

Predicting bifurcation points via extreme learning machine methods trained on time-series datasets and parameters in discrete dynamical systems

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This paper proposes a data-driven method for obtaining the bifurcation points of a system utilizing a method based on Extreme Learning Machine with input parameter channels (ELM-IPC) trained on the time-series datasets from discrete dynamical systems. The input parameter channels permit the proposed method to align the original parameter spaces of the time-series datasets with the derived parameter spaces of the ELM-IPC model; this results in the accurate analysis of the bifurcation points of a system. To demonstrate the effectiveness of this approach, we apply it to two discrete dynamical systems: the two-dimensional Hénon map and the four-dimensional system comprising two conservatively coupled Hénon maps. Our results indicate that it is possible to obtain the bifurcation points of a system even in parameter spaces where multiple solutions coexist or periodic solutions with periods exceeding 100 are present.

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I. INTRODUCTION

Dynamical systems are ubiquitous across a wide range of fields; they appear frequently in mechanical, electrical, communication, biological, and ecological systems. In general, dynamical systems consist of state variables and parameters. Variations in these parameters can give rise to bifurcation phenomena, which can induce significant changes in the behavior of the state variables. Consequently, deriving the bifurcation points of a system, i.e., the specific parameter values at which bifurcations occur, within a broad parameter space is essential for the understanding of the qualitative behavior of dynamical systems.

Several methods for deriving the bifurcation points of a system have been proposed [1–5], and the selection of the appropriate method depends on the type of bifurcation phenomena under investigation; such methods typically rely on underlying mathematical models based on simplifications and assumptions. Constructing such models for application related to real-world problems is often complex and time consuming, making the derivation of bifurcation points a challenging task. Here, we focus on a novel approach: the derivation of the bifurcation points of a system based on data-driven mathematical models constructed via machine learning techniques, including neural networks (NNs). As such, the use of data-driven methods represents a promising approach for obtaining the bifurcation points of a system, especially given

that existing methods have numerous limitations [1–5]. This is especially true in environments where time-series datasets are available but standard mathematical modeling proves challenging.

NNs are typically trained on time-series datasets for a known parameter and then used to predict the future states of the system for the same parameter value. NNs demonstrate efficacy in both short- and long-term time-series forecasting [6–8]. There have also been numerous studies focused on predicting the behavior of a system with unknown parameters using NNs trained on time-series datasets with several known parameters [9–24]. The methods developed in these studies can be broadly categorized into two classes: Principal component analysis (PCA) [25]-based methods [9–15] and non-PCA-based methods [16–24].

One significant early contribution to PCA-based methods was made by Tokunaga, who proposed a method using a feedforward NN (FFNN) [26] trained on time-series datasets for several parameter values [9]. A characteristic of the FFNN method is the use of PCA on the connection weights and biases to replicate the behavior of a dynamical system with unknown parameter values. In 2017, extreme learning machine (ELM) [27] methods were developed; these methods were considered to be an improvement on FFNNs [14], which they have largely superseded. ELM is associated with a fast learning speed and good performance in terms of generalization. Hereafter, we refer to the ELM method used in Ref. [14] as “ELM-PCA”. The ELM-PCA method simplified and accelerated the algorithm [9] using common connection weights and biases across the hidden layers of all ELMs and applying the PCA only to the connection weights of the output layers. Applications of the ELM-PCA method to the Hénon map [28] and the Rössler equations [29] have demonstrated accuracy that is comparable to that of the method developed by Tokunaga [9] while reducing the training time by a factor of 1000.

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Moreover, the method has shown high accuracy in the prediction of the Lyapunov exponents [30], further demonstrating its effectiveness when applied to nonlinear time-series analysis.

The majority of non-PCA-based methods are based on the NN with input parameter channels (channels for inputting known parameters) [16–19]. The input parameter channels are applied to the input of the NNs and enable NNs to train known parameters. These non-PCA-based methods [16–19] introduce the input channels into each NN, enabling it to replicate the behavior of the system with unknown parameter values. The approach used in Ref. [16] employs a feed-forward radial basis function network [31] and has been extended to recurrent NNs, such as the echo state network [32] and next generation reservoir computing [33], which are known to be effective for time-series predictions, in Refs. [17–19]. These non-PCA-based methods predict the behavior of a system with unknown parameters without relying on PCA using the input parameter channel. These non-PCA-based methods have been applied to a broad range of dynamical systems, including three-dimensional autonomous systems, such as the Rössler equations, the chaotic food chain model [34], and bistable systems [35], as well as four-dimensional autonomous systems, such as the power system model [36] and multistable systems [37]. Notably, the method presented in Ref. [16] was successfully applied to time-series datasets in circuit experiments on the Colpitts oscillator [38], where it demonstrated high accuracy in reproducing the behavior of the system. It is notable that the presence of noise in the experimental data did not significantly affect the accuracy of the results, highlighting the robustness of the method. Furthermore, training on time-series datasets that include transient states further improved reproducibility [16]. In Ref. [17], the same method was found to work well in problems related to the information entropy and reproducibility of basins in systems with unknown parameters, further demonstrating the effectiveness of the method in problems related to nonlinear time-series analysis.

For instance, D. E. Ozan [19] proposed a method for a parameter sensitivity analysis based solely on the time-series data obtained from dynamical systems; this method utilizes echo state networks with input parameter channels. Specifically, this data-driven parameter sensitivity analysis method involves the use of the first derivatives of the output of the reservoir layer with respect to the initial conditions and parameters in echo state networks trained on time-series data, and it then utilized these derivatives in the adjoint method. Furthermore, it was demonstrated that this method can efficiently and accurately identify the optimal parameter values that lead to a minimization of the acoustic energy in thermoacoustic time-delay systems. We note that the aforementioned research reports on PCA-based and non-PCA-based methods focus on verifying the feasibility of predicting the behavior of dynamical systems at unknown parameters [9–18,20–24] and their parameter sensitivity analysis [19].

