

Critical scaling and supercritical coarsening in active model B+

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We study critical dynamics and phase-ordering kinetics in active model B (AMB) and its minimal extension, active model B+ (AMB+), using deterministic simulations in two dimensions. At criticality $r_c = 0$, both models display identical mean-field scaling despite nonequilibrium currents, with order-parameter decay with time as $m(t) \sim t^{-\alpha}$, with $\alpha = \frac{1}{4}$, and the dynamical exponent being $z = 4$. A generalized equal-area construction yields the binodal densities and phase diagram of AMB+. For supercritical quenches, domain size grows as $L(t) \sim t^{1/3}(1 + c/\ln t)$, revealing logarithmic corrections to the classic $t^{1/3}$ growth law; moreover, it is consistent with the functional renormalization group predictions for marginal activity in $d = 2$. The logarithmic corrections are clearly evident in both AMB and AMB+ in the macrophase-separated regime. However, in AMB+, domain growth is arrested in the parameter regime where the active current opposes the formation of macroscopic clusters, leading to long-lived microphase-separated states.

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I. INTRODUCTION

Active matters are nonequilibrium systems in which individual agents continuously consume energy from the environment to self-propel [1,2]. Examples span a wide range of scales, including schools of fish [3], flocks of birds [4], active colloidal suspensions [5–11], the cytoskeleton of cells [12–15], molecular motors [16–20], and actin filaments [21]. Because energy is injected locally at the microscopic scale, these systems generically violate detailed balance, allowing additional symmetry-permitted terms in their coarse-grained dynamics. The resulting local breaking of time-reversal symmetry leads to novel collective phenomena absent in equilibrium systems. A prominent example of this kind of system is motility-induced phase separation (MIPS), in which self-propelled particles spontaneously phase-separate into dense and dilute phases due to density-dependent motility [22–24]. This mechanism has been extensively confirmed in simulations [24–28] and experiments [29,30], and has motivated the development of continuum field theories describing phase separation in active matter systems [23,31].

Motivated by the apparent similarity between MIPS and equilibrium liquid-vapor phase separation, early theoretical efforts sought to extend equilibrium phase-separation formalisms to active systems [22,23,32], speculating that

time-reversal symmetry (TRS) is restored in the steady state at the macroscopic level [6,33,34]. The equilibrium liquid-vapor phase separation with a conserved scalar order parameter ϕ is described by the Cahn-Hilliard equation [35], also known as model B in the classification of Hohenberg and Halperin [36]. Here, the dynamics obeys detailed balance and the chemical potential derives from a free-energy functional, $\mu_{\text{eq}} = \delta\mathcal{F}/\delta\phi$. Consequently, both critical behavior and late-stage coarsening are well understood: the upper critical dimension is $d_c = 4$, static critical exponents belong to the Ising universality class, and domain growth after a deep quench follows the Lifshitz-Slyozov law $L(t) \sim t^{1/3}$ for $d \geq 2$ [37]. Importantly, within the renormalization group framework, equilibrium model B possesses no marginal operators in $d = 2$, so logarithmic corrections to scaling are not expected.

Despite these equilibrium-inspired descriptions of MIPS, accumulating evidence indicates that active phase separation cannot, in general, be reduced to an effective equilibrium process. Subsequent numerical investigations of repulsive active Brownian particles (ABPs) [27] revealed the emergence of a steady-state *bubble* phase, providing clear evidence of broken TRS with clusters stabilizing at a mesoscopic scale instead of spanning across the system. Interestingly, this behavior of ABPs contrasts with experimental observations in self-propelled colloids [29,38] where aggregation leads to the formation of *clusters*.

The simplest extension of equilibrium model B, which accounts for this broken TRS, is active model B (AMB), obtained by adding a gradient-nonlinear term to the equilibrium chemical potential, $\mu = \delta\mathcal{F}/\delta\phi + \lambda(\nabla\phi)^2$ [39]. This term cannot be derived from a free-energy functional and therefore explicitly breaks detailed balance. While AMB captures key nonequilibrium features of MIPS, renormalization group (RG) analysis shows that it is not closed under coarse graining, as additional active terms are generated, necessitating a

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more general model. The minimal RG-consistent extension is active model B+ (AMB+), first emphasized by Tjhung *et al.* [40], with a dynamics given by

$$\begin{aligned}\partial_t \phi &= -\nabla \cdot (\mathbf{J} + \sqrt{2DM}\mathbf{\Lambda}), \\ \frac{\mathbf{J}}{M} &= -\nabla \left[\frac{\delta \mathcal{F}}{\delta \phi} + \lambda |\nabla \phi|^2 + \frac{\nu}{2} \nabla^2(\phi^2) \right] \\ &\quad + \zeta (\nabla^2 \phi) \nabla \phi,\end{aligned}\quad (1)$$

where $\mathbf{\Lambda}(\mathbf{r}, t)$ is Gaussian white noise with zero mean and unit variance. M is mobility and D is the thermal diffusion coefficient. The negative gradient term in current $\mathbf{J}$ can be thought of as an effective chemical potential $\mu = \mu_{\text{eq}} + \mu_a$, where the nonequilibrium chemical potential is generated by the active terms, $\mu_a = \lambda |\nabla \phi|^2 + \frac{\nu}{2} \nabla^2(\phi^2)$. The equilibrium chemical potential $\mu_{\text{eq}} = \delta \mathcal{F} / \delta \phi$, however, is derived from a free-energy functional of model B,

$$\mathcal{F}[\phi] = \int \left[f(\phi) + \frac{\kappa}{2} |\nabla \phi|^2 \right] d\mathbf{r}, \quad (2)$$

with $f(\phi) = \frac{r}{2} \phi^2 + \frac{u}{4} \phi^4$, being the bulk free-energy density. The term $\zeta (\nabla^2 \phi) \nabla \phi$ in $\mathbf{J}$ is the active current which cannot be derived from a chemical potential, and it explicitly breaks TRS.

Together, all the active couplings ζ, λ, ν break the $\phi \rightarrow -\phi$ invariance, and produce nonequilibrium currents. Interestingly, they all share the same canonical scaling dimension $(2-d)/2$, and are therefore marginal at $d=2$. A recent functional renormalization group (FRG) analysis by Fejős *et al.* [41] clarifies the implications of this marginality. In two dimensions, the beta functions for λ and ν lack linear terms and begin at quadratic order, so the RG flow does not approach a conventional infrared fixed point. Instead, the couplings evolve logarithmically slowly with scale: $\zeta(\ell), \lambda(\ell), \nu(\ell) \sim 1/\ell$, with $\ell = \ln(L/\xi)$. Consequently, the effective mobility acquires scale dependence [i.e., $M \rightarrow M_{\text{eff}}(L)$], producing logarithmic corrections to correlation functions, as expected for systems governed by marginal operators at their upper critical dimension. We propose that the implication of this marginality extends beyond criticality into the phase-separated regime and affects the coarsening dynamics substantially. A logarithmically slow evolution of couplings with scale may lead to effective interfacial mobility as $M_{\text{eff}}(L) = M_0(1 + \frac{b_0}{\ln(L/\xi)})$, which results in a modified growth law: $L(t) = L(0)t^{1/3}(1 + \frac{c}{\ln t})$.

