

Long-tailed dissipationless hydromechanics: Weak thermalization and ergodicity breaking

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We analyze the dynamic properties of dissipationless generalized Langevin equations in the presence of fluid inertial kernels possessing power-law tails, $k(t) \sim t^{-\kappa}$. For $\kappa > 1$ no thermalization occurs, and particle motion is ballistic. Since the long-term statistical properties of velocity depend on the initial preparation of the system, this case falls in the realm of nonergodic behavior as momentum transfer is concerned. Conversely, new phenomena arise for $0 < \kappa < 1$. In this case, a form of weak thermalization appears in the presence of thermal/hydrodynamic fluctuations and attractive potentials. However, the absence of dissipation clearly emerges once an external constant force is applied: an asymptotic settling velocity cannot be achieved as the expected value of the particle velocity diverges.

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I. INTRODUCTION

Starting from the work by Kubo [1,2], fluctuation-dissipation theory has been extended to fluctuating dynamics possessing memory effects, and the theory of generalized Langevin equations (GLE) has progressively acquired an ever more prominent role in statistical physics [3–6]. Amongst the rich phenomenology characterizing the dynamics of GLEs [7], the occurrence of ergodicity breaking and its characterization has attracted considerable attention [8–12]. The ergodicity breaking in GLE occurs under slightly different conditions with respect to the kinematic counterpart, e.g., continuous random walk models [13–15] or related models for particle displacement [16–18]. In the latter case, this is mainly due to long-range correlations and diverging transition time. The focus in the kinematic case is oriented towards the statistics of the particle displacements, even in the case where an explicit GLE is considered [16]. A classification of different phenomenologies has been introduced, distinguishing between weak ergodicity breaking when ensemble and time-averaged mean-square displacements do not coincide, and strong ergodicity breaking when the unicity of the stationary invariant measure associated with the Markov operator of the dynamics is lost, and particles visit disjoint subsets of the phase space depending on the initial condition. An ergodicity breaking parameter has been introduced [16–18] to quantify statistically the deviations of time-averaged mean-square displacements from the ensemble averages in the weak case.

In the case of GLE, two main breakup mechanisms have been so far identified: the lack of dissipation [8–11], and

the breakup of dissipative stability in the meaning addressed in Ref. [19]. The second case occurs even in the presence of a non vanishing positive and bounded friction factor [20] and finds in the Plyukhin model [12] a simple and beautiful example.

Although GLEs arise in a variety of physical systems, particle hydromechanics in a fluid medium, in the operating conditions at which thermal/hydrodynamic fluctuations are relevant (micrometric particles), is probably the paradigmatic physical problem in which GLE theory applies [2,7,19,21]. Consequently, it is not surprising that particle hydromechanics can be used as the backbone of a theory of memory effects in GLE, discriminating and dissecting their different physical origins.

In the case of particle motion in macroscopically still fluids (Brownian motion), memory effects arise from two different physical phenomena: viscoelasticity and fluid inertia. Viscoelasticity is associated with dissipative memory effects in the constitutive equations relating shear stress to deformation tensors [22,23]. The second mechanism involves fluid inertial effects [24] deriving from the description of medium fluid hydrodynamics via the time-dependent Stokes equation, made necessary by the high value of the Strouhal number deriving by the high frequency of oscillations characterizing thermal fluctuations [25]. In the Newtonian and incompressible case, memory fluid inertial effects appear in the form of the memory Basset force (characterized by a $1/\sqrt{t}$ -memory kernel) [24].

In this article we analyze the dynamic properties of dissipationless GLEs characterized by fluid inertial kernels (see further Sec. II) possessing long-lasting power-law tails

$$k(t) \sim t^{-\kappa} \quad (1)$$

for large t , as new, and partially unexpected features arise for small values of the exponent $\kappa < 1$.

There are several motivations for considering this case, i.e., the absence of dissipation: (i) it corresponds to the problem of a nonrelativistic particle moving in a superfluid [26,27]; (ii) it provides new insights on the role of dissipation (by

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considering the absence of it) in irreversible processes [28]; (iii) it completes the analysis of ergodicity breaking occurring in GLE dynamics [8–12,20] by including the case of long-tailed dissipationless memory kernels. It should be noted in passing that, even in the case of Newtonian fluids, the fluid inertial (Basset) kernel possesses a power-law scaling with $\kappa = 1/2$.

The ergodicity breaking associated with dissipationless GLE is different from the corresponding phenomena occurring in kinematic models, as it is related to the dependence of the long-term momentum thermalization on the initial conditions of the system, i.e., on its statistical properties as regards the initial velocity distribution of the particle ensemble. For this reason, the widely used ergodicity breaking parameters defined with respect to mean-square displacement statistics cannot be used, as the quantification of this phenomenon is essentially grounded on the moments of the velocity variables as a function of the initial preparation of the system. Therefore, in the present case the properties of the mean-square displacement are enslaved to the momentum relaxation dynamics that constitutes the physical core of the statistical anomalies.

With reference to the class of dissipationless kernels possessing the long-term property defined by Eq. (1), the most remarkable feature is that a sharp transition in the long-term thermalization properties occurs as a function of the exponent κ . For $\kappa > 1$, ergodicity is broken, the system does not display thermalization as regards momentum transfer, and particle motion is, for most of the initial conditions, purely ballistic. Conversely, for $\kappa < 1$ some form of weak thermalization (defined in Sec. III G) occurs, and this is associated with a superdiffusive motion. Moreover, the change from ergodic to nonergodic behavior near the critical point $\kappa_c = 1$ corresponds to a bifurcation in the sense of dynamical system theory associated with the exchange of stability (see Sec. III G).

In fact, the occurrence of some form of thermalization for $\kappa < 1$ motivates a finer characterization of the behavior of physical systems in the presence of different forces (stochastic thermal forces, conservative forces stemming from a potential, bounded nonconservative forces, etc.), and a deeper analysis on the role of dissipation in irreversible processes and on the singular limits characterizing the dissipationless case, as discussed in Sec. III G.

The article is organized as follows. Section II defines the mathematical and hydromechanic setting of the problem and its modal formulation consistent with Kubo fluctuation-dissipation theory. Section III develops the part associated with the qualitative dynamic behavior (ergodicity and its breaking) depending on the exponent κ entering Eq. (1), addressing in detail the characterization of the bifurcation occurring at $\kappa = 1$. Moreover, the role of dissipation and irreversibility in the presence of thermal fluctuations, external forces and potentials is carefully addressed, and the difference between weak thermalization and strong dissipative behavior is thoroughly discussed.

II. SETTING OF THE PROBLEM

Consider the hydromechanics of a spherical particle of mass m in a still fluid (sphericity essentially implies the

isotropy of fluid-particle interactions). The equation of motion for the particle velocity $v(t)$ reads as [25]

$$m \frac{dv(t)}{dt} = - \int_0^t h(t-\tau) v(\tau) d\tau - \int_0^t k(t-\tau) \times \left(\frac{dv(\tau)}{d\tau} + v(0) \delta(\tau) \right) d\tau + R(t), \quad (2)$$

where $R(t)$ is the thermal and/or hydrodynamic fluctuational force, defined by the Kubo fluctuation-dissipation relations [1,2], which essentially means the fulfillment of the Langevin condition $\langle R(t) v(\tau) \rangle_{\text{eq}} = 0$ for $\tau \leq t$, where $\langle \cdot \rangle_{\text{eq}}$ indicates the expected value with respect to the equilibrium probability measure.

The GLE Eq. (2) is characterized by two kernels: the dissipative kernel $h(t)$, and the fluid-inertial kernel $k(t)$ [25]. The dissipative kernel involved in fluid-particle interactions is non-negative [20], i.e., $h(t) \geq 0$ (for other physical phenomenologies this condition can be relaxed [12]), and this stems from rheology [22,23]. Its integral $\eta_\infty = \int_0^\infty h(t) dt$, if bounded and different from zero, enters the global Stokes-Einstein relation for the long-term particle diffusivity D at constant temperature T , $D \eta_\infty = k_B T$, where k_B is the Boltzmann constant. Conversely, the fluid inertial term, defined by the kernel $k(t)$ does not contribute to dissipation. It modulates the shape of the velocity autocorrelation function [29,30]. From hydrodynamic analysis, i.e., for the few cases in which $k(t)$ is analytically available (Newtonian fluids, Maxwell fluids) [24,31], it can be assumed that $k(t) \geq 0$, albeit there is no conclusive result in this regard.

Equation (2) is the most general representation of rigid particle hydromechanics in a still unconfined fluid in the limit of negligible Reynolds number, in the absence of external velocity fields or potentials acting on the particle [2,32–34]. It is obtained by solving the hydrodynamics of the solvent fluid, coupled with the boundary condition at the external wetted surface of the particle imposing, in the no-slip case, that the fluid at the particle boundary possesses at any time instant the particle velocity itself [35,36]. This boundary condition can be generalized in the presence of slippage at the particle external surface [35]. Equation (2) can be derived also from stochastic hydrodynamics [37,38], also in the presence of solvent fluid compressibility [39].

The theory of suspensions, including hydrodynamic interactions amongst particles, is grounded on this class of equations [40,41]. While the presence of a dissipative term dates back to the works by Einstein and Langevin (Stokesian friction) on Brownian motion [42,43], the importance of fluid inertial effects, expressed by the second integral at the right-hand side of Eq. (2), deriving from a time-dependent linear description of incompressible hydrodynamics of the fluid medium, has been confirmed experimentally in highly resolved measurements of Brownian trajectories [29,30,44], determining a transition for the Stokesian exponential decay of the particle velocity autocorrelation function (in the absence of fluid inertial effects), to a power-law decay. Moreover, fluid-inertial effects have been proved relevant in the study of Brownian particles in viscoelastic fluids [34]. For the sake of historical correctness, it should be mentioned that the first inclusion of fluid inertial effects dates back to

a 1945 paper by Vladimírsky and Terletzky written in Russian [45]. This belongs to a series of important contributions of the Soviet school of statistical physics that remained almost unknown in the western scientific community for long time, similarly to the first formulation of the fluctuation-dissipation theorem that, as observed by Tothova *et al.* [46,47], is due to Vladimírsky [48], two decades before the seminal paper by Kubo [1].

