

Weak chaos, anomalous diffusion, and weak ergodicity breaking in systems with delayTony Albers^{1,*}, Lukas Hille^{1,†}, David Müller-Bender^{1,‡} and Günter Radons^{1,2,§}¹*Institute of Physics, Chemnitz University of Technology, 09107 Chemnitz, Germany*²*ICM - Institute for Mechanical and Industrial Engineering, 09117 Chemnitz, Germany* (Received 12 July 2024; revised 13 June 2025; accepted 3 September 2025; published 14 October 2025)

We demonstrate that standard delay systems with a linear instantaneous and a delayed nonlinear term show weak chaos, asymptotically subdiffusive behavior, and weak ergodicity breaking if the nonlinearity is chosen from a specific class of functions. In the limit of large constant delay times, anomalous behavior may not be observable due to exponentially large crossover times. A periodic modulation of the delay causes a strong reduction of the effective dimension of the chaotic phases, leads to hitherto unknown types of solutions, and the occurrence of anomalous diffusion already at short times. The observed anomalous behavior is caused by nonhyperbolic fixed points in function space.

DOI: [10.1103/qqs-7pll](https://doi.org/10.1103/qqs-7pll)

Anomalous diffusion is a frequently observed transport mechanism, where the mean-squared displacement increases nonlinearly in time [1]. It occurs in physical, biological, and chemical systems such as charge-carrier transport in amorphous semiconductors [2], macromolecule motion in the cytoplasm of living cells [3], and monomer dynamics of long polymers [4]. While often described by stochastic models like fractional Brownian motion [5] or continuous-time random walks [6], anomalous diffusion also arises in deterministic chaotic systems. This so-called anomalous chaotic diffusion has been investigated intensively, but typically only for low-dimensional systems such as iterated maps [7–13] or Hamiltonian systems [14–18]. In the latter case, the appearance of so-called accelerator modes [19–21], i.e., sticky but transporting KAM islands embedded in the chaotic sea of a mixed phase space, typically leads to superdiffusion. There are only few works dealing with chaotic diffusion in higher-dimensional or even infinite-dimensional systems whose dynamics are ruled by delay differential equations [22–30] or partial differential equations [31–34]. Except for two cases in the latter class [31,32], the diffusion there was always normal. In this Letter, we show that anomalous diffusion can also be found in a standard family of systems with delay. Furthermore, the observed dynamics shows weak chaos [35,36], i.e., the maximal Lyapunov exponent vanishes [37]. We also find weak ergodicity breaking [38–40], implying an inequivalence of ensemble and time averages. To our knowledge, such phenomena have not been observed in

formally infinite-dimensional delay systems, which have many applications in physics, biology, and engineering [41–44]. Concrete examples are chaos control [45], population dynamics [46], turning and milling processes [47], and nonlinear optics [48,49].

We consider a typical class of time-delayed feedback systems with a linear instantaneous and a nonlinear delayed term described by the delay differential equation (DDE)

$$\frac{1}{\Theta} \dot{x}(t) = -x(t) + f(x[R(t)]). \quad (1)$$

The parameter Θ sets the overall timescale, where the case $\Theta \gg 1$ corresponds to the large delay limit and Eq. (1) then belongs to the class of singularly perturbed DDEs [50–56]. The retarded argument $R(t) = t - \tau(t)$ is determined by a time-varying delay $\tau(t)$, where we choose a periodic delay modulation $\tau(t) = \tau_0 - A/(2\pi) \sin(2\pi t)$ with mean delay τ_0 and amplitude A . For $A = 0$, one recovers the well-studied case of constant delay $\tau = \tau_0$ [50–52,57]. Equation (1) includes famous examples for specific choices of the nonlinearity $f(x)$ such as the Mackey-Glass equation [58] and the Ikeda equation [48,50]. In the following, we consider a certain class of nonlinearities such that the resulting DDE has unstable hyperbolic fixed-point solutions (HFPSs), but also unstable nonhyperbolic fixed-point solutions (NHFPSs) in function space. For the numerical investigations of Eq. (1), we use as a specific example the so-called double-sine nonlinearity

