqs. (5) and (10), so the parameter γ plays no further role.

Typical simulations run up to $t = 5 \times 10^7$, a duration sufficient to reach a state that exhibits the main characteristics of true stationarity. The strongest evidence for this convergence is that significantly different initial conditions lead to similar states on these timescales. For instance, the dynamical process shown in the top panel of Fig. 3, initiated from a vertically segregated state, yields a final configuration comparable to those starting from horizontally segregated or white-noise initial conditions. Additional standard stationarity probes are detailed in Appendix B. We note, however, that once convection cells form, they become remarkably robust. On much longer timescales, the number of cells is expected to fluctuate, a process too slow to be witnessed in our simulations. Consequently, the scaling of the number of convection cells with system size (Sec. III D) is a result that could be affected by these possible long-term dynamics.

B. Convection patterns

Consider the temperature profile of Eq. (2), with system sizes up to $L_x = 200$ and $L_y = 2L_x$. In the subcritical region,

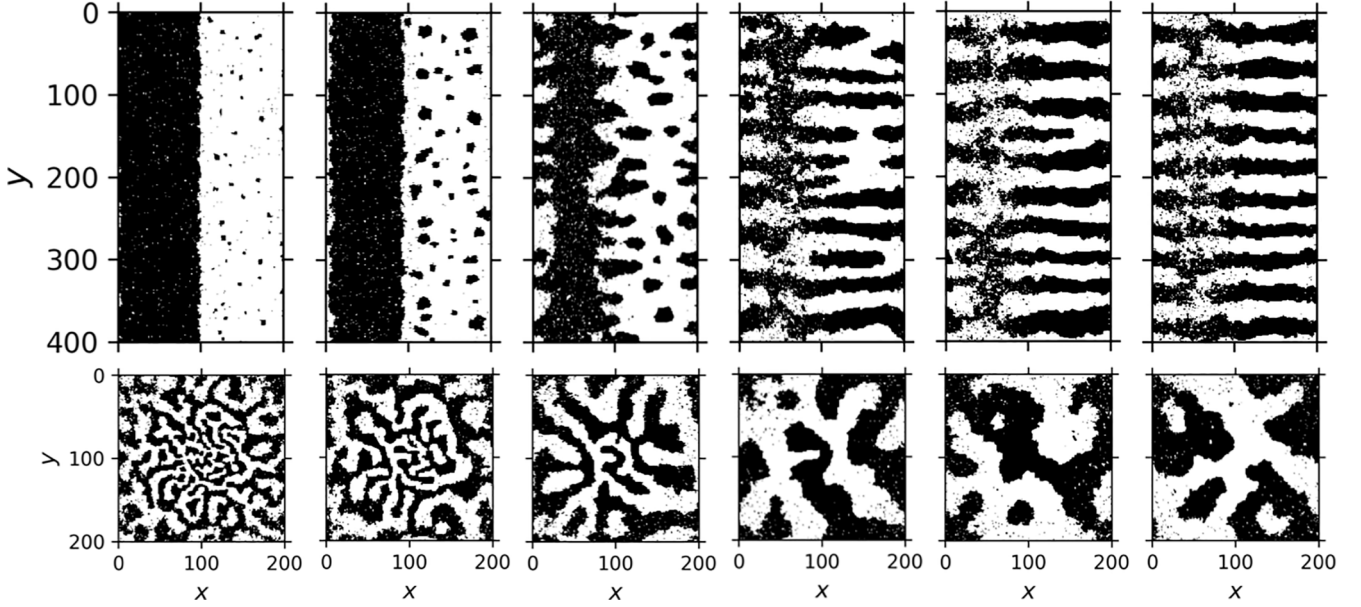


FIG. 3. Time evolution of the density field. Snapshots of the particle density at 0.1%, 0.4%, 1.6%, 6.3%, 25%, and 100% of the total simulation time $t = 5 \times 10^7$, for a single realization at filling factor $\bar{\rho} = 1/2$. Upper row: x -dependent temperature profile as in Eq. (2) with $T_{\text{mean}} = 0.4$, $T_{\text{amp}} = 0.2$, and system size $L_y = 2L_x = 400$. Lower row: dip temperature profile as in Eqs. (3) and (14) with $T_{\text{mean}} = 0.4$, $T_{\text{amp}} = 0.2$, and system size $L_x = L_y = 200$. The system is initialized from an infinite-temperature (white-noise) configuration.

the dynamics generates regularly arranged convection cells manifesting as stripes of alternating high and low particle density. This occurs for all tested initial conditions: an infinite-temperature configuration, a phase-separated state with a single vertical interface at $x = L_x/2$, and a phase-separated state with one or several horizontal interfaces. See the left panel in Fig. 1 and the upper panels of Fig. 3.

The convection cells are robust. Once formed, we never observe changes in the number of stripes for large systems. Such events arise only for very small lattices and become highly unlikely as the system size grows. The position of the cells do fluctuate on somewhat shorter timescales, though. Indeed, performing a time-average over longer times than in Fig. 1 did reveal some more blurring for some realizations. This is consistent with the existence of a unique steady state that is invariant under translations along the y direction.

This robustness is due to the persistent particle currents circulating within each cell, as illustrated in Fig. 1. The vector field shown there corresponds to the bond currents $(j_{x,x+e_1}, j_{x,x+e_2})$, see Eq. (8), time-averaged over a certain period. In practice, it is computed by counting the number of particle jumps from $\mathbf{x}$ to $\mathbf{y}$ minus those from $\mathbf{y}$ to $\mathbf{x}$ over a fixed integration time, and dividing by that time. Averages are taken over the same time window used for the density profiles. This window is chosen large enough to remove fluctuations but short enough so that the circulating currents do not reverse as the pattern move due to the eventual restoration of translational symmetry. The presence of these currents is consistent with basic thermodynamics: energy is absorbed from the bath in the hot region, through the creation of domain walls, and released into it in the cold one through their destruction. The constant flux of particles makes this process possible.

The emergence of periodic conduction patterns depends partly on the temperature profile in Eq. (2). To assess this dependence, we consider alternative radial profiles such as Eq. (3) (see also Appendix A). We first examine

$$\varphi_1(r) = \begin{cases} \cos(2\pi\alpha_1 r), & \alpha_1 r \leq 1, \\ 1, & \alpha_1 r \geq 1, \end{cases} \quad (13)$$

with $\alpha_1 \simeq 0.88$, which ensures that φ_1 has zero average. Simulations are performed with $L_x = L_y = 200$. This profile produces a Mexican-hat-like temperature landscape: the temperature peaks at the center, decreases at intermediate r , and rises again at larger r . The resulting time-averaged density and current fields, displayed on the lower panel of Fig. 1, appear as a distorted version of the reference pattern on the upper panel, indicating that the same phenomenology is at play.

A qualitatively different behavior arises for

$$\varphi_2(r) = \begin{cases} -\cos(2\pi\alpha_2 r), & \alpha_2 r \leq 1/2, \\ 1, & \alpha_2 r \geq 1/2, \end{cases} \quad (14)$$

with $\alpha_2 \simeq 0.32$. This profile features a single central dip. Here no stable pattern is formed. Snapshots are shown on the lower panel of Fig. 3. They feature transient, star-like distortions along radial directions, and time averaging yields much faster homogenization than in the other cases.

C. Regularity of the convection patterns

The convection patterns form a regular, approximately periodic arrangement; see Figs. 1–3. A natural explanation is that each stripe carries a particle current, and current conservation requires that these currents balance at interfaces. Stripes of equal width satisfy this condition automatically.