On the other hand, the present work not only estimates attractors but also combines bifurcation theory and neural networks trained on time-series datasets, resulting in directly predicting the parameters at which bifurcations occur (i.e., bifurcation points). In this work, our objective is to directly predict bifurcation points; this is clearly distinct from the research goals of previous studies. Moreover, apart from the

work presented in Ref. [39], no methods for identifying the bifurcation points of a system using NNs have been proposed. The method proposed in Ref. [39] for predicting the bifurcation points of a system is based on the ELM-PCA method [14]. This method has been applied to the Hénon map, and it was demonstrated that the ELM-PCA method trained on time-series datasets with several parameters can effectively derive the bifurcation points for the period-doubling and saddle-node bifurcations in unknown parameter spaces. However, the derived parameter spaces obtained via the ELM-PCA method were found to be out of alignment with the original parameter spaces of the time-series datasets used in training the ELM-PCA model. This is because the principal component spaces are employed as substitutes for the original parameter spaces, and these two spaces are scaled differently. Moreover, compressing the high-dimensional connection weights of the ELM-PCA model into low-dimensional principal components via PCA results in the loss of information. This often leads to distortions in the derived sets of bifurcation points (bifurcation curves).

This paper proposes a data-driven method for deriving the bifurcation points of a system based on ELM with input parameter channels (ELM-IPCs) that avoids dimensionality reduction, ensuring that the original parameter spaces directly align with the derived parameter spaces. First, we implement the input parameter channels in the standard ELM model [27] to construct the ELM-IPC model. The ELM-IPC model incorporates and permits training on known parameters. Second, we then develop a method for deriving the bifurcation points of a system based on ELM-IPC. Third, we demonstrate the effectiveness of the proposed method by applying it to the Hénon map and a model comprising two conservatively coupled Hénon maps [40], the latter of which exhibits multi-periodic solutions exceeding 100 periods. Finally, we discuss the strengths of the proposed method and compare the model developed here with the method presented in Ref. [39]. This discussion presents the primary advantage of the proposed method: the capacity of the model to derive the bifurcation points of the system while maintaining alignment between the original parameter spaces of the dynamical system and the derived parameter spaces of the ELM-IPC model.

II. THE ELM-IPC MODEL

Figure 1 shows a schematic of the ELM-IPC model. The ELM-IPC model is an FFNN composed of three layers: an input layer, a hidden layer, and an output layer. The input layer contains the channels for training on the known parameters. The output of the ELM-IPC model is expressed by the following discrete dynamical system:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{f}(\mathbf{x}(k), \boldsymbol{\lambda}), \\ &= \boldsymbol{\beta} \cdot \mathbf{h}(\mathbf{x}(k), \boldsymbol{\lambda}), \end{aligned} \quad (1)$$

where $\mathbf{x}(k) = [x_1(k), \dots, x_N(k)]^\top \in \mathbb{R}^N$ represents the state variables at a discrete time k , and $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_P]^\top \in \mathbb{R}^P$ represents the known parameters of the dynamical system. The matrix $\boldsymbol{\beta} \in \mathbb{R}^{N \times M}$ is made up of the connection weights of the output layer, and the vector function $\mathbf{h}(\cdot, \cdot) = [h_1, \dots, h_M]^\top \in \mathbb{R}^M$ is the output of the hidden layer. $N \in \mathbb{N}$

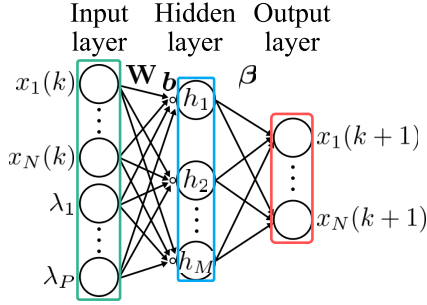


FIG. 1. A schematic of the ELM-IPC model. $\mathbf{x}(k) = [x_1(k), \dots, x_N(k)]^\top$ is the state variables at a discrete time k . $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_P]^\top$ is the known parameters of the dynamical systems and correspond with the input parameter channels. The vector function $\mathbf{h}(\cdot, \cdot) = [h_1, \dots, h_M]^\top$ represents the output of the hidden layer. $\mathbf{W}$ is the connection weights matrix of the hidden layer, and $\mathbf{b}$ is the biases vector. $\boldsymbol{\beta}$ is the connection weights matrix of the output layer.

is the dimension of the state variables, $P \in \mathbb{N}$ represents the dimension of the known parameters, and $M \in \mathbb{N}$ is the number of hidden neurons. The vector function $\mathbf{h}(\cdot, \cdot)$ is described as follows:

$$\mathbf{h}(\mathbf{x}(k), \boldsymbol{\lambda}) = \frac{\zeta}{1 + e^{-\rho \cdot \mathbf{z}}} - \sigma, \quad (2)$$

$$\mathbf{z} = \mathbf{W} \cdot \begin{bmatrix} \mathbf{x}(k) \\ \boldsymbol{\lambda} \end{bmatrix} + \mathbf{b}. \quad (3)$$

Equation (2) represents the sigmoid function, which operates element-wise on the vector $\mathbf{z} = [z_1, \dots, z_M]^\top \in \mathbb{R}^M$. The ELM-IPC model receives $\mathbf{x}(k)$ and $\boldsymbol{\lambda}$, such that $[\mathbf{x}(k)^\top, \boldsymbol{\lambda}^\top]^\top = [x_1(k), \dots, x_N(k), \lambda_1, \dots, \lambda_P]^\top \in \mathbb{R}^{N+P}$, as described in Eq. (3), and can produce outputs for any given $\boldsymbol{\lambda}$. The parameters of the sigmoid function $\zeta, \sigma, \rho \in \mathbb{R}$ are determined by the minimum and maximum values of the state variables $\mathbf{x}(k)$ in the target data. The matrix $\mathbf{W} \in \mathbb{R}^{M \times (N+P)}$ and the vector $\mathbf{b} \in \mathbb{R}^M$ represent the connection weights and biases of the hidden layer, respectively. Both are initialized utilizing uniform random values in the range $[-1, 1]$.

