

Resetting of underdamped Brownian motion

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We study underdamped Brownian motion under Poissonian resetting, where we apply two different full resetting procedures. The first one resets both position and velocity to zero, while the second one resets the displacement to zero but takes the velocity after resetting to follow a Maxwell distribution. In both cases, the marginal probability density function (PDF) of the displacement in the stationary state shows exponential tails for large displacements, but the singularities at the mode of the PDF are very different and specific for the corresponding resetting prescription: in the first case, one has $p(x) \sim |x|^{-1/3}$, while in the second one, one gets $p(x) \sim -\ln|x|$. The convergence of the PDF to a stationary one is investigated by examining its excess kurtosis as a function of time. Applying two sets of initial conditions for the second resetting procedure, we show that the behavior of the excess kurtosis for short times is highly sensitive to these.

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I. INTRODUCTION

The concept of resetting can be found in many aspects of everyday life, ranging from the search for lost items, where one usually restarts an unsuccessful search from the position the item was last seen, to restarting one's computer when some procedure gets stuck, to interpreting coming home in the evening as a reset in the process of day-to-day life.

In physics, stochastic resetting was first considered as an exactly solvable example for a nonequilibrium steady state [1]. In other disciplines, the interest in resetting arose while considering search processes. Some of these works preceded the discussion in physics for a long time. In computer science, for example, the resetting of stochastic algorithms can speed up the search [2,3] by hindering the algorithm from wandering far into solution-free regions. Birth-death processes may serve as another example, where the death of a population [4] or a vanishing queue [5] serves as a reset, after which a new population/customers may arrive again.

In biology, the flight of a butterfly from flower to flower can be described as a stochastic process under velocity resetting with refractory periods [6]. In chemistry, it was shown that substrate unbinding in Michaelis–Menten enzymatic reactions can benefit enzymatic turnover [7,8], where substrate unbinding can be interpreted as resetting. Other search processes involving resetting are, for example, animal foraging [9] or the search by enzymes along DNA strands [10]. Further, resetting can minimize the mean hitting time for search processes with

several targets [11]. For an overview of stochastic resetting in physics, see [12–14].

In the following, we will consider Brownian motion (BM) under resetting. Concerning overdamped BM, it is known that the stationary probability density function (PDF) of overdamped BM under Poissonian resetting is a Laplacian (two-sided exponential) one [1]. This result was experimentally reproduced [15]. Further, several generalizations have been made for overdamped BM, such as the introduction of a partially absorbing target [16], the extension to arbitrary dimensions with and without an absorbing target [17], or the introduction of an arbitrary potential [18]. Moreover, the variants of resetting problems with refractory periods [19], and with different deterministic behaviors during the return phase [20,21] were considered. Note also that several works discuss variants of non-Poissonian resetting, for example, the one with power-law-distributed waiting times [11,22] or with resetting at a constant pace [21,23], which is interesting because it may provide the optimal scheme for minimizing first passage times [11].

Far less is known about the behavior of the underdamped Brownian motion, even for the Poissonian case. Concerning underdamped BM, Ref. [24] considers the situation when the position of the particle is reset to the origin, while the velocity is left invariant. The main attention in this work is paid to the behavior of the stationary PDF in its tails, and to the convergence to stationarity in the body and in the tails of the distribution. In Ref. [6], velocity resetting is considered, while the position is left invariant under resets. Here, a recursive relation is derived for all moments, and the Fourier-Laplace transform of the marginal PDF of the position is provided.

Our present discussion builds on the one by Gupta in Ref. [24]. There, the initial condition for the velocity corresponds to starting from a value v_0 taken from an equilibrium distribution corresponding to the same temperature at which the further evolution of the system takes place. In what

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follows, we consider some additional details of underdamped BM under Poissonian resetting, concentrating on the role of the resetting prescription for the velocity and of initial conditions.

First, we go into more detail about the stationary distribution, and consider two situations:

(i) In the first procedure, which we denote by $(x, v) \mapsto (0, 0)$, both the coordinate and the velocity are reset to sharp values (taken as zero).

(ii) In the second one, denoted as $x \mapsto 0, v \mapsto \text{Maxwell}$, the velocity is reset to a random value chosen from the Maxwell distribution.

Since the resetting procedure in x alone erases the memory on the previous evolution of the coordinate, and x corresponds to the integration of the velocity process starting from the velocity taken from the equilibrium distribution, the stationary distribution of the coordinate in this second case is the same as in the case discussed in Ref. [24]. The situation considered here corresponds, however, to a slightly different experimental setup than the one implied by Gupta's analysis. In the case discussed in Ref. [24], one considers an uninterrupted trajectory of the particle's motion, and at resetting epochs takes the corresponding position of the particle to be a new origin of coordinates, and looks for its further displacement. Our situation corresponds to the full resetting procedure, where the particle, which is tracked between the resetting epochs, is transported to the initial position, let thermalize at this position, and then released. The transport/thermalization time is then censored from the trajectory record, as is done in the experiment in Ref. [15]. We are not aware of experiments implementing the $(x, v) \mapsto (0, 0)$ resetting scheme, but this scheme is still highly interesting from the theoretical point of view, showing that for the underdamped dynamics, starting from a nonequilibrium initial condition considerably affects the shape of the stationary distribution of the displacement in its central part. Moreover, this situation may be emulated in model macroscopic experiments with rarefied active baths. Having this in mind, we also consider the case when the particle starts at the beginning with zero velocity but is reset to the state with the equilibrium one when discussing the approach to the stationarity.

Before turning to our problem, we remark on similar results concerning the stationary PDF. We have already commented on Ref. [24], in which exponential tails of the stationary PDF are predicted when the position is reset to zero, and the velocity remains unchanged under the resetting. The work [25] gives a generalization of Ref. [24] by introducing a non-Markovian thermal bath.

In Ref. [26], a random acceleration process (RAP) is considered with two different resetting procedures. The first one is the same as our first resetting procedure, where both position and velocity are reset to zero. In this case, Ref. [26] predicts a singularity at the mode of type $p(x) \sim |x|^{-1/3}$ and for the tails $p(x) \sim \frac{1}{\sqrt{|x|}} e^{-\sqrt{a}|x|}$, where the prefactor a depends only on the mean time between resetting epochs t_r and the diffusion coefficient D_v of the velocity. The second resetting procedure considered is the same as that of Ref. [24]. For this second case, no stationary PDF exists for RAP. The Langevin

equations for underdamped BM and RAP differ only by the damping term ($-v/t_r$ in Eq. (4) below), which is absent for RAP. Thus, for small velocities, the processes are very similar. Hence, it will be worth comparing our results for the resetting procedure $(x, v) \mapsto (0, 0)$ to those of Ref. [26].

Finally, active BM in two dimensions under Poissonian resetting is discussed in Ref. [27]. A state is given by the coordinates (x, y) and the orientation φ , where $\varphi = 0$ coincides with being oriented along the x axis. Three reset procedures are considered where (x, y) or φ or both are reset to zero. For each, the stationary PDF is determined for two cases: one with very high and another with low resetting rates. For high resetting rates, the periods between resets are short, and one expects the particle to be bound near the origin, thus complying with the approximations for small $|x|$. Applying the same argument, the results for low resetting rates will be compared with approximations for the tails. The first resetting procedure in Ref. [27] resets (x, y) to zero and the orientation to be along the x axis. Thus, for the coordinate y , this corresponds to our first resetting procedure $(x, v) \mapsto (0, 0)$. For high resetting rates, Ref. [27] gives $p(y) \sim |y|^{-1/3}$ and, for low rates, an exponential distribution. The second resetting procedure, which is considered, resets (x, v) to zero but leaves the orientation unchanged. The projection onto one of the coordinates would coincide with the resetting procedure $(x, v) \mapsto (0, v)$ in Ref. [24], with the difference that the velocity is bounded. As discussed above, we compare the resetting procedure $(x, v) \mapsto (0, v)$ in Ref. [24] with our second one. For high resetting rates, the result is $p(y) \sim -\log|y|$, and for low rates, an exponential distribution.