As clearly stated in the title and discussed in Sec. I, we analyze the dissipationless case, corresponding to Eq. (2) in the absence of the kernel $h(t)$. Henceforth, let us consider Eq. (2) in dimensionless form, by rescaling time and velocity so that $\langle v^2 \rangle_{\text{eq}} = 1$ would correspond to the Maxwellian equipartition relation, and $h(t) = 0$, i.e., no dissipation. If one further assumes $k(t = 0)$ bounded, as for any viscoelastic fluids [31], the fluid-inertial term can be reworked by parts, so that the dimensionless version of Eq. (2) [in dimensionless form $m = 1$, and we keep the notation t , $v(t)$, $R(t)$, and $k(t)$ to indicate the dimensionless time, velocity, random force and inertial kernel], takes the form

$$\frac{dv(t)}{dt} = -k(0)v(t) + \int_0^t \frac{dk(t-\tau)}{d\tau} v(\tau) d\tau + R(t). \quad (3)$$

It is well known from rheology that the relaxation modulus $G(t)$ of a complex viscoelastic fluid can be expressed as a finite or countable combination of exponentially decaying function of time, (referred to as Prony series), each of which characterized by a different relaxation rate [22,23]. This property is inherited by the dissipative kernel $h(t)$ owing the direct proportionality between $G(t)$ and $h(t)$. In the case of a spherical particle of radius R , in the Stokes regime $G(t)$ simply replaces the Newtonian viscosity μ in the expression for the friction factor, thus $h(t) = 6\pi R G(t)$. Similarly, we can use the same representation for the kernel $k(t)$ in order to account for the long-term power-law scaling expressed by Eq. (1). To this end, one can consider the class of kernels defined by the analytic function,

$$g_\kappa(t) = \sum_{n=1}^{\infty} \frac{1}{2^{n\kappa}} e^{-t/2^n} \sim t^{-\kappa} \quad (4)$$

with $\kappa > 0$, i.e., $k(t) = g_\kappa(t)$. The truncated representations

$$g_\kappa(t; N) = \sum_{n=1}^N \frac{1}{2^{n\kappa}} e^{-t/2^n} \quad (5)$$

for N large enough, but still order of 10, provide accurate power-law scalings over several decades.

Specifically, Fig. 1 depicts the behavior of $g_\kappa(t; N)$ for $N = 40$ at two different values of κ . It can be observed that the power-law scaling holds up to $t = 10^9$, and this value is the upper integration limit in the simulations of the dynamics reported in the remainder (see Sec. III). Henceforth we assume $k(t) = g_\kappa(t; N)$ with N large enough to ensure the power-law scaling of the kernel over the timescales involved. Set

$$\gamma_n = \frac{1}{2^{n\kappa}}, \quad \mu_n = \frac{1}{2^n}, \quad \Gamma = \sum_{n=1}^N \gamma_n \quad (6)$$

$n = 1, \dots, N$, so that $k(t) = \sum_{n=1}^N \gamma_n e^{-\mu_n t}$. Equation (3) with the kernel $k(t)$ given by Eq. (4) can be recast in the modal ex-

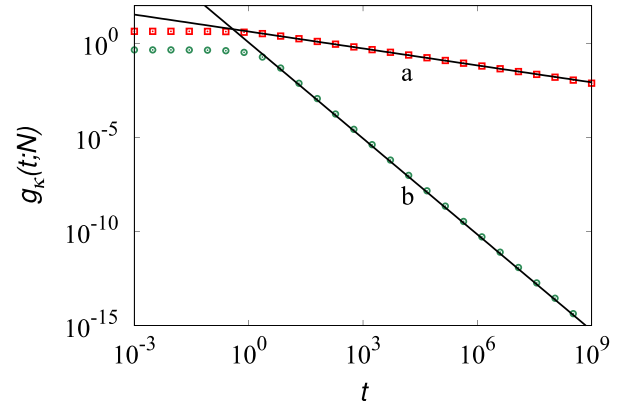


FIG. 1. Function $g_\kappa(t; N)$ vs t defined by Eq. (5) with $N = 40$. Symbols ($\square$) refer to $\kappa = 0.3$, ($\circ$) to $\kappa = 1.7$. Solid lines (a) and (b) represent the scalings $g_\kappa(t; N) \sim t^{-\kappa}$ for the two values of the exponent κ considered.

pansion (Markovian embedding) [19] involving the auxiliary variables $z_n(t)$, $n = 1, \dots, N$,

$$\begin{aligned} \frac{dv(t)}{dt} &= -\Gamma v(t) + \sum_{n=1}^N \gamma_n \mu_n z_n(t) + \sqrt{2} \sum_{n=1}^N d_n \xi_n(t) \\ \frac{dz_n(t)}{dt} &= -\mu_n z_n(t) \\ &\quad + v(t) + \sqrt{2} c_n \xi_n(t), \quad n = 1, \dots, N. \end{aligned} \quad (7)$$

The expression for the coefficients c_n and d_n modulating the intensity of the Wiener stochastic forcings can be derived from the Langevin condition in the limit for vanishing dissipation (see Sec. II A).

Structure of the thermal stochastic forcing

In order to derive the expression for the coefficients c_n and d_n entering Eq. (7), consider the hydromechanic model for fluid-particle interactions characterized by a constant Stokesian friction factor ε and by a fluid inertial kernel $k(t)$ of the form defined by Eq. (5). Its modal expansion reads

$$\begin{aligned} \frac{dv(t)}{dt} &= -(\varepsilon + \Gamma) v(t) + \sum_{n=1}^N \gamma_n \mu_n z_n(t) \\ &\quad + \sqrt{2} a \xi'(t) + \sqrt{2} \sum_{n=1}^N d_n \xi_n(t) \\ \frac{dz_n(t)}{dt} &= -\mu_n z_n(t) + v(t) + \sqrt{2} c_n \xi_n(t), \end{aligned} \quad (8)$$

where $\xi'(t)$, $\xi_n(t)$, $n = 1, \dots, N$ are distributional derivatives of mutually independent Wiener processes, and the coefficients a , d_n , c_n should be determined by fluctuation-dissipation theory.

For $\varepsilon > 0$ the system is ergodic: there exists a unique stationary density of the joint process $[v(t), z_1(t), \dots, z_N(t)]$ for any values of the coefficients a , d_n , c_n , $n = 1, \dots, N$. These coefficients can be determined by enforcing the first Kubo fluctuation-dissipation relation, namely that the velocity autocorrelation function $C_{vv}(t)$ should be the solution of the

equation

$$\frac{dC_{vv}(t)}{dt} = -(\varepsilon + \Gamma)C_{vv}(t) + \sum_{n=1}^N \gamma_n \mu_n e^{-\mu_n t} * C_{vv}(t), \quad (9)$$

where “*” indicates convolution, and $C_{vv}(0) = \langle v^2 \rangle_{\text{eq}} = 1$. Consider the dynamics of the second-order moments $m_{vv}(t) = \langle v^2(t) \rangle$, $m_{vz_n}(t) = \langle v(t)z_n(t) \rangle$, $m_{z_n z_m}(t) = \langle z_n(t)z_m(t) \rangle$ associated with Eq. (8), and the corresponding velocity autocorrelation function. It is easy to see that $C_{vv}(t)$ satisfies Eq. (9) if and only if $m_{vv}^* = 1$, and the second mixed moments at equilibrium (indicated by an asterisk) satisfy the conditions

$$m_{vz_n}^* = 0, \quad n = 1, \dots, N \quad (10)$$

that represent the detailed Langevin conditions in the modal expansion. Enforcing Eq. (10) for the moment values at equilibrium [49], the expression for the coefficients a , d_n , c_n , $n = 1, \dots, N$ can be obtained, namely

$$a = \sqrt{\varepsilon}, \quad d_n = \sqrt{\gamma_n}, \quad c_n = -\frac{1}{\sqrt{\gamma_n}}, \quad n = 1, \dots, N. \quad (11)$$

The dissipationless case consistent with the Kubo fluctuation-dissipation theory is obtained from Eqs. (8) and (11) in the limit for $\varepsilon \rightarrow 0$, namely $a = 0$ and

$$d_n = \sqrt{\gamma_n}, \quad c_n = -\frac{1}{\sqrt{\gamma_n}}, \quad n = 1, \dots, N. \quad (12)$$

III. MAIN RESULTS

This section reports the scaling theory and the main qualitative properties of the GLE dynamics Eqs. (3) or (7). To begin with, for $\kappa > 1$ the long-term statistical properties of the particle velocity depend on the initial conditions, and in this sense they show nonergodic thermalization properties. This can be easily checked by considering different initial conditions as regards the values of $v(t)$ and $z_n(t)$, $n = 1, \dots, N$ at $t = 0$, and observing that in general

$$\lim_{t \rightarrow \infty} \langle v^2(t) \rangle_0 \neq \langle v^2 \rangle_{\text{eq}} = 1, \quad (13)$$

where $\langle \cdot \rangle_0$ refers to the expected value referred to an initial joint density $p_0(v_0, \mathbf{z}_0)$, where $\mathbf{z}_0 = (z_{1,0}, \dots, z_{N,0})$. Henceforth we will indicate with $\langle v^2(t) \rangle$ or $\langle v^2(t) \rangle_0$ the square velocity variance as a function of time, starting from a given initial statistical setting specified case by case. And this should not be confused with $\langle v^2 \rangle_{\text{eq}}$ that is the equilibrium value equal to $k_B T/m$ in the dimensional formulation and to 1 in the present nondimensional setting.

In a similar way, the velocity autocorrelation function $C_{vv}^0(t)$, starting from specific initial conditions defined by the density $p_0(v_0, \mathbf{z}_0)$, is given by

$$C_{vv}^0(t) = \int dv \int v v_0 p(v, t | v_0, 0) p_0(v_0) dv_0, \quad (14)$$

where $p(v, t | v_0, 0)$ is the conditional probability density of having the velocity v at time t if the initial velocity at time $t = 0$ is v_0 , and $p_0(v_0) = \int p_0(v_0, \mathbf{z}_0) d\mathbf{z}_0$ is the initial marginal velocity density. In the nonergodic case, $C_{vv}^0(t)$ may be different from the Kubo velocity autocorrelation function

at equilibrium, indicated as $C_{vv}(t)$ [see further Eq. (17) for its definition].

Conversely, for $\kappa < 1$ a regular relaxation towards $\langle v^2 \rangle_{\text{eq}}$ is observed, as well the collapse of $C_{vv}^0(t)$ towards $C_{vv}(t)$ for any initial conditions, suggesting the validity of the ergodic hypothesis as regards momentum relaxation.