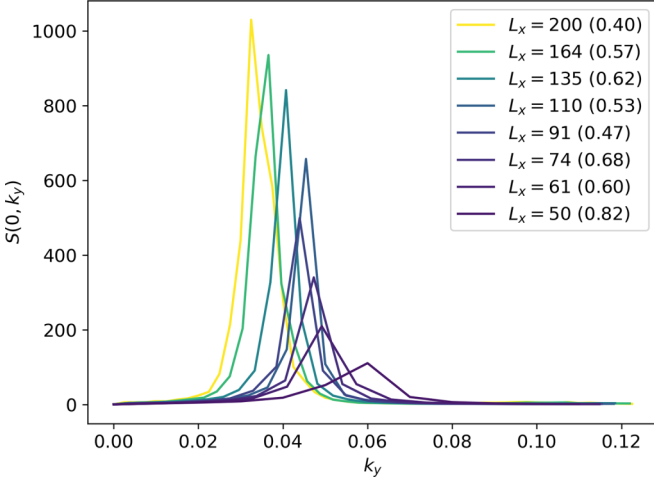


FIG. 4. Structure factor. Structure factor $S(0, k_y)$ as a function of k_y for several system sizes L_x , at fixed aspect ratio $L_y/L_x = 2$ and filling factor $\bar{\rho} = 1/2$. The temperature varies along x as in Eq. (2). The value of $|\tilde{n}(\mathbf{k})|^2$ is averaged over the final 25% of a simulation of duration $t = 5 \times 10^7$, and over a proportion of 40 realizations, indicated in parentheses for each L_x .

However, as the system size increases, fluctuations may cause distant stripes to acquire different widths, potentially degrading global periodicity.

To quantify this, following the approach of Ref. [46], we compute the structure factor

$$S(\mathbf{k}) = \frac{1}{L_x L_y} \langle |\tilde{n}(\mathbf{k})|^2 \rangle, \quad \tilde{n}(\mathbf{k}) = \sum_{\mathbf{x}} e^{2i\pi \mathbf{k} \cdot \mathbf{x}} (n(\mathbf{x}) - \bar{\rho}),$$

with filling factor $\bar{\rho}$, wave vectors $\mathbf{k} = (k_x, k_y)$ given by $k_z = m_z/L_z$ ($m_z = 0, \dots, L_z - 1$ for $z = x, y$), and $\langle \cdot \rangle$ a form of steady-state average chosen to preserve the convection structure while suppressing short-scale fluctuations. For a homogeneous configuration, $S(\mathbf{k}) = \mathcal{O}(1)$; for a periodically ordered state, $S(\mathbf{k}_{\text{peak}}) \sim V$ with $V = L_x L_y$, indicating long-range periodic order at $\mathbf{k}_{\text{peak}}$.

Figure 4 shows S as a function of $\mathbf{k} = (0, k_y)$ for $\bar{\rho} = 1/2$ and the temperature profile of Eq. (2) with $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$, for various system sizes and given aspect ratio $L_y/L_x = 2$. To avoid washing out the signal, we do not average over realizations with different numbers of cells at a given value of L_x . Instead, for each realization we sample 25 configurations equally spaced in time over the last 25% of the simulation and determine, for that realization, the most frequently occurring number of cells (typically overwhelmingly dominant, with rare deviations due to short-lived bridges between neighboring cells). This assigns a unique cell number to each realization. We then consider 40 realizations and retain only those whose assigned cell number matches the most probable value across realizations. Finally, $S(\mathbf{k})$ is averaged over all 25 sampled configurations of the retained realizations, and $S(0, k_y)$ is reported.

Two features emerge. First, $S(\mathbf{k})$ exhibits a clear peak at $k_{y,\text{peak}}$, with $k_{y,\text{peak}} \rightarrow 0$ as L_x increases, consistent with the sublinear growth of the cell number. Second, although $S(\mathbf{k}_{\text{peak}})$ increases with system size, it grows slower than V . This indicates that periodicity may be visible over

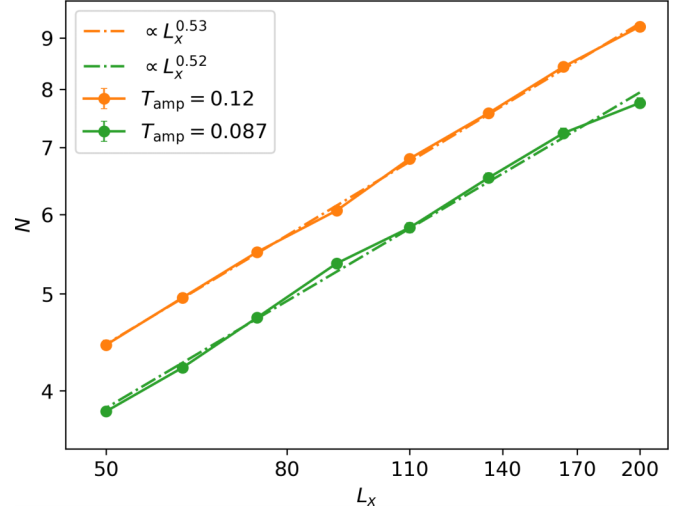


FIG. 5. Number of high-density stripes $\mathcal{N}$ as a function of the system size L_x for fixed aspect ratio L_y/L_x and filling factor $\bar{\rho} = 1/2$. The temperature varies along x as in Eq. (2) for $T_{\text{mean}} = 0.42$, and $T_{\text{amp}} = 0.12$ (orange) or $T_{\text{amp}} = 0.087$ (green). Average over 40 samples and over the last 25% of the total time $t = 1.33 \times 10^7$. See main text.

intermediate distances, but true periodic long-range order is eventually destroyed by fluctuations.

D. Scaling of the number of convection cells

We investigate how the number of convection cells, or equivalently of high-density stripes $\mathcal{N}$, scales with system size at fixed aspect ratio. We restrict ourselves to the stripes that appear in the low temperature region for the temperature profile in Eq. (2) with filling factor $\bar{\rho} = 1/2$. Some remarks are useful to frame the problem. First, one expects a scaling $\mathcal{N} \sim L^a$ with $0 < a < 1$, as both $a = 0$ and $a = 1$ are incompatible with local equilibrium, as will be discussed in Sec. III E. Other forms, such as $\mathcal{N} \sim \log L$, cannot be excluded *a priori* but are not supported by our data. Second, the dynamics admits a relatively broad range of stripe numbers that remain stable on simulation timescales. Starting from perfectly striped configurations with $1 \leq \mathcal{N} \leq 24$, we find that they give rise to stable patterns with the same number of stripes for at least $8 \leq \mathcal{N} \leq 17$ during the simulation time $t = 5 \times 10^7$. Finally, as stressed already in Sec. III A, it is possible that the figure would be modified if we could track the dynamics for much longer timescales.

To extract the most intrinsic scaling, we initialize the system at infinite temperature (white noise). Figure 5 shows results for $T_{\text{mean}} = 0.42$ with $T_{\text{amp}} = 0.12$ and 0.087 , ensuring that the temperature remains subcritical throughout the system. The method used for counting stripes is reported in Appendix C. The data are consistent with the scaling $\mathcal{N} \sim L_x^{1/2}$.

The interpretation of the data is complicated by a threshold effect arising from the fact that the number of stripes is an integer, which is not suppressed by increasing the number of samples. To mitigate this, we simulate systems with aspect ratio $L_y/L_x = 6$, for $50 \leq L_x \leq 200$, instead of the ratio

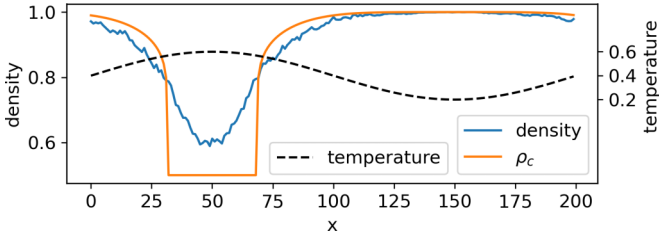


FIG. 6. Time- and y-averaged density within high density lines as a function of x . Temperature as in Eq. (2) with $T_{\text{mean}} = 0.4$, $T_{\text{amp}} = 0.2$, system size $L_y = 2L_x = 400$, $\bar{\rho} = 1/2$. Blue line: time average taken over the final 75% of a simulation of duration $t = 5 \times 10^7$ and spatial average in the y direction taken over a fraction of the sites with y coordinate corresponding to maximal density at $x = L_x/2$ (see main text). Single realization, starting from an initial configuration with 13 equally spaced stripes. Orange line: equilibrium spontaneous density at the local temperature.