Our work is structured as follows: We begin by giving the general expressions for stochastic resetting in Sec. II. We continue by considering the stationary PDF for the respective resetting procedure in Sec. III. Approximations are given for the tails of the stationary PDFs of displacements and, moreover, for displacements close to the PDF's mode at the origin. The behavior observed for the stationary PDF in its central part strongly resembles that observed in other models with similar velocity resetting prescriptions, and thus can be considered as a signature thereof. In Sec. IV, we turn to the second moment and the excess kurtosis in the respective cases, which serve as tools to discuss the approach to the stationarity. Further, two different initial conditions are considered. As we proceed to show, the velocity resetting prescription strongly influences the shape of the stationary PDF in its central part, and also the convergence to its stationary shape. The conclusions are given in Sec. V.

II. PRELIMINARIES

The probability density function (PDF) of a random process under resetting with waiting time distribution $\psi(t)$ and survival probability $\Psi(t) = \int_t^\infty \psi(t') dt'$ is given by

$$p(x|t) = p_0(x|t)\Psi(t) + \int_0^t p_{lr}(x|t, t') p_-(t'|t) dt', \quad (1)$$

see, e.g., [22]. Here, $p_0(x|t)$ is the PDF of the undisturbed underlying process, and $p_{lr}(x|t, t')$ is the PDF to find the particle at displacement x at time t under the condition that the last

reset took place at time t' . Further, $p_-(t'|t)$ denotes the PDF of a backward recurrence time, i.e., the PDF that the last reset before time t took place at time t' . The first term in Eq. (1) gives the contribution of the trajectories which do not undergo resetting until observation at time t , while the second term corresponds to all trajectories which experienced at least one reset. In the present paper, we consider Poissonian resetting, i.e., stochastic resetting with exponentially distributed waiting times,

$$\psi(t) = \frac{1}{t_r} e^{-\frac{t}{t_r}},$$

with mean reset time $t_r > 0$. The survival probability reads $\Psi(t) = e^{-t/t_r}$, and p_- is $p_-(t'|t) = \psi(t - t')$. Inserting Ψ and p_- into Eq. (1), we obtain

$$p(x|t) = p_0(x|t) e^{-\frac{t}{t_r}} + \frac{1}{t_r} \int_0^t p_{lr}(x|t, t') e^{-\frac{t-t'}{t_r}} dt'. \quad (2)$$

Note that if $p_{lr}(x, t|t') = p_{lr}(x, t - t')$ holds, we may substitute $\tau = (t - t')/t_r$ in the integral. For the stationary distribution which is attained for $t \rightarrow \infty$, we then obtain

$$p(x) = \int_0^\infty \tilde{p}_{lr}(x, \tau) e^{-\tau} d\tau, \quad (3)$$

where $\tilde{p}_{lr}(x, \tau) = p_{lr}(x, t_r \tau)$ follows from $p_{lr}(x, t - t')$ by the change of variables. After discussing the underlying process, we pass to dimensionless variables, Eqs. (7) and (8), and suppress the tilde for the rest of the work.

For the underlying process, we chose a free underdamped BM, whose evolution is given by the Langevin equation

$$\dot{x} = v \quad \text{and} \quad \dot{v} = -\frac{v}{t_v} + \sqrt{2D_v} \xi(t). \quad (4)$$

Here, $\xi(t)$ is a Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(t') \rangle = \delta(t - t')$, D_v denotes the velocity's diffusion coefficient, and $t_v = m/\gamma$ is the velocity's relaxation time, with m being the particle's mass and γ the Stokes' friction coefficient. Note that the velocity process is an Ornstein–Uhlenbeck one, and in the absence of resetting, the distribution of velocity tends to a Gaussian (Maxwell) one, with variance $\sigma_v^2 = \langle v^2 \rangle = D_v t_v$, which in the case of a thermal bath corresponds to $\langle v^2 \rangle = k_B T/m$. The behavior of $x(t)$ in the absence of resetting is diffusive at long times, the diffusion coefficient of the coordinate is $D = D_v t_v^2 = k_B T/\gamma$. The width of the Maxwell distribution is therefore

$$\langle v^2 \rangle = \frac{k_B T}{m} = \frac{k_B T}{\gamma} \frac{\gamma}{m} = \frac{D}{t_v}. \quad (5)$$

Dimensionless parameters. All in all, we have three parameters that describe our system: The diffusion coefficient D , the velocity relaxation time t_v , and the mean resetting time t_r . We moreover introduce the characteristic length scale $L = \sqrt{D t_r}$ (which turns out to be the scale parameter of the exponential decay in the tails of the stationary PDF), and the dimensionless quotient

$$\theta = \frac{t_v}{t_r}, \quad (6)$$

of the two characteristic times. In our further discussion, the parameters L and t_r will be set as units of length and time.

In these units, the diffusion coefficient is unity, and the unit of velocity is L/t_r . Wanting to use dimensionless variables instead of the displacement x and the time t , we introduce the dimensionless length and time

$$\chi = \frac{x}{L} \quad \text{and} \quad \tau = \frac{t}{t_r}. \quad (7)$$

We moreover introduce the dimensionless velocity

$$v = v \frac{t_r}{L}. \quad (8)$$

The variance of the equilibrium dimensionless velocity (following the Maxwell distribution) is

$$\langle v^2 \rangle = \theta^{-1}.$$

Note that in what follows, all dimensional variables are denoted by Latin letters, and all dimensionless ones by Greek letters.

The marginal probability density in χ is given by Eq. (171) of [28], which in our notation reads

$$p_0(\chi|\tau, \chi_0, v_0) = \frac{1}{\sqrt{2\pi} \theta \phi_1(\tau/\theta)} e^{-\frac{[\chi - \chi_0 - v_0 \theta (1 - e^{-\tau/\theta})]^2}{2\theta \phi_1(\tau/\theta)}}, \quad (9)$$

where we abbreviate

$$\phi_1(z) = 2z - 3 + 4e^{-z} - e^{-2z}. \quad (10)$$

Here, χ_0, v_0 denote the dimensionless versions of the initial displacement and initial velocity.

III. APPROXIMATIONS FOR THE STATIONARY DENSITIES

As previously mentioned, we consider two resetting procedures: The first resets both position and velocity to zero, while the second resets the position to zero but takes the reset velocity to follow a Maxwell distribution. We derive approximations for the stationary PDFs for large and small displacements for these two resetting schemes.

A. First resetting procedure: $(\chi, v) \mapsto (0, 0)$

Considering the resetting procedure $(\chi, v) \mapsto (0, 0)$, the probability density p_{lr} of the last reset equals p_0 , since virtually all memory from before the reset is lost, and the process is restarted for initial conditions $(\chi_0, v_0) = (0, 0)$. Inserting $p_{lr}(\chi|\tau, \tau') = p_0(\chi|\tau - \tau', 0, 0)$ into Eq. (3), we obtain the stationary PDF

$$\begin{aligned} p(\chi) &= \int_0^\infty p_0(\chi|\tau, 0, 0) e^{-\tau} d\tau \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi} \theta \phi_1(\tau/\theta)} e^{-\frac{\chi^2}{2\theta \phi_1(\tau/\theta)} - \tau} d\tau. \end{aligned} \quad (11)$$

We denote the exponential's argument as

$$f_1(\chi, \tau) = \frac{\chi^2}{2\theta \phi_1(\tau/\theta)} + \tau. \quad (12)$$

In Fig. 1, the stationary PDF is displayed, where we compare the numerical integration of Eq. (11) to the density obtained by simulation. To simulate the resetting process, the resetting times are drawn from an exponential distribution

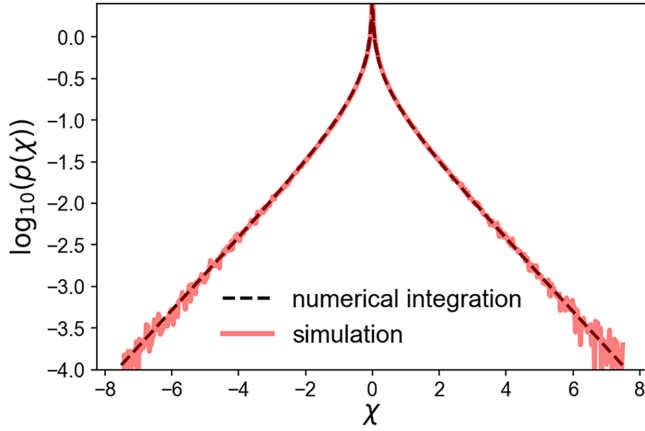


FIG. 1. Simulation based on the Langevin equation against numerical integration of Eq. (11) for the parameter $\theta = 0.6$.

