

Random walks across dimensions: Exploring simplicial complexes

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(Received 22 January 2026; revised 9 July 2026; accepted 9 September 2026; published 5 October 2026)

We introduce an operator to describe a random walk process on a simplicial complex. Walkers are allowed to wander across simplices of various dimensions, bridging nodes to edges, and edges to triangles, via a nested organization that hierarchically extends to higher structures of arbitrarily large, but finite, dimension. The asymptotic distribution of the walkers provides a natural ranking to gauge the relative importance of higher-order simplices. Optimal search strategies in the presence of stochastic teleportation are addressed, and the peculiar interplay of noise with higher-order structures is unraveled.

DOI: [10.1103/ppm1-mxhz](https://doi.org/10.1103/ppm1-mxhz)

I. INTRODUCTION

In the last few decades, complex networks (CNs) have proven essential in understanding a wide range of natural and technological problems, via captivating representations of the binary interactions among constituting units [1,2]. Examples range from biological applications (such as the patterns of interactions between proteins and/or genes) to neuroscience, via engineering-related problems (as power grid stability and synchronization), socioeconomic phenomena, and the spreading of epidemics [3–10]. In all examined settings, nontrivial interconnections between the observed dynamics and the underlying support emerge as solid evidence [11–13].

A further step in understanding the collective dynamics of systems composed of mutually entangled agents can be achieved by explicitly accounting for higher-order interactions (HOIs). These latter are naturally incorporated in simplicial complexes and/or hypergraphs, which thus define a modeling frontier to challenge the onset of emergent phenomena in complex environments populated by many agents in simultaneous interactions [14–18]. Notable applications of such a generalized paradigm range from neuroscience to sociology, and potentially embrace a large gallery of distinct disciplinary realms [19–22]. HOI models offer indeed an ideal platform to elaborate on the nontrivial interplay between dynamics and topology, as highlighted in [17,23–26]. Among the most fundamental dynamical processes that can be deployed on heterogeneous networks and used to unravel the hidden features of the latter, random walks are worth mentioning. Random walks have been thoroughly studied on standard networks built on pairwise interactions [27] and incessantly

invoked as a fundamental modeling ingredient. Their use can be found in the study of network topology via centrality measures, ranking and community detection [28–30], network exploration [31], and competing agentic frameworks [32], to mention a few. Furthermore, random walks have historically been successfully employed as general models of stochastic processes, providing a discrete framework for diffusion and Brownian motion [33,34] and have found applications in finance and pricing models [35,36], biology [37], and other fields. Starting from these premises, we aim to fill an evident gap by extending the elemental random walks recipe to higher-order simplices. At variance with past attempts [38,39], we leverage the generalized connectivity as naturally defined by the boundary operators to drive topological hopping across simplices of contiguous dimensions. This scheme applies to settings relevant, e.g., at the human scale; think, for instance, about information (or ideas) that can propagate from individuals to cliques and back again.

Further applications arise in contexts where dynamics evolve on and are shaped by higher-order interactions. In neuroscience, the brain is increasingly modeled as a simplicial complex, where neurons (0-simplices) are connected via synapses (1-simplices) and influenced by higher-order interactions composed by neural cliques of cofiring neurons [19,40–42].

Moreover, protein or gene regulation interactions often involve more than pairwise relationships, motivating descriptions in terms of simplicial complexes or hypergraphs [43–45]. In this context, a dynamical process that transitions across dimensions, such as among higher-order functional clusters, provides a natural framework to model how signals propagate between different topological scales.

More generally, the proposed cross-dimensional random walk can be interpreted as a topological probing mechanism for empirical complex systems, including social, biological, and technological networks enriched with clique complexes. In such settings, it enables the community detection of higher-order organization beyond dyadic interactions. In particular, communities are not only characterized by dense edge

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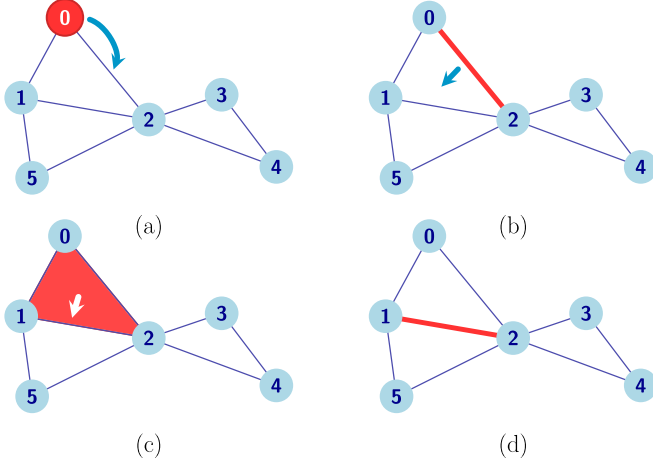