$L_y/L_x = 2$ used so far. However, holding L_x fixed, we verify that $\mathcal{N}$ grows linearly with L_y . The values reported in Fig. 5 are obtained by dividing the measured stripe counts by 3, and are expected to provide a faithful representation of the average number of stripes for the aspect ratio $L_y/L_x = 2$.

We have also examined cases where the temperature crosses the critical point. Here exponents significantly larger than $1/2$ appear, depending on T_{amp} , though the curves exhibit noticeable downward bending for large system sizes. No universal exponent was found in this regime; the corresponding data are presented in Appendix D.

E. Local equilibrium

Local equilibrium means that the expectation of local observables coincides with their value in equilibrium at the local temperature and some local chemical potential in the large volume limit. The snapshots in the top row of Fig. 3 furnish a first indication of local equilibrium: as defects migrate toward the cold part, they nucleate bigger clusters, consistent with the fact that, in equilibrium, the density of defects decreases with decreasing temperature.

Verifying local equilibrium in the cold region is delicate due to the proximity of interfaces. In Fig. 6, we consider the temperature profile of Eq. (2) at filling factor $\bar{\rho} = 1/2$. We compute the time-averaged density $\langle n_x \rangle$ over the interval specified in the figure caption. On the vertical line $x = L_x/2$, where the temperature is subcritical, we select $1/8$ of the sites with the largest values of $|2\langle n_x \rangle - 1|$, corresponding to densities close to 0 or 1, and retain only those with $\langle n_x \rangle \simeq 1$. This procedure yields approximately $1/16$ of the sites on the line. We then average the density over the y direction along the corresponding rows. These rows lie well within the bulk of the high-density stripes in the low-temperature region.

Since the chemical potential is expected to remain zero throughout the system (see Sec. IV C), we compare the measured densities with Onsager's exact result [50] for the equilibrium spontaneous density at the local temperature, $\rho_c(T(\mathbf{x})) \geq 1/2$. Overall agreement is good. Importantly, the density does not fall significantly below the spontaneous density, which would signal the presence of a metastable state.

Slight deviations to this are observed, but we checked that this effect decreases when sampling deeper inside the stripes (here restricting the average to the central $1/8$ of the high-density region), consistent with a finite-size effect. In the high-temperature region the density exceeds $1/2$. This does not violate global symmetry, since the average is restricted to high-density lines. It is also compatible with local equilibrium, although it differs from the value expected at $\mu = 0$. We attribute this discrepancy to finite-size effects and expect the condition $\mu = 0$ to hold throughout in the thermodynamic limit, a point that warrants further investigation.

F. Scaling of the particle current

At low temperature and within each phase, particle fluxes are driven by density variations, which in turn stem from temperature variations; see Fig. 6. When L_x increases at fixed macroscopic temperature profile, the microscopic temperature gradient scales as $1/L_x$ and, away from criticality, the local density variation also scales as $1/L_x$. Given the diffusive transport in this system (Fick's law, see Sec. IV A), the current is expected to be proportional to the density gradient, and to scale as $1/L_x$.

We use the temperature profile of Eq. (2) at filling factor $\bar{\rho} = 1/2$ and measure the time-averaged current in the x direction across bonds intersecting $x = L_x/2$, where the current is approximately maximal. As in the analysis of local equilibrium, care must be taken to restrict measurements to regions well inside the high and low-density stripes, where the current is negative and positive, respectively. Accordingly, we compute the time-averaged density $\langle n_x \rangle$ over a given time window and select the $1/8$ of sites with the largest values of $|2\langle n_x \rangle - 1|$. The current through each horizontal bond originating from these sites is then averaged over the same time interval. Finally, we average these currents, assigning negative sign to those in high-density regions ($\langle n_x \rangle \simeq 1$) and positive sign to those in low-density regions ($\langle n_x \rangle \simeq 0$).

The results are shown in Fig. 7 for values of T_{mean} and T_{amp} such that the temperature remains subcritical throughout the system. The data indicate that the expected $1/L_x$ decay has not yet been reached at the accessible system sizes. Nevertheless, the pronounced downward curvature relative to a pure power-law fit is consistent with this asymptotic behavior. While the absolute magnitude of the measured current depends on details of the measurement protocol, the overall scaling behavior is robust. For instance, probing the current less deeply inside the stripes by replacing the fraction $1/8$ introduced above with $1/2$ yields power-law fits that differ by only a few percent. Additional measurements for temperature profiles that are not everywhere subcritical are reported in Appendix E.

G. Filling factor $\bar{\rho} \neq 1/2$

We now examine how the phenomenology changes for filling factors $\bar{\rho} > 1/2$. We focus on systems with $L_x = L_y/2 = 200$ and the temperature profile of Eq. (2).

A stationary density profile for $\bar{\rho} = 0.8$ is shown in the right panel of Fig. 2. Stripes form in the coldest region of the system. As mentioned in the summary of the results, this steady-state is markedly different from the equilibrium state

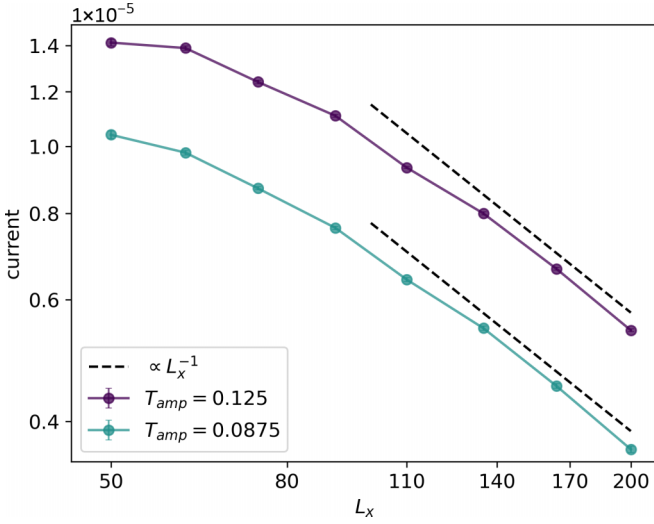


FIG. 7. Particle current in the x direction as a function of the system size L_x for fixed aspect ratio L_y/L_x and filling factor $\bar{\rho} = 1/2$. The temperature varies along x as in Eq. (2) for $T_{\text{mean}} = 0.42$, and $T_{\text{amp}} = 0.12$ (purple) or $T_{\text{amp}} = 0.087$ (green). Average over 40 samples and over the last 25% of the total time $t = 1.33 \times 10^7$. See main text.

obtained for the rates in Eq. (10) under the same temperature profile, shown in the left panel of Fig. 2. A related contrasting behavior was reported in a one-dimensional model in Ref. [51]. At the dynamical level, the key difference is that in the nonequilibrium case defects propagate diffusively within each phase, while this motion is suppressed for the equilibrium dynamics, with defects being kept away from the cold temperature part, as is visible in the left panel of Fig. 2; see Sec. IV A.

A further indication of nonequilibrium behavior is the geometry of the interfaces: they deviate strongly from the minimal-curvature shapes expected in equilibrium. The stripes are elongated rather than circular, with even a dumbbell-like profile. This reflects the fact that interface shapes are controlled by nonequilibrium currents that dominate over much weaker mean-curvature ones; see Sec. IV B.

In Sec. IV C, we derive the macroscopic chemical-potential profile $\mu(\mathbf{x})$. This quantity is simpler to analyze than $\rho(\mathbf{x})$ because it is insensitive to interfaces. For the temperature profile as in Eq. (2), μ is expected to be y -independent. The theory in Sec. IV C yields a direct prediction for the vertically averaged density

$$\tilde{\rho}(x) = \frac{1}{L_y} \sum_y \rho(x, y); \quad (15)$$

see Eq. (23) as well as the discussion at the end of Sec. IV C. For $\bar{\rho} > 1/2$, $\tilde{\rho}(x)$ should equal $1/2$ on a symmetric interval around $x = 3L_x/4$, and equal the spontaneous density at the interval's boundary elsewhere. The width of this plateau is fixed by the value of $\bar{\rho}$.