with rate $\frac{1}{t_r}$ and take time steps of $\Delta t = 0.01$. Between two resetting events, the velocity is simulated by applying Heun's algorithm to the Langevin equation in Eq. (4). The displacement is obtained from the velocity by the trapezoid rule. We reset at the first time step that exceeds the resetting time. The velocity and displacement are then set to zero. Further, a new resetting time is drawn and added to the previous one to obtain the absolute time at which the next reset occurs. The simulation is performed using the dimensional parameters D , t_v and t_r , and shown for the values $D = 7$, $t_v = 3$ and $t_r = 5$, corresponding to $\theta = 0.6$. The results presented correspond to averaging over the ensemble of 10^4 trajectories, and over the time between $t = 8 \times 10^3$ and $t = 10^4$ for each trajectory. To compare the resulting PDF with Eq. (11), the displacement x is translated to its dimensionless version χ . As seen in Fig. 1, the simulation and the numerical integration of Eq. (11) coincide.

Let us now turn to approximations of the stationary PDF for large and small χ .

1. Asymptotic behavior for large χ

For large displacements, we apply the Laplace method to the integral in Eq. (11). The position of the minimum of f_1 in τ corresponding to $\partial_\tau f_1(\chi, \tau) = 0$, follows the equation

$$|\chi| \theta^{-1} = e^{-\frac{\tau}{\theta}} - 3 + \frac{2}{1 - e^{-\frac{\tau}{\theta}}}. \quad (13)$$

Note that the right-hand side of Eq. (13) is strictly increasing in τ . Assuming $|\chi| \gg \theta$, we may consequently neglect the exponential contributions $e^{-\tau/\theta}$. The local minimum of $f_1(\chi, \tau)$ then approximately corresponds to $z = \frac{\tau}{\theta} \simeq \frac{1}{2\theta}(3\theta + |\chi|)$. Applying the Laplace's method, we get

$$p(\chi) \simeq \frac{e^{-\frac{3}{2}\theta}}{2} \exp(-|\chi|). \quad (14)$$

Thus, the tails show an exponential decay.

Note that the discussion above makes clear the role of θ as a parameter which separates the domain of large $|\chi|$ ("tails" of the distribution, $|\chi| \gg \theta$) from the central part of the distribution $|\chi| \ll \theta$. The same interpretation applies to other cases discussed in the present work.

If we reintroduce the dimensional parameter x in Eq. (14), the characteristic scale of the exponential decay would be $L = \sqrt{D t_r}$, which is the same scale as in Ref. [1]. Thus, apart from the factor $e^{-\frac{3}{2}\theta}$, Eq. (14) coincides with the stationary PDF for overdamped BM [1]. In the overdamped case, the prefactor of the exponential is the normalization factor. This is not the case here. The fact that the long-time asymptotics is not normalized by itself indicates that the behavior of the PDF at the central part of the distribution, close to its mode, in the underdamped case is different from that for overdamped dynamics. The extra prefactor disappears for $\theta \ll 1$, i.e., for $t_v \rightarrow 0$, exactly in the overdamped limit. Note that the exponential behavior in the tails also appears for underdamped BM under Poissonian resetting with $(x, v) \mapsto (0, v)$ [24] and also for active BM [27], and by itself is not a wonder, since large displacements are dominated by large intervals between resetting epochs during which the displacements' statistics achieve diffusive behavior. The prefactors are, however, model-specific, indicating that the behavior of the PDF in its central part depends on the details of the system.

2. Asymptotic behavior for small χ

Looking at Fig. 1, we notice a sharp peak (singularity) at $\chi = 0$. In fact, the integral in Eq. (11) is dominated by small τ for $\chi = 0$, where $\phi_1(z) \simeq \frac{2}{3}z^3$. Substituting this, we see that the integral diverges at the lower limit.

Proceeding with the approximations for small displacements $|\chi|$, we insert $\phi_1(z) \simeq \frac{2}{3}z^3$ into Eq. (11) and obtain

$$\begin{aligned} p(\chi) &\simeq \sqrt{\frac{3\theta^2}{4\pi}} \int_0^\infty \tau^{-\frac{3}{2}} \exp\left(-\frac{3\theta^2\chi^2}{4}\tau^{-3}\right) e^{-\tau} d\tau \\ &= \frac{|\chi|^{-\frac{1}{3}}}{\sqrt{\pi}} \left(\frac{\theta^2}{36}\right)^{\frac{1}{3}} \int_0^\infty \vartheta^{-\frac{5}{6}} e^{-\vartheta} e^{-c(\chi)\vartheta^{-1/3}} d\vartheta, \end{aligned} \quad (15)$$

where we substitute $\vartheta = \frac{3}{4}\theta^2\chi^2\tau^{-3}$ and abbreviate $c(\chi) = (\frac{3}{4}\theta^2\chi^2)^{\frac{1}{3}}$ in the last step. We now consider the remaining integral

$$I_1(c) = \int_0^\infty \vartheta^{-\frac{5}{6}} e^{-\vartheta} e^{-c\vartheta^{-1/3}} d\vartheta, \quad (16)$$

for small $c \propto \chi^{\frac{2}{3}}$, and note that it has a finite limit for $c \rightarrow 0$, namely, $I_1(0) = \Gamma(\frac{1}{6})$. Returning to our initial notation, we obtain

$$p(\chi) \simeq \frac{1}{\sqrt{\pi}} \left(\frac{\theta^2}{36}\right)^{\frac{1}{3}} \Gamma\left(\frac{1}{6}\right) |\chi|^{-\frac{1}{3}}. \quad (17)$$

The PDF $p(\chi)$ possesses at $\chi = 0$ a singularity of the type $|\chi|^{-1/3}$. A similar behavior was found for active BM [27] and for the RAP [26]. Let us remark on the similarity of the Langevin equation for small velocities in all three cases. We have already commented on this similarity for underdamped and random accelerated BM in Sec. II. Moreover, applying the small-angle approximation for the y -coordinate in Ref. [27] renders the Langevin equation for random accelerated BM, explaining this similarity as well.

In Fig. 2, the approximations are plotted against the numerical integration of Eq. (11). To highlight the exponential nature of the tails, we use log-linear scales for large χ ,

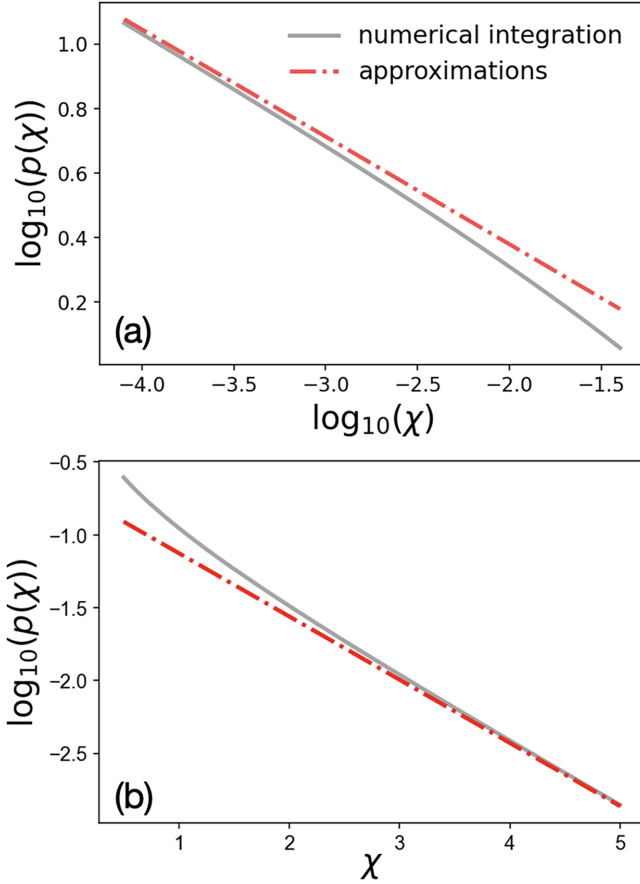


FIG. 2. Comparison of the asymptotic behavior with the numerical integration of Eq. (11) for the parameter $\theta = 0.6$. (a) Behavior for small displacements, close to the mode of the distribution shown on the double-logarithmic scales. The approximation shown is given by Eq. (17). (b) Behavior at the tail of the distribution shown on log-linear scales. Here, the approximation is given by Eq. (14).

while the double-logarithmic scales are used for small χ to highlight the power-law nature of the singularity. As can be seen, both approximations converge toward the graphs of the numerical integration. Note that the results of the approximations, Eqs. (14) and (17), are shown as they are, without any fitting or additional corrections. The same is true for all other approximate results reported in the work.