In this work, we revisit coarsening dynamics in AMB and AMB+ and investigate, through extensive numerical simulations, the possible emergence of logarithmic corrections to the growth law $L(t)$. We systematically analyze the interplay between different active terms and their collective influence on coarsening dynamics, demonstrating that the apparent deviation from the $t^{1/3}$ growth law can be consistently explained by the inclusion of suitable logarithmic corrections. Our results, therefore, indicate that activity introduces marginal corrections to conserved diffusive coarsening rather than modifying the underlying scaling law itself.

The article is organized as follows. In Sec. II, we study the critical behavior of AMB and AMB+ in the absence of any thermal noise and thereby verify mean-field predictions by

extracting critical exponents through finite-size-scaling (FSS) analysis. Here, we show that our order parameter faithfully reproduces the coexistence lines of phase separation transitions predicted theoretically. In Sec. III, we study the coarsening dynamics of both AMB+ and AMB, and find the existence of a slow logarithmic correction to the well-known $t^{1/3}$ power law, arising from the marginality of the activity parameters in $d=2$. We conclude with the results in Sec. IV, with a short discussion on possible future directions.

II. CRITICAL BEHAVIOR

First, we investigate the critical relaxation dynamics of both AMB and AMB+ in the absence of thermal noise ($D=0$). Without noise, and in the absence of active terms, the dynamics is purely deterministic and governed by the mean-field equations of motion [36]. As a result, the critical behavior in $d=2$ follows the *classical* mean-field exponents [42,43]. We ask how the active terms (λ, ν, ζ) modify the *dynamical* critical exponent z even at the mean-field level, or whether the critical relaxation remains identical to that of equilibrium model B. By performing deterministic numerical simulations at the critical point $r_c=0$, we can isolate the effects of the nonlinear gradient terms on the growth and saturation of the order parameter.

A. Scaling and critical exponents

At the Gaussian (mean-field) fixed point, the static critical exponents take their classical values:

$$\beta = \frac{1}{2}, \quad \gamma = 1, \quad \eta = 0, \quad \nu = \frac{1}{2}. \quad (3)$$

The order parameter and the susceptibility scale as $\phi \sim |r|^\beta$ and $\chi \sim |r|^{-\gamma}$ for $r < 0$. The critical exponents η and ν are associated with the equal-time correlation function and the correlation length, respectively. The active terms (ζ, λ, ν) alter the coarsening properties of the system by modifying the dynamics of the interfaces that separate the high and low-density regimes [34,44], and one does not expect any change in the static critical exponents. Our aim would be to investigate if the dynamical critical exponents [45] of AMB and AMB+ are modified. Critical behavior and finite-size scaling (FSS) follow standard scaling theory [46–48]. In equilibrium model B ($\lambda=0=\nu, \zeta=0$), the purely relaxational deterministic equation at the critical point [37],

$$\partial_t \phi = -\kappa \nabla^4 \phi + \mathcal{O}(\phi^3, \nabla^2 \phi^3, \dots), \quad (4)$$

gives the scaling relation $[t] = [x]^4$, resulting in a growth of correlation length ξ as $t^{1/z}$ with the dynamic exponent being $z=4$.

Another critical exponent associated with the dynamics is ν_t , which governs the scaling of the relaxation time ξ_t near criticality as $\xi_t \sim |r|^{-\nu_t}$ and obeys a scaling relation $\nu_t = z\nu$ [46,47]. Thus from Eq. (3) we have $\nu_t = \frac{1}{2}$. The relaxation time ξ_t diverges near criticality as $\xi_t \sim |r|^{-\nu_t}$, where the temporal exponent satisfies the scaling relation $\nu_t = z\nu$ [46,47]. Using Eq. (3), we obtain $\nu_t = 4 \times (1/8) = 1/2$. Thus the dynamical critical exponents of the transition are

$$z = 4, \quad \nu_t = 1/2. \quad (5)$$

To study the critical behavior in AMB and AMB+, we need to define a suitable order parameter. Since the scalar field $\phi(\mathbf{r}, t)$ can be interpreted as the deviation of the coarse-grained local density $\rho(\mathbf{r}, t)$ from the conserved global density $\bar{\rho}$ of the system [40,49]

$$\phi(\mathbf{r}, t) = \rho(\mathbf{r}, t) - \bar{\rho}, \quad (6)$$

clearly an inhomogeneous density profile results in $\phi \neq 0$: the high and low-density regions are represented by $\phi > 0$ and $\phi < 0$, respectively. Thus one can define the order parameter of the system to be

$$m(t, L) \equiv \frac{1}{L^d} \int d^d r |\phi(\mathbf{r}, t)|, \quad (7)$$

which vanishes in the mixed phase and picks up nonzero values in the phase-separated state. A similar order parameter is defined for MIPS transition in lattice models of run and tumble particles [50].

In a finite system of linear size L , the critical fluctuations at $r_c = 0$ are cut off by the finite system size, leading to a finite-size scaling (FSS) of the order parameter [46,51],

$$m(t, L) = t^{-\alpha} f(t/L^z), \quad (8)$$

z being the dynamical exponent at criticality. Since the correlation length ξ and time ξ_t scales at criticality as $\xi \sim \varepsilon^{-\nu}$ and $\xi_t \sim \varepsilon^{-\nu_t}$, respectively, with $\varepsilon = r_c - r$, one expects $\xi_t \sim \xi^{\nu_t/\nu}$. Thus, for a finite system, we have $t \sim L^{\nu_t/\nu}$, resulting in $z = \nu_t/\nu$. Also, in the supercritical steady state (i.e., in $t \rightarrow \infty$ limit) m must saturate to a value m_s , proportional to $|\varepsilon|^\beta \equiv L^{\beta/\nu}$. Thus

$$m(t, L) = \begin{cases} t^{-\alpha} & \text{as } L \rightarrow \infty \\ L^{-\beta/\nu} & \text{as } t \rightarrow \infty \end{cases}. \quad (9)$$

Now, for the scaling function $f(\cdot)$ in Eq. (8) to be consistent with the above, it must obey a form

$$f(x) = \begin{cases} \text{constant} & \text{for } x \rightarrow 0 \\ x^{\beta/(z\nu)} & \text{for } x \rightarrow \infty \end{cases}, \quad (10)$$

with $\alpha = \frac{\beta}{z\nu} = \frac{\beta}{\nu_t} = \frac{1}{4}$ [from Eqs. (3) and (5)].