A. Analysis

Given the modal representation of the GLE Eq. (7), the associated Fokker-Planck equation for the density $p(v, \mathbf{z}, t)$, where $\mathbf{z} = (z_1, \dots, z_N)$, reads

$$\begin{aligned} \frac{\partial p}{\partial t} = & \Gamma \frac{\partial(v p)}{\partial v} - \sum_{n=1}^N \gamma_n \mu_n z_n \frac{\partial p}{\partial v} + \sum_{n=1}^N \mu_n \frac{\partial(z_n p)}{\partial z_n} \\ & - \sum_{n=1}^N v \frac{\partial p}{\partial z_n} + \sum_{n=1}^N d_n^2 \frac{\partial^2 p}{\partial v^2} \\ & + 2 \sum_{n=1}^N c_n d_n \frac{\partial^2 p}{\partial v \partial z_n} + \sum_{n=1}^N c_n^2 \frac{\partial^2 p}{\partial z_n^2} \end{aligned} \quad (15)$$

from which the evolution equations of the second-order moments follow

$$\begin{aligned} \frac{dm_{vv}(t)}{dt} = & -2\Gamma m_{vv}(t) + 2 \sum_{n=1}^N \gamma_n \mu_n m_{vz_n}(t) + 2 \sum_{n=1}^N d_n^2 \\ \frac{dm_{vz_m}(t)}{dt} = & -\Gamma m_{vz_m}(t) + \sum_{n=1}^N \gamma_n \mu_n m_{z_n z_m}(t) \\ & - \mu_m m_{vz_m} + m_{vv}(t) + 2c_m d_m \\ \frac{dm_{z_m z_n}(t)}{dt} = & -(\mu_m + \mu_n) m_{z_m z_n}(t) + m_{vz_m}(t) \\ & + m_{vz_n}(t) + 2c_n^2 \delta_{mn} \end{aligned} \quad (16)$$

$m, n = 1, \dots, N$. Observe that, by Eqs. (12), the last term at the right-hand side of the first Eq. (16) simplifies as $2 \sum_{n=1}^N d_n^2 = 2\Gamma$, and the last term at the right-hand side of the second Eq. (16) equals $2c_n d_n = -2$. The Kubo velocity autocorrelation function follows directly from the modal expansion Eq. (7) and from the Langevin condition, $\langle \xi(t)v(0) \rangle_{\text{eq}} = 0$,

$$\begin{aligned} \frac{dC_{vv}(t)}{dt} = & -\Gamma C_{vv}(t) + \sum_{n=1}^N \gamma_n \mu_n C_{z_n v}(t) \\ \frac{dC_{z_n v}(t)}{dt} = & -\mu_n C_{z_n v}(t) + C_{vv}(t) \end{aligned} \quad (17)$$

equipped with the initial conditions $C_{vv}(0) = \langle v^2 \rangle_{\text{eq}} = 1$, $C_{z_n v}(0) = 0$. Figure 2 depicts the behavior of $\langle v^2(t) \rangle = m_{vv}(t)$, starting from $m_{vv}(0) = m_{vz_n}(0) = m_{z_n z_m}(0) = 0$, $n, m = 1, \dots, N$, for different values of the exponent κ . The simulations of the system of equations Eq. (7) are performed choosing $N = 40$ as regards the number of modes and using an Euler-Langevin scheme with a temporal step size $\Delta t = 10^{-3}$ considering an ensemble of 10^5 – 10^6 realizations.

For $\kappa < 1$ thermalization is achieved, and $\langle v^2(t) \rangle$ decays in a power-law way towards the equipartition result. For $\kappa > 1$ the long-term limit of $\langle v^2(t) \rangle$ converges towards a different

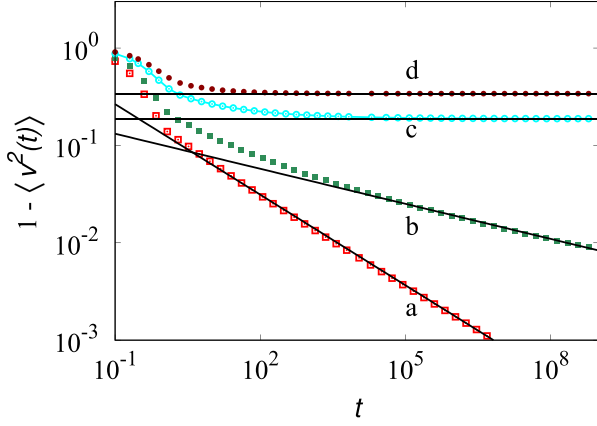


FIG. 2. $1 - \langle v^2(t) \rangle$ vs t for the dissipationless model Eq. (7) obtained from the moment dynamics Eq. (16) (symbols) starting from vanishing initial conditions $m_{vv}(0) = m_{vz_n}(0) = m_{z_n z_m}(0) = 0$, $n, m = 1, \dots, N$. Symbols $\square$ and line (a) and $\blacksquare$ and line (b) refer to $\kappa = 0.7, 0.9$, respectively, while $\circ$ and line (c) and $\bullet$ and line (d) to $\kappa = 1.3, 1.6$, respectively.

(and smaller) value. Figure 3 depicts the behavior of $\langle v^2(t) \rangle$, in the case $z_n(0) = 0$ $n = 1, \dots, N$, and $v(0)$ is a Gaussian random variable with zero mean and variance σ_0 for GLEs within and outside the ergodicity region ($\kappa < 1$). Another set of initial conditions is depicted in Fig. 4. In this case $v(0)$ is still Gaussian with variance σ_0 , while $z_n(0)$ $n = 1, \dots, N$ are Gaussian random variables sampled from their equilibrium distribution: a Gaussian distribution with zero mean and variance $1/\sqrt{\gamma_n \mu_n}$, $n = 1, \dots, N$. In the nonergodic case [$\kappa = 1.5$, Fig. 4(b)] if $\sigma_0 = 1$, i.e., if the initial condition coincides with the thermodynamic equilibrium conditions the system remains in equilibrium. In all the other cases $\sigma_0 \neq 1$, the limiting value of $\langle v^2(t) \rangle$ is different from $\langle v^2 \rangle_{\text{eq}} = 1$.

A qualitatively similar result is obtained for the velocity autocorrelation function, as depicted in Fig. 5. These data are obtained from the stochastic simulation of Eq. (7) considering an ensemble of $N = 10^5$ realizations and analyzing the correlation properties after a sufficiently long transient order of $t = 10^4$.

Below $\kappa = 1$, the velocity correlation properties coincide with those predicted by fluctuation dissipation theory while, for $\kappa > 1$, these properties depend on the initial conditions. In

the latter case, we observe for $C_{vv}^0(t)$ a translation of the Kubo autocorrelation function, i.e., $C_{vv}^0(t) = C_{vv}(t) + K$ where the constant K depends on the initial conditions.

The results on the transition occurring at $\kappa = 1$ can be compactly summarized by considering the long-term stationary value of the squared velocity variance $\langle v^2 \rangle_s = \lim_{t \rightarrow \infty} \langle v^2(0) \rangle$ vs the initial variance σ_0 , where $\sigma_0^2 = \langle v^2(0) \rangle$, depicted in Fig. 6. The initial velocity distribution is Gaussian with zero mean, thus completely specified by σ_0 , while the other auxiliary variables are initially vanishing, i.e., $z_n(0) = 0$, $n = 1, \dots, N$.

For $\kappa < 1$, $\langle v^2 \rangle_s = 1$ independently of σ_0 , while $\langle v^2 \rangle_s$ deviates from the equipartition result for $\kappa > 1$, and this deviation is more significant as κ increases. The comparison of these results with the analysis of GLE performed by Deng and Barkai clearly reveals the differences in the models and, as a consequence the different nature of ergodicity breaking. In Ref. [16], the GLE is purely dissipative, corresponding to the hydrodynamic picture of a viscoelastic fluid with a power-law dissipative kernel $h(t) \sim t^{2H-2}$, $1/2 < H < 1$, while $k(t) = 0$. This generates a breaking of ergodicity for $H \rightarrow 1$ as regards the statistical properties of the particle mean-square displacement. In the present case $h(t) = 0$, while $k(t)$ possesses a power-law tail Eq. (1) inducing nonergodic properties in the momentum relaxation dynamics compactly expressed by the data reported in Fig. 6.

B. Scaling analysis and anomalous diffusion

A different diffusive behavior characterizes the system below and above the critical value κ_c . To this end, let us supplement Eq. (7) with the kinematic equation

$$\frac{dx(t)}{dt} = v(t) \quad (18)$$

for the particle position $x(t)$. The analysis of the diffusive anomalies can be viewed as the extension of Mason's generalized Stokes-Einstein relation (GSER) for dissipative viscoelastic fluids [50,51] to the dissipationless case. The GSER connects the rheological properties of a complex viscoelastic fluid, and specifically the Fourier transform of its relaxation modulus, to the Fourier transform of the mean-square displacement (viewed as a function of time) in the case the particle motion displays anomalous subdiffusive scaling

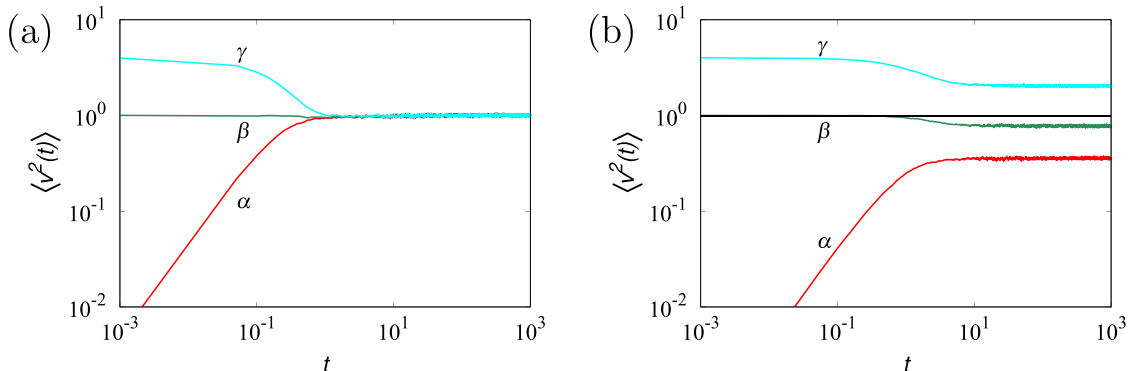


FIG. 3. $\langle v^2(t) \rangle$ vs t for initial conditions $z_n(0) = 0$ $n = 1, \dots, N$, and $v(0)$ normally distributed with variance σ_0 . (a) refers to $\kappa = 0.5$, (b) to $\kappa = 2.5$. Lines α, β, γ correspond to $\sigma_0 = 0, 1, \sqrt{2}$, respectively.

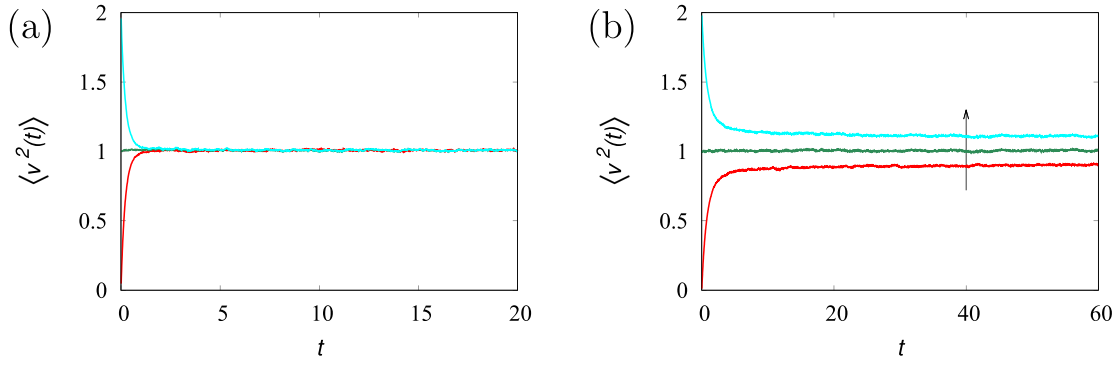


FIG. 4. $\langle v^2(t) \rangle$ vs t for initial conditions where $z_n(0) = 0$ $n = 1, \dots, N$ are normally distributed with zero mean and variances equal to their equilibrium variances, and $v(0)$ normally distributed with variance σ_0 . (a) refers to $\kappa = 0.5$, (b) to $\kappa = 1.5$. Lines α , β , γ correspond to $\sigma_0 = 0, 1, \sqrt{2}$, respectively.