B. Second resetting procedure: $\chi \mapsto 0$, $v \mapsto \text{Maxwell}$

To proceed with the second resetting procedure, where the displacement is reset to 0 and the velocity is reset to v_0 following the Maxwell distribution $\rho(v_0) = \frac{1}{\sqrt{2\pi\theta^{-1}}} \exp(-\frac{v_0^2}{2\theta^{-1}})$, the probability density $p_{lr}(\chi|\tau, \tau')$ must be determined. As before, the memory of events before a reset is lost, and we need to average over all possible reset velocities:

$$\begin{aligned} p_{lr}(\chi|\tau) &= \int_{-\infty}^{\infty} p_0(\chi|\tau, 0, v) \rho(v) dv \\ &= \frac{1}{\sqrt{4\pi^2\phi_1}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{v^2}{2\theta^{-1}} - \frac{[\chi - v\theta(1 - e^{-\tau/\theta})]^2}{2\theta\phi_1} \right\} dv \end{aligned}$$

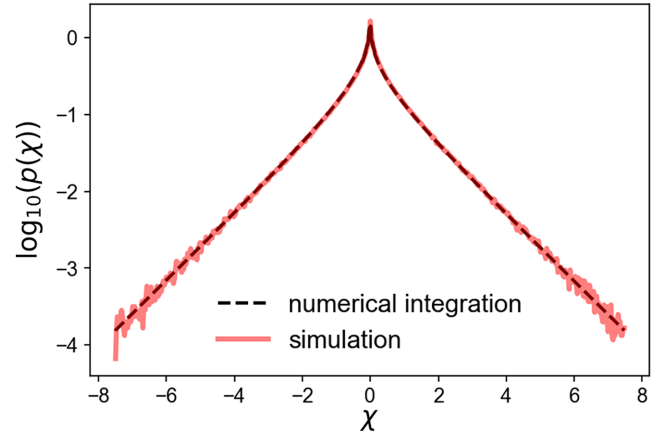


FIG. 3. Simulation based on the Langevin equation against the numerical integration of Eq. (20) for the parameter $\theta = 0.6$.

$$\begin{aligned} &= \frac{\exp(-\frac{\chi^2}{2\theta\phi_1})}{\sqrt{4\pi^2\phi_1}} \times \\ &\int_{-\infty}^{\infty} \exp \left\{ -v^2 \frac{\theta}{2} \left[1 + \frac{(1 - e^{-\tau/\theta})^2}{\phi_1} \right] - \frac{\chi v(1 - e^{-\tau/\theta})}{\phi_1} \right\} dv, \end{aligned}$$

where, for the time being, we omitted the argument of the function $\phi_1 = \phi_1(\tau/\theta)$. Now we introduce

$$\phi_2(z) = \frac{1}{2}[\phi_1(z) + (1 - e^{-z})^2] = z - 1 + e^{-z}, \quad (18)$$

so that

$$\begin{aligned} p_{lr}(\chi|\tau) &= \frac{\exp(-\frac{\chi^2}{2\theta\phi_1})}{\sqrt{4\pi^2\phi_1}} \times \\ &\int_{-\infty}^{\infty} \exp \left(-\frac{v^2}{2\theta^{-1}} \frac{\phi_2}{\phi_1} - \frac{\chi v(1 - e^{-\tau/\theta})}{\phi_1} \right) dv. \end{aligned}$$

Performing integration, we get

$$\begin{aligned} p_{lr}(\chi|\tau) &= \frac{1}{\sqrt{4\pi\phi_2}} \exp \left\{ -\frac{\chi^2[2\phi_2 - (1 - e^{-\tau/\theta})^2]}{4\phi_1\phi_2\theta} \right\} \\ &= \frac{1}{\sqrt{4\pi\theta\phi_2(\tau/\theta)}} \exp \left(-\frac{\chi^2}{4\theta\phi_2(\tau/\theta)} \right), \end{aligned} \quad (19)$$

where in the last line we used Eq. (18) again.

The stationary PDF given by Eq. (3) now reads

$$p(\chi) = \int_0^{\infty} \frac{1}{\sqrt{4\pi\theta\phi_2(\tau/\theta)}} e^{-f_2(\chi, \tau)} d\tau, \quad (20)$$

with

$$f_2(\chi, \tau) = \frac{\chi^2}{4\theta\phi_2(\tau/\theta)} + \tau. \quad (21)$$

In Fig. 3, the numerical integration of the stationary PDF (20) is displayed against the density of the simulated process. Apart from the fact that the velocity is now reset to values drawn from the Maxwell distribution, the simulation and its parameters are the same as for the first resetting procedure.

1. Asymptotic behavior for large χ

Following the same line of argumentation as for the previous resetting procedure, we assume the condition $|\chi| \gg \theta$ while estimating the local minimum of the exponential's argument f_2 to apply the Laplace method. After neglecting subleading corrections, we arrive at

$$p(\chi) \simeq \frac{e^{-\theta}}{2} \exp(-|\chi|). \quad (22)$$

Note that we arrive at the same approximation as in the first case up to the prefactor, which now reads $e^{-\theta}$ as opposed to the previous prefactor $e^{-\frac{3}{2}\theta}$. Following the same argumentation, in contrast to the overdamped case, we expect to find a non-Laplacian behavior in the central part of the distribution. Furthermore, we note that Eq. (22) is exactly the result found in Ref. [24] for the tails when resetting follows the $(x, v) \mapsto (0, v)$ procedure.

2. Asymptotic behavior for small χ

As in the previous resetting case, we may not apply the Laplace method for small χ and will take a similar approach to the previous one. By Taylor expanding $\phi_2(\tau) \simeq \frac{1}{2}\tau^2$ in Eq. (20), we have

$$\begin{aligned} p(\chi) &\simeq \sqrt{\frac{\theta}{2\pi}} \int_0^\infty \tau^{-1} \exp\left(-\frac{\theta}{2} \chi^2 \tau^{-2} - \tau\right) d\tau \\ &= \sqrt{\frac{\theta}{8\pi}} \int_0^\infty \vartheta^{-1} e^{-\vartheta} e^{-c \vartheta^{-\frac{1}{2}}} d\vartheta, \end{aligned} \quad (23)$$

where we substitute $\vartheta = \frac{\theta}{2} \chi^2 \tau^{-2}$ in the first step and abbreviate $c = \sqrt{\frac{\theta}{2}} |\chi| > 0$. Now we aim to estimate the integral

$$I_2(c) = \int_0^\infty \vartheta^{-1} e^{-\vartheta} e^{-c \vartheta^{-\frac{1}{2}}} d\vartheta, \quad (24)$$

for small $c \propto |\chi|$. In contrast to $I_1(c)$ for the previous resetting procedure, the integral $I_2(c)$ is divergent for $c \rightarrow 0$. Let us take the Laplace transform of this integral with respect to c :

$$\mathcal{L}\{I_2(c)\}(u) = \int_0^\infty I_2(c) e^{-uc} dc = \frac{1}{u} \int_0^\infty \frac{e^{-\vartheta}}{\vartheta^{\frac{1}{2}} + u^{-1}} \frac{d\vartheta}{\vartheta^{1/2}}. \quad (25)$$

Changing the variable of integration to $z = \vartheta^{\frac{1}{2}} + u^{-1}$ we may write

$$\mathcal{L}\{I_2(c)\}(u) = \frac{2}{u} e^{-u^2} \int_{u^{-1}}^\infty \frac{1}{z} e^{-z^2} e^{2zu^{-1}} dz. \quad (26)$$