We first analyze the time evolution of the order parameter $m(t, L)$ to extract the critical exponent α . The numerical procedure for integrating Eq. (1) is described in Appendix. For a system size of $L = 256$, we plot $m(t, L)$ versus t (in log-log scale) at different r values, viz., at $r = -0.01, -0.005, 0, 0.01, 0.02$ in Fig. 1. For $r > 0$, the curves decay faster than power laws, while for $r < 0$ they saturate, indicating a critical point at $r_c = 0$. Using the critical scaling relation $m(t, L) \sim t^{-\alpha}$ at r_c , a linear fit in log scale yields $\alpha = 0.25$. The exponent values obtained for both AMB ($\zeta = 0, \lambda = -2, \nu = 0.5$) in Fig. 1(a) and AMB+ ($\zeta = 1, \lambda = -1, \nu = 0.5$) in Fig. 1(b) are the same.

We must mention that the critical value r_c also suffers from the finite-size effects; the finite-size critical value $r_c(L)$ is slightly smaller than its thermodynamic limit $r_c(\infty) = 0$. This shift follows directly from the linear dispersion relation

$$\omega(k) = -k^2(r + \kappa k^2), \quad (11)$$

which implies that the homogeneous density profile is linearly stable only when $r + \kappa k^2 > 0$. In a finite system with periodic

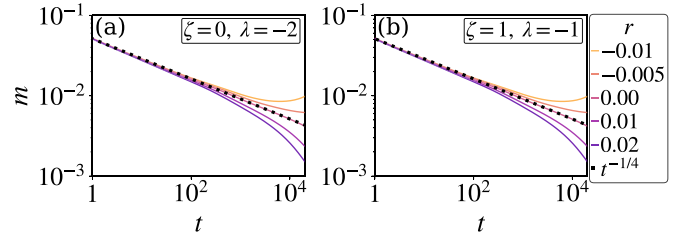


FIG. 1. Decay of order parameter $m(t)$ for (a) AMB: $\zeta = 0, \lambda = -2, \nu = 0.5$, and (b) AMB+: $\zeta = 1, \lambda = -1, \nu = 0.5$. In the supercritical regime ($r < 0$), m saturates to a steady value m_s whereas it vanishes in the subcritical ($r > 0$) region. At the critical threshold, i.e., $r_c = 0$, $m(t) \sim t^{-\alpha}$, both AMB and AMB+ exhibit a critical decay exponent $\alpha = \frac{1}{4}$ (which is the slope of the dashed line). Here, the system size considered is $L = 256$.

boundary conditions, the wave vectors are discrete, and the smallest nonzero mode is $k_{\min} = 2\pi/L$. Stability must therefore hold when $r + \kappa k_{\min}^2 > 0$, yielding

$$r_c(L) = -\kappa \left(\frac{2\pi}{L} \right)^2. \quad (12)$$

Thus the shift of the critical point follows the FSS form $r_c(L) - r_c(\infty) \sim L^{-1/\nu}$, with $\nu = 2$.

Equation (12) shows that a finite system at $r = 0$ is effectively subcritical, since the true critical point is shifted to $r_c(L) < 0$. Consequently, the order parameter $m(t, L)$ ultimately decays faster than a power law at long times. An extended algebraic regime showing $m(t) \sim t^{-\alpha}$ is therefore observed only for sufficiently large L (like $L = 256$ in Fig. 1) or when dynamics is analyzed relative to $r_c(L)$.

To determine the order-parameter exponent β , we analyze the supercritical regime $r < r_c$, where the order parameter $m(t, L)$ saturates at long times to a finite value,

$$m_s \sim \varepsilon^\beta, \quad \varepsilon = (r_c - r). \quad (13)$$

We perform simulations at system size $L = 128$, for which $r_c = -(2\pi/128)^2$ (with $\kappa = 1$), and measure the steady-state value m_s for several values of r . In Fig. 2(a) we show the saturation of $m(t, L)$ with t in log-log plot for both AMB

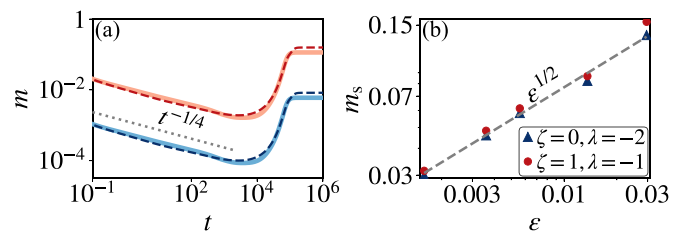


FIG. 2. (a) Log-log plot of order parameter m with time t in the supercritical region ($r < 0$), for AMB ($\zeta = 0, \lambda = -2, \nu = 0.5$ in the lower curves) and AMB+ ($\zeta = 1, \lambda = -1, \nu = 0.5$ in the upper curves). We consider two system sizes: $L = 64$ (solid lines), and 128 (dashed lines) at $r = -0.016$, which is well below their respective $r_c(L)$ given in Eq. (12). Data for AMB+ is multiplied by a factor for better visual clarity. (b) Log-log plot of m_s vs $\varepsilon = r_c(L) - r$ for both AMB and AMB+, $m_s \sim \varepsilon^\beta$ gives $\beta = \frac{1}{2}$. Here, we consider a system size $L = 128$.

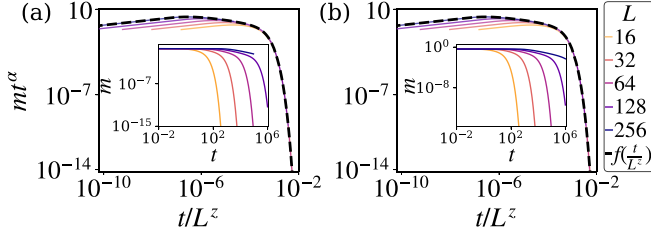


FIG. 3. (a) Scaling collapse in AMB ($\zeta = 0, \lambda = -2, \nu = 0.5, r = 0$): log-log plot of $m(t)t^\alpha$ as a function of t/L^z for different L collapse onto each other when $\alpha = \frac{1}{4}, z = 4$. (b) The same for AMB+ ($\zeta = 1, \lambda = -1, \nu = 0.5$). For both AMB and AMB+, the collapsed curve matches the corresponding scaling function of equilibrium model B (dashed line) after suitable rescaling of the x axis.

($\zeta = 0, \lambda = -2, \nu = 0.5$) and AMB+ ($\zeta = 1, \lambda = -1, \nu = 0.5$) at an arbitrary value of $r = -0.016$, which is well below the respective $r_c(L)$ for $L = 64$ and 128 . For both cases, the curves initially decay as $t^{-1/4}$, but before saturation, they exhibit undershooting. A log-scale plot of this m_s as a function of $\varepsilon = (r_c - r)$, with $r_c > r$, is shown in Fig. 2(b) for both AMB and AMB+. In both cases, we find that $\beta = \frac{1}{2}$ fits the data quite well.