During the training phase of the ELM-IPC model, the weights $\mathbf{W}$ and biases $\mathbf{b}$ are set to their initial values, and only the output layer weights $\boldsymbol{\beta}$ are optimized based on the following equation:

$$\begin{aligned} \boldsymbol{\beta} &= \mathbf{X}(k+1) \cdot \mathbf{H}^\dagger, \\ &= \mathbf{X}(k+1) \cdot (\mathbf{H}^\top \cdot \mathbf{H})^{-1} \cdot \mathbf{H}^\top, \end{aligned} \quad (4)$$

where $\mathbf{X}(k+1) = [\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(k+1), \dots, \mathbf{x}(L)] \in \mathbb{R}^{N \times L}$ represents the target data of length $L \in \mathbb{N}$. $\mathbf{H} \in \mathbb{R}^{M \times L}$ denotes the output matrix of the hidden layer based on the training data. $\mathbf{H}^\dagger$ represents the Moore-Penrose pseudoinverse of the matrix $\mathbf{H}$. The matrix $\mathbf{H}$ is expressed as

$$\mathbf{H} = \frac{\zeta}{1 + e^{-\rho \cdot \mathbf{Z}}} - \sigma, \quad (5)$$

where

$$\mathbf{Z} = \mathbf{W} \cdot \begin{bmatrix} \mathbf{X}(k) \\ \boldsymbol{\Lambda} \end{bmatrix} + \mathbf{B}. \quad (6)$$

Equation (5) represents the sigmoid function, similar to that in Eq. (2), acting element-wise on each component of the matrix $\mathbf{Z} \in \mathbb{R}^{M \times L}$. In Eq. (6), the term $[\mathbf{X}(k)^\top, \boldsymbol{\Lambda}^\top]^\top \in \mathbb{R}^{(N+P) \times L}$ represents the training data of length L , consisting of the state variable matrix at time k , $\mathbf{X}(k) = [\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(k), \dots, \mathbf{x}(L-1)] \in \mathbb{R}^{N \times L}$, and $\boldsymbol{\Lambda}$ is the known parameter matrix given by $\boldsymbol{\Lambda} = [\boldsymbol{\lambda}, \dots, \boldsymbol{\lambda}] \in \mathbb{R}^{P \times L}$. The matrix $\mathbf{B} = [\mathbf{b}, \dots, \mathbf{b}] \in \mathbb{R}^{M \times L}$ is the bias matrix. We note that Eqs. (4), (5), and (6) describe the learning rule used to predict the state variable $\mathbf{x}(k+1)$ based on the value of the state variable $\mathbf{x}(k)$ and the parameter $\boldsymbol{\lambda}$. By substituting the optimized $\boldsymbol{\beta}$ obtained from Eqs. (4), (5), and (6) into Eq. (1), the ELM-IPC model can be accurately represented by an N -dimensional discrete dynamical system.

III. A METHOD FOR DERIVING THE BIFURCATION POINTS OF DISCRETE DYNAMICAL SYSTEMS BASED ON ELM-IPC

The ELM-IPC model represents the N -dimensional discrete dynamical system described in Eq. (1). The solution satisfying the initial condition $\mathbf{x}(k)$ is given by

$$\mathbf{x}(k+1) = \boldsymbol{\varphi}(1, \mathbf{x}(k), \boldsymbol{\lambda}). \quad (7)$$

We note that solutions that return to the initial value $\mathbf{x}(k)$ after m iterations of the map are referred to as m -periodic points. An m -periodic point satisfies the expression

$$\boldsymbol{\varphi}(m, \mathbf{x}(k), \boldsymbol{\lambda}) - \mathbf{x}(k) = \mathbf{0}. \quad (8)$$

The characteristic equation at a given m -periodic point is defined as follows:

$$\chi(\mu) = \det\left(\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}(k)}(m, \mathbf{x}(k), \boldsymbol{\lambda}) - \mu \mathbf{I}_N\right) = 0, \quad (9)$$

where $\mathbf{I}_N \in \mathbb{R}^{N \times N}$ is the identity matrix, and $\mu \in \mathbb{R}$ is the characteristic multiplier. Local bifurcation points at the m -periodic points can be computed by solving Eqs. (8) and (9) simultaneously; this can be done via numerical methods, such as Newton's method. These bifurcation conditions can be encapsulated by the following expression:

$$\mathbf{F}(\mathbf{x}(k), \boldsymbol{\lambda}) := \begin{bmatrix} \boldsymbol{\varphi}(m, \mathbf{x}(k), \boldsymbol{\lambda}) - \mathbf{x}(k) \\ \chi(\mu) \end{bmatrix} = \mathbf{0}. \quad (10)$$

When solving Eq. (10) using Newton's method, the Jacobian matrix $D\mathbf{F}(\mathbf{x}(k), \lambda_i)$ is required. $D\mathbf{F}(\mathbf{x}(k), \lambda_i)$ is given by

$$D\mathbf{F}(\mathbf{x}(k), \lambda_i) = \begin{pmatrix} \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}(k)} - \mathbf{I}_N & \frac{\partial \boldsymbol{\varphi}}{\partial \lambda_i} \\ \frac{\partial \chi}{\partial \mathbf{x}(k)} & \frac{\partial \chi}{\partial \lambda_i} \end{pmatrix}, \quad (11)$$

where λ_i represents the bifurcation parameter and is the i -th component of $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_P]^\top \in \mathbb{R}^P$. Equation (11) includes the derivatives of the m times composite map, $\boldsymbol{\varphi}(m, \mathbf{x}(k), \boldsymbol{\lambda})$, with respect to the initial conditions and the bifurcation parameter. These derivatives can be obtained recursively from other derivatives of the mapping with respect to the initial conditions and the bifurcation parameter, and this recursive process is utilized m times. Here, the mapping refers to the ELM-IPC output. A method for composing these derivatives has been established in Ref. [41]. The derivatives

with respect to the bifurcation parameter of the ELM-IPC output are derived in detail in the Appendix of this work. Additionally, because the derivatives with respect to the initial conditions of the ELM-IPC output are expressed in the same form as those of the ELM-PCA, we refer the reader to Ref. [14] for further information on those derivatives. We note that the derivatives of the ELM-IPC output can also be computed using automatic differentiation, as implemented in numerous programming environments, including Python (e.g., TensorFlow, JAX, and PyTorch), Julia (e.g., Flux, Lux, Enzyme, and Zygote), and other languages that support automatic differentiation.

The above defines a method for deriving bifurcation points based on the ELM-IPC model. By incorporating known parameters into the input of the ELM-IPC model, the proposed method permits the derivation of the bifurcation points of a system while maintaining alignment between the original parameter spaces of the dynamical system and the derived parameter spaces of the ELM-IPC model.

IV. NUMERICAL EXPERIMENTS

This section derives the bifurcation points of two discrete dynamical systems using the novel method proposed here. Here, the method is applied to the Hénon map [28] (a well-known two-dimensional discrete dynamical system) and a system composed of two conservatively coupled Hénon maps [40] (a four-dimensional discrete dynamical system that exhibits multi-periodic solutions with periods exceeding 100). Here, we demonstrate that this model can be used to obtain the period-doubling and saddle-node bifurcation curves observed in both dynamical systems.