Figure 8 shows $\tilde{\rho}(x)$ for $\bar{\rho} = 0.8$ and various T_{mean} and T_{amp} . The dashed blue curve corresponds to $\tilde{\rho}(x)$ for the stationary density in the right panel of Fig. 2. Here the spontaneous density at the stripe boundary is very close to 1, and

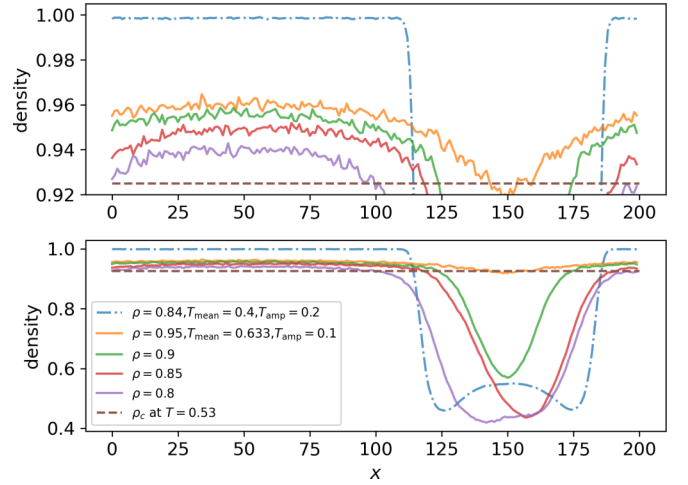


FIG. 8. Time- and y -averaged density profile $\tilde{\rho}(x)$ at filling factor $\bar{\rho} = 0.8$. The temperature varies along x as in Eq. (2), with $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$ (blue dashed line), and $T_{\text{mean}} = 0.633$ and $T_{\text{amp}} = 0.1$ (solid lines). The horizontal brown dashed line indicates the critical density $\rho_c(T = 0.53)$. The system size is $L_y = 2L_x = 400$. Time is averaged over 75% of the total simulation time $t = 5 \times 10^7$ for a single realization. The lower panel shows the full profile, while the upper panel highlights the region where $\tilde{\rho}(x) > \rho_c(T = 0.53)$.

the agreement with the macroscopic prediction outside the stripe region is good. Inside the stripe region, however, $\tilde{\rho}(x)$ varies noticeably around $1/2$, consistent with the dumbbell-like stripe shape. It remains unclear whether this variation persists at larger system sizes.

We also explore temperature profiles yielding smaller spontaneous densities in the cold region, giving rise to the remaining curves in Fig. 8. A notable observation is that $\tilde{\rho}(x)$ systematically exceeds the spontaneous density ρ_c in the coldest part by a few percent. As far as we could test, this effect does not diminish with increasing system size, and we currently do not have an explanation for it.

IV. MACROSCOPIC DESCRIPTION

We now turn to a macroscopic description of the nonequilibrium system introduced in Sec. II A. The framework relies on two fundamental physical principles: local equilibrium in the steady-state and diffusive transport of the conserved quantity (Sec. IV A). This setup yields three key results. First, out-of-equilibrium steady currents, rather than interface curvature, primarily shape the out-of-equilibrium interfaces (Sec. IV B). Second, it determines the chemical potential profiles, matching our numerical observations (Sec. IV C). Finally, it explains the physical mechanism behind the observed proliferation of stripes in the large-volume limit (Sec. IV D).

Before moving on, let us note that the temperature profiles considered so far may induce symmetries that are not broken microscopically. For example, if the temperature depends only on x as in Eq. (2), the stationary density remains homogeneous in y , reflecting the unfixed vertical position of the stripes. As we saw though, particle configurations exhibit well-defined stripes whose positions are remarkably stable, with motion timescales that grow rapidly with system size.

The present macroscopic theory retains only currents scaling as $1/L$ or larger, thus neglecting dynamics occurring on timescales beyond diffusion. As a result, it can treat as stationary configurations that are microscopically quasistationary, rather than their broad superpositions. Accordingly, throughout this section, the density ρ refers to that of a locally pure state, not of a mixture.

Throughout this section, the system size is rescaled by $1/L$ so that $L_x/L = a$ and $L_y/L = b$ for fixed constants $a, b > 0$ as $L \rightarrow \infty$. The corresponding macroscopic coordinates are $\mathbf{u} = \mathbf{x}/L \in [0, a] \times [0, b]$ (with periodic boundary conditions).

A. Macroscopic constraints

We explicitly state the physical principles constraining the macroscopic steady state: local equilibrium in the bulk and at the interfaces [Eqs. (16) and (17)], particle number conservation [Eqs. (18) and (19)], and macroscopic diffusive transport away from the interfaces [Eq. (20)]. See also Refs. [42,43].

1. *Local equilibrium.* Local equilibrium holds throughout the system: the expectation of any local observable approaches that of an equilibrium system at the local temperature $T(\mathbf{u})$ as $L \rightarrow \infty$. Our numerical results are consistent with local equilibrium, see Sec. III E. This originates from the fact that the rates in Eq. (5) break detailed balance by an amount that scales as $1/L$ for $L \rightarrow \infty$.

Local equilibrium precludes metastable states, implying that for any $\mathbf{u}$,

$$\rho(\mathbf{u}) \notin (1 - \rho_c(\mathbf{u}), \rho_c(\mathbf{u})) \quad (\text{everywhere}), \quad (16)$$

where

$$\rho_c = \rho_c(T(\mathbf{u})) \geq 1/2$$

is the spontaneous density, as given by Onsager's formula [50]. Near interfaces, where the local density averages to $1/2$, local equilibrium further predicts

$$\rho_+(\mathbf{u}) = \rho_c(\mathbf{u}), \quad \rho_-(\mathbf{u}) = 1 - \rho_c(\mathbf{u}) \quad (\text{interfaces}), \quad (17)$$

with $\rho_{\pm}$ denoting the densities on each side of the interface.

2. *Conservative dynamics.* Since the dynamics conserves the particle number at the microscopic level, Eq. (6), the macroscopic density field likewise satisfies a continuity equation everywhere. Let $J(\mathbf{u})$ denote the macroscopic density current at position $\mathbf{u}$. In the bulk, away from interfaces separating high and low-density phases,

$$\nabla \cdot J(\mathbf{u}) = 0 \quad (\text{bulk}), \quad (18)$$

while at interfaces the condition reduces to the static Stefan relation,

$$(J_+(\mathbf{u}) - J_-(\mathbf{u})) \cdot \mathbf{n} = 0 \quad (\text{interface}), \quad (19)$$

where $J_{\pm}$ are the currents on each side of the interface and $\mathbf{n}$ is the unit normal vector.

3. *Fick's law.* The microscopic continuity equation is given in Eq. (6), with the corresponding microscopic in Eq. (8). Naively interpolating this expression at the macroscopic level yields the following expression for the macroscopic current $J = (J_x, J_y)$:

$$J_z = -\partial_z \rho + 2\rho(1 - \rho) \tanh[(5/2T) \partial_z \rho],$$

$z = x, y$, where we have set $4J = 1$, as previously. Crucially, this current vanishes when $\nabla \rho = 0$, indicating that the large-scale dynamics is diffusive. However, we do not expect the diffusion coefficient to be read directly from this expression; instead, we allow for a more general constitutive relation of the form

$$J(\mathbf{u}) = -D(T(\mathbf{u}), \rho(\mathbf{u})) \nabla \rho(\mathbf{u}) \quad (\text{bulk}), \quad (20)$$

i.e., Fick's law, where D denotes the diffusion coefficient. The coefficient $D(T, \rho)$ remains finite and bounded away from zero as long as $\rho(\mathbf{u}) \geq \rho_c(\mathbf{u})$ and as we keep away from the critical point. This is consistent with the observation that, at the microscopic level and in the low temperature region, defects diffuse freely within the majority phase in first good approximation. See also Ref. [52]. Equation (20) is expected to hold away from interfaces.

Note that the current in Eq. (12) for the equilibrium dynamics with an inhomogeneous Hamiltonian does not yield diffusive dynamics, since $\Delta_{\mathbf{x},\mathbf{y}}^T$ in Eq. (11) does not vanish at constant density.