Using the critical exponents $\beta = \nu = \frac{1}{2}$ and $\nu_t = 2$, we test the scaling relation in Eq. (8). We compute $m(t, L)$ for different system sizes and plot mt^α vs t/L^z with $\alpha = \frac{1}{4}$ and $z = 4$. Excellent data collapse is observed for both AMB ($\zeta = 0, \lambda = -2, \nu = 0.5$) [Fig. 3(a)] and AMB+ ($\zeta = 1, \lambda = -1, \nu = 0.5$) [Fig. 3(b)]. The dashed lines in Figs. 3(a) and 3(b) are the scaling function $f(\cdot)$ obtained for the equilibrium model B, which matches well with that of the AMB and AMB+. This confirms that (i) $m(t)$, defined in Eq. (7), serves as an appropriate order parameter for the phase-separation transition, and (ii) the transition in AMB and AMB+ belongs to the same universality class as equilibrium model B. This finding is consistent with lattice models [50,52] of active particles, which exhibit Ising universality in two dimensions.

B. Phase diagram: Coexistence line

In this section, we determine the binodal densities ϕ_- (gas or vapor phase) and ϕ_+ (liquid or dense phase), defined as the coexisting steady-state densities across a flat interface. With $\phi > 0$ representing the high-density region and $\phi < 0$ the low-density region, the corresponding areas can be computed numerically as

$$A_\pm = \frac{1}{L^d} \int d^d r \Theta(\pm \phi(\mathbf{r}, t)), \quad (14)$$

where $\Theta(\cdot)$ is the Heaviside step function. Now the binodal densities $\phi_\pm$ can be defined as

$$\phi_\pm = \frac{\Phi_\pm}{A_\pm}; \quad \Phi_\pm = \int d^d r \phi(\mathbf{r}, t) \Theta(\pm \phi(\mathbf{r}, t)). \quad (15)$$

To calculate $\phi_\pm$ analytically, we follow the approach prescribed in Refs. [39,44,53]. For the noiseless ($D = 0$) case of Eq. (1), if we consider a flat interface, then clearly the density profile depends only on the normal coordinate x (say). Now, in this one-dimensional ($d = 1$) system, using the

identities (i) $\frac{\nu}{2} \partial_{xx} \phi^2 = \nu (\partial_x \phi)^2 + \nu \phi \partial_{xx} \phi$ and (ii) $\partial_x (\partial_x \phi)^2 = 2 \partial_x \phi \partial_{xx} \phi$, the coefficients of the terms $\partial_{xx} \phi$ and $(\partial_x \phi)^2$ in Eq. (1) become $K(\phi) = \kappa - \nu \phi$ and $\tilde{\lambda} = \lambda + \nu - \zeta/2$, respectively. The problem thus reduces to a one-dimensional gradient current $J = -\partial_x \mu$, with (setting $M = 1$) an effective nonequilibrium chemical potential

$$\mu = f'(\phi) - K(\phi) \partial_{xx} \phi + \tilde{\lambda} (\partial_x \phi)^2, \quad (16)$$

where $f'(\phi) = r\phi + u\phi^3$. This chemical potential has the same form as in Ref. [40], with $\tilde{\lambda}$ replacing $\lambda - \zeta/2$.

We seek a stationary profile $\phi(x)$ (with $u = 1$) satisfying $\phi(-\infty) = \phi_-$ and $\phi(+\infty) = \phi_+$, describing a smooth interface centered at $x = 0$. In the bulk, homogeneity implies $J = 0$, so the chemical potential should remain constant there, say $\mu = \mu_0$, yielding the first coexistence condition,

$$f'(\phi_-) = f'(\phi_+) = \mu_0. \quad (17)$$

This relation alone does not determine the binodals. Following [44,53], we introduce a pseudodensity $R(\phi)$ and consider the integral $I = \int_{x_1}^{x_2} \mu \partial_x R(\phi) dx$. Using $\mu(x) = \mu_0$, the integral takes the implicit form

$$I = \mu_0 (R(\phi_+) - R(\phi_-)). \quad (18)$$

Due to the presence of the gradient terms in μ as in Eq. (16), computing I explicitly is generally difficult. For the specific form of the pseudodensity $R(\phi)$, such that the relation

$$KR'' = -(2\tilde{\lambda} + K')R' \quad (19)$$

holds, where $' \equiv \partial_\phi$, this can be done. Then the integral explicitly gives

$$I = g(\phi_+) - g(\phi_-), \quad (20)$$

where the pseudopotential $g(\phi)$ satisfies the relation $\frac{\partial g}{\partial R} = \frac{\partial f}{\partial \phi}$.

Then, Eqs. (18) and (20) enforce the equality of the pseudopressure, defined as $P = R \frac{\partial g(\phi)}{\partial R} - g(\phi)$ at the interface between the two phases, giving a second relation,

$$R(\phi_-) \mu_0(\phi_-) - g(\phi_-) = R(\phi_+) \mu_0(\phi_+) - g(\phi_+). \quad (21)$$

With the specific form of the chemical potential as in Eq. (16), the explicit form of R and g is calculated for the case of AMB+, to be

$$R = -\frac{1}{\nu} \frac{y^{1+a}}{(1+a)}, \quad (22)$$

$$g = \frac{1}{\nu^4(4+a)} y^{4+a} - \frac{3\kappa}{\nu^4(3+a)} y^{3+a} + \frac{3\kappa^2/\nu^4 + r/\nu^2}{(2+a)} y^{2+a} - \frac{\kappa^3/\nu^4 + r\kappa/\nu^2}{(1+a)} y^{1+a}, \quad (23)$$

where $y = \kappa - \nu\phi$, and $a = \frac{b}{\nu}$ with $b = 2\lambda + \nu - \zeta$.

The binodal densities are determined as stationary solutions ($\phi_\pm$) by enforcing equality of μ_0 and P across the two coexisting phases. Numerically, it can be obtained by simultaneously solving Eqs. (17) and (21), which is equivalent to a common-tangent construction on $\phi(R)$.