FIG. 1. Example of a random walk process across the dimensions of a simplicial complex. Here, the walker is initialized on the node 0 [panel (a)]. Then, it reaches the connected link $[0,2]$ [panel (b)], heads towards the triangle $[0,1,2]$ [panel (c)], and finally lands on link $[1,2]$ [panel (d)].

connectivity, but also by regions enriched in triangles, tetrahedra, and higher-dimensional simplices, creating functional topological modules [46,47].

Working within the proposed scheme, we will show that the asymptotic distribution of the walkers returns a sensible ranking for the relative importance of the simplices. Noise—acting as a form of long-range teleportation—is also introduced and shown to significantly potentiate the ability of the walkers to navigate within a complex environment decorated with higher-order structures, more effectively than what happens for the reference case where agents are solely bound to hop across vertices. In the following, we will begin by defining the suggested mathematical model.

II. THE MODEL

In Fig. 1, we provide a pictorial representation of the process that we set out to model, and that is ultimately intended to reproduce the dynamics of a walker wandering across the simplicial complex dimensions. In the displayed cartoon, the walker is initialized on a node, panel (a). Then, in the next step of evolution, it relocates to eventually land on one of the connected simplices (a link), see panel (b). From there, the process is iterated forward, and the walker first reaches an incident triangle, panel (c), before transitioning toward one of the edges that delimits the boundary of the departure face.

To mathematically tackle the above process, we begin by considering a D -simplicial complex $\mathcal{X}$, containing N_k k -simplices, $0 \leq k \leq D$, $\sigma_i^{(k)}$, $i = 1, \dots, N_k$, the latter being composed of $(k+1)$ nodes, i.e., 0-simplices. Once an orientation has been set for the simplices, the simplicial complex is completely defined by the incidence matrices $\mathbf{B}_k \in \mathbb{R}^{N_{k-1} \times N_k}$, $k = 1, \dots, D$ that set the underlying topological structure, namely, how simplices are assembled together, by taking into account their relative orientation [48,49]. We shall use in particular the unsigned incidence matrices $\mathbf{A}_k \in \mathbb{R}^{N_{k-1} \times N_k}$, whose entries equal 1 if the $(k-1)$ -simplex is incident, i.e., it is a

face, of the k -simplex, and 0 otherwise. In formulas,

$$A_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ iff } \sigma_i^{(k-1)} \subset \sigma_j^{(k)}. \quad (1)$$

For $k = 1, \dots, D$, we can introduce the “upper” degree of the $(k-1)$ -simplex $\sigma_i^{(k-1)}$, namely, the number of k -simplices to which it belongs, by

$$d_i^{(k-1)} = \sum_{j=1}^{N_k} A_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}). \quad (2)$$

In the case $k = 1$, i.e., the simplex $\sigma_i^{(0)}$ is a node, then $d_i^{(0)}$ returns the standard (network) node degree. Observe that we can set $d_i^{(D)} = 0$ for all $i = 1, \dots, N_D$, because in the D -simplicial complex, by definition, all the maximal simplices $\sigma_i^{(D)}$ are not faces of any larger simplices.

We can also introduce the “lower” degree of the k -simplex, $\sigma_j^{(k)}$, $k = 1, \dots, D$, namely, the number of $(k-1)$ -simplices contained in it; however, because of the closure condition of simplicial complexes, we can trivially state $\delta_j^{(k)} = \sum_{i=1}^{N_{k-1}} A_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) \equiv k+1$. Let us observe that since nodes do not accommodate smaller simplices, $\delta_j^{(0)} = 0$, for all $j = 1, \dots, N_0$. As stated above, hops occur between incident or adjacent simplices, namely, only steps toward elements characterized by one dimension higher or lower are allowed. Further, we assume time evolves by discrete steps, i.e., at each time step, the walker performs a hop from one simplex to another one. We define $p_t(\sigma_i^{(k)})$ as the probability of finding the walker at time t on the simplex $\sigma_i^{(k)}$ and assume an unbiased random walk. For all $0 < k < D$, the master equation ruling the time evolution of the above-defined probability reads