$$f_\mu(x) = x - \mu(\sin(2\pi x) + \frac{1}{2} \sin(4\pi x)), \quad (2)$$

which is a natural extension of the previously studied climbing-sine nonlinearity with only unstable HFPSs in the corresponding DDE which results in normal chaotic diffusion [29]. It has the reflection symmetry $f_\mu(-x) = -f_\mu(x)$ and the discrete translational symmetry $f_\mu(x+1) = f_\mu(x) + 1$, where the latter allows one to divide the coordinate axis into unit cells. Equation (1) with Eq. (2) possesses a lattice of NHFPSs $x_{\text{nh}}^*(t) = 1/2 + n \in \mathbb{Z}$ at the half integers and HFPSs $x_h^*(t) = n \in \mathbb{Z}$ at the integers. A corresponding linear stability

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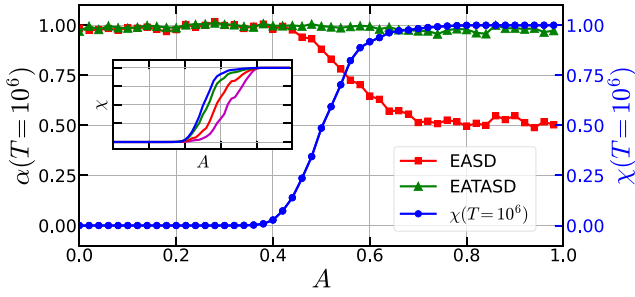


FIG. 1. EASD exponents α (red squares) show a transition from normal diffusion ($\alpha = 1$) to subdiffusion ($\alpha = 1/2$) as the modulation amplitude A increases. EATASD exponents (green triangles) remain equal to 1, indicating weak ergodicity breaking. This correlates with the mean fraction χ of plateau phases (blue circles), increasing from 0 (only turbulent phases, see Fig. 2 left or right) to 1 (only plateau phases). Due to nonstationarity, all results depend on the observation time T (see inset for fraction χ for $T = 10^3, 10^4, 10^5$, and 10^6 from bottom to top; ensemble size $N = 1024$).

analysis of the DDE shows that for the NHFPSs one stability exponent is equal to zero while all the others have negative real parts [59,60]. For the HFPSs, all stability exponents are complex with a finite number having positive real parts while all others have negative real parts. For constant delay, one knows in addition that these eigenvalues form a pseudocontinuous spectrum [55,61,62]. We demonstrate that the resulting time-delayed system, Eq. (1) with Eq. (2), can show weak chaos, chaotic subdiffusion, and weak ergodicity breaking, where the variable $x(t)$ can be interpreted as an unbounded phase variable. We point out, however, that the problem is formally infinite dimensional because Eq. (1) can be solved by the method of steps [63] leading to an iteration of functions, the so-called solution segments $x_n(t)$ defined on state intervals $(t_{n-1}, t_n]$ with $t_{n-1} = R(t_n)$. These solution segments can show diffusion but because they are of finite variation, it is sufficient for diffusion studies to consider only one value of each segment.

We quantify the observed diffusion by the ensemble-averaged squared displacement (EASD) $\langle \Delta x^2(t) \rangle_E := \langle [x(t) - x(0)]^2 \rangle_E \sim t^\alpha$ with diffusion exponent α and by the time-averaged squared displacement (TASD) $\langle \Delta x^2(t) \rangle_T := 1/(T - t) \int_0^{T-t} [x(t' + t) - x(t')]^2 dt'$. Because the latter can be a random variable even for $T \rightarrow \infty$, we consider its ensemble average (EATASD). To study the influence of the time-varying delay, we fix the mean delay, $\tau_0 = 1$, and we vary the amplitude of the delay modulation in the range $0 \leq A \leq 1$ such that the retarded argument $R(t)$ remains invertible. Furthermore, we set $\Theta = 50$ for all figures because we are interested in the large delay limit. While in Fig. 1 the numerically observed diffusion exponent from the EASD varies between 1 and $1/2$, the exponent from the EATASD is found to be 1 for every A , indicating a nonequivalence of ensemble and time averages. This means that we find anomalous diffusion and weak ergodicity breaking in our delay system.