[50,51]. In this sense it represents a generalization of the Stokes-Einstein relation, connecting properties diffusive properties and mechanical dissipation in the anomalous case.

Let $\widehat{C}_{vv}(s)$ be the Laplace transform of $C_{vv}(t)$, where s is the Laplace variable (henceforth we use the hatted notation for Laplace transforms of generic functions of time). From Eq. (2) with $h(t) = 0$, or equivalently from Eq. (3), expressed in nondimensional form, we obtain

$$\widehat{C}_{vv}(s) = \frac{\langle v^2 \rangle_{\text{eq}}}{s + s \widehat{k}(s)}. \quad (19)$$

From Eq. (4), for $\kappa < 1$, we have for small s close to $s = 0$, $\widehat{k}(s) \sim 1/s^{1-\kappa}$, and thus

$$\widehat{C}_{vv}(s) \sim \frac{\langle v^2 \rangle_{\text{eq}}}{s + s^\kappa} \sim \frac{\langle v^2 \rangle_{\text{eq}}}{s^\kappa}, \quad \text{for } s \rightarrow 0 \quad (20)$$

that implies

$$C_{vv}(t) \sim t^{-(1-\kappa)}, \quad t \rightarrow \infty \quad (21)$$

indicating that, for $\kappa < 1$ a superdiffusive diffusional scaling occurs

$$\langle x^2(t) \rangle \sim t \int_0^t C_{vv}(\tau) d\tau \sim t^{1+\kappa}. \quad (22)$$

Conversely, for $\kappa > 1$, it is reasonable to argue that, for generic initial conditions, the scaling of the mean-square displacement should be ballistic, $\langle x^2(t) \rangle \sim t^2$. This is indeed the case as shown in Fig. 7. However, it should be observed that in the nonergodic case there may exist initial conditions providing different scalings, at least at intermediate times, as depicted in Fig. 7(b) curve (α), starting from vanishing initial conditions for the velocity.

In the ergodic case ($\kappa < 1$), the statistics of velocity and normalized particle displacement $\Delta(t) = (x(t) - x_0)/\sigma_x(t)$, where $\sigma_x^2(t) = \langle (x(t) - x_0)^2 \rangle$ is Gaussian in the long term, as depicted in Fig. 8 for $\kappa = 0.5$.

C. Transitional properties

The qualitative change of behavior occurring at $\kappa_c = 1$ from ergodic to nonergodic dynamics resembles a second-order phase transition. For this transition we can take the normalized difference ϕ_v as the order parameter

$$\phi_v = \frac{\langle v^2 \rangle_{\text{eq}} - \lim_{t \rightarrow \infty} \langle v^2(t) \rangle_0}{\langle v^2 \rangle_{\text{eq}}}, \quad (23)$$

i.e., the normalized difference between the equilibrium value (in the nondimensional case $\langle v^2 \rangle_{\text{eq}} = 1$), and the long-term

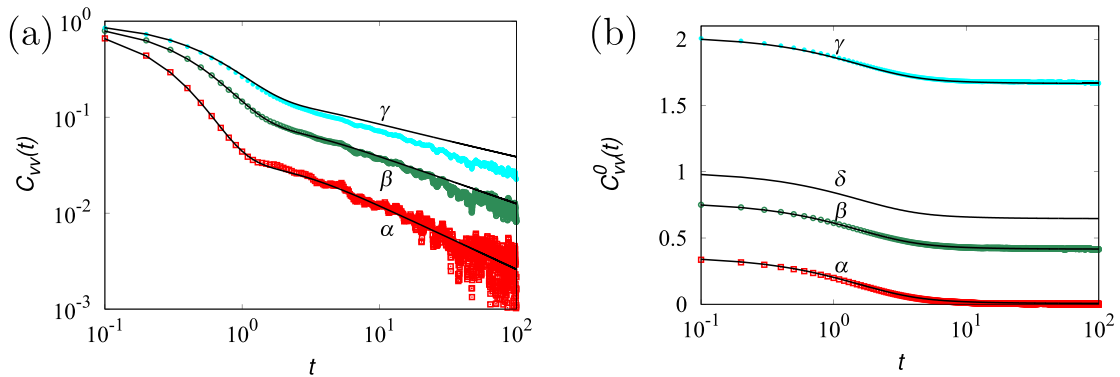


FIG. 5. Long-term velocity autocorrelation function obtained from the averaging over an ensemble of realizations of Eq. (7). The initial conditions are $z_n(0) = 0$, $n = 1, \dots, N$, while $v(0)$ is a Gaussian random variable with variance σ_0 . (a) reports data for kernels with $\kappa < 1$, $\sigma_0 = 0$. Symbols refer to stochastic simulations with $\kappa = 0.3, 0.5, 0.7$, from line α to line γ , lines corresponds to the Kubo velocity autocorrelation function. (b) refers to $\kappa = 2.5$. Symbols and lines, from α to γ , correspond to different values of $\sigma_0 = 0, 1, 2$. Line (δ) corresponds to the Kubo velocity autocorrelation function.

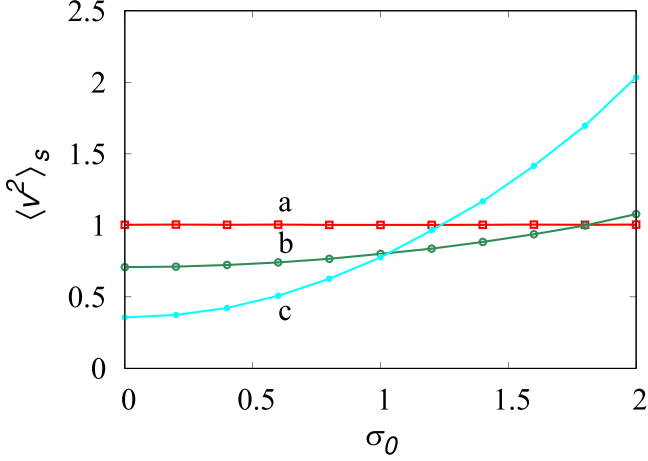


FIG. 6. $\langle v^2 \rangle_s$ vs σ_0 for different values of the exponent κ . Line (a) and symbols ($\square$) refer to $\kappa = 0.5$, line (b) and ($\circ$) to $\kappa = 1.5$, line (c) and ($\bullet$) to $\kappa = 2.5$.

limit of the squared velocity variance starting from vanishing initial conditions $v(0) = z_n(0) = 0$, $n = 1, \dots, N$. The subscript “0” indicates just this particular initial setting. Of course, other initial statistical settings, different from the equilibrium configuration, can be chosen as well. Near κ_c , the order parameter behaves as

$$\phi_v = \begin{cases} 0 & \kappa \leq \kappa_c \\ (\kappa - \kappa_c)^\zeta & \kappa > \kappa_c \end{cases} \quad (24)$$

Figure 9 depicts the behavior of ϕ_v as a function of $\kappa - \kappa_c$ for $\kappa > \kappa_c$. Data refer to the solution of the moment equations (16) starting from $m_{vv} = m_{vz_n} = m_{z_n z_m} = 0$, $n, m = 1, \dots, N$, (see Fig. 2) up to a maximal time order of 10^9 . Symbols ($\circ$) refer to situations where reliable estimates of the saturation value were possible. From the scaling of the latter class of data a value $\zeta = 0.8$ of the critical exponent defined by Eq. (24) is obtained.

The occurrence of this ergodic or nonergodic transition can be predicted from the dynamical properties of the system. Consider Eq. (7) in the absence of the stochastic forcings, and express it in matrix form introducing the joint variable $\mathbf{y} = (v, z_1, \dots, z_N)^T$, where “ T ” indicates transpose,

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A} \mathbf{y}(t), \quad (25)$$

where

$$\mathbf{A} = \begin{pmatrix} -\Gamma & \gamma_1 \mu_1 & \gamma_2 \mu_2 & \cdots & \gamma_N \mu_N \\ 1 & -\mu_1 & 0 & \cdots & 0 \\ 1 & 0 & -\mu_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & -\mu_N \end{pmatrix}. \quad (26)$$

Consider the spectral properties of the coefficient matrix $\mathbf{A}$, i.e., its right $\mathbf{A} \mathbf{e}_i = -\lambda_i \mathbf{e}_i$, and left $\mathbf{e}_i^* \mathbf{A} = -\lambda_i \mathbf{e}_i^*$, $i = 0, \dots, N$ eigenvectors. The matrix $\mathbf{A}$ possesses a vanishing eigenvalue $\lambda_0 = 0$, while the others are real and negative, i.e., $\lambda_i > 0$, Fig. 10.

The left eigenvectors constitute the dual basis, and by definition they are normalized as

$$\mathbf{e}_i^* (\mathbf{e}_j) = \sum_{k=0}^N e_{i,k}^* e_{j,k} = \delta_{ij}, \quad (27)$$

where $e_{j,k}$ and $e_{i,k}^*$, $k = 0, \dots, N$ are the entries of $\mathbf{e}_j$ and $\mathbf{e}_i^*$, respectively. The right and left eigenvectors associated with $\lambda_0 = 0$ are given by

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ \mu_1^{-1} \\ \mu_2^{-1} \\ \vdots \\ \mu_N^{-1} \end{pmatrix}, \quad \mathbf{e}_0^* = C (1, \gamma_1, \gamma_2, \dots, \gamma_N), \quad (28)$$

where the constant C is defined by the normalization condition Eq. (27), i.e., $\mathbf{e}_0^* (\mathbf{e}_0) = 1$, and thus

$$C = \left(1 + \sum_{n=1}^N \frac{\gamma_n}{\mu_n} \right)^{-1}. \quad (29)$$

Next, consider the evolution of $\mathbf{y}(t)$, solution of Eq. (25) starting from a generic initial condition $\mathbf{y}(0) = \mathbf{y}_0$,

$$\mathbf{y}(t) = e^{\mathbf{A}t} \mathbf{y}_0 = \sum_{n=0}^N \mathbf{e}_n^* (\mathbf{y}_0) e^{-\lambda_n t} \mathbf{e}_n. \quad (30)$$

In the long-term limit, $\mathbf{y}(t)$ converges to a point lying in the central eigenmanifold spanned by $\mathbf{e}_0$,

$$\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{e}_0^* (\mathbf{y}_0) \mathbf{e}_0 \quad (31)$$

depending on the initial condition. From Eqs. (28) and (29), setting $\mathbf{y}_0 = (v_0, z_{1,0}, \dots, z_{N,0})$, we have

$$\mathbf{e}_0^* (\mathbf{y}_0) = \frac{v_0 + \sum_{n=1}^N \gamma_n z_{n,0}}{1 + \Phi_N}, \quad (32)$$

where

$$\Phi_N = 1 + \sum_{n=1}^N \frac{\gamma_n}{\mu_n} = 1 + \sum_{n=1}^N \frac{1}{2^{n(\kappa-1)}}. \quad (33)$$

Two situations occur in the limit for $N \rightarrow \infty$: (i) if $\kappa < 1$, $\lim_{N \rightarrow \infty} \Phi_N = \infty$. Consequently, $\mathbf{e}_0^* (\mathbf{y}_0) = 0$, and the long-term limit of the solution Eq. (30), is independent of the initial condition $\mathbf{y}_0$. This is the case providing the occurrence of an ergodic behavior; (ii) if $\kappa > 1$, $\lim_{N \rightarrow \infty} \Phi_N = \Phi_\infty < \infty$, and the long-term limit Eq. (31) does depend on the initial conditions, corresponding to a nonergodic behavior.