B. Role of interface curvature

Interface curvature plays no role in the macroscopic description presented above. We now explain why these curvature effects, which are vital in shaping equilibrium interfaces, become secondary to out-of-equilibrium currents when temperature gradients are imposed.

In equilibrium at uniform subcritical temperature, with a global filling factor between $1 - \rho_c$ and ρ_c , steady profiles consist of a single bubble of the minority phase with minimal curvature, and the density is critical everywhere. Equations (16)–(20) are then satisfied. However, these conditions alone do not select the observed profiles. This reflects the absence of any term involving interface curvature in our macroscopic equations.

Such curvature terms are excluded because only currents of order $1/L$ are retained in the present description. In a more refined treatment, the interfacial densities in Eq. (17) acquire corrections proportional to the inverse radius of curvature, i.e., of order $1/L$. These small variations generate currents scaling as $1/L^2$, giving rise to slow mean-curvature motion [53].

When the temperature profile is not uniform, microscopic steady currents appear that scale as $1/L$, see Sec. III F. Because these dominate over mean-curvature contributions, curvature effects play little role in shaping macroscopic nonequilibrium steady profiles, see, e.g., the right panels of the right panel of Figs. 2 and 1, where interface shapes clearly deviate from mean-curvature behavior.

C. Chemical potential profiles

Under local equilibrium, we can define the chemical potential $\mu(\mathbf{u})$ through

$$\rho(\mathbf{u}) = \langle n \rangle_{T(\mathbf{u}), \mu(\mathbf{u})}.$$

Determining the chemical potential profile provides more fundamental information than the density profile, as it does not require identifying the position and shape of the interfaces within the $\mu = 0$ region. In fact, the macroscopic theory

allows us to determine it completely, and the result can be described as follows. In general, an isothermal line at a subcritical temperature can separate the domain into two regions. In the region where the temperature is higher than this line, denoted as the set Ω_0 in Eq. (22) below, the density is constant and equal to the spontaneous density on the isothermal line. In the complement of Ω_0 , where the temperature is lower, $\mu = 0$, and this region may be filled with stripes. The position of this isothermal line is dictated by the overall density. Of course, degenerate cases may occur where no such line exists, meaning Ω_0 is either empty ($\mu = 0$ everywhere, as in all our simulations at half filling) or occupies the entire space (which occurs when the density exceeds a specific threshold).

Let us derive these claims. Combining Eqs. (18) and (20) gives

$$\nabla \cdot J = \nabla \cdot (D(T(\mathbf{u}), \rho(\mathbf{u})) \nabla \rho(\mathbf{u})) = 0, \quad (21)$$

valid away from interfaces. The only free parameter is the overall filling factor $\bar{\rho}$, which by symmetry can be taken in $[1/2, 1]$.

Using the chain rule, $\nabla \rho = (\partial \rho / \partial T) \nabla T + (\partial \rho / \partial \mu) \nabla \mu$, Eq. (21) can be reformulated as an equation for μ . When $\mu \neq 0$, the system is in a single phase and Eq. (21) applies. We impose that μ varies continuously across the boundary separating the $\mu \neq 0$ and $\mu = 0$ regions.

This is sufficient to determine $\mu(\mathbf{u})$ throughout the system. Let $\mathbf{u}_0$ denote a point of maximal temperature and define

$$\Omega_0 = \{\mathbf{u} : \rho_c(T(\mathbf{u})) < \rho(\mathbf{u}_0)\}. \quad (22)$$

We then set

$$\mu(\mathbf{u}) = \begin{cases} \text{s.t. } \langle n \rangle_{T(\mathbf{u}), \mu(\mathbf{u})} = \rho(\mathbf{u}_0), & \mathbf{u} \in \Omega_0, \\ 0, & \mathbf{u} \notin \Omega_0. \end{cases} \quad (23)$$

Inside Ω_0 , Eq. (23) is reexpressed more simply as

$$\rho(\mathbf{u}) = \rho(\mathbf{u}_0).$$

Hence, $\nabla \rho(\mathbf{u}) = 0$, satisfying Eq. (21). On the boundary of Ω_0 , the definition of Ω_0 ensures $\rho(\mathbf{u}) = \rho_c(T(\mathbf{u}))$, so that $\mu = 0$ there, and $\mu(\mathbf{u})$ is continuous everywhere.

Two limiting cases are immediate. If Ω_0 spans the whole volume, then $\rho(\mathbf{u})$ is constant, which occurs when the temperature is everywhere supercritical or when the filling factor satisfies $\bar{\rho} > \rho_c(T_{\min})$, with $T_{\min}$ the minimum temperature in the system. Conversely, if $\Omega_0 = \emptyset$, then $\mu = 0$ everywhere, corresponding to half-filling.

We now consider a temperature profile depending only on x , as in Eq. (2), with aspect ratio $L_y/L_x = 2$ such that the macroscopic domain is $[0, 1] \times [0, 2]$. For this geometry, $\Omega_0 = I_0 \times [0, 2]$, where I_0 is an interval centered around $x = 1/4$ in macroscopic units (periodic boundary conditions). The width of I_0 is set by the filling factor $\bar{\rho}$. The theory predicts that $\rho(\mathbf{u})$ remains constant within Ω_0 , with a value equal to the spontaneous density at the boundary of I_0 ($\mu = 0$). This prediction is confirmed by the observations on Fig. 8, up to some deviations discussed in Sec. III G. Further, it provides a full explanation to the main difference between the left and right panels of Fig. 2.

D. Stripe formation

We now address the emergence of stripe patterns. The central physical insight is that, at the macroscopic level, a point of minimal, subcritical temperature cannot lie within a macroscopic region of high or low density. This restriction underlies the formation of stripes observed for the geometry predominantly considered here, the number of which must grow unbounded with the system size.

Below, we formulate this property as a mathematical fact and discuss its implications for interpreting our numerical results. Before doing so, however, let us present an explicit construction to illustrate the mechanism most directly.

1. *Explicit construction of steady profiles.* In Fig. 9, we construct profiles that exactly satisfy the macroscopic constraints in Eqs. (17)–(20) but (weakly) violate the local equilibrium condition in Eq. (16) near the minimal temperature region; see the green line in panels (c) and (d), which displays a density within the forbidden interval $(1 - \rho_c, \rho_c)$. As the number of stripes increases [compare panel (c) with panel (d)], meaning the single-phase domains around the lowest temperature line become less macroscopic, this violation diminishes. We identify this as the underlying mechanism driving the proliferation of stripes with increasing system size.

Note also that the stripe configurations in Fig. 9 are exactly periodic, ensuring that currents are balanced at interfaces, i.e., that Eq. (19) is satisfied. This periodicity is consistent with the patterns observed in microscopic simulations; see Fig. 4 and the discussion in Sec. III C.

For the simulations in Fig. 9, we set $D = 1$ in Eq. (20) for simplicity. The steady state is obtained by integrating the spatially discretized heat equation on a rectangular lattice Λ of $L_x \times L_y = 200 \times 400$ sites. The system is evolved long enough to reach stationarity within each domain delimited by a phase interface [see panels (a) and (b) in Fig. 9]. For each of them, we impose fixed boundary conditions along the interfaces, where the density equals the positive or negative spontaneous density at the corresponding temperature.

2. *Rigorous foundation.* Going beyond these specific simulations, we show that a point of minimal, subcritical temperature cannot lie within a macroscopic high- or low-density region. This restriction is a direct consequence of the tension between local equilibrium, on the one hand [Eqs. (16) and (17)], and macroscopic diffusive transport, on the other hand [Eq. (20)].

Let us consider temperature profiles as in our simulations, i.e., as in Eq. (2) or Eq. (3), though the claim is more general. We also assume that the profile stays subcritical throughout, as crossing the critical temperature may lead the diffusion coefficient D in Eq. (20) to vanish.

Claim. Assume that $\rho(\mathbf{u})$ is a steady macroscopic density profile satisfying Eqs. (16)–(20). Then either $\rho(\mathbf{u})$ is constant, or no point where the temperature is minimal can lie in the interior of a high or low-density region.