To understand the transition from normal to anomalous diffusion, we first consider a single solution of the DDE, which

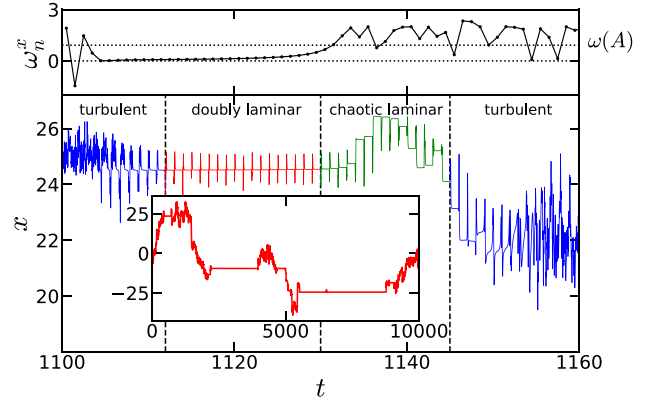


FIG. 2. A single solution $x(t)$ of the DDE on a short timescale for $A = 0.6$ shows the coexistence of doubly laminar, chaotic laminar, and turbulent phases ($\Theta = 50$, $\tau_0 = 1$, and $\mu = 0.9$). Plateau phases begin (end) when the instantaneous expansion rate ω_n^x in the coordinate direction is smaller (larger) than the expansion rate $\omega(A)$ on the time axis for a longer phase. The inset shows the same solution on a larger timescale, which due to its long waiting times (doubly laminar phases) is reminiscent of a subdiffusive continuous-time random walk.

was obtained numerically [64] for an intermediate value of the modulation amplitude A (see Fig. 2). On a short timescale (main figure), this time series shows a switching between two types of laminar and turbulent phases [65]. Turbulent phases are characterized by chaotic high-frequency fluctuations similar to turbulent chaos [48] as known from systems with constant delay ($A = 0$). Both forms of the laminar phases are characterized by a sequence of nearly constant plateaus (interrupted by irregular bursts). Therefore, we refer to them collectively as plateau phases. The special case of chaotic laminar phases has recently been discovered [66,67] and investigated experimentally [49,68–70] for delay systems as in Eq. (1). They are called laminar due to the laminar motion within the plateaus, and they are chaotic because the heights of the plateaus are mapped from one plateau to the next by the one-dimensional (1D) map $x' = f(x)$, with $f(x)$ from Eq. (2), which is chaotic if its Lyapunov exponent is positive $\lambda_f > 0$. For unbounded motion of the plateaus this can lead to (normal) laminar chaotic diffusion [29]. The condition for the development of plateaus is $\lambda_f + \lambda_R < 0$, where λ_R is the Lyapunov exponent of the 1D map $t' = R(t)$. This inequality just expresses that the stretching of the signal $x_n(t)$ in the t direction [due to the action of the Koopman operator associated with $R(t)$] in each step is on average stronger than the enlargement of fluctuations of $x_n(t)$ [due to the action of the nonlinear operator associated with $f(x)$]. For details, see Refs. [66,67]. A second type of laminar phases features plateaus whose heights change little between state intervals. They are laminar within and across plateaus, hence doubly laminar. This regular, nonchaotic motion dominates the long-term dynamics. Their appearance can be understood by noting that the map $x' = f_\mu(x)$ [due to Eq. (2) called the double-sine map], belongs to the well-studied class of Pomeau-Manneville maps [71–73] because it has marginally unstable fixed points at $x^* = 1/2 + n \in \mathbb{Z}$ where the map for $x \rightarrow x^*$ behaves as $f_\mu(x) \simeq x + 4\pi^3\mu(x - x^*)^3$. According to the theory in