Since we approximate for small c , consequently large u , we can put $e^{-u^2} \simeq 1$ and moreover, expand $e^{2zu^{-1}} \simeq 1 + 2zu^{-1} + \dots$. We obtain

$$\begin{aligned} \mathcal{L}\{I_2(c)\}(u) &\simeq \frac{2}{u} \left[\int_{u^{-1}}^\infty \frac{1}{z} e^{-z^2} dz + \frac{2}{u} \int_{u^{-1}}^\infty e^{-z^2} dz + \dots \right] \\ &= \frac{1}{u} \Gamma(0, u^{-2}) + \frac{2\sqrt{\pi}}{u^2} \operatorname{erfc}(u^{-1}) + \dots \end{aligned} \quad (27)$$

For u large, the argument of the complementary error function is close to zero, so that $\operatorname{erfc}(u^{-1}) \simeq 1$, and

$$\mathcal{L}\{I_2(c)\}(u) \simeq \frac{1}{u} \Gamma(0, u^{-2}) + \frac{2\sqrt{\pi}}{u^2}. \quad (28)$$

The inverse Laplace transform of the second term is simply $2\sqrt{\pi}c$. To obtain the inverse Laplace transform of the first term, we expand the incomplete gamma function Γ . Combining Eqs. (5.1.11) and (6.5.15) of Ref. [29], we get $\Gamma(0, w) = -\gamma - \ln w - \sum_{k=1}^\infty \frac{(-w)^k}{k k!} \simeq -\gamma - \ln w + w + \dots$ (for $w > 0$), so that

$$\begin{aligned} \frac{1}{u} \Gamma(0, u^{-2}) &= -\frac{\gamma}{u} + 2 \frac{\ln u}{u} + u^{-3} \dots \\ &= 2 \frac{\gamma + \ln u}{u} - 3 \frac{\gamma}{u} + u^{-3} \dots, \end{aligned} \quad (29)$$

where γ denotes the Euler–Mascheroni constant. Now we note that $\mathcal{L}\{\ln |x|\}(u) = -u^{-1}(\gamma + \ln u)$, and obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{u} \Gamma(0, u^{-2})\right\}(c) \simeq -2 \ln |c| - 3\gamma + \frac{c^2}{2}. \quad (30)$$

The last term can be safely neglected compared to the inverse Laplace transform of the second term in Eq. (28). Putting everything together, we obtain

$$p(\chi) \simeq \sqrt{\frac{\theta}{8\pi}} \left(-2 \ln |\chi| + \ln \frac{2}{\theta} - 3\gamma \right) + \frac{\theta}{2} |\chi|. \quad (31)$$

A similar logarithmic divergence was found in Ref. [27] for active BM with a comparable resetting procedure. In Fig. 4, the approximations are compared to the numerical integration of the stationary PDF. For small χ we plot $p(\chi)$ as a function of $\ln(\chi)$ since in this case the logarithmic term is the leading one. The natural logarithmic scale is used since it allows us to plot Eq. (31) as is, without unnecessary transformation coefficients. For the tails, we plot $\log(p(\chi))$ as a function of χ since the main dependence is exponential. The approximation for large displacements converges well to the result of the numerical integration. The approximation for small displacements is, however, off by a constant. This fact is due to the very weak dominance of the logarithmic term compared to the constant contributions, which are not fully captured by the approximation. The cause can be found when comparing the integrand in Eq. (23) using the Taylor expansion of ϕ_2 and the actual integrand in Eq. (20). The approximation is good for small values of the variable of integration, but already for moderate values of the variable of integration of the order of unity the difference in the integrands is large enough to evoke a shift between the approximation and the actual integral. In Fig. 5, the difference $\Delta p(\chi)$ between the value of $p(\chi)$ as obtained by numerical integration of Eq. (20) and the asymptotic result in Eq. (31) is plotted for three different values of θ . As can be seen, for small χ , the difference $\Delta p(\chi)$ between the results is approximately constant. For large values of θ , that is, for smaller arguments of ϕ_2 , the difference becomes smaller, so that the quality of the approximation improves.

IV. APPROACH TO THE STATIONARY PDF

To discuss the approach to the stationary densities, let us first look at the second moments of χ for the respective resetting procedures, and then discuss the excess kurtosis

$$\kappa(\tau) = \frac{\langle \chi^4(\tau) \rangle}{\langle \chi^2(\tau) \rangle^2} - 3. \quad (32)$$

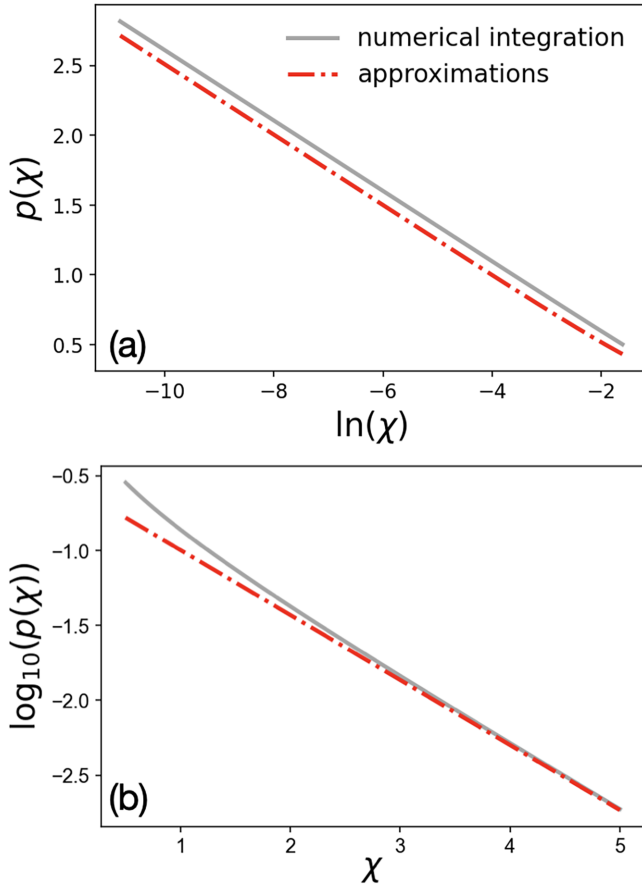


FIG. 4. Comparison of estimated asymptotic behavior with the numerical integration of Eq. (3) with p_{lr} given by Eq. (19) for $\theta = 0.6$. (a) Behavior for small displacements, close to the mode. The approximation is given by Eq. (31). Note the linear-log scales. (b) Behavior in the tail of the distribution, shown on a log-linear scale, as opposed to the previous one. Here, the approximation is given by Eq. (22).

The excess kurtosis is a measure of how much probability mass lies in the tails and close to the mode of the distribution. A large excess kurtosis corresponds to a distribution whose probability mass is concentrated in its tails and close to its mode [30]. Note that a Gaussian probability density has an excess kurtosis of 0; in particular, the underlying process has an excess kurtosis of 0. Further, a Laplacian probability distribution yields an excess kurtosis of 3. Consequently, the stationary PDF for the overdamped BM under Poissonian resetting has an excess kurtosis of 3.

The moments can be easily obtained from Eq. (2) and transformed to dimensionless variables:

$$\langle \chi^n \rangle(\tau) = \langle \chi^n \rangle_0(\tau) e^{-\tau} + \int_0^\tau \langle \chi^n \rangle_{lr}(\tau') e^{-\tau'} d\tau', \quad (33)$$

where $\langle \chi^n \rangle_0(\tau)$ is the corresponding moment of $p_0(\chi|\tau)$ and $\langle \chi^n \rangle_{lr}(\tau)$ is the moment of $p_{lr}(\chi|\tau)$. Note that the first term in Eq. (33) incorporates the influence of the initial conditions, which we lose when considering the stationary case.