To obtain $\phi_\pm$ numerically, we use Eqs. (14) and (15). In steady state, we compute the area fractions $A_\pm$ as the number of grid points with $\phi(\mathbf{r}, t) \geq 0$, and $\Phi_\pm$ as the corresponding

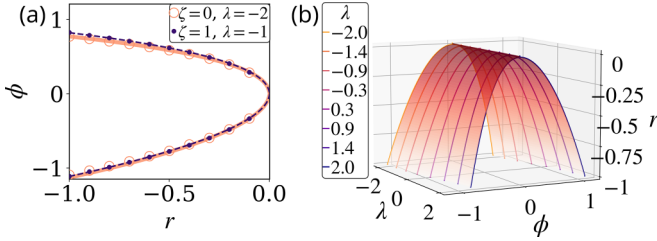


FIG. 4. (a) Binodal diagram in the r – ϕ plane for AMB ($\zeta = 0$, $\lambda = -2$, $\nu = 0.5$) and AMB+ ($\zeta = 1$, $\lambda = -1$, $\nu = 0.5$). The solid line (AMB) and the dashed line (AMB+) show the exact analytical values of ϕ_+ (upper branch) and ϕ_- (lower branch), obtained from Eqs. (17) and (21). Corresponding points are obtained from simulating the noiseless partial differential equation numerically on a 128×128 grid and using Eqs. (14) and (15). (b) Full three-dimensional phase diagram of phase separation transition in AMB+ in the ϕ – λ – r plane for $\zeta = 1$, $\nu = 0.5$.

sums of $\phi(\mathbf{r}, t)$. Then the coexistence densities can be obtained as $\phi_{\pm} = \Phi_{\pm}/A_{\pm}$. A binodal diagram in the r – ϕ plane is shown in Fig. 4(a) for AMB ($\zeta = 0$, $\lambda = -2$, $\nu = 0.5$) and AMB+ ($\zeta = 1$, $\lambda = -2$, $\nu = 0.5$), showing excellent agreement with analytical predictions, therefore validating the order parameter defined in Eq. (15). Figure 4(b) displays the binodal surface in the extended ϕ – λ – r space at fixed $\zeta = 1$. Each curve represents $\phi_{\pm}(r)$ for a given λ , forming a smooth phase-separated surface. Varying λ mainly changes the magnitude of $\phi_{\pm}$, i.e., the binodal width, rather than shifting the critical point. The apex near $r = 0$ remains nearly unchanged, while symmetry about $\phi = 0$ is broken, reflecting the breaking of time-reversal symmetry in AMB and AMB+ models.

We close the section with the following comments. The pseudodensity R in Eq. (22) is ill defined in the limit $\nu = 0$, and this case must therefore be treated separately. The corresponding results for $\nu = 0$ have already been derived in Ref. [40], where R exhibits an exponential dependence rather than the algebraic form obtained here. This difference arises because $K(\phi) = \kappa - \nu\phi$ becomes a constant when $\nu = 0$, reducing Eq. (19) to the simple linear equation $R'' = -\frac{2\tilde{\lambda}}{K} R'$, whose solution is exponential. The equations presented here therefore constitute a nontrivial generalization of the $\nu = 0$ case.

III. SUPERCRITICAL QUENCH

To understand the coarsening process in AMB+, first we look at how the dynamics in Eq. (1) scales under the naive renormalization group approach: $x \rightarrow bx$, $t \rightarrow b^z t$, and $\phi \rightarrow b^x \phi$. The dominant linear term near criticality remains $\partial_t \phi \sim -\kappa \nabla^4 \phi$ and matching their scaling dimensions gives the Gaussian dynamical exponent $z = 4$. The field scaling follows from the quadratic free-energy functional part $\mathcal{F} \sim \int d^d x (\nabla \phi)^2$, yielding $\chi = \frac{2-d}{2}$. The active nonlinearities $\nabla^2 (\nabla \phi)^2$, $\nabla^2 (\phi \nabla^2 \phi)$ scale as $\sim b^{2\chi-4}$ and comparing it with the scaling of linear terms $b^{\chi-4}$ shows that the scaling dimension of the couplings ζ , λ , and ν are $[\zeta] = [\lambda] = [\nu] = -\chi = \frac{d-2}{2}$. Hence at the Gaussian fixed point they are marginal in $d = 2$, irrelevant for $d > 2$, and relevant for $d < 2$. Thus naive power counting for AMB+ leads to the fact that the

activity enters as a marginal perturbation in two dimensions; only higher-order fluctuations can determine whether these operators are marginally relevant or irrelevant.

Marginal perturbations typically do not modify universal power-law exponents but generate multiplicative logarithmic corrections to scaling. Classic examples include the ϕ^4 interaction at the upper critical dimension $d_c = 4$ [47,54], logarithmic corrections in the two-dimensional XY model near the Kosterlitz-Thouless transition [55], and conserved phase-ordering kinetics at the marginal dimensions [56]. Such logarithmic factors reflect slow renormalization group (RG) flow near the fixed point rather than the emergence of new scaling exponents.

To understand coarsening in AMB+, we first recall the passive case of equilibrium model B, with a ϕ^4 free energy. Coarsening is driven by interfacial energy reduction, where the equilibrium chemical potential $\mu_{\text{eq}} = \delta \mathcal{F} / \delta \phi$ is dominated by the Laplace pressure contribution, with scaling with the curvature as $\kappa \sim 1/L$ [56,57]. For conserved dynamics $\partial_t \phi = \nabla^2 \mu_{\text{eq}} = -\nabla \cdot \mathbf{J}$, the current $\mathbf{J} = -M \nabla \mu_{\text{eq}}$ scales as $\mathbf{J} \sim M/L^2$, with M being the mobility. The change of domain size is now driven by the current, leading to $dL/dt \sim M/L^2$, or equivalently $L(t) \sim t^{1/3}$, the Lifshitz-Slyozov (LS) growth law.

For AMB and AMB+ in two dimensions, we expect the growth law to be determined by a balance

$$\frac{dL}{dt} \sim \frac{M_{\text{eff}}(L)}{L^2}, \quad (24)$$

where $M_{\text{eff}}(L)$ is an effective, scale-dependent mobility, in contrast to the constant M of equilibrium model B. The active couplings are marginal in $d = 2$; their one-loop beta functions are quadratic, giving the asymptotic scaling $\tilde{\lambda}(\ell)$, $\tilde{\nu}(\ell)$, $\tilde{\zeta}(\ell) \sim 1/\ell$ with $\ell = \ln(L/\xi)$ [58]. Consequently, the anomalous dimension of the mobility scales as $\eta_M(\ell) \sim 1/\ell^2$, resulting in the RG flow: $d(\ln M)/d\ell \sim 1/\ell^2$. This can be integrated to yield the logarithmic correction to the mobility:

$$M_{\text{eff}}(L) = M_0 \left[1 + \frac{b_0}{\ln(L/\xi)} + \dots \right], \quad (25)$$

where ξ is a microscopic cutoff. The effective mobility $M_{\text{eff}}(L)$ governs the curvature-driven dynamics of the interface. To a leading order in L we can write

$$\frac{dL}{dt} \sim \frac{1}{L^2} \left[1 + \frac{3c}{\ln(L)} \right], \quad (26)$$

where the constant $3c$ absorbs b_0 and ξ . This can be solved asymptotically by writing $L(t) = t^{1/3} \Psi(t)$, where $\Psi(t)$ is a slowly varying function of t . Expanding at long times, one finds to leading order in $\ln t$

$$L(t) = L(0) t^{1/3} \left(1 + \frac{c}{\ln t} \right). \quad (27)$$

For very large t one recovers the standard LS growth law $L(t) \sim t^{1/3}$.