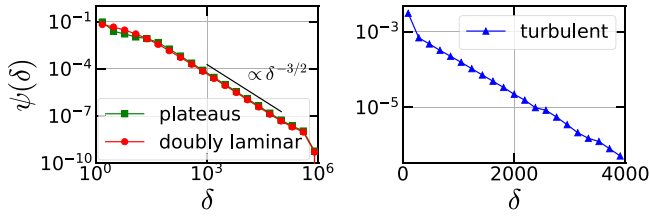


FIG. 3. The distributions $\psi_l(\delta)$ of durations of plateau phases and $\psi_{dl}(\delta)$ of doubly laminar phases are both heavy tailed, decaying asymptotically as $\delta^{-3/2}$. In contrast, the distribution $\psi_t(\delta)$ of the durations of turbulent phases is exponential (same parameters as in Fig. 2).

Ref. [7], this leads to long residence times Δ close to x^* with probability distribution $\psi(\delta) \sim \delta^{-3/2}$. Hence, the mean residence time close to x^* , where $f'_\mu(x^*) = 1$, diverges. As a consequence, the standard Lyapunov exponent λ_f of the map in the long time limit is equal to 0, i.e., the map shows weak chaos [35,36]. Furthermore, for our parameter ranges, $\lambda_R = \ln(1 - A)$ is always negative except for constant delay ($A = 0$), where $\lambda_R = 0$, such that the abovementioned condition, extended to weak laminar chaos, is fulfilled for every $0 < A < 1$. However, such Pomeau-Manneville-like maps are known to show weak ergodicity breaking, typically implying strong sample-to-sample fluctuations of time-averaged quantities such as the Lyapunov exponent [35,36] even in the long-time limit and especially on a shorter timescale, resulting in the switching behavior in Fig. 2. In more detail, we assign to each solution segment $x_n(t)$ taken at the attractive fixed points t_n^* of $R(t) + 1$ an instantaneous expansion rate $\omega_n^x = \ln |f'_\mu(x_n(t_n^*))|$ in the coordinate direction and an instantaneous expansion rate on the time axis $\omega_n^t = \ln |(R^{-1})'(t_n^*)| = -\ln |R'(t_n^*)| = -\ln(1 - A) = \omega(A)$, which is independent of the number n of the segment $x_n(t)$. Plateaus can develop if $\omega_n^x < \omega(A)$ for a longer period. This is the case if $x_n(t_n^*)$ comes close to the marginally unstable fixed points of the double-sine map (see Fig. 2). For laminar chaos, the dynamics of the plateau heights is governed by the iterated map $x' = f_\mu(x)$ [see Fig. 4(c)], which in our case shows intermittency with long residence times close to the marginally unstable fixed points (resulting in the delay system in the doubly laminar phases) interrupted by chaotic bursts (resulting in chaotic laminar phases). During the latter, ω_n^x can be larger than $\omega(A)$ for a longer period, leading to a destruction of the chaotic

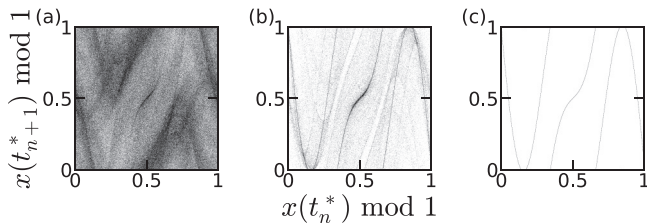


FIG. 4. Return maps for different values of the modulation amplitude A ($A = 0.5, 0.7$, and 0.9 from left to right). For larger values of A , one recognizes the structure of the double-sine map $x_{n+1} = f_\mu(x_n)$, but for decreasing values of A , the picture becomes more and more blurred due to high-dimensional chaotic bursts (see Fig. 5).