A formal proof is deferred to Appendix F; here, we outline the core of the argument. The primary analytical tool is the maximum principle (see, e.g., Ref. [54]) applied to the steady state of our diffusive system [Eqs. (18) and (20)]. This principle states that a steady density profile $\rho(\mathbf{u})$ must reach its maximum on the boundary of any domain that does not

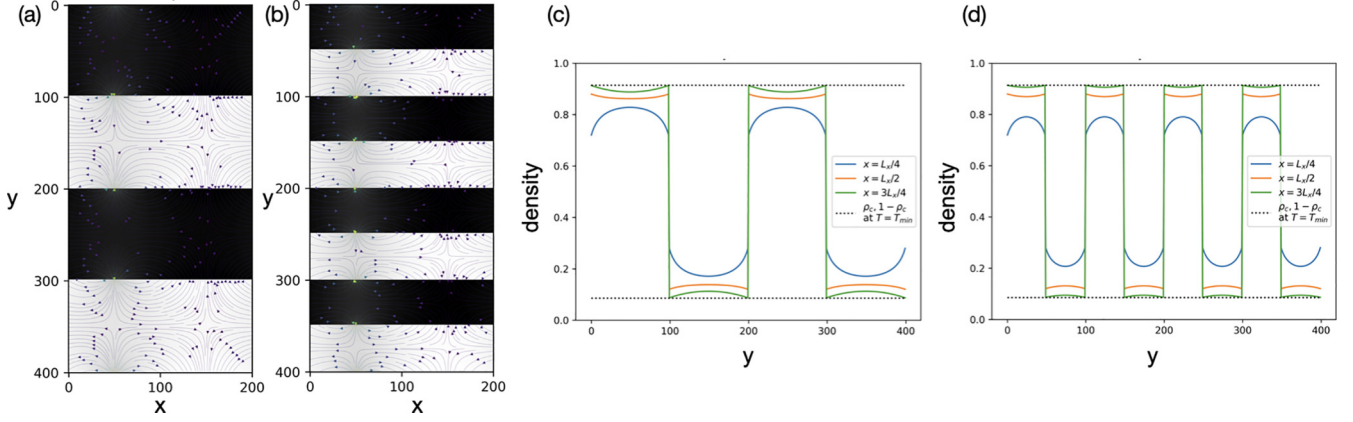


FIG. 9. Two steady density profiles satisfying Eqs. (17)–(20). Panels (a) and (b) show the spatial density (grayscale) and particle currents (colored arrows), featuring two and four high-density stripes, respectively. Panels (c) and (d) show the density along the y direction at $x = L_x/4$ (blue), $x = L_x/2$ (orange), and $x = 3L_x/4$ (green); panel (c) corresponds to the solution in panel (a), and panel (d) corresponds to the one in panel (b). The horizontal dotted lines mark the spontaneous density in the coldest region ($x = 3L_x/4$).

cross any phase interface (i.e., where the high-density phase transitions to the low-density phase).

We assume that $\rho(\mathbf{u})$ is not constant and, to establish a contradiction, suppose there exists a domain containing a point of minimal temperature in its interior, with the phase interface as its boundary. By Eq. (17), the density on this boundary must equal the critical density. However, the local equilibrium constraint in Eq. (16) requires that the density ρ at the coldest point be greater than or equal to the spontaneous density ρ_c at that specific temperature. Because $\rho_c(T)$ increases monotonically as temperature decreases, ρ at the coldest point must be larger than the critical density value fixed on the warmer boundary. This requirement implies that the maximum is reached in the interior of the domain, yielding a contradiction.

3. Implications for microscopic simulations. We examine how this constraint is reflected, or circumvented, in the microscopic systems studied numerically. For the x -dependent temperature profile of Eq. (2), the minimum temperature lies along $x = 3L_x/4$. For filling factors close to $1/2$, interfaces must form, and the observation above implies that, in the macroscopic limit, no point on this line can lie deep inside a single phase. Microscopic simulations indeed show that the system fragments into an increasing number of stripes as the system size grows; see Fig. 5. Our observation implies that the number of stripes must indeed increase with system size, so that no point becomes truly internal to a phase in the macroscopic limit. Similar reasoning applies to the “Mexican-hat” profiles in Eqs. (3) and (13).

V. DISCUSSION AND OUTLOOK

We have studied how the phase-separated low-temperature state of a lattice gas with Ising interactions is modified when driven out of equilibrium by a macroscopic temperature gradient, in a regime where local equilibrium is maintained throughout the system. We find that, at the macroscopic scale, the ordered phase reorganizes into a regular array of convection cells whose geometry is remarkably robust in time. These nonequilibrium patterns differ markedly from the steady states

obtained under equilibrium dynamics at the corresponding local temperatures, highlighting the nontrivial impact of weak nonequilibrium driving. The observations can be qualitatively explained within a macroscopic, or hydrodynamic, description of the system.

Our results raise several questions concerning the nature of nonequilibrium steady states and their fluctuations. For example, can one develop a quantitative theory of macroscopic fluctuations that is sufficiently predictive to account for the scaling of the number of current vortices? Are the observed patterns and their stability specific to open systems, here realized by coupling to a heat bath throughout the bulk, or do analogous structures arise in closed systems? Can the shape and organization of the steady-state profiles be understood in terms of fundamental properties of entropy production and, more fundamentally, could they point to organizing principles that replace free-energy minimization in nonequilibrium settings? We leave these questions and many others for future work.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [55].

APPENDIX A: ALTERNATIVE DYNAMICS

To stress the robustness of our main findings, we consider three variants of the dynamics studied in the main text:

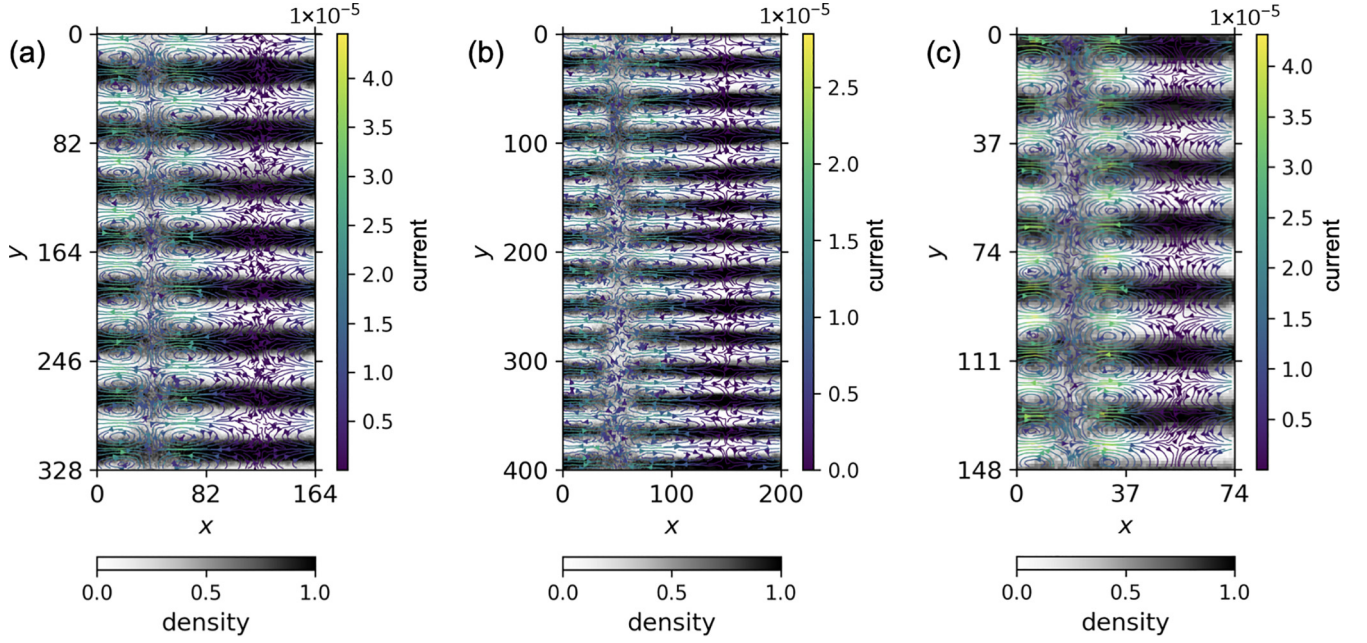


FIG. 10. Time-averaged densities (grayscale) and particle currents (colored lines) for a single realization and half filling ($\bar{\rho} = 1/2$). Averaging is performed over the final 25% of the total simulation time ($t = 5 \times 10^7$). (a) x -dependent temperature profile as in Eq. (A1) with $L_y = 2L_x = 328$. (b) x -dependent temperature profile as in Eq. (A2) with $L_y = 2L_x = 400$. (c) x -dependent temperature profile as in Eq. (2) and temperature dependent γ as in Eq. (A3), with $L_y = 2L_x = 148$.