To calculate the excess kurtosis for either resetting procedure, the second and the fourth moments of the underlying process [Eq. (9)] are needed. Note that the excess kurtosis is

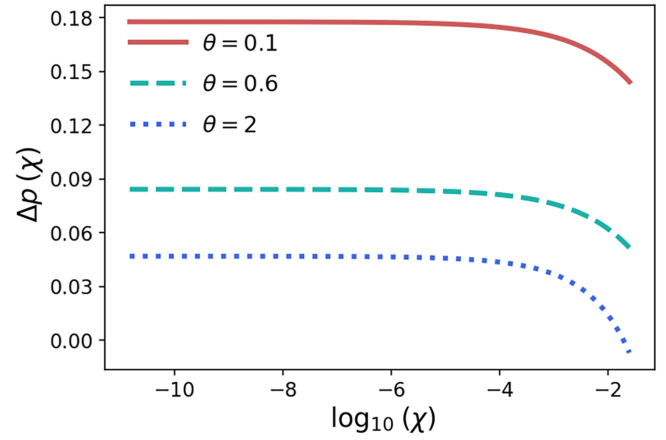


FIG. 5. The difference $\Delta p(\chi)$ between the values of $p(\chi)$ as obtained from Eqs. (20) and (31) for different values of θ .

the same whatever unit of length is used, i.e., it is the same when considering the dimensionless displacement χ and the displacement x in natural units.

In Fig. 6, we plot the second moments for the respective resetting procedures and the initial condition above. These are given by Eqs. (36) and (43) below. Both stagnate rather quickly, which hints at the approach to the stationary distribution.

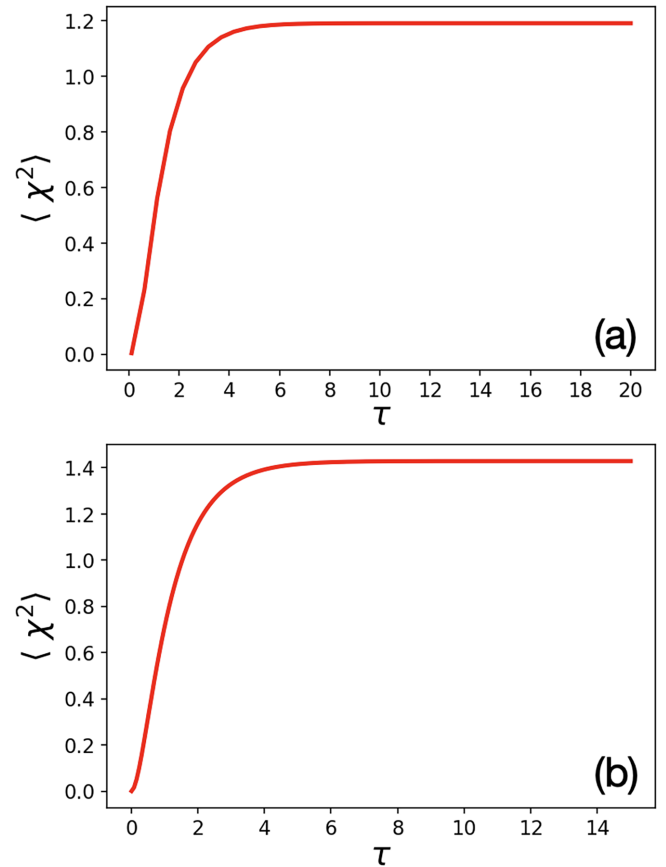


FIG. 6. The second moment for the first and second resetting procedure for $\theta = 0.4$. Both resetting procedures correspond to $\chi \mapsto 0$. (a) The case $\nu \mapsto 0$. (b) The case when ν has a Maxwell distribution.

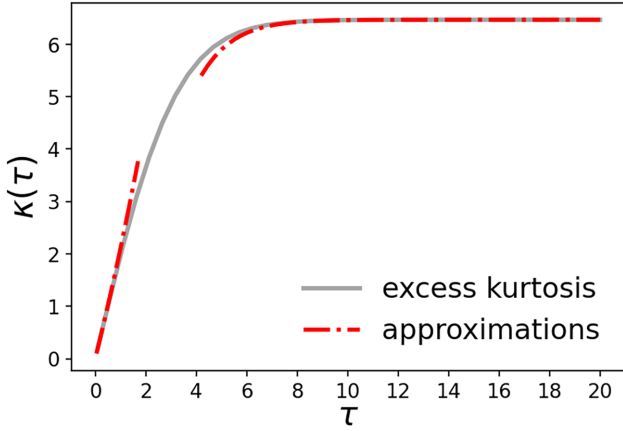


FIG. 7. Excess kurtosis for the resetting procedure $(\chi, v) \mapsto (0, 0)$ with initial conditions $(\chi_0, v_0) = (0, 0)$ and the parameter $\theta = 0.4$.

A. First resetting procedure: $(\chi, v) \mapsto (0, 0)$

As the initial condition, we adopt $(\chi_0, v_0) = (0, 0)$. For this initial condition, we have

$$\langle \chi^2 \rangle_0(\tau) = \theta \left(2\frac{\tau}{\theta} - 3 + 4e^{-\frac{\tau}{\theta}} - e^{-2\frac{\tau}{\theta}} \right), \quad (34)$$

$$\langle \chi^4 \rangle_0(\tau) = 3\theta^2 \left(2\frac{\tau}{\theta} - 3 + 4e^{-\frac{\tau}{\theta}} - e^{-2\frac{\tau}{\theta}} \right)^2. \quad (35)$$

As previously argued while considering the asymptotic approximations, for the first resetting procedure, $p_{lr}(\chi|\tau, \tau')$ is given by $p_0(\chi|\tau - \tau')$, with the last one given by Eq. (9), and therefore $\langle \chi^2 \rangle_{lr}(\tau) = \langle \chi^2 \rangle_0(\tau)$, $\langle \chi^4 \rangle_{lr}(\tau) = \langle \chi^4 \rangle_0(\tau)$. Using Eq. (33), we get

$$\begin{aligned} \langle \chi^2 \rangle(\tau) = & 2\theta \left(\frac{2}{\theta(1+\theta)(2+\theta)} - \theta^{-1}e^{-\tau} \right. \\ & \left. + \frac{2}{1+\theta}e^{-\tau\frac{1+\theta}{\theta}} - \frac{1}{2+\theta}e^{-\tau\frac{2+\theta}{\theta}} \right) \end{aligned} \quad (36)$$

and

$$\begin{aligned} \langle \chi^4 \rangle(\tau) = & 12\theta^2 \left(\frac{8(12+25\theta+10\theta^2)}{\theta^2(1+\theta)^2(2+\theta)^2(3+\theta)(4+\theta)} \right. \\ & - \frac{2\tau+2-3\theta}{\theta^2}e^{-\tau} \\ & + \frac{4\tau(1+\theta)-2\theta(3+5\theta)}{\theta(1+\theta)^2}e^{-\tau\frac{1+\theta}{\theta}} \\ & + \frac{2(\theta(11+6\theta)-\tau(2+\theta))}{\theta(2+\theta)^2}e^{-\tau\frac{2+\theta}{\theta}} \\ & \left. - \frac{6}{3+\theta}e^{-\tau\frac{3+\theta}{\theta}} + \frac{1}{4+\theta}e^{-\tau\frac{4+\theta}{\theta}} \right). \end{aligned} \quad (37)$$

Inserting Eqs. (36) and (37) into Eq. (32), we get $\kappa(\tau)$, which is shown in Fig. 7.

Let us consider the behavior of $\kappa(\tau)$ for short and long times, which allows us to grasp how the PDF departs from the initial state, and how it asymptotically converges to a stationary one. For short τ , the Taylor expansion of $\kappa(\tau)$ up to

the second order reads

$$\kappa(\tau) \simeq \frac{27}{14}\tau \left[1 + \left(\frac{11}{40} - \frac{3}{40\theta} \right) \tau \right] + O(\tau^3). \quad (38)$$

This value departs from zero (characteristic for a Gaussian PDF) linearly in τ , and the dependence on θ appears only in the second order in τ . We will return to this result when discussing the short-time behavior of excess kurtosis in our second resetting procedure.