$$p_{t+1}(\sigma_i^{(k)}) = \sum_{j=1}^{N_{k-1}} p_t(\sigma_j^{(k-1)}) \frac{A_k(\sigma_j^{(k-1)}, \sigma_i^{(k)})}{d_j^{(k-1)} + \delta_j^{(k-1)}} + \sum_{l=1}^{N_{k+1}} p_t(\sigma_l^{(k+1)}) \frac{A_{k+1}(\sigma_l^{(k+1)}, \sigma_i^{(k)})}{d_l^{(k+1)} + \delta_l^{(k+1)}}, \quad (3)$$

where the leftmost term on the right-hand side stands for hops from $(k-1)$ to the k -simplices, while the rightmost one is for hops from a $(k+1)$ -to the k -simplices. The special cases $k = 0$ and $k = D$ are given by

$$p_{t+1}(\sigma_i^{(0)}) = \sum_{j=1}^{N_1} p_t(\sigma_j^{(1)}) \frac{A_1(\sigma_j^{(1)}, \sigma_i^{(0)})}{d_j^{(1)} + 2},$$

$$p_{t+1}(\sigma_i^{(D)}) = \sum_{j=1}^{N_{D-1}} p_t(\sigma_j^{(D-1)}) \frac{A_D(\sigma_j^{(D-1)}, \sigma_i^{(D)})}{d_j^{(D-1)} + D}. \quad (4)$$

Introduce for $k = 0, \dots, D$ the vectors $\vec{p}_t^{(k)} = (p_t(\sigma_1^{(k)}), \dots, p_t(\sigma_{N_k}^{(k)})) \in \mathbb{R}^{N_k}$ and the transition matrices $\tilde{\mathbf{M}}_k$ and $\tilde{\mathbf{M}}_{k+1}$ whose elements are, respectively, $\tilde{M}_k(j, i) = \frac{A_k(\sigma_j^{(k-1)}, \sigma_i^{(k)})}{d_j^{(k-1)} + \delta_j^{(k-1)}}$ and $\tilde{M}_{k+1}(i, l) = \frac{A_{k+1}(\sigma_l^{(k+1)}, \sigma_i^{(k)})}{d_l^{(k+1)} + \delta_l^{(k+1)}}$. Then, the above process can be recast in the more compact form:

$$\vec{p}_{t+1} = \vec{p}_t \mathbf{M}, \quad (5)$$

where $\bar{p}_t$ is the (row) vector $(\bar{p}_1^{(0)}, \dots, \bar{p}_t^{(D)}) \in \mathbb{R}^N$ and $\mathbf{M} \in \mathbb{R}^{N \times N}$, with $N = N_0 + \dots + N_D$ reads

$$\mathbf{M} = \begin{pmatrix} \mathbf{O}_0 & \hat{\mathbf{M}}_1 & \dots & \dots & \dots & \dots & \dots \\ \tilde{\mathbf{M}}_1^\top & \mathbf{O}_1 & \hat{\mathbf{M}}_2 & \dots & \dots & \dots & \dots \\ \vdots & \tilde{\mathbf{M}}_2^\top & \mathbf{O}_2 & \hat{\mathbf{M}}_3 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ \vdots & \vdots & \vdots & \tilde{\mathbf{M}}_k^\top & \mathbf{O}_k & \hat{\mathbf{M}}_{k+1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (6)$$

with $\mathbf{O}_k$ being the null $N_k \times N_k$ matrix. It is straightforward to prove that the above discrete-time dynamical system (5) preserves the total probability, namely, $\sum_{k=0}^D \sum_{i=1}^{N_k} p_t(\sigma_i^{(k)}) = 1 \quad \forall t \geq 0$. Hence, the largest eigenvalue of the stochastic matrix $\mathbf{M}$ is $\lambda_{\max} = 1$, and it is associated to the normalized eigenvector

$$\vec{u} = \frac{1}{\hat{N}} (d_1^{(0)}, \dots, d_{N_0}^{(0)}, d_1^{(1)} + 2, \dots, d_{N_1}^{(1)} + 2, \dots, d_1^{(D-1)} + D, \dots, d_{N_{D-1}}^{(D-1)} + D, D + 1, \dots, D + 1), \quad (7)$$

with

$$\begin{aligned} \hat{N} &:= 2N_1 + \dots + (D + 1)N_D + \sum_{k=0}^{D-1} \sum_{i=1}^{N_k} d_i^{(k)} \\ &= \sum_{k=1}^D 2(k + 1)N_k, \end{aligned} \quad (8)$$

(see [50]). The vector $\vec{u}$ represents the asymptotic occupation probability as obtained by iterating forward the dynamics via the map (5).