The coarsening behavior of AMB and its extension AMB+ has been investigated previously by several authors [26,39,40,59,60]. These works established AMB and

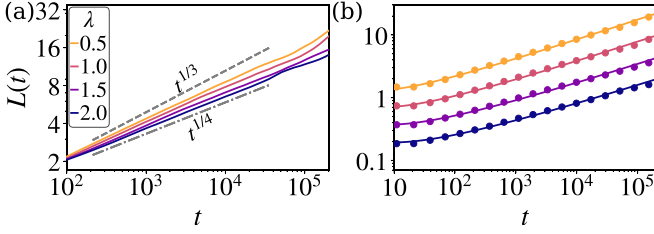


FIG. 5. Active model B ($\zeta = 0$) shows logarithmic corrections. (a) The coarsening length $L(t)$ as a function of time for various λ values. The log-log plots of the data appear to follow a power law t^θ , with the exponent θ varying continuously with λ and approaching $\frac{1}{3}$ as $\lambda \rightarrow 0$. (b) The same data (symbols) fits to theoretical predictions (lines) as in Eq. (27) with $c = 2.325, 3.661, 5.688, 7.75$ for $\lambda = 0.5, 1, 1.5, 2$, respectively. The curves are vertically shifted for clarity. Here $\zeta = 0$, $\nu = 0.5$, $r = -1$, and the system size is $L = 1024$.

AMB+ as continuum field theories for scalar active matter [see Eq. (1)] that break time-reversal symmetry while conserving density. Numerical simulations revealed domain growth broadly consistent with diffusion-limited coarsening, but accompanied by unusually slow relaxation and strong corrections to scaling, which obscured a clear identification of the asymptotic growth law [26,39].

Subsequent large-scale numerical studies [59,60] reported that the characteristic domain size appeared to follow an effective power law $L(t) \sim t^{1/z_c}$, with a dynamic exponent z_c varying continuously with activity parameters. This behavior suggested a possible violation of classical LS scaling for conserved coarsening. Here, since the critical behavior of AMB and AMB+ coincides with that of equilibrium model B, rather than invoking a new scaling law, we propose that the observed drift of effective exponents originates from strong logarithmic corrections to LS scaling. We demonstrate that the numerical data are well described by Eq. (27), which preserves the LS asymptotic exponent while naturally producing extended preasymptotic regimes that mimic continuously varying effective exponents.

A. Coarsening in active model B ($\zeta = 0$)

Numerical evidence for the presence of logarithmic correction in the coarsening length $L(t)$ in the case of AMB is presented in Fig. 5. To measure $L(t)$, we compute the equal-time two-point correlation function,

$$\begin{aligned} C(\mathbf{r}, t) &= \langle \phi(\mathbf{R}, t) \phi(\mathbf{R} + \mathbf{r}, t) \rangle - \langle \phi(\mathbf{R}, t) \rangle \langle \phi(\mathbf{R} + \mathbf{r}, t) \rangle \\ &\equiv C(r, t). \end{aligned} \quad (28)$$

Then the corresponding equal-time structure factor $S(\mathbf{k}, t)$ is computed by taking the Fourier transform of $C(\mathbf{r}, t)$. The characteristic domain size $L(t)$ is then defined as

$$L(t) = \frac{\int S(k, t) dk}{\int k S(k, t) dk}, \quad (29)$$

where $k = \|\mathbf{k}\|$ is the modulus of wave vector $\mathbf{k}$ and $S(k, t) = \langle \phi(\mathbf{k}, t) \phi(-\mathbf{k}, t) \rangle_k$ is the spherically averaged structure factor.

The measured values of $L(t)$ for different λ (the numerical details are discussed in Appendix) are shown as a function of

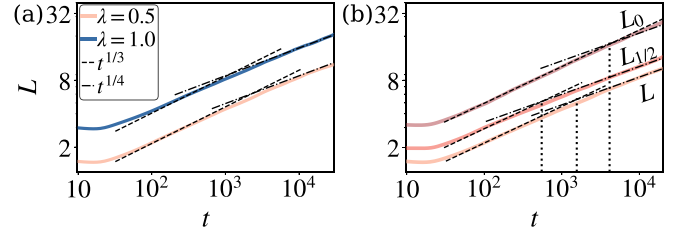


FIG. 6. (a) Growth of the coarsening length $L(t)$ for AMB with $\lambda = 0.5$ and 1 (solid lines). The data for $\lambda = 1$ are vertically shifted for clarity. The growth appears algebraic, $L(t) \sim t^\theta$, with an effective exponent θ crossing over from $\theta = 1/3$ at short times to $\theta = 1/4$ at long times, consistent with earlier observations in Ref. [59]. (b) Comparison of three different measures of the coarsening length, L_0 , $L_{1/2}$, and L (see text for definitions). All three exhibit an apparent crossover; however, as shown in Fig. 5(b), the data are well described by Eq. (27) over a much broader temporal range.

t in Fig. 5(a) on a log scale. At short times, the data appear to follow an algebraic growth law, $L(t) \sim t^\theta$, with an effective exponent θ that varies continuously with λ , consistent with earlier observations in Ref. [39]. As expected, θ approaches $\frac{1}{3}$ as $\lambda \rightarrow 0$. However, we find that the same data are described remarkably well over a much broader temporal range by the scaling form given in Eq. (27). Figure 5(b) compares the numerical data (symbols) with this logarithmically corrected growth law. Treating c as a fitting parameter, we find a best fit to the data at $c = 2.325, 3.661, 5.688, 7.75$, for $\lambda = 0.5, 1, 1.5, 2$, respectively. The excellent agreement between the data and Eq. (27) supports the RG prediction that the λ term is marginally relevant, with its principal effect being to introduce a slow logarithmic correction to the classical $t^{1/3}$ growth law.

We have carefully examined the system-size dependence of the coarsening dynamics. For a smaller system size, $L = 512$, the coarsening length $L(t)$ exhibits an apparent crossover of the effective growth exponent $1/z_{\text{eff}}$ from $1/3$ to $1/4$, qualitatively similar to the behavior reported in Ref. [59]. This is illustrated in Fig. 6(a) for $\lambda = 0.5$ and 1.0 , respectively. For an even smaller system, $L = 256$, the initial $t^{1/3}$ growth region persists to later times, while finite-size saturation sets in relatively early, obscuring any subsequent slower-growth regime. Thus the apparent extent of the $1/3$ regime decreases with increasing system size. For the largest system studied, $L = 1024$, the initial $1/3$ regime becomes extremely short, if not altogether absent, and the data are instead characterized by a single effective exponent satisfying $1/4 < 1/z_{\text{eff}} \leq 1/3$ [Fig. 5(a)], consistent with the coarsening behavior reported in Ref. [39].