For long times, we get

$$\kappa_{\text{lim}} = \lim_{\tau \rightarrow \infty} \kappa(\tau) = \frac{36 + 129\theta + 57\theta^2}{(3+\theta)(4+\theta)}. \quad (39)$$

Let us consider the limit of this last expression for $\theta \rightarrow 0$. Physically, we note that $\theta \rightarrow 0$ corresponds to two different situations: The first one is $t_v \rightarrow 0$, which is the overdamped limit, and the second one is $t_r \rightarrow \infty$, i.e., when no resetting takes place at all. Equation (39) indeed reproduces $\kappa_{\text{lim}} = 3$ corresponding to a Laplace distribution for the overdamped case, and does not reproduce the second situation.

When no resetting takes place, the process has to behave like the underlying process, which is Gaussian. Thus, one could expect $\kappa \rightarrow 0$. For finite times, one indeed has $\lim_{\theta \rightarrow 0} \kappa(\tau) = 0$, but the limits $\tau \rightarrow \infty$ and $\theta \rightarrow 0$ (i.e., now $t_r \rightarrow \infty$ for fixed t_v) do not commute, which is due to the definition of $\tau = \frac{t}{t_r}$. For this reason, the limit $t_r \rightarrow \infty$ cannot be read off from Eq. (39).

The convergence of $\kappa(\tau)$ to the limit given by Eq. (39) for $\tau \gg \theta$ (and $\theta > 0$) is approximately exponential:

$$\kappa(\tau) \simeq \kappa_{\text{lim}} - \frac{3}{2}(1+\theta)^2(2+\theta)^2 \tau e^{-\tau}. \quad (40)$$

To obtain this expression, one starts from Eq. (32), uses Eqs. (36) and (37) for the moments, and separates the constant contributions from the ones depending on $q = e^{-\tau}$ in the numerator and denominator. The value of q is then considered as a small parameter, and only the terms of zeroth and the first order in q are kept in the final expression. Thus, the kurtosis converges to its limit for large τ roughly as $\tau e^{-\tau}$.

B. Second resetting procedure: $\chi \mapsto 0, v \mapsto \text{Maxwell}$

For this case, we discuss two different kinds of initial conditions: The first initial condition is like in the previous case: $\chi_0 = 0$ and $v_0 = 0$ (which in this case is different from the resetting prescription), while the second initial condition sets $\chi_0 = 0$ and v_0 to follow the Maxwell distribution (i.e., the initial condition is the same as the resetting prescription). Before turning to the calculation of excess kurtosis, let us look at the time evolution of the PDF of the particle's displacements. We expect the influence of the initial conditions to be strong at short times, when a resetting is yet unlikely to occur. On the other hand, for long times, the stationary PDF is attained, which does not depend on initial conditions. The PDFs $p(\chi|\tau)$ for these two initial conditions are shown in Fig. 8 for $\chi > 0$ and different values of τ . The Figure comes from the numerical integration of Eq. (2) in dimensionless units with $p_{lr}(\chi|\tau)$ given by Eq. (19), and the initial PDF $p_0(\chi, \tau)$ given either by Eq. (9) or by $p_0(\chi, \tau) = p_{lr}(\chi, \tau)$, respectively.

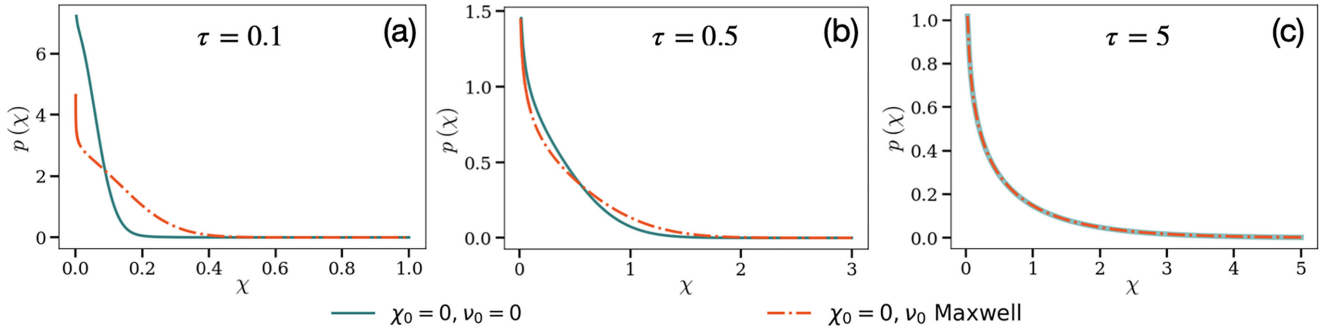


FIG. 8. Time evolution of PDF $p(\chi|\tau)$ with the second resetting procedure for the two initial conditions for $\theta = 0.4$. The panels (a), (b), and (c) correspond to $\tau = 0.1, 0.5$, and 5 , respectively. Full curves correspond to $\nu_0 = 0$ and the dot-dashed curves to ν_0 taken from the Maxwell distribution, see text for details.

The shapes of the corresponding curves at short times are indeed quite different. The peak close to the mode of the PDF for the initial condition $(\chi_0, \nu_0) = (0, 0)$ is sharper than for the second one. The reason for this is as follows. For $\chi_0 = 0$ and $\nu_0 = 0$, at short times all particles that did not experience a reset so far are likely to possess small velocities and thus reside in the vicinity of the origin. The few particles that did experience a reset within the same time are likely to possess larger velocities after the reset, and therefore display larger displacements. Hence, this initial condition favors a slow softening of the peak for small times, while the resets favor the growth of the tails. The different behaviors are mirrored in the time evolution of the excess kurtosis.

The moments are calculated as in the previous case, but with p_{lr} as in Eq. (19), whose moments are

$$\langle \chi^2 \rangle_{lr}(\tau) = 2\theta \left(\frac{\tau}{\theta} - 1 + e^{-\frac{\tau}{\theta}} \right), \quad (41)$$

and

$$\langle \chi^4 \rangle_{lr}(\tau) = 12\theta^2 \left(\frac{\tau}{\theta} - 1 + e^{-\frac{\tau}{\theta}} \right)^2. \quad (42)$$

1. Initial conditions $\chi_0 = 0$ and $\nu_0 = 0$

The second and fourth moments for the initial conditions $\chi_0 = 0$ and $\nu_0 = 0$ are

$$\begin{aligned} \langle \chi^2 \rangle(\tau) = & \theta \left(\frac{2}{\theta(1+\theta)} - (1+2\theta^{-1})e^{-\tau} \right. \\ & \left. + 2 \left(1 + \frac{1}{1+\theta} \right) e^{-\tau \frac{1+\theta}{\theta}} - e^{-\tau \frac{2+\theta}{\theta}} \right), \end{aligned} \quad (43)$$

and

$$\begin{aligned} \langle \chi^4 \rangle(\tau) = & 3\theta^2 \left(\frac{8(2+3\theta)}{\theta^2(1+\theta)^2(2+\theta)} - 8e^{-\tau \frac{3+\theta}{\theta}} \right. \\ & + \left(5 - \frac{4\tau}{\theta} - \frac{8\tau}{\theta^2} - 8\theta^{-2} + 8\theta^{-1} \right) e^{-\tau} \\ & + 8 \left(\frac{\theta^2}{(1+\theta)^2} + \frac{2+\theta}{\theta(1+\theta)} \tau - 3 \right) e^{-\tau \frac{1+\theta}{\theta}} \\ & \left. + 2 \left(11 - \frac{2\tau}{\theta} - \frac{2\theta}{2+\theta} \right) e^{-\tau \frac{2+\theta}{\theta}} + e^{-\tau \frac{4+\theta}{\theta}} \right). \end{aligned} \quad (44)$$

The corresponding excess kurtosis is shown in Fig. 9(a).

