

Internal reliability and antireliability in dynamical networks

Tommaso Matteuzzi ^{1,*} Franco Bagnoli ^{1,2,†,‡} Michele Baia ^{1,2,§} Stefano Iubini ^{3,2,||} and Arkady Pikovsky ^{4,¶}

¹*Department of Physics and Astronomy and CSDC, University of Firenze, via G. Sansone 1, I-50019 Sesto Fiorentino, Italy*

²*Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, via G. Sansone 1, I-50019 Sesto Fiorentino, Italy*

³*Istituto dei Sistemi Complessi, Consiglio Nazionale delle Ricerche, via Madonna del Piano 10, I-50019 Sesto Fiorentino, Italy*

⁴*Department of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Strasse 24/25, 14476 Potsdam-Golm, Germany*



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We consider finite dynamical networks and define internal reliability based on the synchronization properties of a replicated unit or a set of units. If the states of the replicated units coincide with their prototypes, they are reliable; otherwise, if their states differ, they are antireliable. Quantification of reliability using the transversal Lyapunov exponent enables a straightforward analysis of different models. For a Kuramoto model of globally attractively coupled phase oscillators with a distribution of natural frequencies, we show that before the onset of synchronization peripheral frequency units (at the edges of the distribution) are antireliable, while central ones (with natural frequencies close to the mean frequency) are reliable. For repulsive coupling, the complementary configuration occurs, with central antireliable units and peripheral reliable ones. For this model, reliability can be expressed through phase correlations in a sort of fluctuation-dissipation relation. Sufficiently large subnetworks in the Kuramoto model are always antireliable; the same holds for a recurrent neural network, where individual units are always reliable. Furthermore, we show that other coupled oscillator models (Winfree-type coupled phase oscillators, coupled rotators, and coupled Stuart-Landau oscillators) demonstrate patterns of reliability and antireliability similar to those of the Kuramoto setup.

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I. INTRODUCTION

The stability properties of dynamical states lie at the core of characterizing nonlinear dynamical systems. Sensitive dependence of the dynamics on initial conditions, quantified by positive Lyapunov exponents (LEs), serves as a definition of chaos [1]. In chaotic systems, the memory of the initial state is lost as the dynamics unfolds. If one prepares two identical systems (two replicas) but with slightly different initial conditions, then the states of the replicas will not be correlated after a transient period. The notions of reliability and antireliability [2–5] are used to describe the response of replicas to nontrivial external forces, in particular to external noise. Here, if the largest LE is negative, two replicas driven by the same noisy force show exactly the same dynamics and thus respond to this force reliably. In contrast, in the case of

a positive largest LE, the states of the two replicas differ. In the context of noisy forcing, one often speaks about the synchronization of replicas by common noise [6–9]. However, the same concept applies to a chaotic driving force. A reliable response of a nonlinear system to chaotic driving corresponds to a generalized synchronization phenomenon [10]; it is also best characterized by following two replicas of the driven system. Replicas are also used in the Pecora-Carroll synchronization setup [11].

In this paper, we apply the concepts of reliability and antireliability to the internal dynamics of deterministic (in particular, oscillatory) networks. Complex networks often demonstrate chaotic behavior. The question we address is the following: in such a regime, does a particular unit (or, more generally, a subgroup of the units) reliably follow the driving of the other units? As discussed above, to answer this question, we need to create a replica of the unit and observe whether this replica exhibits the same dynamics as the prototype (i.e., replicated) unit. In the following, we consider a range of typical models and demonstrate that in some of them all units are reliable, while in other models some units are reliable and some are not (to this class belongs the famous Kuramoto model, as well as some other models of coupled oscillators, in particular phase oscillators with Winfree-type coupling, a network of Stuart-Landau oscillators, and a network of rotators). We establish a relation between reliability and correlations for the Kuramoto model of globally coupled phase oscillators, which can be interpreted as a dissipation-fluctuation relation. Furthermore, we extend the notion of internal reliability to replicated subnetworks.