(1) The dynamics is as described in Sec. II A, but the temperature variation in Eq. (2) is replaced by a sinusoidal inverse temperature profile:

$$\frac{1}{T(u_x, u_y)} = \beta_{\text{mean}} - \beta_{\text{amp}} \sin(2\pi u_x), \quad (\text{A1})$$

with $\beta_{\text{mean}} = 1.4\beta_c$ and $\beta_{\text{amp}} = 0.5\beta_c$ ($\beta_c \approx 1.76$).

(2) The dynamics is as described in Sec. II A, but the temperature variation in Eq. (2) is replaced by a sawtooth profile:

$$T(u_x, u_y) = \begin{cases} 4T_{\text{amp}}u_x + T_{\text{mean}}, & 0 \leq u_x \leq 1/4, \\ -4T_{\text{amp}}(u_x - 1/2) + T_{\text{mean}}, & 1/4 \leq u_x \leq 3/4, \\ 4T_{\text{amp}}(u_x - 1) + T_{\text{mean}}, & 3/4 \leq u_x \leq 1, \end{cases} \quad (\text{A2})$$

with $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$.

(3) The temperature profile remains as defined in Eq. (2) (with $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$), but the rate γ in Eq. (5) is made temperature-dependent:

$$\gamma_{x,y} = \left(\frac{T_{x,y}}{T_{\text{max}}} \right) \gamma, \quad (\text{A3})$$

where $T_{\text{max}} = T_{\text{mean}} + T_{\text{amp}}$. This ensures $0 < \gamma_{x,y} \leq \gamma$ and models the slowing of dynamics in colder regions.

Figure 10 shows the average density and currents for these cases. Averages are taken over the final 25% of a simulation of total duration $t = 5 \times 10^7$, starting from an infinite-temperature state at half filling ($\bar{\rho} = 1/2$). The aspect ratio is $L_y = 2L_x$. The resulting steady states are qualitatively analogous to that shown on the left panel of Fig. 1, confirming

that the salient features of the nonequilibrium steady states are not sensitive to details of the model.

APPENDIX B: ASSESSING STATIONARITY

We perform two tests to check that the dynamics have reached configurations representative of the stationary state. As noted in the main text, once convection cells form, their number may fluctuate over time, but the timescales for these processes are too slow to be seen in our simulations.

The first test is to measure the energy density, $E(\eta(t))/V$, with $V = L_x L_y$ and $E(\eta)$ the energy given in Eq. (1). This quantity is expected to become approximately constant once the stationary regime is reached. The results are shown in the top panels of Fig. 11 for a system with $T_{\text{mean}} = 0.4$, $T_{\text{amp}} = 0.2$, and aspect ratio $L_y/L_x = 2$. Starting from an initial configuration at infinite temperature for every system size considered, the energy density is plotted as a function of time for a short window $[0, 2 \times 10^5]$ in Fig. 11(a), and for a long window $[0, 5 \times 10^7]$ in Fig. 11(b). The latter is the window used for most results in the main text. The results in Fig. 11 show that the energy density stabilizes on timescales that are only a few percent of the total simulation time.

The second test measures how the number of convection cells (or high-density stripes $\mathcal{N}$ as considered in Sec. III D) varies with time. See Appendix C for details on how this is measured. The results are shown in the bottom panels of Fig. 11 for the same system, initial conditions, and time windows as the energy. These results also show a stabilization of the number of cells on timescales similar to those of the energy.

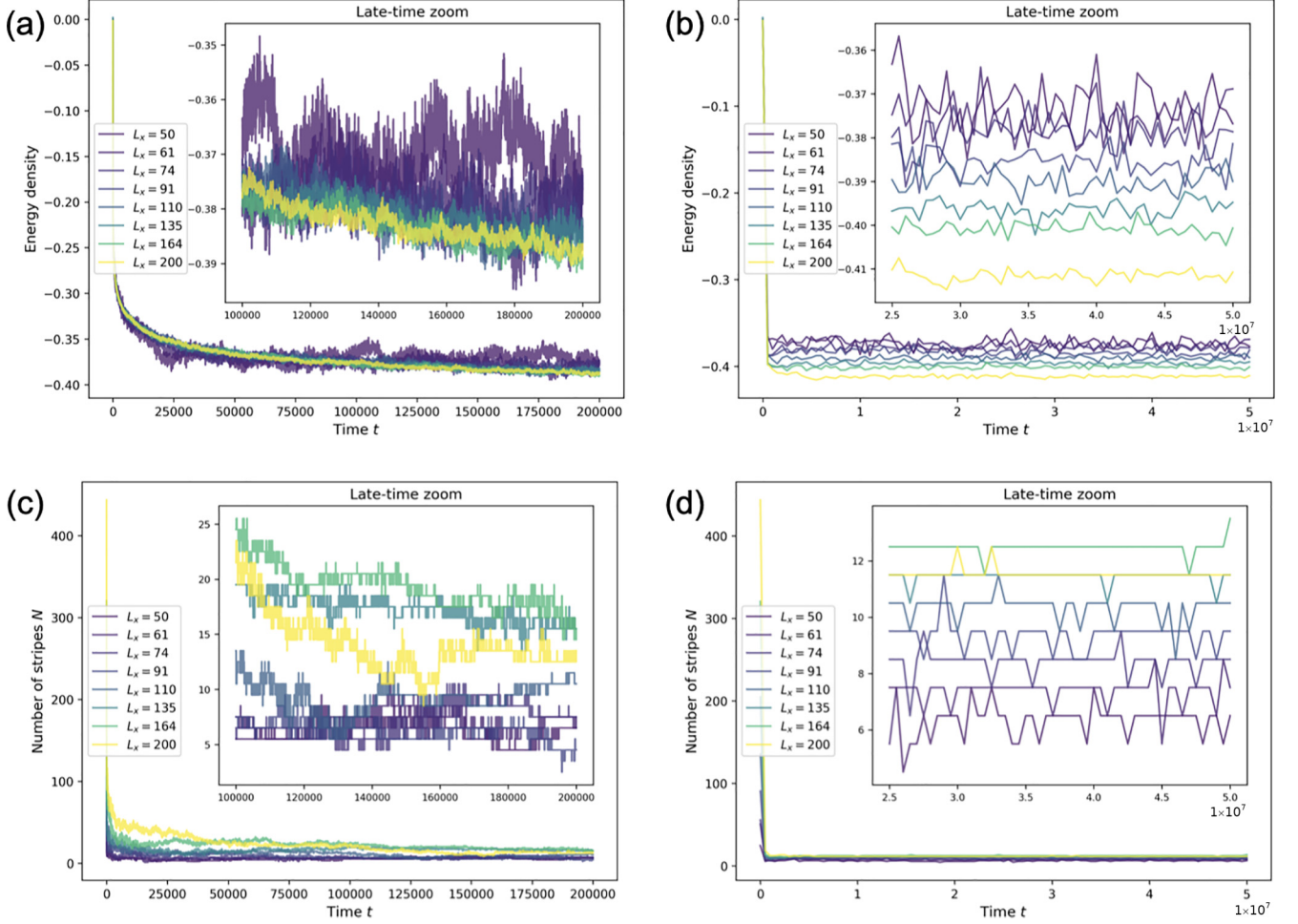


FIG. 11. Time dependence of the energy density (top panels) and the number of stripes $\mathcal{N}$ (bottom panels). The data shown are for a single realization starting from an infinite-temperature configuration at half filling ($\bar{\rho} = 1/2$), with the temperature profile given in Eq. (2) and an aspect ratio $L_y/L_x = 2$. Different system sizes L_x are indicated in the inset. Panels (a) and (c) show a short time window $[0, 2 \times 10^5]$ with 2×10^4 measurements. Panels (b) and (d) show the long time window $[0, 5 \times 10^7]$ with 100 measurements, which corresponds to the window used for the majority of results in the main text.