It can be observed that the condition on Φ_N in the limit for $N \rightarrow \infty$ corresponds to the integrability condition for the inertial kernel $k(t)$. Indicating with $k_N(t)$ the truncated approximation of $k(t)$ up to N modes, we have

$$\Phi_N = 1 + \int_0^\infty k_N(t) dt. \quad (34)$$

D. Quasi-steady-state approximation

There is also another way to detect the ergodicity transition, adopting a quasi-steady-state approximation of the dynamics. Consider the dynamics of the second-order

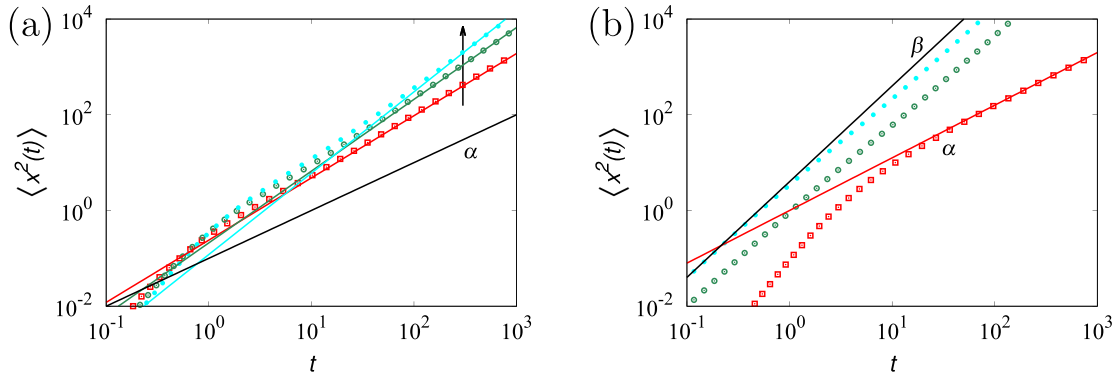


FIG. 7. $\langle x^2(t) \rangle$ vs t , obtained from stochastic simulations (symbols) of Eqs. (7) and (18) for different values of the exponent κ , and for different initial conditions starting from $x(0) = 0$. (a) refers to the ergodic case ($\kappa < 1$). The arrow indicates increasing values of $\kappa = 0.3, 0.5, 0.7$. The solid lines represent Eq. (22) for the corresponding values of κ . Line α corresponds to $\langle x^2(t) \rangle \sim t$ in order to mark the difference with respect to the regular diffusive scaling. The initial conditions are $v(0) = z_n(0) = 0$, $n = 1, \dots, N$, but in the ergodic case the initial setting is immaterial. (b) refers to $\kappa = 2.5$, starting from vanishing initial condition for the auxiliary variables $z_n(0) = 0$, $n = 1, \dots, N$ and with $v(0)$ Gaussian random variable with zero mean and variance σ_0 . Symbols ($\square$) refer to $\sigma_0 = 0$, symbols ($\circ$) and ($\bullet$) to $\sigma_0 = 1, 2$, respectively. Line (β) corresponds to the ballistic scalings $\langle x^2(t) \rangle \sim t^2$, while line (α) to $\langle x^2(t) \rangle \sim t^{1.1}$.

moments Eq. (16), and perform a quasi-steady-state approximation for m_{vv} and $m_{z_m z_p}$, corresponding to the assumption that $dm_{vv}/dt = dm_{z_m z_p}/dt = 0$. This leads to the following algebraic conditions,

$$m_{vv} = \frac{1}{\Gamma} \sum_{n=1}^N \gamma_n \mu_n m_{vz_n} + \frac{1}{\Gamma} \sum_{n=1}^N d_n^2 \quad (35)$$

$$m_{z_m z_n} = \frac{m_{vz_m} + m_{vz_n} + 2c_n^2 \delta_{mn}}{\mu_m + \mu_n}. \quad (36)$$

Substituting these expressions into the dynamics for the mixed moments $m_{vz_n}(t)$ [second set of equations in Eq. (16)], and enforcing the definition of the coefficients d_n, c_n Eq. (12), one ends up with the reduced dynamics

$$\begin{aligned} \frac{dm_{vz_n}}{dt} = & -(\Gamma + \mu_n) m_{vz_n} + \sum_{m=1}^N \frac{\gamma_m \mu_m}{\mu_n + \mu_m} m_{vz_n} \\ & + \sum_{m=1}^N \frac{\gamma_m \mu_m}{\mu_n + \mu_m} m_{vz_m} + \frac{1}{\Gamma} \sum_{m=1}^N \gamma_m \mu_m m_{vz_m}. \end{aligned} \quad (37)$$

Let $\mathbf{B}$ the constant $N \times N$ coefficient matrix associated with the reduced model Eq. (37) and let $\nu_{\max}$ be the maximal real part of its eigenvalues. The graph of $\nu_{\max}$ vs κ is depicted in Fig. 11. For $\kappa < 1$, all the eigenvalues have negative real part. At $\kappa = \kappa_c = 1$ the reduced system loses stability and $\nu_{\max}$ becomes positive for $\kappa > 1$. Although there is no specific dynamic motivation underlying the quasi-steady-state assumption, the physical interpretation of this result is rather clear. Since for $\kappa < 1$ all the eigenvalues of $\mathbf{B}$ are negative, the mixed moments $m_{vz_n}(t)$ are vanishing in the long term. The vanishing of these mixed moments is the condition (detailed Langevin condition) ensuring the validity of Kubo fluctuation-dissipation relations, and in turn $\langle v^2(t) \rangle$ would converge to the equilibrium value $\langle v^2 \rangle_{\text{eq}} = 1$, independently of the initial conditions (ergodicity). Conversely, the instability characterizing the reduced system Eq. (37) for the mixed moments with $\kappa > 1$ indicates that in this case $m_{vz_n}(t)$ cannot be vanishing in the long term, and consequently the dynamics of the system behaves in a way not consistent with the fluctuation-dissipation predictions. The latter analysis of the transition occurring at

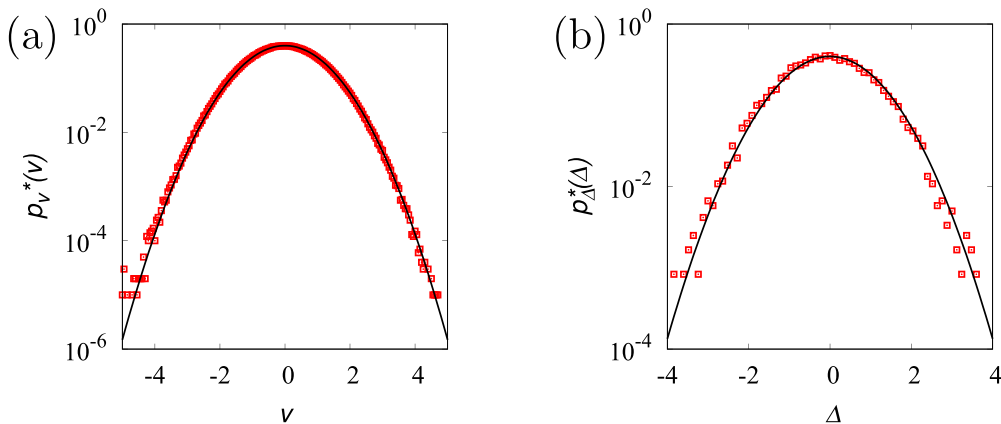


FIG. 8. (a) Equilibrium velocity distribution $p_v^*(v)$ for $\kappa = 0.5$. (b) Long-term distribution $p_\Delta^*(\Delta)$ for the normalized displacements $\Delta(t)$. Symbols represent the results of stochastic simulations, the solid line in both panels is the normalized Gaussian density with zero mean and unit variance.

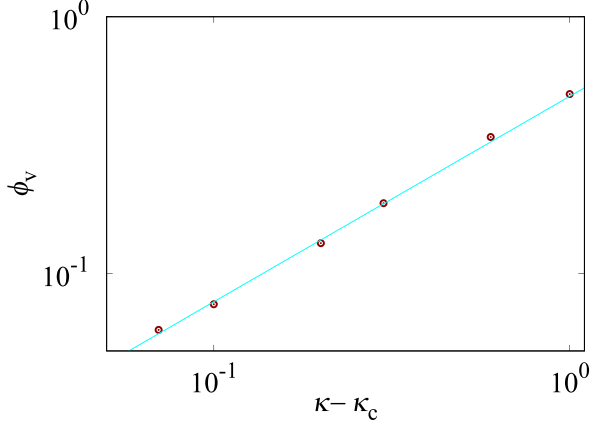


FIG. 9. Behavior of the order parameter ϕ_v defined by Eq. (23) as a function of $\kappa - \kappa_c$. Symbols are the results of moment analysis up to a maximal time order of 10^9 . Symbols ($\circ$) correspond to reliable data in this temporal window. The solid line represent the scaling $\phi_v \sim (\kappa - \kappa_c)^\xi$ with $\xi = 0.8$.

$\kappa = \kappa_c$ recasts the problem in the form of a bifurcational issue associated with the loss of stability of an equilibrium point for the second-order reduced moment dynamics.