It is important to note that several characteristic length scales can be extracted from the equal-time correlation function $C(r, t)$ defined in Eq. (28). Besides the coarsening length $L(t)$, one may consider the half-height length $L_{1/2}$, defined through $C(L_{1/2}, t) = C(0, t)/2$, and the first zero-crossing length L_0 , defined by $C(L_0, t) = 0$. While L_0 often admits a clear geometric interpretation as a measure of the typical domain spacing, and $L_{1/2}$ is generally more robust against statistical fluctuations, both are operational definitions tied to particular points of the correlation function. By contrast,

$L(t)$ is the fundamental length scale entering the dynamic scaling form $C(r, t) = g(\frac{r}{L(t)})$, and therefore characterizes the morphology as a whole. In the scaling regime, $L_{1/2}$, L_0 , and other reasonable length measures are all proportional to $L(t)$ and consequently yield the same asymptotic growth exponent, differing only by constant prefactors. However, $L(t)$ remains the preferred measure since it is directly linked to the scaling hypothesis and does not rely on arbitrary threshold choices such as the half-height condition or the first zero crossing.

Figure 6(b) compares $L_{1/2}$, L_0 , and $L(t)$ as functions of time for $\lambda = 0.5$ and system size $L = 512$. Although all three quantities exhibit qualitatively similar growth, their quantitative behavior differs noticeably, particularly in the location of the apparent crossover time t^* at which the effective exponent $1/z_{\text{eff}}$ changes. The strong dependence of t^* on the chosen definition of the length scale demonstrates that it cannot be regarded as a genuine physical timescale associated with the coarsening process. Rather, the observed crossover is a consequence of the particular operational measure used to characterize domain growth and should therefore be viewed as a pseudocrossover. Most importantly, our analysis shows that the apparent slowing down of coarsening can be consistently understood within a logarithmically corrected scaling form, without invoking a true transition between distinct growth regimes.

B. Coarsening in active model B+

We study the coarsening dynamics of active model B+ (AMB+), where the additional activity parameter ζ generates higher-order, nonintegrable interfacial currents that break time-reversal symmetry (TRS) beyond AMB [39]. These active currents compete with diffusive transport responsible for classical Cahn-Hilliard coarsening and can qualitatively modify phase-separation kinetics, potentially reversing Ostwald ripening and stabilizing microphase-separated states with a finite characteristic length scale. In this work, we focus on the positive ζ regime, fixing $\zeta = 1$, where coarsening persists over long times while retaining clear nonequilibrium signatures. This parameter range allows us to probe how activity modifies domain growth laws and to test deviations from classical Lifshitz-Slyozov (LS) scaling without entering the fully arrested bubbly phase as mentioned in Ref. [40].

To characterize the coarsening behavior, we numerically integrate the AMB+ equation, i.e., Eq. (1), on a $L \times L$ square grid with $L = 1024$, keeping $\nu = 0.5$ and $\zeta = 1$ fixed while varying λ in the range $-2 \leq \lambda \leq 2$. The characteristic domain size $L(t)$ is extracted from the structure factor according to Eq. (29) and shown on logarithmic scales in Fig. 7.

We first consider the case $\lambda < 0$. As shown in Fig. 7(a), for $t \gtrsim 10^2$ the domain size $L(t)$ follows the classical $t^{1/3}$ growth law with only weak deviations. These small departures are consistent with the logarithmic correction proposed in Eq. (27). Indeed, Fig. 7(b) demonstrates that the data are well described by the form of Eq. (27) over the entire temporal range, including relatively early times. The fitted coefficients $c = 1.3, 1.2, 1.1$, and 0.8 for $\lambda = -0.5, -1, -1.5$, and -2 , respectively, are significantly smaller than those obtained for AMB, indicating weaker corrections to LS scaling. This suggests that the presence of the ζ term stabilizes

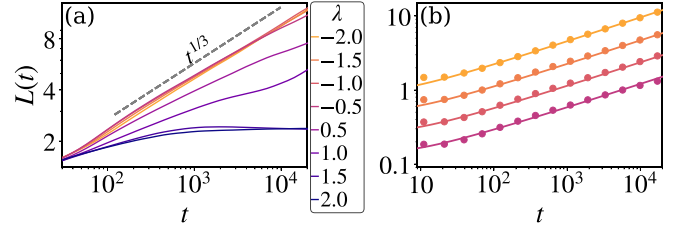


FIG. 7. Active model B+ ($\zeta = 1$) exhibits weak logarithmic corrections to coarsening. (a) Log-log plot of $L(t)$ vs t for different values of λ . For $\lambda < 0$, the growth remains close to the Lifshitz-Slyozov law $L(t) \sim t^{1/3}$, with small deviations attributable to logarithmic correction. For large positive λ , $L(t)$ rapidly saturates, indicating a microphase-separated state associated with reverse Ostwald ripening. Intermediate λ values show anomalous growth, requiring larger system sizes for reliable characterization. (b) Data for $\lambda < 0$ (symbols) are well described by Eq. (27) with $c = 1.3, 1.2, 1.1$, and 0.8 for $\lambda = -0.5, -1, -1.5$, and -2 , respectively. Parameters are $\zeta = 1$, $\nu = 0.5$, $r = -1$; system size $L = 1024$; and curves are vertically shifted for clarity.

curvature-driven coarsening by reducing the slow transient effects induced by the λ term. We therefore hypothesize that for $\lambda < 0$ the combined action of the λ and ζ terms suppresses anomalous interfacial currents, restoring nearly classical diffusive coarsening.

The behavior for $\lambda > 0$ with $\zeta = 1$ is qualitatively different, however. For small positive λ , $L(t)$ initially exhibits apparent power-law growth, whereas for larger λ the domain size eventually saturates. This transition is controlled by the effective coupling parameter $b = 2\lambda + \nu - \zeta$, which measures the strength of active interfacial currents relative to passive interfacial stiffness. Since b is positive for $\lambda > 1/4$ (when $\zeta = 1$, $\nu = 1/2$), in this regime we observe strongly suppressed growth. At intermediate positive activity (e.g., $\lambda = 1$), the system instead displays an extended transient regime before approaching saturation, as illustrated in Fig. 7(a).