A. Hénon map

The Hénon map can be expressed as follows:

$$\begin{aligned} x_1(k+1) &= 1 - \lambda_1 x_1(k)^2 + x_2(k), \\ x_2(k+1) &= \lambda_2 x_1(k), \end{aligned} \quad (12)$$

where $\mathbf{x}(k) = [x_1(k), x_2(k)]^\top \in \mathbb{R}^2$ represents the state variables, and $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^\top \in \mathbb{R}^2$ represents the parameters describing the dynamical system. In the following, we focus on the bifurcation phenomena in the parameter space $(\lambda_1, \lambda_2) \in [1.2, 1.6] \times [0.1, 0.2]$. Based on the results presented in Ref. [9], the parameters used to train the ELM-IPC model are taken to be

$$\lambda_1^{(v)} = 0.2 \sin\left(\frac{2\pi(v-1)}{8}\right) + 1.4, \quad (v = 1, \dots, 8), \quad (13)$$

$$\lambda_2^{(v)} = 0.05 \cos\left(\frac{2\pi(v-1)}{8}\right) + 0.15, \quad (v = 1, \dots, 8), \quad (14)$$

where $\boldsymbol{\lambda}^{(v)} = [\lambda_1^{(v)}, \lambda_2^{(v)}]^\top \in \mathbb{R}^2$ is the known training parameters for the v -th point; we note that this experiment uses the eight training parameters $\boldsymbol{\lambda}^{(v)}$, $(v = 1, \dots, 8)$. For each parameter $\boldsymbol{\lambda}^{(v)}$, we consider 1000 time-series data points of $\mathbf{x}(k)$ in the steady state. Thus, the training and target data for the ELM-IPC model are made up of 8000 time-series data points of the known parameters. The number of input, hidden, and output neurons of the ELM-IPC model was set to 4, 35,

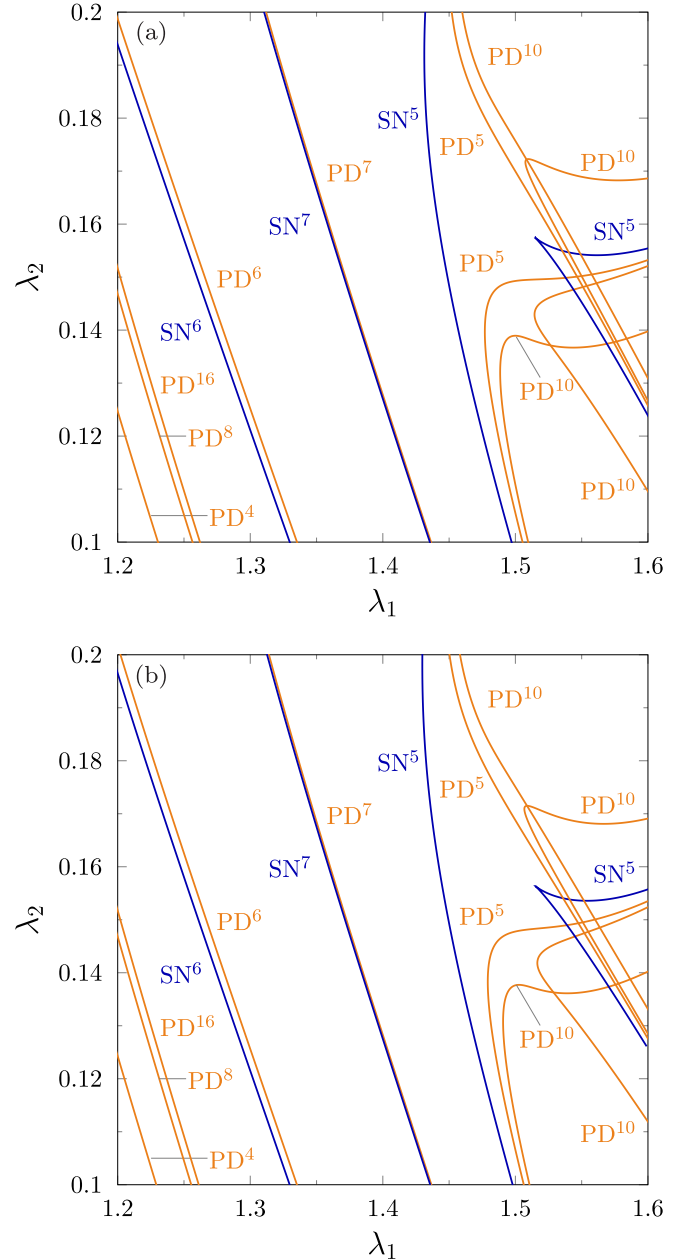


FIG. 2. Two-parameter bifurcation diagrams for the Hénon map obtained using the method proposed in Ref. [2] and that proposed in this work. It is noted that extremely close agreement between the obtained results is observed. (a) Results obtained considering the difference equation. (b) Results obtained utilizing the method proposed here.

and 2, respectively. The parameters of the sigmoid function were adjusted to $\zeta = 2.3$, $\sigma = 1.1$, and $\rho = 1.0$ based on the maximum and minimum values of $x_1(k)$ and $x_2(k)$ in the target dataset. Figure 2 shows the two-parameter bifurcation diagrams of this system. Panel (a) illustrates the bifurcation curves obtained from the difference equation in Eq. (12), while panel (b) represents the bifurcation curves derived using the proposed method. The orange lines are period-doubling bifurcation curves (labeled PD), and the blue lines are saddle-node bifurcation curves (labeled SN). The superscripts on

the labels PD and SN indicate the number of periods in the solutions. In the case of the period-doubling bifurcations, the characteristic multiplier μ in Eq. (9) is set to $\mu = -1$, while for the saddle-node bifurcations, $\mu = 1$. The bifurcation curves are obtained by solving the bifurcation condition given in Eq. (10) using Newton's method. We note that Fig. 2(a) was computed using the method for deriving bifurcation points proposed in Ref. [2]. The method proposed here derives the bifurcation points in the parameter space by training the ELM-IPC model using the time-series datasets related to only eight parameter points. One key advantage of this approach is that the ELM-IPC model utilizes known parameters, ensuring that the original parameter spaces of the dynamical system are a direct match with the parameter spaces of the ELM-IPC model. Additionally, it is observed that the bifurcation curves obtained via the method presented here exhibit no significant distortions; this is because, in contrast with PCA-based methods, the method proposed here does not use dimensional reduction. We note that the region on the right of Fig. 2 shows multiple bifurcation curves intersecting (i.e., a coexistence region) and that the proposed method accurately captures these bifurcation curves.