E. Presence of attractive potentials

Another interesting situation involves the presence of attractive potentials $U(x)$ such that $\int_{-\infty}^{\infty} e^{-U(x)} dx < \infty$, ensuring the existence of a steady state. Observe that in the nondimensional formulation $k_B T = 1$. In this case, the first Eq. (7) for $dv(t)/dt$ should be substituted by

$$\begin{aligned} \frac{dv(t)}{dt} = & -\Gamma v(t) + \sum_{n=1}^N \gamma_n \mu_n z_n(t) \\ & - \frac{dU(x(t))}{dx} + \sqrt{2} \sum_{n=1}^N d_n \xi_n(t) \end{aligned} \quad (38)$$

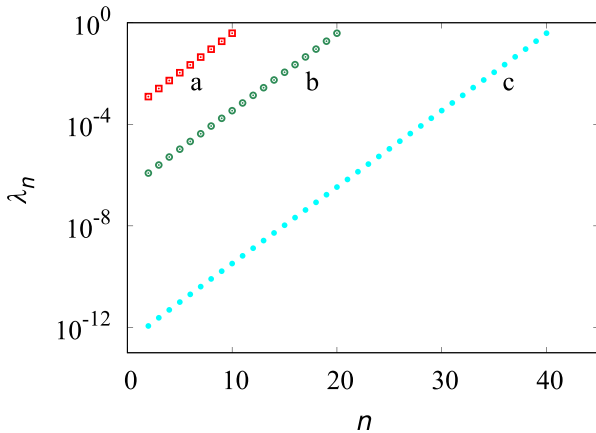


FIG. 10. Eigenvalues (with reversed sign) λ_n vs n associated with the kernel $k(t)$ defined by the matrix $\mathbf{A}$ Eq. (26) at $\kappa = 0.5$ for different values of the truncation order N : (a) $N = 10$, (b) $N = 20$, (c) $N = 40$.

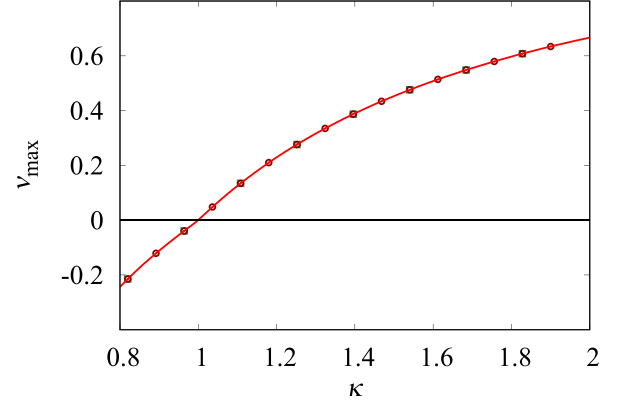


FIG. 11. Maximal real part $\nu_{\max}$ of the eigenvalues associated with the coefficient matrix of the linear reduced system Eq. (37) as the exponent κ varies.

coupled with the kinematic equation Eq. (18). We consider two prototypical attractive potentials: (i) the harmonic potential

$$U(x) = \frac{x^2}{2} \quad (39)$$

and the double-well potential

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2}. \quad (40)$$

In both cases, the long-term steady-state solution for the marginal velocity and position densities are the Maxwellian and the Boltzmannian respectively. This is shown in Fig. 12. This result is independent of the initial conditions and is valid for any value of κ as depicted in Fig. 13.

Indeed, this result is valid in the presence of any dissipationless inertial kernel. To prove this, we can consider the simplest possible case, namely an exponential kernel $k(t) = \gamma e^{-\mu t}$ defined by a single mode, in the presence of a harmonic potential. The modal equations for this setting read

$$\begin{aligned} \frac{dx(t)}{dt} &= v(t) \\ \frac{dv(t)}{dt} &= -\gamma v(t) + \gamma \mu z(t) - x(t) + \sqrt{2} d \xi(t) \\ \frac{dz(t)}{dt} &= -\mu z(t) + v(t) + \sqrt{2} c \xi(t), \end{aligned} \quad (41)$$

where $d = \sqrt{\gamma}$ and $c = -1/\sqrt{\gamma}$. If we consider the dynamics of the second-order moments we obtain at steady state (steady-state moments are indicated with an asterisk),

$$m_{vv}^* = m_{xx}^* = 1, \quad m_{zz}^* = \frac{1}{\gamma \mu}, \quad m_{xv}^* = m_{xz}^* = m_{vz}^* = 0 \quad (42)$$

corresponding to the equilibrium solution. This result can be straightforwardly generalized to the presence of an arbitrary number of modes. This means that the presence of an attractive potential (the theoretical analysis is limited to the harmonic case) provides a form of thermalization even in the absence of dissipative effects deriving from fluid-particle interactions.

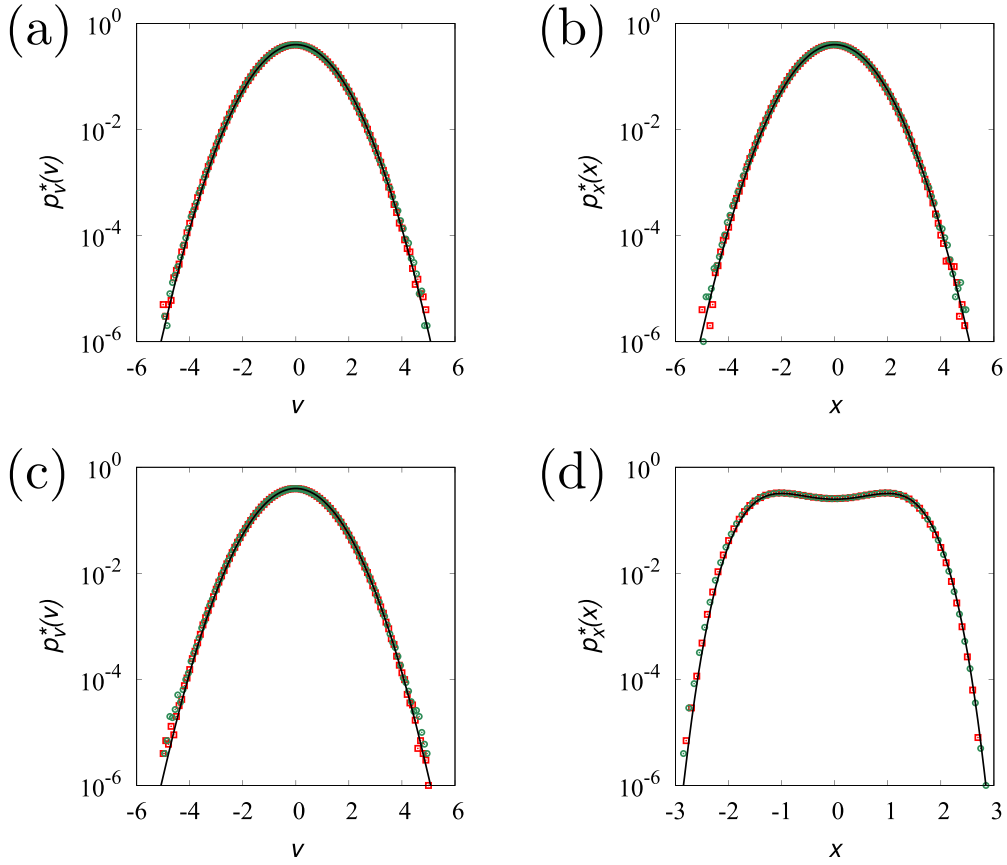


FIG. 12. Equilibrium velocity $p_v^*(v)$ [(a) and (c)], and position $p_x^*(x)$ [(b) and (d)] marginal densities in the presence of attractive potentials. Symbols represent the results of stochastic simulations: ($\square$) correspond to $\kappa = 0.5$, ($\circ$) to $\kappa = 2.5$. (a) and (b) refer to the harmonic potential, (c) and (d) to the double-well potential. Solid lines in (a), (b), and (c) correspond to the Gaussian distribution with zero mean and unit variance. Solid line in (d) is the Boltzmann distribution $p_x^*(x) = Ae^{-U(x)}$, where A is the normalization constant and $U(x) = x^4/4 - x^2/2$ is the double-well potential.

The explanation of this phenomenon is purely of dynamic nature. Consider again the one-mode model Eq. (41), which is governed by a linear dynamic for $\mathbf{y} = (x, v, z)$ controlled by

the coefficient matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -\gamma & \gamma\mu \\ 0 & 1 & -\mu \end{pmatrix}. \quad (43)$$

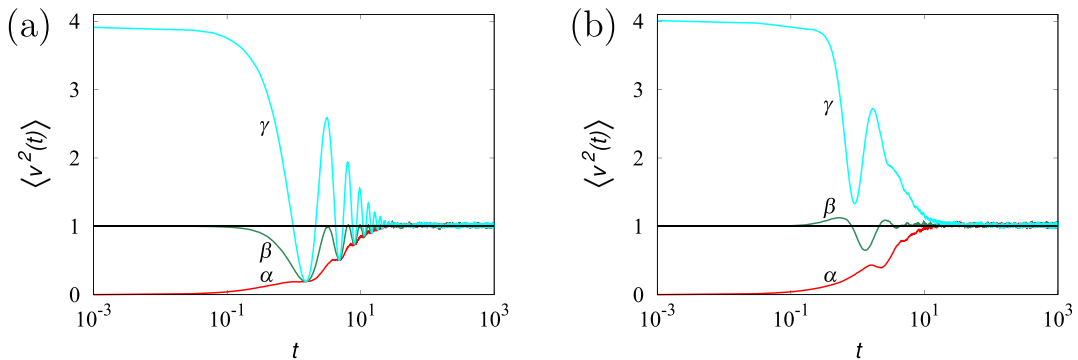


FIG. 13. $\langle v^2(t) \rangle$ vs t in the presence of attractive potentials, $\kappa = 2.5$, for initial conditions $z_n(0) = 0$ $n = 1, \dots, N$, and $v(0)$ normally distributed with variance σ_0 . (a) refers to the harmonic potential, (b) to the double-well potential. Lines α , β , γ correspond to $\sigma_0 = 0, 1, 2$, respectively.

From the analysis of the characteristic polynomial associated with the matrix $\mathbf{A}$, it follows that all the eigenvalues admit negative real part for $\gamma, \mu > 0$. Therefore, the presence of a harmonic potential introduces dynamically a stabilizing behavior enabling the achievement of a thermalizing action of thermal fluctuations even in the absence of dissipative fluid inertial interactions. This result is consistent with the analysis reported in Ref. [9].

F. Presence of steady forces

As addressed in Sec. III A, there are some features of the class of inertial kernel defined by Eqs. (3) and (4) that make them acquire some irreversible behavior leading to thermalization and to an equilibrium solution independently of the initial conditions in the presence of attractive potentials.

This stems essentially from the spectral structure of the resulting GLE defined by the matrix $\mathbf{A}$ Eq. (26): it possesses a vanishing eigenvalue while the remaining ones are strictly negative (see Fig. 10). The presence of a harmonic potential destroys the central eigenspace, turning the central eigenvalue from vanishing to negative.

In this regard, there is a profound difference with respect to the kernels arising from linear heat-bath models (Zwanzig heat-baths) [21,52,53] involving a finite but arbitrarily large numbers of bath particles, for which all the eigenvalues of the coefficient matrix are purely imaginary, and the superposition of an external potential acting on the tagged particle does not change the qualitative (spectral) properties of the dynamics.