IV. CONCLUSION AND DISCUSSIONS

In this work, we present a comprehensive numerical study of the critical dynamics and supercritical coarsening in active model B (AMB) and its minimal extension, active model B+ (AMB+), focusing on the deterministic limit in two dimensions. Our investigation was motivated by recent functional renormalization group (FRG) predictions [41], which identified the active couplings λ , ν , and ζ as marginally relevant perturbations at the Gaussian fixed point in $d = 2$, potentially introducing logarithmic corrections to the classical Lifshitz-Slyozov (LS), $t^{1/3}$ coarsening law. Through extensive simulations across a wide range of parameters, we have systematically characterized how these nonequilibrium terms modify phase-ordering kinetics and have uncovered a rich interplay between different active contributions that can either enhance or suppress deviations from equilibrium scaling.

At the thermodynamic critical point $r_c = 0$, both AMB and AMB+ exhibit mean-field critical behavior identical to equilibrium model B, with exponents $\alpha = 1/4$, $z = 4$, $\beta =$

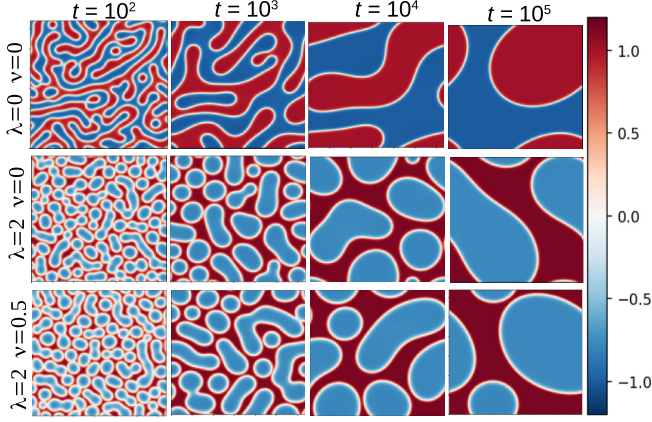


FIG. 8. Snapshot of the time evolution of model B (top), and AMB ($\lambda = 2$, $\nu = 0, 0.5$) at different time points, starting from a random initial state. The system size considered is $L = 128$.

$1/2$, and $\nu = 1/2$, providing a baseline for future fluctuation-induced effects. We also derived analytical expressions for the pseudodensity and pseudopotential in AMB+, validated numerically, yielding a complete phase diagram in the ϕ - λ - r plane.

For supercritical quenches, domain growth in AMB and AMB+ follows $L(t) \sim t^{1/3}(1 + c/\ln t)$, confirming λ as a marginal perturbation that introduces only logarithmic corrections without altering the LS scaling, precisely matching FRG predictions. While the correction term c is large in AMB, it is suppressed in AMB+. In AMB+, the ζ term suppresses the logarithmic deviations further; for $\zeta > 0$, the growth law remains close to $t^{1/3}$ with smaller log corrections. For large values of the coupling parameter $|b| = |2\lambda + \nu - \zeta|$, with $\lambda, \zeta > 0$ (first quadrant) and $\lambda, \zeta < 0$ (third quadrant) in the λ - ζ plane, the active current reverses Ostwald ripening, driving the system towards a microphase-separated state where coarsening arrests over extended time windows. Collectively, our results show that marginal activity modulates phase ordering via logarithmic corrections in AMB, while AMB+ allows cancellation of the marginal combination through b , enabling a return to pure $t^{1/3}$ scaling or qualitatively new arrested behavior.

The physical origin of this marginality in AMB can be understood from the structure of the active current. The λ term generates an interfacial current proportional to the local curvature, with a direction determined by the orientation of the density gradient. Importantly, this contribution does not introduce a new intrinsic length scale; instead, it modifies the efficiency of mass transport along and across interfaces. As a result, early-time coarsening can differ from the equilibrium case, reflecting activity-induced transients, whereas at late times the merging of large domains proceeds similarly to equilibrium model B dynamics. Figure 8 compares the evolution of AMB on a 128×128 grid for $\lambda = 2$ with $\nu = 0$ and 0.5 , alongside the equilibrium case ($\lambda = 0$). Despite noticeable differences at short times, the late-stage coarsening morphology and growth dynamics remain qualitatively similar, consistent with the marginal character of the active terms in two dimensions.

Overall, our results provide a unified picture of how marginal activity modifies phase-ordering kinetics in two-dimensional conserved scalar field theories. In AMB, the active parameters λ and ν generate logarithmic corrections. In AMB+, additional active terms allow cancellation of the marginal contribution through the effective coupling $b = 2\lambda + \nu - \zeta$, resulting in an effective growth exponent close to $1/3$ in appropriate parameter regimes. When this cancellation is incomplete or reversed, active currents can oppose Ostwald ripening and stabilize finite-size domains over extended times.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: NUMERICAL SIMULATIONS

We integrate the dynamics of AMB+ as in Eq. (1) with a pseudospectral method employing periodic boundary conditions in two dimensions and an Euler-time update scheme. In all our simulations, we set $u = 1$, $\kappa = 1$, and we take spatial discretization $\Delta x = \Delta y = 1$ and time step $dt = 10^{-2}$. We checked that our results are stable upon decreasing Δx , Δy , and dt .

Order parameter m . For a $L \times L$ square grid we take the absolute of $\phi(i, j)$ (with $i = j = 1, 2, \dots, L$) values at each grid point at time t and calculate

$$m(\mathbf{r}, t) = \frac{1}{L^2} \sum_{i=1}^L \sum_{j=1}^L |\phi(i, j, t)|,$$

for the estimation of order parameter $m(\mathbf{r}, t)$ as in Eq. (7).

Binodal density. On a $L \times L$ square grid, we calculate $\phi_{\pm}$, by measuring all the $\phi(i, j, t) > 0$ and $\phi(i, j, t) < 0$ separately at all grid points and at time t . Then we calculate the binodal densities as

$$\phi_{\pm} = \frac{1}{N_{\pm}} \sum_{i=1}^L \sum_{j=1}^L \phi(i, j, t) \Theta(\pm \phi(i, j, t)),$$

$$\text{where } N_{\pm} = \sum_{i=1}^L \sum_{j=1}^L \Theta(\pm \phi(i, j, t)).$$

Supercritical quench. To implement a supercritical quench in the numerical simulations of AMB and AMB+, we initialize the field at $(t = 0)$ as

$$\phi(i, j, 0) = \eta(i, j) - \frac{1}{L^2} \sum_{i=1}^L \sum_{j=1}^L \eta(i, j),$$

where $\eta(i, j)$ are independent random numbers drawn uniformly from the interval $(-q, q)$ with $q = 0.05$. This construction ensures that $\langle \phi(i, j, 0) \rangle = 0$. Since we report the evolution of $L(t)$ from single runs for large system sizes

over long times, we use exactly the same initial condition for different parameter values so that the measured $L(t)$ can be compared meaningfully without introducing variations arising from different initial configurations.

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