The cross-dimensional dynamical process expressed by the operator shown in Eq. (6) can be further enriched with jumps across various dimensions. We elaborate on that in the Appendix.

Before continuing, let us note that the proposed framework can be naturally extended to more general structures, such as finite regular CW-complexes. In particular, a similar discussion of walkers hopping among incident cells [in the spirit of Eq. (6)] can be carried out for polytopal cell complexes by defining the proper lower degree of each cell as the number of its incident codimension-one faces (4 for a square, 6 for a cube, 5 for a pentagon, etc.) [48,51]. A systematic formulation of such topological random walks is left for future discussions.

To challenge numerically the dynamical scheme introduced above, we need to preliminarily define a suitable algorithm for simplicial complex generation. In the following, we briefly comment on the solution that we have adopted [52] and which represents the ideal extension of the preferential attachment mechanism to higher-order graphs. In particular, at each generation step:

- (1) a new node enters the structure;
- (2) at this point, the dimension $k + 1$ of the simplex to be formed is chosen with probability p_{k+1} , with $\sum_{k=0}^{D-1} p_{k+1} = 1$;
- (3) then, the simplex $\sigma_j^{(k)}$ of dimension k is selected with probability $\Pi_j^{(k)} = \Delta_j^{(k)} / \sum_{h=1}^{N_k} \Delta_h^{(k)}$. Here, $\Delta^{(k)}$ stands for a

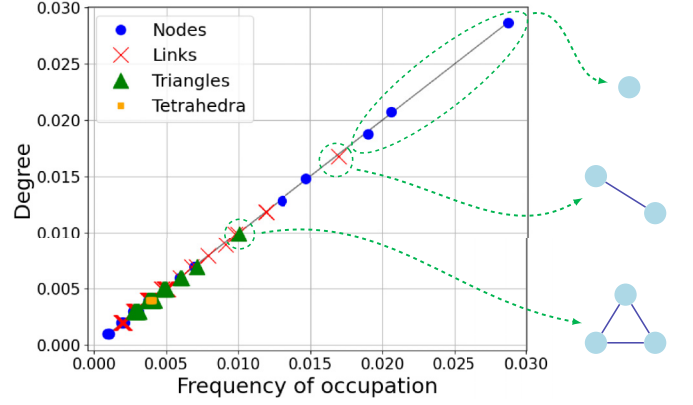


FIG. 2. Correspondence between the asymptotic frequency of occupation on the simplices (symbols) and the corresponding generalized degree (solid line). The degree centrality, corresponding to the occupation probability of the simplices, naturally mixes the various dimensions. Here, we set $p_1 = p_2 = p_3 = 1/3$ and $N_0 = 50$.

proxy of the degree, which can be set either to $d^{(k)} + \delta^{(k)}$, or, as we chose to do in this case, to $d^{(k)}$ to avoid dimensional biases. The selected simplex $\sigma_j^{(k)}$ is the module to which the incoming node attaches to create a new simplex of dimension $k + 1$ with all the subsimplices required by the closure property.

In the experiments reported throughout this work, we have chosen to solely adjust the triplet (p_1, p_2, p_3) , under the normalization condition $p_1 + p_2 + p_3 = 1$. Hence, all p_k with $k > 3$ are identically equal to zero and $D = 3$. Further details on the generation algorithm are provided in [52]. As a side comment, it is worth stressing that the above scheme allows us to deal with simplexes of different dimensions and aligns with established mechanisms for simplicial complex building that are, however, limited to considering simplexes of a given dimension at the time: by setting $p_D = 1$ and $\Delta^{(k)} = d^{(k)}$, one recovers the framework presented in [53] and analyzed in [54,55], with the so-called flavor value set to unit. Other examples of methods for SC generation are reported in [56,57], while, in parallel, a comprehensive collection of real-world datasets is provided in [58].