*Contact author: tommaso.matteuzzi@unifi.it

†Present address: Team Project IMAGES - UMR Espace-Dev, University of Perpignan via Domitia UPVD, 52 Avenue Paul Alduy, 66000 Perpignan, France

‡Contact author: franco.bagnoli@unifi.it

§Contact author: michele.baia@unifi.it

||Contact author: stefano.iubini@cnr.it

¶Contact author: pikovsky@uni-potsdam.de

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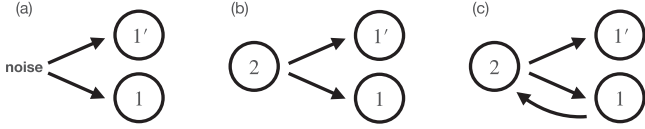


FIG. 1. Sketch of three setups: (a) synchronization by common noise (reliability), (b) generalized synchronization via auxiliary system method, and (c) internal reliability setup. Systems 1 and 1' are identical, identically driven systems; system 2 generally differs from system 1.

II. RELIABILITY, GENERALIZED SYNCHRONIZATION, AND INTERNAL RELIABILITY

Here, we discuss the relation between internal reliability, usual reliability, and generalized synchronization.

Reliability [2] is the same as synchronization by common noise [6–8], illustrated in Fig. 1(a). Here, the same noise acts on two identical systems 1 and 1'. Because the systems are identical and the driving is identical, a synchronous solution exists where the states coincide. This state can be stable or unstable. In the former case, the driven systems synchronize even if their initial states are different, so one can say that the response to the noisy driving is fixed; in this sense, the system reliably responds to the noise.

If one replaces the noisy driving with chaotic driving, then one obtains a setup of generalized synchronization [10,12] [Fig. 1(b)]. Here, a signal from a chaotic system 2 unidirectionally drives another dynamical system 1. To test for generalized synchronization between systems 1 and 2, one takes an auxiliary system 1' that is an exact copy (replica) of system 1 and checks if the states of 1 and 1' synchronize [13]. Thus, generalized synchronization is equivalent to reliability in the presence of chaotic (instead of noisy) driving. Note that systems 1 and 2 can be completely different, for example, having different dimensions. Many examples of generalized synchronization are discussed in Refs. [14,15].

In the generalized synchronization setup, a unidirectional coupling $2 \rightarrow 1$ is assumed. If the coupling is bidirectional $2 \leftrightarrow 1$ (but not necessarily symmetric), a natural way to look at whether system 1 “reliably” responds to system 2 is to make a replica 1' of system 1 and drive it with the same signal as 1 is driven, although there is no action from 1' back on 2 [Fig. 1(c)]. If the states of 1 and 1' coincide, we call the dynamics of 1 “internally reliable,” and we will call it “antireliable” if the states of 1 and 1' do not synchronize. “Internal” in this context means that no external driving is applied to the closed system of coupled subsystems [Eqs. (1) and (2)]. The subject of this paper is the exploration of the property of internal reliability, illustrated in Fig. 1(c).

Example of internal reliability in two coupled chaotic systems

Here, we give an example of the setup shown in Fig. 1(c). Following the original paper on generalized synchronization [13], we consider the following system of equations:

$$\begin{aligned}\dot{x} &= -y - z - au_1, \\ \dot{y} &= x + 0.2y,\end{aligned}$$

$$\begin{aligned}\dot{z} &= 0.2 + z(x - 5.7), \\ \dot{u}_1 &= 16(v_1 - u_1) + g(x - u_1), \\ \dot{v}_1 &= 45.92u_1 - v_1 - u_1w_1, \\ \dot{w}_1 &= u_1v_1 - 4w_1, \\ \dot{u}_2 &= 16(v_2 - u_2) + g(x - u_2), \\ \dot{v}_2 &= 45.92u_2 - v_2 - u_2w_2, \\ \dot{w}_2 &= u_2v_2 - 4w_2.\end{aligned}\quad (1)$$

Here variables (x, y, z) constitute a Roessler system [system 2 in Fig. 1(c)]. It drives two identical Lorenz systems (u_1, v_1, w_1) and (u_2, v_2, w_2) [systems 1 and 1' in Fig. 1(c), correspondingly] with coupling strength g . Lorenz system 1 drives the Roessler system with strength a .

In Fig. 2 we illustrate two cases, one without generalized synchronization and internal reliability [unidirectional coupling $2 \rightarrow 1$, $a = 0$, and $g = 4.55$ (in Ref. [13], generalized synchronization is reported for this setup for a larger coupling $g = 10$), Figs. 2(a)–2(c)] and one with internal reliability [$a = 0.29$ and $g = 4.55$, Figs. 2(d)–2(f)]. In both cases, the dynamics is chaotic.