B. Two conservatively coupled Hénon maps

Two conservatively coupled Hénon maps can be described by the following equations:

$$\begin{aligned} x_1(k+1) &= \lambda_1 - x_1(k)^2 - cx_2(k) + \varepsilon(x_1(k) - x_3(k)), \\ x_2(k+1) &= x_1(k), \\ x_3(k+1) &= \lambda_2 - x_3(k)^2 - cx_4(k) + \varepsilon(x_3(k) - x_1(k)), \\ x_4(k+1) &= x_3(k), \end{aligned} \quad (15)$$

where $\mathbf{x}(k) = [x_1(k), x_2(k), x_3(k), x_4(k)]^\top \in \mathbb{R}^4$ represents the state variables, and $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, c, \varepsilon]^\top \in \mathbb{R}^4$ represents the parameters describing the dynamical system. Here, we consider the case of $c = 0.321$ and $\varepsilon = 0.4$ and focus on the parameter space $(\lambda_1, \lambda_2) \in [0.89, 0.95] \times [2.015, 2.035]$. The parameters used to train the ELM-IPC model are

$$\begin{aligned} \lambda_1^{(v)} &= 0.03 \sin\left(\frac{2\pi(v-1)}{8}\right) + 0.92, \quad (v = 1, \dots, 8), \\ \lambda_2^{(v)} &= 0.01 \cos\left(\frac{2\pi(v-1)}{8}\right) + 2.025, \quad (v = 1, \dots, 8), \end{aligned} \quad (16)$$

where $\boldsymbol{\lambda}^{(v)} = [\lambda_1^{(v)}, \lambda_2^{(v)}]^\top \in \mathbb{R}^2$ represents the v -th point in the set of known training parameters. The training and target data used for the ELM-IPC model consisted of 1000 time-series data points of $\mathbf{x}(k)$ in the steady state for each training parameter. Thus, the training and target data consisted of 8000 time-series data points and their corresponding known parameters. The number of input, hidden, and output neurons was set to 4, 150, and 2, respectively. The parameters of the sigmoid function were set to $\zeta = 1.2$, $\sigma = 0.2$, and $\rho = 1.0$ based on the maximum and minimum values of $x_1(k)$, $x_2(k)$, $x_3(k)$, and $x_4(k)$ in the target data.

Figure 3 shows the two-parameter bifurcation diagrams describing this system. Panel (a) shows the bifurcation curves

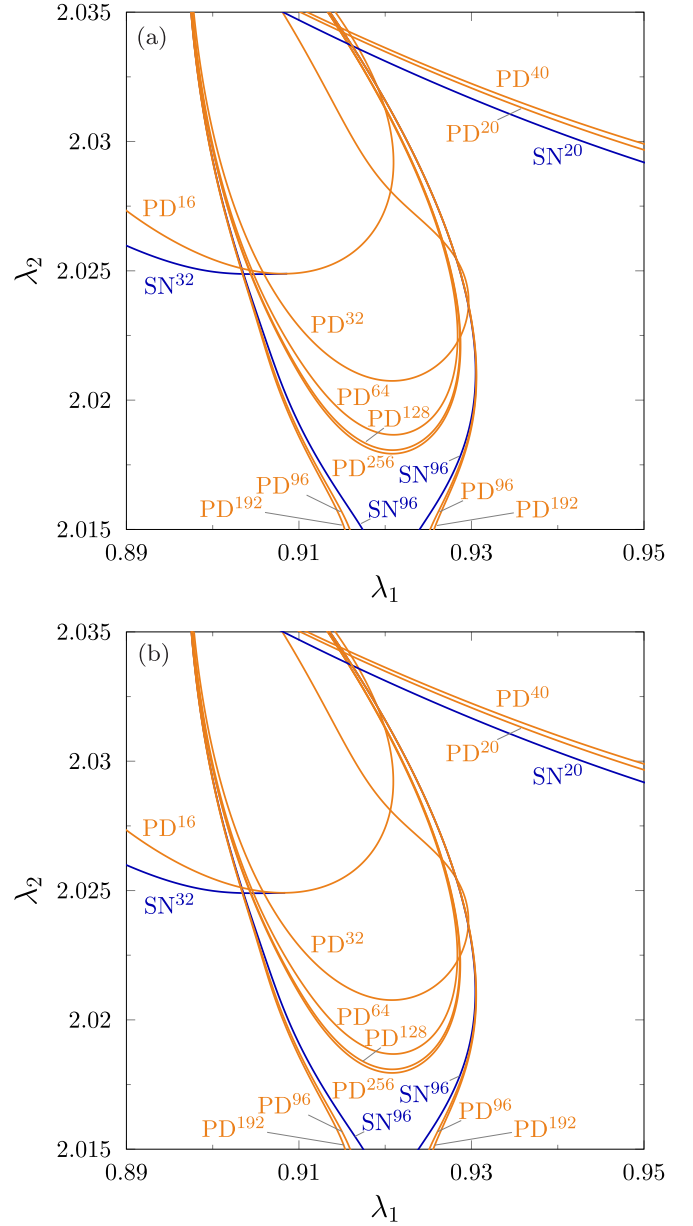


FIG. 3. Two-parameter bifurcation diagrams for the two conservatively coupled Hénon maps obtained using the method proposed in Ref. [2] and proposed in this work. It is noted that extremely close agreement between the two sets of results is observed. (a) Results obtained via the difference equation. (b) Results obtained using the method proposed here.

obtained from the difference equation given by Eq. (15), and panel (b) represents the bifurcation curves obtained using the method proposed here. In the figure, the orange lines represent the period-doubling bifurcation curves, and the blue lines represent the saddle-node bifurcation curves. It can be seen that the proposed method accurately identifies the bifurcation points in the parameter space. The method is seen to be effective even when the parameter space exhibits bifurcation phenomena involving solutions with periods of up to 256. Remarkably, this is achieved despite the model being trained on time-series datasets of only eight parameter points. As