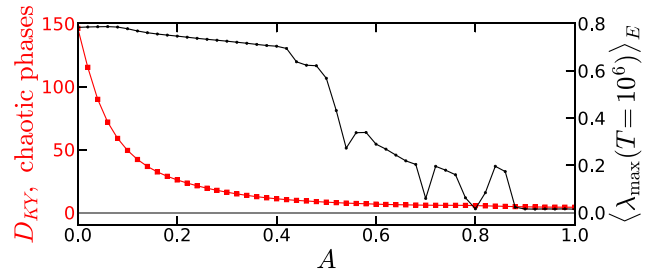


FIG. 5. The Kaplan-Yorke dimension D_{KY} calculated only from the chaotic (turbulent and chaotic laminar) phases shows a strong reduction of their effective dimension as the modulation amplitude A is increased. According to the fraction χ of plateau phases shown in Fig. 1, the maximal finite-time Lyapunov exponent $\langle \lambda_{\max}(T) \rangle_E$ is nearly 0 for ranges of A , where only plateau phases are present.

laminar phase and to the beginning of a new turbulent phase. This switching between doubly laminar, chaotic laminar, and turbulent phases within one single time series is a new type of dynamical behavior.

Next, we characterize this behavior statistically. First, we consider the distributions of durations of turbulent and plateau phases for one value of the modulation amplitude A (see Fig. 3). The durations of the doubly laminar phases are found to be asymptotically equal in distribution to the residence times close to the marginally unstable fixed points of the double-sine map. Therefore, the distribution $\psi_{dl}(\delta)$ of doubly laminar phases is heavy tailed and decays asymptotically as $\delta^{-3/2}$. Because plateau phases consist of doubly laminar phases and chaotic laminar phases, where the latter are exponentially distributed, doubly laminar and plateau phases show the same heavy tails in their distributions. Turbulent phase durations are also exponentially distributed. These basic properties appear to hold for all values of A , even for $A = 0$.

A variation of the modulation amplitude A , however, drastically changes the parameters of the distributions, which has a strong impact on the diffusive behavior of the system. To understand this in more detail, we first consider the mean fraction χ of plateau phases, i.e., the sum of all durations of chaotic laminar and doubly laminar phases in one single time series divided by the length of the time series averaged over an ensemble of trajectories (see Fig. 1). This fraction strongly depends on the modulation amplitude A and increases from almost 0 (effectively only turbulent phases) to 1 (only plateau phases) as A is varied from 0 to 1 (main figure). Because of the diverging mean of the distribution of plateau phases and the resulting nonstationarity, the fraction $\chi = \chi(T)$ depends on the observation time T (see inset). This fraction is also reflected in the return maps in Fig. 4, which show how the solution taken in the middle of a state interval is correlated with the corresponding value from the next state interval. For larger values of A , where (weak) laminar chaos prevails, one can see the structure of the double-sine map which becomes successively blurred for decreasing values of A . The blurring is caused by high-dimensional excursions of the state function (see Fig. 5). The fraction of plateau phases is related to the diffusive properties of the system as follows. We numerically found that turbulent phases always show normal chaotic diffusion, whereas the plateaus follow

the dynamics of the double-sine map. This map shows chaotic diffusion due to a “transfer range” in each unit cell to one of the neighboring cells [74], which becomes subdiffusive due to the long residence times close to the marginally unstable fixed points [7]. It is known that such kind of dynamics can be described statistically by a subdiffusive continuous-time random walk with instantaneous jumps and heavy-tailed waiting time distributions [9]. The EASD asymptotically increases according to a power law, i.e., $\langle \Delta x^2(t) \rangle_E \sim t^{1/2}$, whereas the EATASD grows linearly in time, which is a manifestation of weak ergodicity breaking [39,40]. This explains why normal diffusion is observed in Fig. 1 if $\chi \approx 0$, whereas subdiffusion is observed in the EASD if $\chi \approx 1$ while the EATASD always shows normal diffusion. Furthermore, our time-delayed system shows weak chaos in the sense that the maximal Lyapunov exponent $\lambda_{\max}(T = \infty)$ vanishes [75,76]. For finite observation times T , $\lambda_{\max}(T)$ also exhibits the apparent transition between normal (positive) and anomalous behavior (zero) as shown in Fig. 5.