III. INTERNAL RELIABILITY OF ONE UNIT IN A NETWORK

We first formulate the conditions for internal reliability in a pretty general form and then consider particular examples. Consider a set $j = 1, 2, \dots, N$ of units described by variables $\mathbf{x}_j(t)$ which have their own dynamics [term $\mathbf{F}_j(\mathbf{x}_j)$] and are coupled:

$$\begin{aligned}\dot{\mathbf{x}}_j &= \mathbf{F}_j(\mathbf{x}_j) + \mathbf{H}_j(\mathbf{x}_j; \mathbf{x}_1, \dots, \mathbf{x}_{j-1}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N), \\ j &= 1, 2, \dots, N.\end{aligned}\quad (2)$$

Let us concentrate on the unit with index k . We prepare a replica $\mathbf{y}_k$ of the unit $\mathbf{x}_k$ (below, we call the replicated unit “prototype”) that receives the same input from other elements as the unit $\mathbf{x}_k$ receives:

$$\dot{\mathbf{y}}_k = \mathbf{F}_k(\mathbf{y}_k) + \mathbf{H}_k(\mathbf{y}_k; \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}, \dots, \mathbf{x}_N). \quad (3)$$

Note that while index j in Eq. (2) runs over all possible values $j = 1, 2, \dots, N$, index k in Eq. (3) takes just one value. So Eqs. (2) and (3) constitute a system of $N + 1$ units. The replica $\mathbf{y}_k$ and its prototype $\mathbf{x}_k$ obey identical equations but have generally different initial states. Of course, if $\mathbf{y}_k(0) = \mathbf{x}_k(0)$, then the states of the prototype and its replica coincide for all future times. For a small deviation $\mathbf{v}_k = \mathbf{y}_k - \mathbf{x}_k$, we obtain, by subtracting Eq. (3) from Eq. (2) and linearizing, the linear system

$$\dot{\mathbf{v}}_k = \left(\frac{\partial \mathbf{F}_k}{\partial \mathbf{x}_k} + \frac{\partial \mathbf{H}_k}{\partial \mathbf{x}_k} \right) \mathbf{v}_k. \quad (4)$$

Solution of this linear equation asymptotically grows, $|\mathbf{v}_k| \sim \exp[\lambda_k t]$, where λ_k is the maximal transversal Lyapunov exponent (TLE) of Eq. (4). Equation (4) generally possesses a set of Lyapunov exponents (if each unit has a dimension larger than 1), but only the largest of them determines transversal stability. Positive λ_k means that the replica diverges from the prototype, and thus the unit k is antireliable; negative λ_k