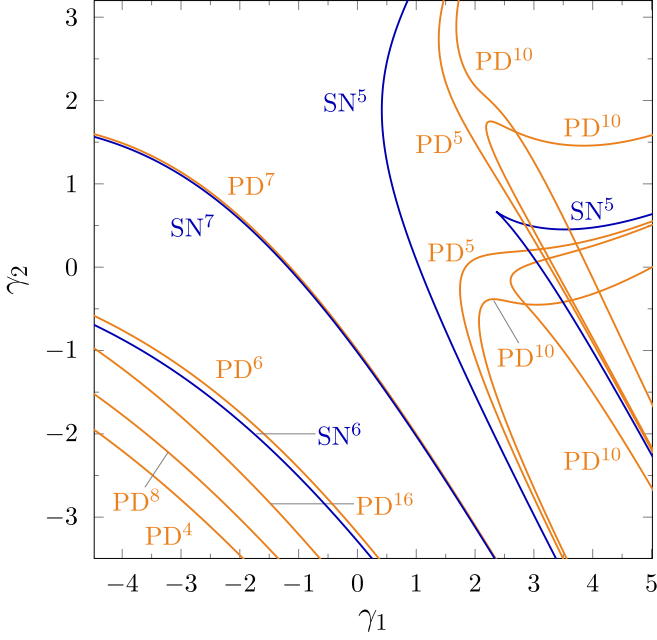


FIG. 4. Two-parameter bifurcation diagram for the Hénon map; the results presented here were derived utilizing the ELM-PCA method presented in Ref. [39]. Here, γ_1 and γ_2 represent the first and second principal components in the PCA.

was the case in the investigation presented in the section, the original parameter spaces of the dynamical system are aligned with the derived parameter spaces of the ELM-IPC model, and no significant distortion is observed in the derived bifurcation curves. In Fig. 3, coexistence regions of multi-periodic solutions can be seen to be scattered throughout the parameter space; despite this complexity, the proposed method is seen to accurately predict the bifurcation curves.

V. DISCUSSION

The principal advantage of the method proposed here is its capacity to derive the bifurcation points of a system while maintaining alignment between the original parameter spaces of the dynamical system and the derived parameter spaces of the ELM-IPC model. This strength is demonstrated by comparing the results of this novel method with those of the only previous study on obtaining the bifurcation points of a system utilizing a data-driven model [39]. Hereafter, the output of the ELM-PCA model (presented in Ref. [39]) is written as $\mathbf{x}(k+1) = \boldsymbol{\phi}(\mathbf{x}(k), \boldsymbol{\gamma})$, where $\boldsymbol{\gamma} = [\gamma_1, \gamma_2]^T$ represents the principal components belonging to the derived parameter spaces of the ELM-PCA model.

To highlight the advantages of the method presented here, the comparative limitations of the previous method are discussed. The bifurcation diagram for the Hénon map (investigated in Sec. IV A of this work) derived using the previous method [39] is shown in Fig. 4. We note that significant distortions can be seen in these results toward the left of the figure compared with the results obtained considering the Hénon map difference equations [see Fig. 2(a)]. Figure 5 shows the trajectory of the known training param-

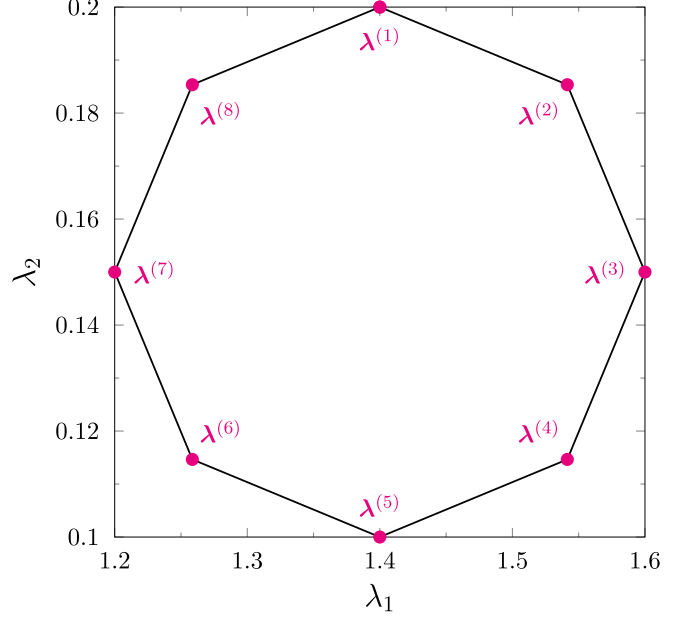


FIG. 5. The bifurcation pass of the Hénon map. The red points are the known training parameters $\lambda^{(v)}$, ($v = 1, \dots, 8$) given in Eqs. (13) and (14).

eters $\lambda^{(v)}$, ($v = 1, \dots, 8$) given in Eqs. (13) and (14). The trajectory is referred to as the bifurcation pass. Figure 6 shows the bifurcation locus, namely the trajectory of the principal components $\boldsymbol{\gamma}^{(v)}$, ($v = 1, \dots, 8$) corresponding to the known training parameters $\lambda^{(v)}$, ($v = 1, \dots, 8$). We note that the more similar the red points of the bifurcation locus and those of the bifurcation pass are, the higher the accuracy of the

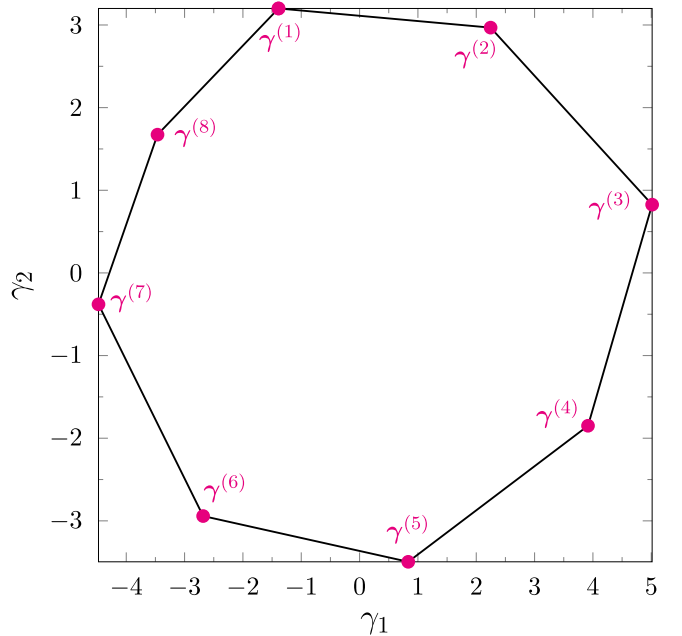


FIG. 6. The bifurcation locus of the Hénon map. The red points are the principal components $\boldsymbol{\gamma}^{(v)}$, ($v = 1, \dots, 8$), which indirectly correspond to the known training parameters $\lambda^{(v)}$, ($v = 1, \dots, 8$).