Also for the extreme case of constant delay, $A = 0$, the occurrence of anomalous diffusion and weak chaos can be explained by an analytical approach using the existence of NHFPSs in function space whose largest stability exponent is equal to 0. The evolution of initial functions close to these NHFPSs can be understood by center manifold theory [77,78]. The center manifold is invariant under the dynamics and is tangential to the center subspace spanned by the eigenfunctions belonging to stability exponents with vanishing real parts. The dynamics on the center manifold in the vicinity of the fixed-point solutions $x_{\text{nh}}^*(t)$ is approximated up to third order by the ordinary differential equation $\dot{y}(t) = \Theta/(\Theta + 1) 4\pi^3 \mu y^3(t)$, with $y(t) = x(t) - x_{\text{nh}}^*(t)$ [79,80], which, except for the prefactor $\Theta/(\Theta + 1)$, is identical to the continuous-time approximation of the dynamics of the double-sine map in the vicinity of the marginally unstable fixed points [7,13]. This leads to identical residence time statistics with a diverging mean leading to anomalous subdiffusion. For $A \rightarrow 0$, the doubly laminar phases lose their plateau character and become classical laminar phases [71–73], whereas chaotic laminar phases become high-dimensional through higher-order generalized laminar chaos [67] and merge with the high-dimensional turbulent bursts. However, the probability that a generic solution of the DDE comes close to the NHFPSs is relatively small because of the large effective dimension of the turbulent phases which we found to be proportional to Θ as in the case of chaotic systems with constant delay [57]. Correspondingly, we numerically found that the distribution of turbulent phases is exponential with a mean value that increases exponentially with their effective dimension. With an appropriate stochastic model [79], one can show that there is a crossover time in the corresponding EASD from normal to anomalous diffusion, which increases exponentially with Θ . Therefore, for large values of Θ (large delay limit), subdiffusion might not be observable as it is the case in Fig. 1. For small values of Θ , however, we were able to observe subdiffusion even in the case of a constant delay. For increasing amplitudes A of the delay modulation, the effective dimension decreases as it is shown in Fig. 5 by means of the Kaplan-Yorke dimension calculated only from turbulent and chaotic laminar phases.

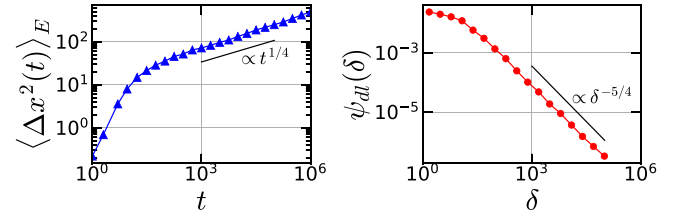


FIG. 6. The EASD $\langle \Delta x^2(t) \rangle_E$ obtained from an ensemble of $N = 1024$ trajectories for the DDE in Eq. (1) with the nonlinearity from Eq. (3) shows asymptotic subdiffusion with diffusion exponent $\alpha = 1/4$. In accordance with the CTRW theory, the distribution of doubly laminar phases $\psi_{\text{dl}}(\delta)$ is heavy tailed with the exponent $-\alpha - 1 = -5/4$ ($\Theta = 50$, $\tau_0 = 1$, $A = 0.98$, $\mu = 4.0$).

This reduction of the dimension allows the observation of anomalous subdiffusion in the large delay limit also for shorter time series.