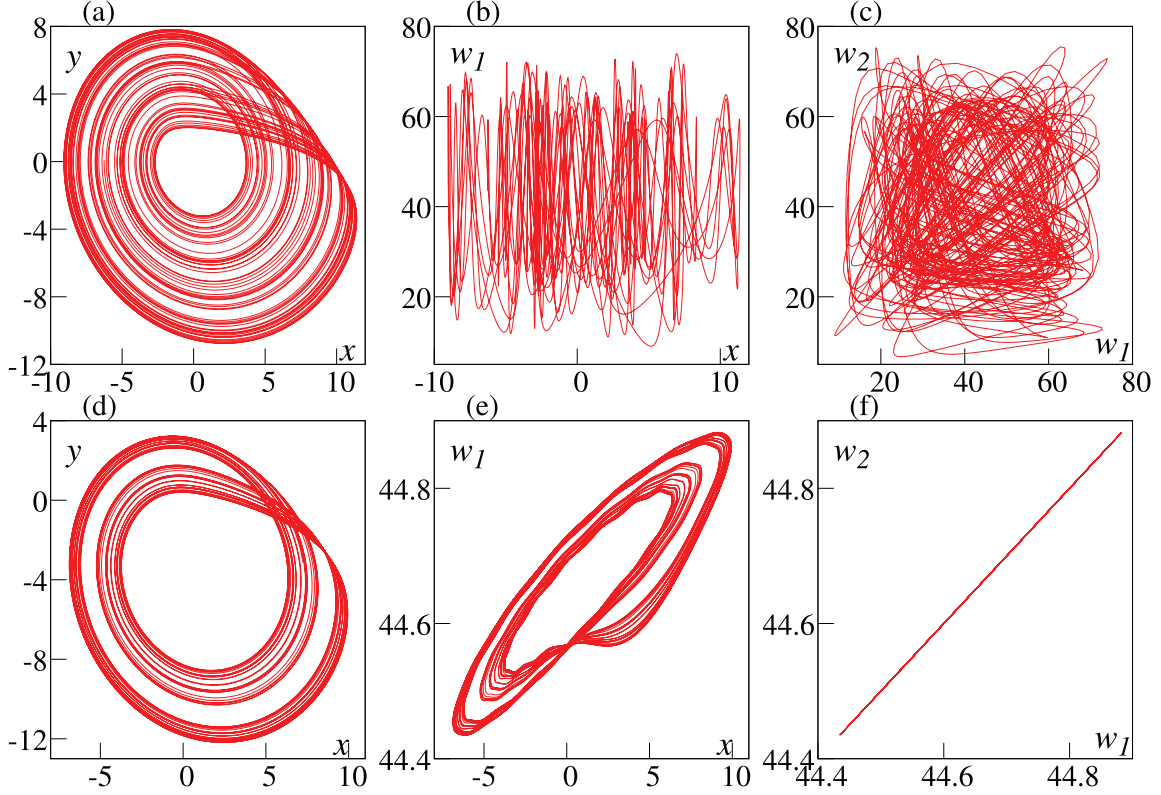


FIG. 2. (a)–(d) Generalized synchronization setup [Fig. 1(b)] for the coupled Roessler and Lorenz systems [Eq. (1)] with $a = 0$, $g = 4.55$. (a) Driving system x vs y . (b) Driving x vs response w_1 . (c) w_1 vs w_2 shows no synchrony between systems 1 and 1'. (d)–(f) Internal reliability setup [Fig. 1(c)] for the coupled Roessler and Lorenz systems [Eq. (1)] with $a = 0.29$, $g = 4.55$. (d) “Driving” system x vs y . (e) Driving x vs response w_1 . (f) w_1 vs w_2 shows complete synchrony between systems 1 and 1', as the trajectory lies on the diagonal.

means reliability: the replica converges and eventually $\mathbf{y}_k(t) = \mathbf{x}_k(t)$ at large times.

As the first relatively trivial example, let us consider a very popular model of a recurrent neural network [16]. Here, x_j is a scalar, and the dynamics reads

$$\dot{x}_j = -x_j + \sum_{i \neq j} J_{ji} \tanh(x_i), \quad j = 1, \dots, N, \quad (5)$$

where J_{ji} are independent random couplings taken from a Gaussian distribution with zero mean. Because the interactions do not depend on the driven unit, we have $\dot{v}_k = -v_k$, which means that all TLEs are -1 and we have complete reliability of all units. Below, we will see that replicated subnetworks of the recurrent neural network [Eq. (5)] of sufficiently large size become antireliable, if the whole network is in a chaotic state (what happens for strong enough coupling [16]).

IV. KURAMOTO MODEL

Below, we will demonstrate that the properties of internal reliability are nontrivial in oscillatory networks. We start with a classical Kuramoto model of N coupled oscillators with natural frequencies ω_k , distributed according to density $g(\omega)$.

Now the system of Eqs. (2) and (3) reads

$$\begin{aligned} \dot{x}_j &= \omega_j + \frac{\varepsilon}{N} \sum_{i \neq j} \sin(x_i - x_j), \quad j = 1, \dots, N, \\ \dot{y}_k &= \omega_k + \frac{\varepsilon}{N} \sum_{i \neq k} \sin(x_i - y_k), \end{aligned} \quad (6)$$

where ε is the coupling strength. In this case, $\mathbf{F}_k = \omega_k$ and $\mathbf{H}_k = \frac{\varepsilon}{N} \sum_{i \neq k} \sin(x_i - x_k)$, thus Eq. (4) reads $\dot{v}_k = -\frac{\varepsilon}{N} \sum_{i \neq k} \cos(x_i - x_k) v_k$. This equation can be rewritten in terms of the logarithm of the perturbation as

$$\frac{d}{dt} \ln v_k = -\frac{\varepsilon}{N} \sum_{i \neq k} \cos(x_i - x_k).$$