However, there is a clear unambiguous indicator of dissipation that marks the difference between truly dissipative (strong thermalizing) and apparently dissipative (weak thermalizing) systems: the long-term velocity dynamics in a presence of a bounded force and specifically a constant one. For this reason, we consider this case, for which the first Eq. (7) for $dv(t)/dt$ becomes

$$\frac{dv(t)}{dt} = -\Gamma v(t) + \sum_{n=1}^N \gamma_n \mu_n z_n(t) + F + \sqrt{2} \sum_{n=1}^N d_n \xi_n(t), \quad (44)$$

where $F > 0$ is constant.

In the dissipative case, the particle velocity relaxes towards a constant value, the settling velocity $v_\infty = F/\eta_\infty$, where $\eta_\infty = \int_0^\infty h(t) dt$ is the integral of the dissipative memory kernel $h(t)$ introduced in Eq. (2).

The saturation towards a constant settling velocity cannot happen in the absence of dissipation, and the particle velocity grows unboundedly in time.

This can be easily seen from Eq. (44), considering the first-order moment, i.e., the expected value for the velocity. In the Laplace domain we have for small s (long times)

$$\langle \hat{v}(s) \rangle = \frac{F}{s[s + s\hat{k}(s)]} \sim \frac{F}{s(s + s^\kappa)}. \quad (45)$$

Also in this case a different scaling occurs below and above the critical value $\kappa_c = 1$. For $\kappa < 1$, the term s^κ within the brackets at the denominator of Eq. (45) is the leading term with respect to s for small values of s , thus $\langle \hat{v}(s) \rangle \sim F/s^{1+\kappa}$, leading to $\langle v(t) \rangle \sim F t^\kappa$, $\langle x(t) \rangle \sim F t^{1+\kappa}$. For $\kappa > 1$, the opposite occurs, $\langle \hat{v}(s) \rangle \sim F/s^2$, and as a consequence $\langle v(t) \rangle \sim$

$F t$, $\langle x(t) \rangle \sim F t^2$. This phenomenon is depicted in Fig. 14 with the aid of data deriving from stochastic simulations of Eq. (44) using an ensemble of 10^5 realizations.

This result is consistent with the analysis developed in Ref. [54] as regards the different dynamical properties of Langevin models compared to conservative models of particle dynamics in a heat bath.

A similar result occurs also in the presence of a superimposed periodic potential. Consider Eq. (44) with F substituted by $F - U'(x)$, $U'(x) = dU(x)/dx$, and $U(x) = -\cos(2\pi x)/2\pi$. The presence of a periodic potential does not alter the scaling of $\langle x(t) \rangle$ and $\langle v(t) \rangle$, as depicted in Fig. 15(a). It is important to observe that the divergence of $\langle v(t) \rangle$ as $t \rightarrow \infty$ is independent of the intensity of the force $F \neq 0$, Fig. 15(b). This is to be expected as, in the fluid inertial case, the structure of the stochastic fluctuations consistent with Kubo's theory involves Wiener forcings acting directly on the velocity dynamics (last term at the right-hand side of Eq. (44)).

G. Weak thermalization and strong dissipation

The results developed in Secs. III A, III E, and III F require a deeper analysis as regards the connection between irreversibility, thermalization and dissipation, as some form of thermalization and irreversibility is exhibited by nondissipative system (associated with power-law inertial kernels with $\kappa < 1$). For the sake of clarity, it is convenient to introduce some definitions taking the GLE associated with particle hydromechanics as a working example. We refer to isothermal conditions.

A GLE is weakly thermalizing if, in the absence of external perturbations (external forces and potentials) other than the equilibrium thermal force, the velocity attains a unique equilibrium distribution independently of the initial state of the system. This implies that $\langle v^2(t) \rangle$ would converge for long times to the equilibrium value $\langle v^2 \rangle_{\text{eq}}$, independently of the initial preparation of the system, i.e., of the values for $v(t)$ and $z_n(t)$ at time $t = 0$.

A GLE is canonically robust if, in the presence of attractive potentials $U(x)$ superimposed to the equilibrium thermal fluctuations, there exists a unique equilibrium distribution, independently of the initial conditions, coinciding with the Maxwell-Boltzmann distribution. This diction stems from the property that the GLE provides equilibrium statistics analogous to those predicted by the equilibrium ensemble theory (canonical ensemble).

A GLE is strongly thermalizing if it is weakly thermalizing and canonically consistent and moreover, for any bounded force $F(x)$ (such that there exists a constant $K > 0$ and $|F(x)| < K$) superimposed to the equilibrium thermal fluctuations, the velocity attains in the long-term limit a bounded value, independently of the initial conditions. More specifically, if the force applied is constant, the velocity should converge towards a constant limit value.

In the light of these definitions the behavior displayed by GLEs characterized by dissipationless inertial kernel with long-term power-law scaling can be summarized as follows:

(1) for $\kappa < 1$ the GLE is both weakly thermalizing and canonically robust;

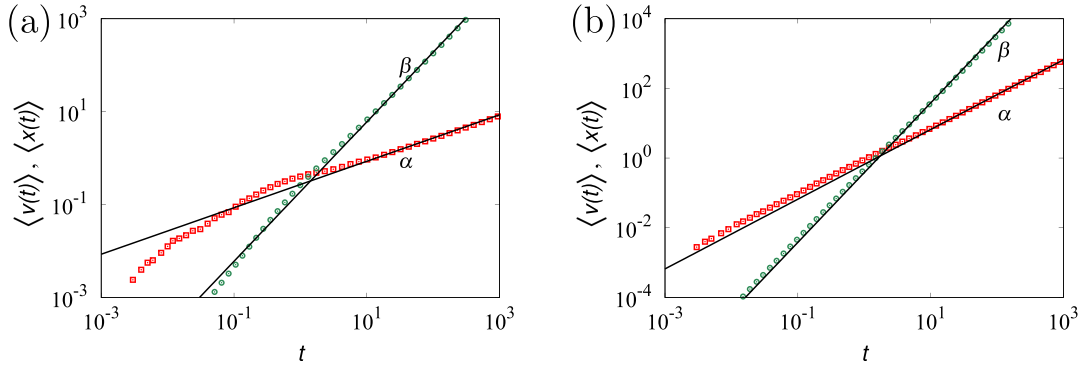


FIG. 14. Mean values for velocity $\langle v(t) \rangle$ [symbols ($\circ$) and lines α] and position $\langle x(t) \rangle$ [symbols ($\square$) and lines β] vs time t in the presence of a constant force $F = 1$. (a) refers to $\kappa = 0.5$, (b) to $\kappa = 2.5$. Symbols are the results of stochastic simulations. Line α in (a) corresponds to $\langle v(t) \rangle \sim t^\kappa$, line β to $\langle x(t) \rangle \sim t^{1+\kappa}$. Line α in (b) corresponds to $\langle v(t) \rangle \sim t$, line β to $\langle x(t) \rangle \sim t^2$.

(2) for $\kappa > 1$ the GLE is canonically robust but the property of weak thermalization is lost;

(3) independently of the value of κ the GLE is not strongly thermalizing.

Therefore, in the absence of dissipation it may happen than a GLE may display thermalization properties analogous to those of dissipative systems, i.e., some form of irreversibility. Nevertheless, the most strict property characterizing dissipative dynamics is their behavior in the presence of bounded forces. And this represents the most cogent test for a correct thermodynamic behavior. To this end, we have shown that dissipationless GLE cannot provide this type of limit behavior. The simplest physical model is the sedimentation of a solid particle in a fluid (with density greater than the density of the fluid medium) under the action of gravity. In the experiments a final settling velocity is achieved, supporting the claim that fluid-particle interactions are strictly dissipative.

It is important to observe that using some mathematical technicalities, it would be possible to justify the occurrence of a thermalizing behavior in dissipationless systems even in situations where thermalization cannot be physically achieved. This is associated with the existence of singular limits, and with the way these limits are performed.

Let us address this topic by means of a simple example consistent with the class of problems analyzed in this article. Consider a GLE characterized by an exponential kernel $k(t) = \gamma e^{-\mu t}$, and by a small friction factor $\epsilon > 0$. With reference to Eq. (2) this implies that the dissipative kernel $h(t)$ is impulsive, $h(t) = \epsilon \delta(t)$. In this case, the modal representation consistent with Kubo fluctuation-dissipation theory reads

$$\begin{aligned} \frac{dv(t)}{dt} &= -(\epsilon + \gamma) v(t) + \gamma \mu z(t) + \sqrt{2\epsilon} \xi'(t) + \sqrt{2} d \xi(t) \\ \frac{dz(t)}{dt} &= -\mu z(t) + v(t) + \sqrt{2} c \xi(t), \end{aligned} \quad (46)$$

where $d = \sqrt{\gamma}$, $c = -1/\sqrt{\gamma}$ and $\xi(t)$, $\xi'(t)$ are distributional derivatives of two independent Wiener processes. For $\epsilon > 0$, the system is dissipative and thus, for any initial conditions (expressed via the statistical properties of $v(0) = v_0$ and $z(0) = z_0$), one obtains

$$\lim_{t \rightarrow \infty} \langle v^2(t) \rangle = 1. \quad (47)$$

Consequently,

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow \infty} \langle v^2(t) \rangle = 1. \quad (48)$$

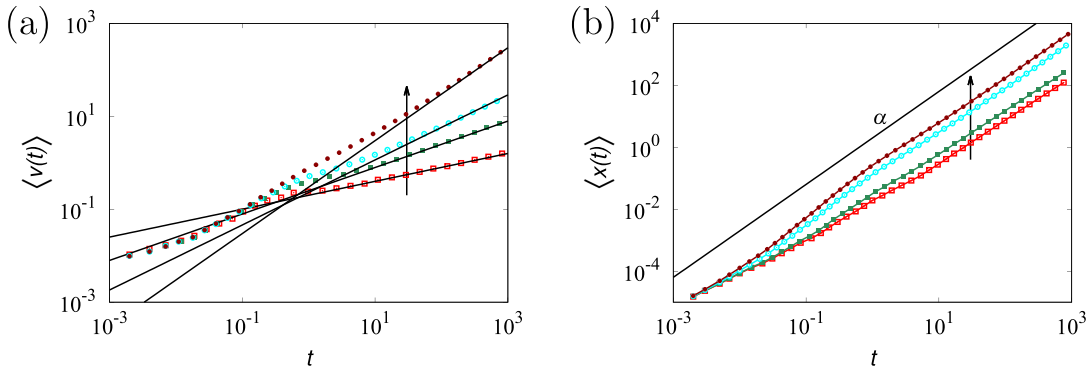


FIG. 15. $\langle v(t) \rangle$ and $\langle x(t) \rangle$ in the presence of a constant force F and of a superimposed potential contribution $U'(x) = \sin(2\pi x)$. (a): $\langle v(t) \rangle$ vs t at $F = 1$ for different values of the exponent κ . The arrows indicate increasing values of $\kappa = 0.3, 0.5, 0.7, 1.5$. Symbols represent the results of stochastic simulations, solid lines correspond to the theoretical scalings, i.e. $\langle v(t) \rangle \sim t^\kappa$ for $\kappa < 1$, and $\langle v(t) \rangle \sim t$ for $\kappa > 1$. (b): $\langle x(t) \rangle$ vs t at $\kappa = 0.5$, for different intensities of the constant force F . The arrow indicates increasing values of $F = 0.05, 0.1, 0.5, 1$. Line (α) corresponds to the scaling $\langle x(t) \rangle \sim t^{1+\kappa}$.