bifurcation points derived by the associated method. This is because when dimensionality reduction based on PCA can appropriately extract the principal components, results in a closer match between the red points of the bifurcation pass and those of the bifurcation locus. However, the red points in Fig. 6 show substantial deviations from the known training parameters $\lambda^{(v)}$, causing the significant distortions observed in Fig. 4. Moreover, the axes of the bifurcation diagram do not align with those of the diagram derived from the Hénon map. These limitations arise because the previous method uses principal components, and the principal component spaces, namely the derived parameter spaces of the ELM-PCA model, do not directly correspond to the original parameter spaces.

Next, the features and strengths of the proposed method are highlighted. The bifurcation diagram derived using the proposed method [shown in Fig. 2(b)] more accurately reproduces the diagram derived from the Hénon map difference equations [Fig. 2(a)]. The derived parameter spaces of the ELM-IPC model are directly aligned with the original parameter spaces of the dynamical system, thus addressing one of the primary issues of the previous method. By incorporating the input parameter channels, the proposed method enables the ELM-IPC model to learn known parameters, making a direct bifurcation analysis possible.

Finally, the differences between the two methods are highlighted in terms of the derivatives used in the ELM-PCA and ELM-IPC methods. Since the ELM-PCA method cannot learn known parameters, it requires the replacement of the bifurcation parameter λ_i , which is the i -th component of λ , with the principal components γ_i , which is the i -th component of γ . Thus, the derived derivatives $\partial\phi/\partial\gamma_i$ and $\partial\chi/\partial\gamma_i$ of the Jacobian matrix used in the application of Newton's method in the ELM-PCA method do not correspond to the original derivatives $\partial\phi/\partial\lambda_i$ and $\partial\chi/\partial\lambda_i$ of the dynamical system being considered. As a result, the derived Jacobian matrix used in the ELM-PCA method no longer aligns with the original Jacobian matrix of the dynamical system, which introduces distortions in the obtained bifurcation diagram. This misalignment makes a direct bifurcation analysis more challenging. By contrast, the method proposed here incorporates the input parameter channels of the system, allowing the ELM-IPC model to directly learn the known parameters λ . This ensures that the derived derivatives $\partial\phi/\partial\lambda_i$ and $\partial\chi/\partial\lambda_i$ based on the ELM-IPC model [Eqs. (A1) and (A2)] directly correspond to the respective derivatives of the dynamical system with respect to λ_i . Consequently, the second column of the derived Jacobian matrix in the ELM-IPC model [Eq. (11)] aligns perfectly with the original Jacobian matrix, permitting a more accurate bifurcation analysis.

In summary, the method proposed here incorporates the input parameter channels into the input of the ELM-IPC model, allowing it to learn known parameters. This makes it possible to compute the derivatives of the output of the model presented here with respect to the bifurcation parameter, permitting the prediction of the bifurcation points of a given system while maintaining alignment of the derived parameter spaces of the ELM-IPC model and the original parameter spaces of the dynamical system being investigated.

VI. CONCLUSION

This paper proposes a novel method for obtaining the bifurcation points of a system based on the ELM-IPC model trained on time-series datasets and the known parameters of the discrete dynamical system. First, we constructed an ELM-IPC model which incorporates the channels for the known parameters as part of its input. Next, we developed a method for deriving the bifurcation points to utilize within the ELM-IPC model. Specifically, we derived the first and second variational equations of the model through symbolic differentiation, which enabled the numerical calculation of the fixed-point and bifurcation conditions. To validate the effectiveness of the proposed method, we applied it to the Hénon map and a system comprising two conservatively coupled Hénon maps, the latter of which exhibits multi-periodic solutions with periods exceeding 100. The obtained results demonstrated that the proposed method accurately obtains the bifurcation points of the systems while maintaining a correspondence between the original parameter spaces of the dynamical system and the derived parameter spaces of the model. Furthermore, as the method does not employ dimensional reduction techniques, no significant distortion was observed in the derived bifurcation curves.

This work opens the possibility for methods based on the ELM-IPC model to be developed that extend the approach presented here to the prediction of the bifurcation points in continuous dynamical systems. As a future challenge, we plan to explore the application of the method proposed here to time-series datasets obtained from chaotic and high-dimensional systems, such as real circuits, nonlinear time-delay oscillators, etc. In addition, we believe that predicting the bifurcation points of a system considering more complex time-series datasets may require the derivation of a method based on recurrent neural networks, which are renowned for their superior predictive performance in attractor estimation and time-series forecasting compared with the ELM. Representative examples of such architectures include gated recurrent neural networks and reservoir computing models.

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DATA AVAILABILITY

Data that are not openly available are available upon reasonable request from the authors.

APPENDIX

In this Appendix, we present expressions for the derivatives that are central to the method presented here; we note that obtaining these derivatives represent a key feature of this work. Specifically, we derive the first derivative with respect to the bifurcation parameter and the second derivative with respect to the initial conditions and the bifurcation parameter of the ELM-IPC output described in Eq. (1). The first derivative of the ELM-IPC output, $\phi(1, x(k+m-r), \lambda)$, with respect to

the bifurcation parameter λ_l can be expressed as follows:

$$\begin{aligned} & \frac{\partial \varphi_i}{\partial \lambda_l}(1, \mathbf{x}(k+m-r), \boldsymbol{\lambda}) \\ &= \frac{\rho}{\zeta} \sum_{j=1}^M W_{j,(N+l)} \beta_{i,j} \{\zeta - (h_j + \sigma)\} (h_j + \sigma), \quad (\text{A1}) \end{aligned}$$

where the subscripts i and j represent the element indices of the vectors and matrices, with $i \in [1, N] \subset \mathbb{N}$ and $j \in [1, M] \subset \mathbb{N}$. The variable r represents the recursion index, with $r \in [1, m] \subset \mathbb{N}$. The second derivative with respect to the initial conditions $\mathbf{x}(k+m-r)$ and the bifurcation

parameter λ_l can be expressed as follows:

$$\begin{aligned} & \frac{\partial^2 \varphi_i}{\partial x_l(k+m-r) \partial \lambda_l}(1, \mathbf{x}(k+m-r), \boldsymbol{\lambda}) \\ &= \frac{\rho^2}{\zeta^2} \sum_{j=1}^M W_{j,l} W_{j,(N+l)} \beta_{i,j} \\ & \cdot \{\zeta - 2(h_j + \sigma)\} \{\zeta - (h_j + \sigma)\} (h_j + \sigma), \quad (\text{A2}) \end{aligned}$$

where the subscript $l \in [1, N] \subset \mathbb{N}$ represents the element index of the vectors and matrices.