III. HIGHER-ORDER RANKING

Random walks evolving on higher-order graphs of the type generated above and following the prescriptions of the dynamics, as ruled by Eq. (5), distribute asymptotically across available dimensions—nodes, links, triangles, for the case addressed here—with a population fraction that reflects the associated generalized degree $\Delta_i^{(k)} = d_i^{(k)} + \delta_i^{(k)}$. This latter quantity (or, equivalently, the measured density of walkers on each substructure, at equilibrium) provides an indirect measure of the relative importance of fundamental simplices, across their hosting simplicial complex. The ensuing ranking, as induced by the operator $\mathbf{M}$, naturally spans across all dimensions, mixing higher- and lower-order structures. Nodes can, for instance, pop out as the most relevant structures, followed by links and triangles, with the interposed presence of additional nodes, as shown in Fig. 2. Symbols stand for direct

simulations of the microscopic dynamics (an ensemble made of noninteracting agents wanders across the dimension of the generated simplex), while the solid line refers to the theoretical solution, as discussed above (the normalized eigenvector of the stochastic matrix $\mathbf{M}$, relative to its largest eigenvalue). Tracking generalized random walkers through time allows us to weight the relative importance of k -dimensional structures of different dimensions and types, extending beyond the limited perspective of 0-dimensional units (the nodes), which define the restricted basin of exploration of their classical counterparts.

IV. EXPLORE ASSISTED BY NOISE

As a next step, we modify the dynamics by accounting for the possibility of stochastic teleportation. More specifically, the dynamics will combine local random walks with long-range jumps toward distant connected structures. Mathematically, this amounts to impose

$$\vec{p}_{t+1} = \vec{p}_t(\alpha\mathbf{M} + (1 - \alpha)\mathbf{D}_s\mathbf{S}), \quad (9)$$

where $\mathbf{S}$ is a random symmetric sparse matrix, with density $\delta = \sum_{i,j} S_{ij}/N^2$, and $\mathbf{D}_s = \text{diag}(\vec{k}_s)^{-1}$ with $k_s = \sum_j S_{ij}$ (see [59]). It is worth remarking that matrix $\mathbf{S}$ bridges simplices of any dimension, thus overcoming the topological constraints that are intrinsic to the proposed random walk recipe. Further, note that the case $\delta = 1$ constitutes a generalization of the celebrated PageRank algorithm [60] to the analyzed setting where higher-order structures are accounted for. The parameter α weights the relative importance of the two combined effects. When $\alpha = 1$, the system performs a pure random walk, and the agent can relocate only by engaging only incident simplices according to the underlying structure. Conversely, when $\alpha = 0$, random jumps are the sole allowed modality to displace in space. Because of the stochastic distribution of the teleportation site, this latter mechanism is conceptualized as a source of noisy perturbation. It is well known that properly tuning the stochastic contributions of the type introduced (via adjusting the control strength α) results in optimal search strategies throughout binary networks [59]. The first passage time T_{ij} , the average number of steps that the walker has to take to reach simplex j , starting from simplex i , is the quantity to be monitored, as a function of α , to shed light on this intriguing phenomenon. Hereafter, we will restrict the starting and final state to 0-dimensional simplices (i.e., the nodes), to enable a direct comparison with what is obtained by operating on ordinary graphs. Following [59], we can analytically compute the terms $T_{ij} = \sum_{h=1}^{N-1} (\mathbf{Z}_j^{-1})_{ih}$, where $\mathbf{Z}_j = \mathbb{I} - \mathbf{P}_j$, with the node's index j identifying the submatrix obtained from $\mathbf{P}$ by removing its j -th row and column. Here, $\mathbf{P} = \alpha\mathbf{M} + (1 - \alpha)\mathbf{D}_s\mathbf{S}$. By averaging T_{ij} over both indices i and j such that $1 \leq i, j \leq N_0$ (namely, all possible departure and arrival nodes), we obtain a scalar measure of the node explorability,

$$\langle T \rangle = \frac{1}{N_0(N_0 - 1)} \sum_{1 \leq i \neq j \leq N_0} T_{ij}. \quad (10)$$

Notice that given a simplicial complex $\mathcal{X}$, one can always define the underlying binary network $\mathcal{G}$ by just extracting