After we developed a deeper understanding for the occurrence of the anomalous behavior in our specific delay system for the constant delay case and the case of a time-varying delay, we want to discuss the general conditions under which this type of dynamics can be observed. We have seen that the classical laminar phases and the doubly laminar phases in the constant and time-varying delay case, respectively, are responsible for the anomalous behavior. They are caused by the existence of NHFPSs in the DDE, which for DDEs of the structure as in Eq. (1) are related to the existence of marginally unstable fixed points in the corresponding iterated map $x' = f(x)$. We expect these fixed points to be a necessary condition for the anomalous behavior in the DDE. In order to substantiate this, we consider a different nonlinearity, the so-called triple-sine nonlinearity defined by

$$\tilde{f}_\mu(x) = x - \mu \left(\frac{1}{2\pi} \sin(2\pi x) + \frac{1.6}{4\pi} \sin(4\pi x) + \frac{0.6}{6\pi} \sin(6\pi x) \right). \quad (3)$$

Analogously to the double-sine map, the corresponding iterated map, the triple-sine map $x' = \tilde{f}_\mu(x)$, has marginally unstable fixed points at the half integers, where the triple-sine map behaves like $\tilde{f}_\mu(x) \simeq x + 16/5\pi^4 \mu (x - x^*)^5$ for $x \rightarrow x^*$. According to the theory in Ref. [7], due to the quintic term [$z = 5$ with $\alpha = 1/(z - 1) = 1/4$], the exponent of the residence time statistics is given by $-\alpha - 1 = -5/4$. Consequently, the distribution of the durations of doubly laminar phases numerically obtained for the corresponding DDE shows the same power-law tail (see the right part of Fig. 6). Also the diffusion exponent $\alpha = 1/4$ of the asymptotic increase of the EASD is confirmed numerically (left part of Fig. 6). This example suggests that every nonlinearity generating chaotic diffusion in the corresponding iterated map with marginally unstable fixed points x^* and asymptotic behavior $f(x) \sim x + (x - x^*)^z$ for $x \rightarrow x^*$ with $z > 2$ leads to anomalous subdiffusion with the diffusion exponent $\alpha = 1/(z - 1)$ in the DDE.

In summary, a coexistence of doubly laminar, chaotic laminar, and turbulent phases alternating within a single time series was found in a typical class of time-delayed systems, where unstable nonhyperbolic fixed-point solutions

play a central role. The nonhyperbolicity is triggered by a stationary state x^* with $f(x^*) = x^*$ and $f'(x^*) = 1$. Such states are typical for bifurcation points and they are related to the occurrence of type-III intermittency [72]. A variety of nonlinear functions $f(x)$ fulfilling the above conditions are well known in the literature [7–9,81]. For a proper description of the nonhyperbolic dynamics, cubic ($z = 3$) corrections of the linearized equations of motion become relevant, which is typical for systems at bifurcation points [82]. So we expect that our results about chaotic subdiffusion, weak chaos, and weak ergodicity breaking generalize to many other systems with $z = 3$, but also to the much wider class of systems with $z > 2$. However, crossover times may depend exponentially on system parameters, making anomalous behavior unobservable on short timescales, as in our constant-delay case. By generalizing anomalous chaotic behavior known from low-dimensional systems to infinite-dimensional delay systems, we opened a new field of research leading to many interesting questions. For instance, as chaotic subdiffusion can occur

in delay systems, is it also possible to find superdiffusion and under which conditions? How do the ergodic properties of known stochastic models transfer to chaotic superdiffusion in infinite-dimensional systems? While the description of the evolution of densities in delay systems is an open problem even for normal behavior [83], the development of a general theory for anomalous behavior in such systems is even more challenging since one has to deal with nonnormalizable densities leading to infinite ergodic theory, which is well developed for one-dimensional systems [35,36,84,85] but almost nothing is known for infinite-dimensional systems.

Note added. While this manuscript was under review, one of the coauthors, Günter Radons, passed away.

The authors gratefully acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Grants No. 438881351 and No. 456546951.

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