Alternatively, one can exchange the order of execution of the limits in Eq. (48), taking first the limit for $\varepsilon \rightarrow 0$, and subsequently the limit for $t \rightarrow \infty$. Doing so, one obtains

$$\lim_{t \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \langle v^2(t) \rangle = f[p_0(v_0, z_0)], \quad (49)$$

where $f[p_0(v_0, z_0)]$ is a functional of the statistical properties of the initial preparation of the system defined by the initial joint probability density function $p_0(v_0, z_0)$. Specifically,

$$f[p_0(v_0, z_0)] = 1 + \mu \frac{2\gamma \mu m_{vz}(0) + \mu m_{vv}(0) + \gamma^2 \mu m_{zz}(0)}{(\gamma + \mu)^2}, \quad (50)$$

where

$$m_{\alpha\beta}(0) = \iint \alpha \beta p_0(v_0, z_0) dv_0 dz_0, \quad \alpha, \beta = v_0, z_0 \quad (51)$$

are the second-order moments of the initial distribution. Eq. (50) stems from elementary moment analysis. We may therefore conclude that

$$\lim_{t \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \langle v^2(t) \rangle \neq \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \langle v^2(t) \rangle. \quad (52)$$

The qualitative properties displayed by the singular limit in Eq. (52) are altogether analogous to other singular limits occurring in the theory of GLEs and dissipation. Specifically, they occur in the definition of the memory kernel associated with Hamiltonian heat bath models [21], where the limit for time tending to infinity and for the number of heat-bath particles tending to infinity may not coincide. In the latter case, and in the case of Eq. (52), it is the physical reasoning that may resolve the apparent ambiguity of the limit, providing the correct interpretation of it.

IV. CONCLUDING REMARKS

In order to complete the analysis of ergodicity breaking in GLEs initiated with the work [20], we have explored in this article the case of dissipationless kernels. In the hydromechanic framework, this corresponds to the presence of purely fluid inertial interactions. The case of inertial kernels with long-term power-law tails is particularly interesting as a critical behavior resembling a phase transition occurs at the critical value $\kappa = 1$ of the exponent characterizing the long-term behavior.

For $\kappa < 1$ the system displays weak thermalization and is ergodic both in the presence of purely thermal fluctuations and in the case attractive potentials are superimposed to it. Weak thermalization is lost for $\kappa > 1$ together with the breakdown of ergodicity. For $\kappa < 1$ a superdiffusive scaling of the mean-square displacement is observed with an exponent equal $1 + \kappa$.

While the results for $\kappa < 1$ seem to suggest that, even in the absence of dissipation, some form of irreversibility may arise, the critical test for the latter claim involves the dynamic properties of momentum transfer in the presence of constant or bounded potentials. And enforcing this test (for a constant force) in the absence of a real dissipative fluid-particle interaction mechanism, the velocity diverges to infinity, and by no means it can reach a constant and bounded settling velocity as observed, e.g., in sedimentation experiments. In other words, even if for $\kappa < 1$ the dissipationless GLE is weak thermalizing, this is not true in the strong sense, and consistently it cannot describe the thermodynamic and energetic properties of an irreversible memory dynamics.

DATA AVAILABILITY

The data that support the findings of this article are openly available at [55].

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- [1] R. Kubo, *Rep. Prog. Phys.* **29**, 255 (1966).
 - [2] R. Kubo, M. Toda, and N. Hashitsune, *Statistical Physics II Nonequilibrium Statistical Mechanics* (Springer-Verlag, Berlin, 1991).
 - [3] H. Mori, *Prog. Theor. Phys.* **33**, 423 (1965).
 - [4] R. M. Hill and L. A. Dissado, *J. Phys. C* **18**, 3829 (1985).
 - [5] L. Stella, C. D. Lorenz, and L. Kantorovich, *Phys. Rev. B* **89**, 134303 (2014).
 - [6] T. Schilling, *Phys. Rep.* **972**, 1 (2022).
 - [7] O. Darrigol, *Eur. Phys. J. H* **48**, 10 (2023).
 - [8] J.-D. Bao and Y.-Z. Zhuo, *Phys. Rev. Lett.* **91**, 138104 (2003).
 - [9] J.-D. Bao, P. Hänggi, and Y.-Z. Zhuo, *Phys. Rev. E* **72**, 061107 (2005).
 - [10] P. Siegle, I. Goychuk, and P. Hänggi, *Europhys. Lett.* **93**, 20002 (2011).
 - [11] F. Ishikawa and S. Todo, *Phys. Rev. E* **98**, 062140 (2018).
 - [12] A. V. Plyukhin, *Phys. Rev. E* **83**, 062102 (2011).
 - [13] G. Bel and E. Barkai, *Phys. Rev. Lett.* **94**, 240602 (2005).
 - [14] R. Metzler, J.-H. Jeon, A. G. Cherstvy, and E. Barkai, *Phys. Chem. Chem. Phys.* **16**, 24128 (2014).
 - [15] Y. Liang, W. Wang, R. Metzler, and A. G. Cherstvy, *Phys. Rev. E* **108**, 034113 (2023).
 - [16] W. Deng and E. Barkai, *Phys. Rev. E* **79**, 011112 (2009).
 - [17] A. G. Cherstvy, S. Thapa, Y. Mardoukhi, A. V. Chechkin, and R. Metzler, *Phys. Rev. E* **98**, 022134 (2018).
 - [18] W. Wang, A. G. Cherstvy, R. Metzler, and I. M. Sokolov, *Phys. Rev. Res.* **4**, 013161 (2022).
 - [19] M. Giona, G. Procopio, and C. Pezzotti, *Phys. Rev. E* **111**, 034105 (2025).
 - [20] G. Procopio, C. Pezzotti, and M. Giona, *Phys. Rev. E* **111**, 034106 (2025).
 - [21] M. Giona, G. Procopio, and C. Pezzotti (2024), <http://dx.doi.org/10.2139/ssrn.5039729>.
 - [22] C. W. Macosko, *Rheology: Principles, Measurements, and Applications* (Wiley-VCH, New York, 1994).
 - [23] J. D. Ferry, *Viscoelastic Properties of Polymers* (John Wiley & Sons, New York, 1970).
 - [24] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1993).
 - [25] M. Giona, G. Procopio, and C. Pezzotti, *Phys. Fluids* **37**, 022042 (2025).

- [26] M. S. Mongiovi, D. Jou, and M. Sciacca, *Phys. Rep.* **726**, 1 (2018).
- [27] A. Schmitt, *Introduction to Superfluidity* (Springer, Cham, 2015).
- [28] M. C. Mackey, *Rev. Mod. Phys.* **61**, 981 (1989).
- [29] R. Huang, I. Chavez, K. M. Taute, B. Lukic, S. Jeney, M. G. Raizen, and E. L. Florin, *Nat. Phys.* **7**, 576 (2011).
- [30] J. Mo and M. G. Raizen, *Annu. Rev. Fluid Mech.* **51**, 403 (2019).
- [31] G. Procopio and M. Giona, *Fluids* **8**, 84 (2023).
- [32] X. Bian, C. Kim, and G. E. Karniadakis, *Soft Matter* **12**, 6331 (2016).
- [33] H.-Y. Yu, D. M. Eckmann, P. S. Ayyaswamy, and R. Radhakrishnan, *Phys. Rev. E* **91**, 052303 (2015).
- [34] M. Grimm, S. Jeney, and T. Franosch, *Soft Matter* **7**, 2076 (2011).
- [35] S. Kim and S. J. Karrila, *Microhydrodynamics, Principles and Selected Applications* (Dover Publications, Mineola, 2005).
- [36] G. Procopio, V. Biagioni, and M. Giona, *Fluids* **9**, 260 (2024).
- [37] R. F. Fox and G. E. Uhlenbeck, *Phys. Fluids* **13**, 1893 (1970).
- [38] E. H. Hauge and A. Martin-Löf, *J. Stat. Phys.* **7**, 259 (1973).
- [39] T. S. Chow and J. J. Hermans, *Physica* **65**, 156 (1973).
- [40] J. F. Morris and E. Guazzelli, *A Physical Introduction to Suspension Dynamics* (Cambridge University Press, Cambridge, 2012).
- [41] J. F. Brady and G. Bossis, *Annu. Rev. Fluid Mech.* **20**, 111 (1988).
- [42] S. Chandrasekhar, *Rev. Mod. Phys.* **15**, 1 (1943).
- [43] P. Langevin, *C. R. Acad. Sci. (Paris)* **146**, 530 (1908).
- [44] T. Li and M. G. Raizen, *Ann. Phys. (Berlin)* **525**, 281 (2013).
- [45] V. Vladimirovsky and Y. A. Terletsky, *Zh. Eksp. Teor. Fiz. (russian)* **15**, 258 (1945).
- [46] J. Tothova, G. Vaszlova, L. Glod, and V. Lisy, *Eur. J. Phys.* **32**, 645 (2011).
- [47] J. Tothova, G. Vaszlova, L. Glod, and V. Lisy, *Eur. J. Phys.* **32**, L47 (2011).
- [48] V. Vladimirovsky, *Z. Eksp. Teor. Fiz.* **12**, 199 (1942).
- [49] M. Giona, G. Procopio, and C. Pezzotti, [arXiv:2412.19166](https://arxiv.org/abs/2412.19166).
- [50] T. G. Mason, K. Ganesan, J. H. van Zanten, D. Wirtz, and S. C. Kuo, *Phys. Rev. Lett.* **79**, 3282 (1997).
- [51] T. M. Squires and T. G. Mason, *Annu. Rev. Fluid Mech.* **42**, 413 (2010).
- [52] R. Zwanzig, *J. Stat. Phys.* **9**, 215 (1973).
- [53] R. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, Oxford, 2001).
- [54] M. Evstigneev and A. Al-Haidari, *J. Phys. A* **52**, 055001 (2019).
- [55] G. Procopio, C. Pezzotti, and M. Giona, Dataset for “Long-tailed dissipationless hydromechanics: weak thermalization and ergodicity breaking” [Data set], Zenodo (2026), doi:[10.5281/zenodo.18550493](https://doi.org/10.5281/zenodo.18550493).