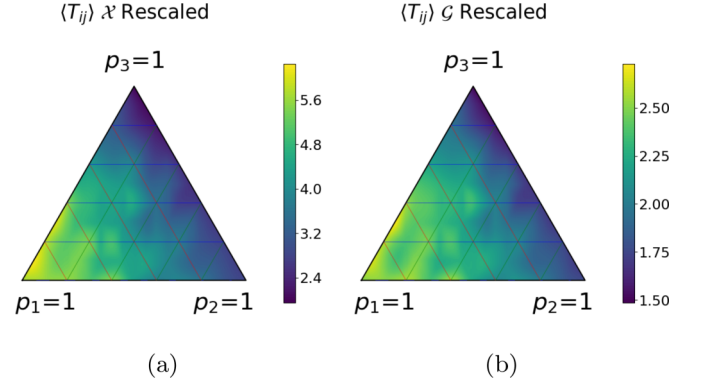


FIG. 3. Each point of these triangles corresponds to a choice of the parameter probabilities defining the topology of the generated simplex $\mathcal{X}$. Panel (a): Mean node FPT for the generated simplicial complex divided by the total number of simplices. Panel (b): Mean FPT for $\mathcal{G}$, divided by the number of nodes (the substructures on which the walker is confined, in this case), fixed at $N_0 = 20$.

the node-to-node interactions encoded by the links. We can therefore apply Eq. (10) to the higher-order support $\mathcal{X}$, on the one side, and to its corresponding graph $\mathcal{G}$, on the other, to compare the respective performance in terms of explorability, the ability to effectively explore the spatial support to which the system is eventually bound. For a fair comparison, the estimated average time $\langle T \rangle$ should be normalized by the extensive parameter that measures the total number of individual structures that the walker can visit when wandering across dimensions. These are the number of nodes for $\mathcal{G}$ and the total number of simplices for $\mathcal{X}$. We begin by analyzing the setting $\alpha = 1$, which corresponds to dealing with pure diffusion. The results are reported in Fig. 3, where the explorability measure $\langle T \rangle$ (normalized by the total number of structures, for either $\mathcal{X}$ and $\mathcal{G}$) is plotted as a function of the generative parameters p_1, p_2, p_3 , under the constraint $p_1 + p_2 + p_3 = 1$ imposed by the constructive algorithm. Apart from a slight difference (see annexed scales) which seems to indicate that the average search time on a simplex is further slowed down, besides obvious extensive scaling, the displayed patterns are pretty similar.

A radically different scenario is found for $\alpha < 1$, i.e., when noise is thus allowed. For any given topological support, the computed value of $\langle T \rangle$, also rescaled, is now influenced by two concurrent dynamical processes: local search and long-ranged teleportation. In Fig. 4, we report on a typical outcome of the analysis for this generalized setting. In panel (a), the properly normalized time $\langle T \rangle$ is plotted as a function of α for a specific choice of the construction parameters p_1, p_2, p_3 . A minimum is clearly displayed for both depicted curves, pointing to the existence of an optimal choice of the control parameter α that gauges the relative balance of the two combined dynamical effects. For large values of α , indicating a predominance of local random walk, the size-normalized explorability $\langle T \rangle$ is larger for simplicial complexes as compared to their respective underlying graph, in agreement with the results reported in Fig. 3. Nevertheless, by progressively strengthening the teleportation term, we observe a more pronounced drop in the (rescaled) search time computed for the