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- [1] E. J. Doedel, *Congr. Numer.* **30**, 265 (1981).
 - [2] H. Kawakami, *IEEE Trans. Circuits Syst.* **31**, 248 (1984).
 - [3] T. Kousaka, T. Ueta, and H. Kawakami, *IEEE Trans. Circuits Syst. II* **46**, 878 (1999).
 - [4] T. Kousaka, T. Ueta, Y. Ma, and H. Kawakami, *IJCTA* **33**, 263 (2005).
 - [5] P. Thota and H. Dankowicz, *SIAM J. Appl. Dyn. Syst.* **7**, 1283 (2008).
 - [6] Z. Wang, Z. Li, H. Pan, G. Liu, Y. Chen, Q. Wu, G. Tyson, and G. Cheng, *IEEE Trans. Parallel Distrib. Syst.* **34**, 1390 (2023).
 - [7] S. Guo, C. Guan, and S. Leng, *Adv. Intell. Syst.* **6**, 2400163 (2024).
 - [8] J. Bi, X. Zhang, H. Yuan, J. Zhang, and M. Zhou, *IEEE Trans. Automat. Sci. Eng.* **19**, 1869 (2022).
 - [9] R. Tokunaga, S. Kajiwara, and T. Matsumoto, *Physica D* **79**, 348 (1994).
 - [10] I. Tokuda, S. Kajiwara, R. Tokunaga, and T. Matsumoto, *Physica D* **95**, 380 (1996).
 - [11] E. Bagarinao Jr., T. Nomura, K. Pakdaman, and S. Sato, *Physica D* **124**, 258 (1998).
 - [12] E. Bagarinao, K. Pakdaman, T. Nomura, and S. Sato, *Phys. Rev. E* **60**, 1073 (1999).
 - [13] E. Bagarinao, K. Pakdaman, T. Nomura, and S. Sato, *Methods Inf. Med.* **39**, 146 (2000).
 - [14] Y. Itoh, Y. Tada, and M. Adachi, *Nonlin. Theor. Appl., IEICE* **8**, 2 (2017).
 - [15] Y. Itoh and M. Adachi, *Nonlin. Theor. Appl., IEICE* **12**, 391 (2021).
 - [16] G. Langer and U. Parlitz, *Phys. Rev. E* **70**, 056217 (2004).
 - [17] M. Roy, S. Mandal, C. Hens, A. Prasad, N. V. Kuznetsov, and M. D. Shrimali, *Chaos* **32**, 101104 (2022).
 - [18] D. Köglmayr and C. R  th, *Sci. Rep.* **14**, 507 (2024).
 - [19] D. E. Ozan and L. Magri, *Phys. Rev. Fluids* **9**, 103902 (2024).
 - [20] E. Bagarinao Jr., K. Pakdaman, T. Nomura, and S. Sato, *Physica D* **130**, 211 (1999).
 - [21] Y. Itoh and M. Adachi, *Phys. Rev. E* **98**, 013301 (2018).
 - [22] Y. Itoh, S. Uenohara, M. Adachi, T. Morie, and K. Aihara, *Chaos* **30**, 013128 (2020).
 - [23] Y. Itoh, *Nonlin. Theor. Appl., IEICE* **14**, 356 (2023).
 - [24] Y. Itoh, *Nonlin. Theor. Appl., IEICE* **14**, 579 (2023).
 - [25] K. Pearson, *Lond. Edinb. Dubl. Phil. Mag. J. Sci.* **2**, 559 (1901).
 - [26] W. S. McCulloch and W. Pitts, *Bull. Math. Biophys.* **5**, 115 (1943).
 - [27] G. B. Huang, Q. Y. Zhu, and C. K. Siew, *Neurocomputing* **70**, 489 (2006).
 - [28] M. H  non, *Commun. Math. Phys.* **50**, 69 (1976).
 - [29] O. E. R  ssler, *Phys. Lett. A* **57**, 397 (1976).
 - [30] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, *Physica D* **16**, 285 (1985).
 - [31] J. Moody and C. J. Darken, *Neural Comput.* **1**, 281 (1989).
 - [32] H. Jaeger, *GMD Report* **148**, 13 (2001).
 - [33] D. J. Gauthier, E. Bollt, A. Griffith, and W. A. Barbosa, *Nat. Commun.* **12**, 5564 (2021).
 - [34] K. McCann and P. Yodzis, *Am. Nat.* **144**, 873 (1994).
 - [35] X. Wang and G. Chen, *Commun. Nonlinear Sci. Numer. Simul.* **17**, 1264 (2012).
 - [36] I. Dobson and H. D. Chiang, *Syst. Control Lett.* **13**, 253 (1989).
 - [37] P. R. Sharma, M. D. Shrimali, A. Prasad, N. V. Kuznetsov, and G. A. Leonov, *Int. J. Bifurcation Chaos* **25**, 1550061 (2015).
 - [38] M. P. Kennedy, *IEEE Trans. Circuits Syst. I* **41**, 771 (1994).
 - [39] Y. Itoh and M. Adachi, *Nonlin. Theor. Appl., IEICE* **10**, 268 (2019).
 - [40] D. V. Savin, A. P. Kuznetsov, A. V. Savin, and U. Feudel, *Phys. Rev. E* **91**, 062905 (2015).
 - [41] Y. A. Kuznetsov and H. G. E. Meijer, *Numerical Bifurcation Analysis of Maps: From Theory to Software* (Cambridge University Press, Cambridge, 2019).