APPENDIX C: MEASURING THE NUMBER OF CONVECTION CELLS

The number of convection cells, or number of stripes $\mathcal{N}$, reported in Figs. 5, 11(c), 11(d), and 12, is obtained as follows. For each particle configuration, we restrict attention to the region $L_x/2 \leq x < L_x$, where the temperature is subcritical and stripes form. We identify connected components of particles, accounting for periodic boundary conditions in the y direction. Connected components containing fewer than 10 particles are discarded. The remaining number of connected components is identified with the number of stripes.

For Figs. 5 and 12, 100 configurations are recorded at uniformly spaced times over the full simulation for each realization. Only configurations within the final 25% of the simulation time are retained, yielding 25 configurations per realization. The reported values of $\mathcal{N}$ are averaged over these configurations and over independent realizations.

APPENDIX D: NUMBER OF CONVECTION CELLS: ADDITIONAL DATA

We report additional data for the number of convection cells, or number of stripes $\mathcal{N}$, obtained with temperature profiles of the form Eq. (2) that intersect the critical temperature, such that only part of the system is subcritical. We fix $T_{\text{mean}} = 0.2$. As for Fig. 5 in the main text, we simulate systems with aspect ratio $L_y/L_x = 6$, for $50 \leq L_x \leq 200$, to mitigate the threshold effect arising from the number of stripes being an integer, and the values for the average number of stripes reported in Fig. 12 are obtained by dividing the measured stripe counts by 3 for consistency. The curve for $T_{\text{amp}} = 0.15$ closely matches that obtained in the main text for $T_{\text{mean}} = 0.42$ and $T_{\text{amp}} = 0.12$, indicating good consistency between the two datasets. As expected, the number of stripes increases further with increasing temperature gradient. We observe a downward bending of the curves $T_{\text{amp}} = 0.2$ and $T_{\text{amp}} = 0.25$, suggesting that the asymptotic scaling regime may not yet be reached.

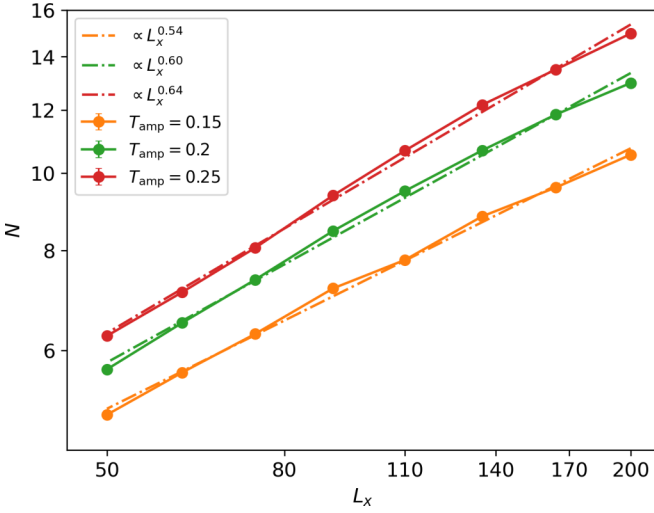


FIG. 12. Average number of high-density stripes $\mathcal{N}$ as a function of the system size L_x for fixed aspect ratio L_y/L_x and filling factor $\bar{\rho} = 1/2$. The temperature varies along x as in Eq. (2), with $T_{\text{mean}} = 0.4$ and several values of the amplitude T_{amp} (see legend). Average obtained from 120 independent realizations and averaged over the final 25% of the simulation time. The total simulation times are $t = 5 \times 10^6$ for $T_{\text{amp}} = 0.15$, $t = 1.33 \times 10^7$ for $T_{\text{amp}} = 0.2$, and $t = 2.66 \times 10^7$ for $T_{\text{amp}} = 0.25$.

APPENDIX E: SCALING OF THE PARTICLE CURRENT: ADDITIONAL DATA

We present additional measurements of the particle current in the x direction for the temperature profile of Eq. (2), focusing on cases where the temperature is not subcritical throughout the system. We examine its scaling with system size for $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$, where the temperature profile crosses the critical point. Results are shown in Fig. 13. Simulations are initialized with a prescribed number of equally spaced stripes (ranging from 12 to 15), corresponding to typical steady-state configurations. The current is measured using the protocol described in Sec. III F. Compared with the data in the main text, obtained for a system where the temperature remains subcritical, the asymptotic $1/L_x$ scaling is not yet observed at the largest sizes simulated. Because the profile crosses the critical point, it is unclear if this scaling will eventually be reached, and we draw no firm conclusions from these data.

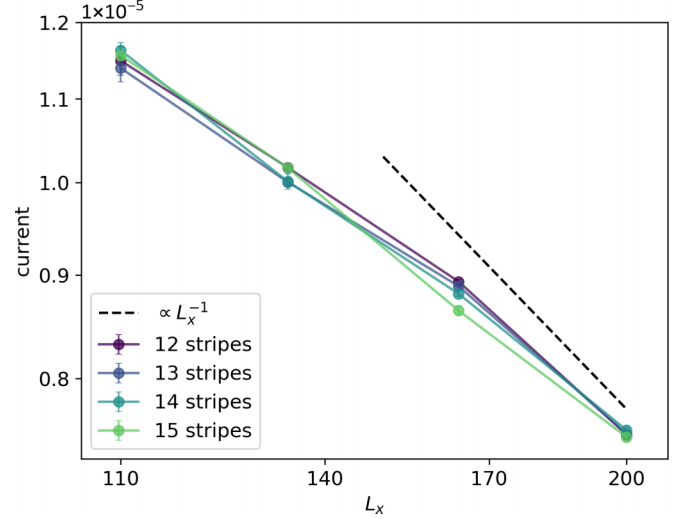


FIG. 13. Particle current in the x direction as a function of system size L_x for fixed aspect ratio $L_y/L_x = 2$ and filling factor $\bar{\rho} = 1/2$. The temperature varies along x as in Eq. (2), with $T_{\text{mean}} = 0.4$ and $T_{\text{amp}} = 0.2$. Simulations are initialized with a fixed number of equally spaced stripes (from 12 to 15). Data are averaged over five realizations for each initial condition, and over the final 25% of a simulation of duration $t = 2 \times 10^8$.

APPENDIX F: PROOF OF THE CLAIM IN SEC. IV D

We provide a more formal proof of the Claim in Sec. IV D. In the absence of interfaces, Eq. (21) holds everywhere, implying that $\rho(\mathbf{u})$ is constant. For a profile with interfaces, let $\Omega_0 \subset [0, a] \times [0, b]$ be a connected domain occupied by one phase. Suppose, for contradiction, that an interior point $\mathbf{u}_0 \in \Omega_0$ corresponds to a temperature minimum. Local equilibrium conditions (16) and (19), together with the monotonic increase of $\rho_c(T)$ with T , give

$$\rho(\mathbf{u}_0) \geq \rho_c(\mathbf{u}_0) \geq \rho_c(\mathbf{u}) = \rho(\mathbf{u}), \quad \forall \mathbf{u} \in \partial\Omega_0,$$

where $\partial\Omega_0$ denotes the boundary of Ω_0 . Hence, $\rho(\mathbf{u})$ attains its maximum inside Ω_0 . Since Eq. (21) holds within Ω_0 , the strong maximum principle [54] implies that $\rho(\mathbf{u})$ must be constant there. For temperature profiles as in our simulations (but it is very generic), the boundary $\partial\Omega_0$ will include points where $T(\mathbf{u}) > T(\mathbf{u}_0)$, implying $\rho_c(\mathbf{u}_0) > \rho_c(\mathbf{u})$. This implies also $\rho(\mathbf{u}_0) > \rho(\mathbf{u})$, in contradiction with ρ being constant.

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