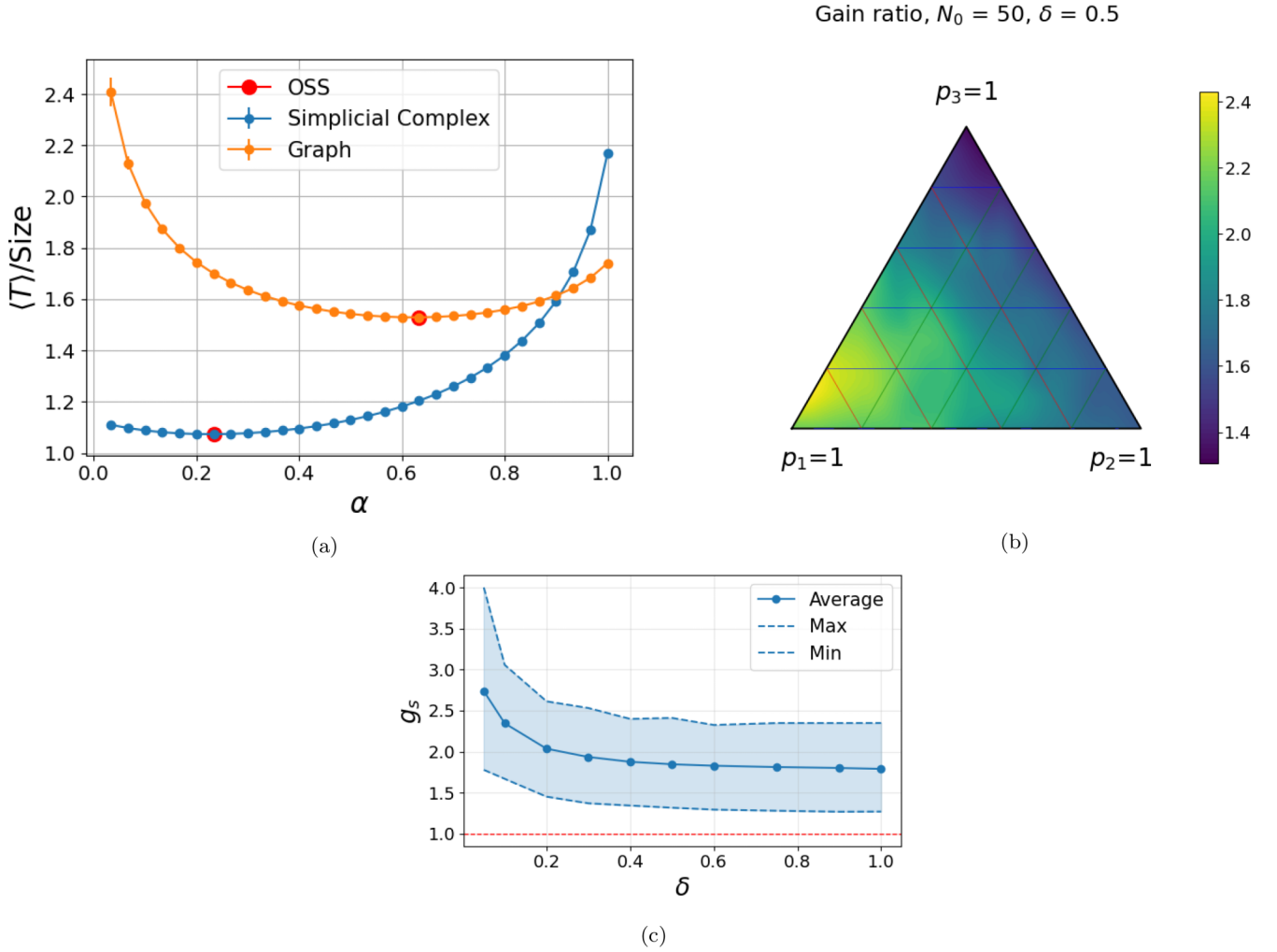


FIG. 4. Panel (a): Size-rescaled explorability for a simplicial complex and its corresponding graph, as a function of α . The red dot identifies the minimum of the reported curves, namely, the optimal search strategies (OSS). Here, we set $p_3 = 1$ and $\delta = 0.05$. Panel (b): Structural gain $g_s = \frac{g_{\mathcal{X}}}{g_{\mathcal{G}}}$, computed as the ratio between the gain of the simplicial complex topology and that of the corresponding graph topology, for $\delta = 0.5$. Panel (c) g_s as a function of δ . The shaded region traces the variability across different realizations of the generated simplices.

simplicial complex. Consequently, the corresponding gain in terms of (noise-assisted) exploration capability $g^a = \frac{(T)_{(\alpha=1)}}{\min_{\alpha}((T))}$, for $a = \mathcal{X}$ or $\mathcal{G}$, is larger for the simplicial complex $\mathcal{X}$, when compared to what is found on the corresponding graph $\mathcal{G}$, see Fig. 4(b) for a specific choice of δ . Simplicial complexes thus prove more responsive to the beneficial effects induced by the imposed stochasticity for effective explorability. Remarkably, the gain that reflects the presence of a global random teleportation term remains consistently greater for simplicial complex structures across all analyzed values of δ , see Fig. 4(c). The gain is particularly pronounced in the region close to $p_1 = 1$, where the simplicial complex topology tends to form more elongated structures that benefit more from the random teleportation.