What is striking when looking at the figure is the divergence for $\tau \rightarrow 0$. Intuitively, for small τ , when resets occur rarely, one would assume that behavior is dominated by the underlying process, which has a Gaussian probability distribution and thus an excess kurtosis of 0. Let us dive into this

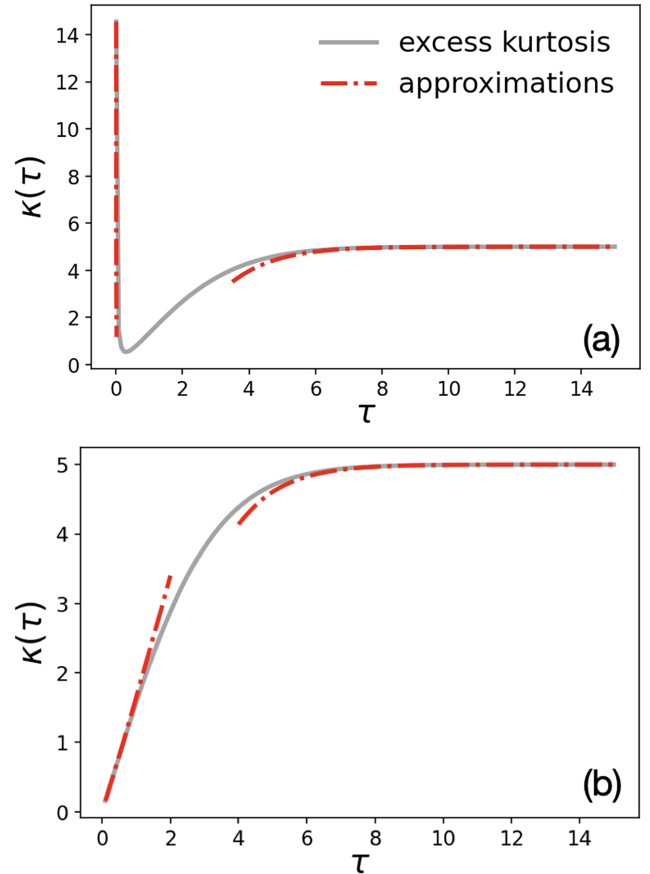


FIG. 9. Excess kurtosis for $\theta = 0.4$ for the displacement resetting $\chi \mapsto 0$ and for two different initial conditions for the velocity: $\nu \mapsto 0$ in panel (a) and ν Maxwell-distributed in panel (b). The approximations shown in panel (a) are given by Eqs. (45) and (49) for short and long times, respectively. The approximations in panel (b) are given by Eqs. (52) and (53).

phenomenon by looking at the Laurent expansion for small τ :

$$\kappa(\tau) \simeq \frac{3\theta^2}{(2+\theta)^2} \tau^{-1} - 3, \quad (45)$$

which diverges as τ^{-1} for $\tau \rightarrow 0$. To explain, consider the first nonvanishing order in Taylor expansions of the moments of the involved densities:

$$\langle \chi^2 \rangle_0(\tau) \simeq \frac{2}{3\theta^2} \tau^3 \text{ and } \langle \chi^4 \rangle_0(\tau) \simeq \frac{4}{3\theta^4} \tau^6, \quad (46)$$

$$\langle \chi^2 \rangle_{lr}(\tau) \simeq \frac{1}{\theta} \tau^2 \text{ and } \langle \chi^4 \rangle_{lr}(\tau) \simeq \frac{3}{\theta^2} \tau^4. \quad (47)$$

Looking at the expressions for the moments of the reset process in Eq. (33), we note that the moments $\langle \cdot \rangle_{lr}$ are integrated in time (with an exponential multiplier which for short times is approximately unity). The integration increases the order of τ for short times in $\langle \cdot \rangle_{lr}$ by one. Hence, the second moments both contribute to a cubic order in τ , whereas $\langle \chi^4 \rangle_{lr}$ dominates the fourth moment with a contribution of τ^5 . Thus, the last-reset density p_{lr} , which has fatter tails than Chandrasekhar's solution, evokes rapidly growing tails of the reset density and, ultimately, the divergence of the excess kurtosis for $\tau \rightarrow 0$.

As for large time scales, we see that the limit is the following:

$$\lim_{\tau \rightarrow \infty} \kappa(\tau) = \frac{6+15\theta}{2+\theta}. \quad (48)$$

As before, we easily see from the limit, Eq. (48), that the overdamped limit, i.e., $t_r \rightarrow 0$, holds true, and one can also show that the limit $t_r \rightarrow \infty$ is satisfied by $\kappa(\tau)$ when assuming finite times τ . The approach to the limit, Eq. (48), is given by

$$\kappa(\tau) \simeq \frac{6+15\theta}{2+\theta} - 3(1+\theta)^2(2+\theta) \tau e^{-\tau}. \quad (49)$$

2. Initial condition $\chi_0 = 0$, v_0 Maxwell-distributed

We have seen that the fatter tails of p_{lr} , compared to those of p_0 favor the fast development of the tails in the system under resetting and thereby cause the divergence of the excess kurtosis for $\tau \rightarrow 0$. Let us verify that this behavior vanishes when applying initial conditions, for which the moments of p_0 are of the same order as those of p_{lr} . Thus, we consider the initial conditions where $\chi_0 = 0$ and v_0 is chosen from the Maxwell distribution. Then, in Eq. (33), we have $\langle \chi^n \rangle_0 = \langle \chi^n \rangle_{lr}$. The moments for these initial conditions are

$$\langle \chi^2 \rangle(\tau) = 2 \left(\frac{1}{(1+\theta)} - e^{-\tau} + \frac{\theta}{1+\theta} e^{-\tau \frac{1+\theta}{\theta}} \right), \quad (50)$$

and

$$\begin{aligned} \langle \chi^4 \rangle(\tau) = & 12 \left(\frac{2(3\theta+2)}{(\theta+1)^2(\theta+2)} - 2(\tau+1-\theta)e^{-\tau} \right. \\ & + \frac{2\theta(\tau(\theta+1)-\theta(2\theta+1))}{(\theta+1)^2} e^{-\tau \frac{\theta+1}{\theta}} \\ & \left. + \frac{2\theta^2}{\theta+2} e^{-\tau \frac{\theta+2}{\theta}} \right). \end{aligned} \quad (51)$$

Looking at Fig. 9(b), we see that for these initial conditions, the excess kurtosis indeed behaves as in the previous resetting procedure. The Taylor expansion around $\tau = 0$ gives

$$\kappa(\tau) \simeq \frac{\tau}{5} \left(8 + \frac{3\theta-1}{2\theta} \tau \right), \quad (52)$$

having a similar structure to Eq. (38). On the other hand, the limit for $\tau \rightarrow \infty$ of $\kappa(\tau)$ coincides with the corresponding limit, Eq. (48), for the previous initial conditions, as one would expect. The approach to this limit is given by

$$\kappa(\tau) \simeq \frac{6+15\theta}{2+\theta} - 6(1+\theta)^2 \tau e^{-\tau}. \quad (53)$$

V. CONCLUSIONS

In this work, we study underdamped Brownian motion under Poissonian resetting in both displacement and velocity. We consider two resetting procedures, which differ in the velocity resetting: the first resets the velocity to zero, while the second resets it to Maxwell-distributed velocities. Both reset the displacement to zero. Approximations for the stationary PDF are, respectively, derived for small and large displacements.

For both resetting procedures, the stationary PDF has a singularity at the mode $\chi = 0$. The divergence is stronger (power-law one) for the first resetting procedure, and weaker (logarithmic) for the second one. Intuitively, this can be explained by smaller velocities at small displacements in the first case. Note that the same type of singularity is found at the mode for both active BM [27] and for randomly accelerated particles [26] with comparable resetting procedures. As for the second resetting procedure, for a comparable resetting procedure for active BM [27], one also finds a logarithmic divergence. This stresses that the behavior in the central part of the distribution is specific for a particular resetting procedure (and can therefore be used as an experimental signature thereof), while the tail behavior is universal (two-sided exponential), up to normalization constants, and similar to the one found in overdamped BM [1]. Note that this behavior was observed earlier, both for underdamped BM, where the displacement is reset to zero, and the velocity is left invariant under resetting [24], and for the active BM [27].

To characterize the approach to the stationarity, when looking at the excess kurtosis, one can verify that the overall behavior of the stationary PDF is not Laplacian, as already shown by the approximations. In all cases, the excess kurtosis converges quite rapidly (essentially exponentially) to its limiting value, which depends on θ only, giving the assessment of a speed of convergence toward the stationary distribution. The behavior of the excess kurtosis at short times strongly depends on initial conditions and is drastically different for cases when the initial condition for the velocity is the same as the resetting condition for it, and when these two prescriptions differ.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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