V. CONCLUSION

In summary, we have presented a natural formulation of a random walk process on simplicial complexes, providing a simple yet conceptually fundamental model to describe cross-dimensional dynamics. This is achieved by introducing

an operator that models hopping across multidimensional structures. The ensuing stationary solution unveils the hidden ranking of composing simplices of different order and kind. We further applied this framework to compute the simplicial complex explorability, restricted to nodes, in the presence of a long-range stochastic relocation term. The effect of noise appears extremely beneficial when the walk is taken on higher-order structures, as compared to what happens on the corresponding binary graph. The codes employed are made publicly available in the repository [61] and maintained on [62].

ACKNOWLEDGMENTS

The work of D. Febbe and D. Fanelli is supported by #NEXTGENERATIONEU (NGEU) and funded by the Ministry of University and Research (MUR), National Recovery and Resilience Plan (NRRP), project MNESYS (Project No. PE0000006) ‘A multiscale integrated approach to the study of the nervous system in health and disease’ (DR. 1553 11.10.2022).

DATA AVAILABILITY

The data that support the findings of this study are openly available at [61].

APPENDIX: LONG-RANGE CROSS-DIMENSIONAL RANDOM WALK

The process described by Eq. (5) defines a random walk on a simplicial complex in which transitions are allowed only between incident simplices whose dimensions differ by one. Such a dynamics provides a simple, yet conceptually fundamental example of cross-dimensional diffusion over higher-order structures. Nevertheless, the same framework can be extended so as to include long-range jumps between simplices of nonadjacent dimensions, provided they are related by inclusion. More precisely, the walker can hop from simplex i to simplex j , or vice versa, iff $\sigma_i^{(m)} \subset \sigma_j^{(n)}$ with $m < n$.

To encode such transitions, we introduce the containment matrix, whose entries are defined as

$$C_{m,n}(\sigma_i^{(m)}, \sigma_j^{(n)}) = \begin{cases} 1, & \text{if } \sigma_i^{(m)} \subset \sigma_j^{(n)}, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A1})$$

that can be expressed in terms of the unsigned incidence matrices

$$C_{m,n}(i, j) = \mathbb{1}(A_{m+1}A_{m+2} \cdots A_n)_{ij}, \quad (\text{A2})$$

where $\mathbb{1}(\cdot)$ denotes the entry-wise indicator function detecting the existence of at least one chain of nested simplices connecting i to j , where, to lighten the notations, we have replaced $\sigma_i^{(m)}$ and $\sigma_j^{(n)}$ with i and j .

By using the matrices $C_{m,n}$, we can define a global block matrix $\mathbf{C} \in \mathbb{R}^{N \times N}$ encapsulating all terms describing the transitions among simplices:

$$\mathbf{C} = \begin{pmatrix} O_0 & C_{0,1} & C_{0,2} & \cdots & C_{0,D} \\ C_{0,1}^\top & O_1 & C_{1,2} & \cdots & C_{1,D} \\ C_{0,2}^\top & C_{1,2}^\top & O_2 & \cdots & C_{2,D} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{0,D}^\top & C_{1,D}^\top & C_{2,D}^\top & \cdots & O_D \end{pmatrix}. \quad (\text{A3})$$

To row-normalize such an operator, we introduce the lower long-range generalized degree of each given simplex $\sigma_i^{(k)}$ as

$$\begin{aligned} \mathfrak{d}_i^{(k)} &= \sum_{m=0}^{k-1} \sum_{j=1}^{N_m} C_{m,k}(\sigma_j^{(m)}, \sigma_i^{(k)}) \\ &= \sum_{m=0}^{k-1} \binom{k+1}{m+1} = 2^{k+1} - 2, \end{aligned} \quad (\text{A4})$$

and the upper long-range generalized degree

$$\mathfrak{d}_i^{(k)} = \sum_{n=k+1}^D \sum_{j=1}^{N_n} C_{k,n}(\sigma_i^{(k)}, \sigma_j^{(n)}). \quad (\text{A5})$$

Now, let $\mathbb{D} \in \mathbb{R}^{N \times N}$ be the diagonal matrix whose diagonal entries are the generalized long-range degrees $\mathfrak{d}_i^{(k)} + \mathfrak{d}_i^{(k)}$. Then, the transition matrix of the multidimensional random walk, generalizing the operator $\mathbf{M}$ defined in Eq. (6), is defined by

$$\mathbb{M} = \mathbb{D}^{-1} \mathbf{C}, \quad (\text{A6})$$

while the probability evolution reads

$$\vec{p}_{t+1} = \vec{p}_t \mathbb{M}. \quad (\text{A7})$$

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