The TLE for unit k is the average growth rate of the logarithm of the perturbation [17]:

$$\lambda_k = -\frac{\varepsilon}{N} \left\langle \sum_{i \neq k} \cos(x_i - x_k) \right\rangle. \quad (7)$$

In the thermodynamic limit $N \rightarrow \infty$, the mean fields that enter Eqs. (6) and (7) are time independent or time periodic, and thus the TLEs either vanish or are negative. However, the dynamics of finite ensembles is nontrivial. In particular, the global largest LE of the Kuramoto system before the synchronization transition is positive for finite N , while it

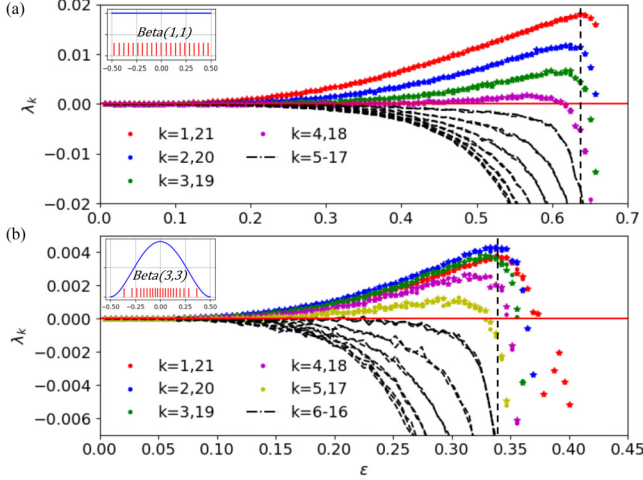


FIG. 3. Transversal LEs of all Kuramoto units vs coupling strength ε for $N = 21$. k denotes the oscillator index. Units that are always reliable are shown with black dashed lines, while those that are antireliable (positive TLEs) in some range are marked by colored dots if $\omega_k > 0$, and stars of the same color for symmetric units with $\omega_k < 0$. In panel (a), frequencies are uniformly distributed, i.e., $B(\omega; 1)$. In panel (b), frequencies are distributed according to $B(\omega; 3)$. The insets show the shape of $g(\omega)$: the vertical red lines are the actual frequencies of the oscillators (regularly sampled, as explained in the main text). The vertical dashed lines mark the theoretical synchronization threshold (in the thermodynamic limit) ε_c .

tends to zero in the thermodynamic limit $N \rightarrow \infty$ [18,19]. As a consequence, the mean fields that enter Eqs. (6) and (7) fluctuate, and the TLEs can, in general, be positive.

We thus focus our analysis on small system sizes. However, when N is small, generating sampled frequencies ω_k according to a specific distribution density poses a challenge, as independent and identically distributed sampling can exhibit significant fluctuations in the frequency spacing. To follow the macroscopic shape of $g(\omega)$ and to avoid the burden of analyzing different samplings, we select the set of frequencies ω_k , $k = 1, \dots, N$, via a regular sampling, such that $G(\omega_k) = (k - 0.5)/N$ where $G(\omega) = \int_{-\infty}^{\omega} g(s)ds$ is the cumulative distribution (see Refs. [19–21]).

In Fig. 3 we show the TLEs for all oscillators in an ensemble of $N = 21$ units vs the coupling strength ε , for two representative distributions of the frequencies. Both are given by a symmetrized Beta distribution on the interval $[-0.5, 0.5]$, $B(\omega; a) = \frac{\Gamma(2a)}{\Gamma^2(a)}[(\omega - 0.5)(0.5 - \omega)]^{a-1}$; $B(\omega; 1)$ is a uniform distribution, and $B(\omega; 3)$ is a bell-shaped one.

One can see that all TLEs tend to zero as $\varepsilon \rightarrow 0$. However, at finite coupling strengths, some units have positive TLEs and are thus antireliable, while the TLEs of other reliable units are negative. This happens at moderate values of ε , before the synchronization onset, where all units have different frequencies and no one is locked. At large ε , all units become reliable due to the synchronization onset (the critical coupling is well defined $\varepsilon_c = \frac{2}{\pi g(0)}$ in the thermodynamic limit, but is slightly shifted and not so well defined for finite N).

Remarkably, reliable and antireliable units coexist across the entire intermediate range of coupling strengths, with pos-

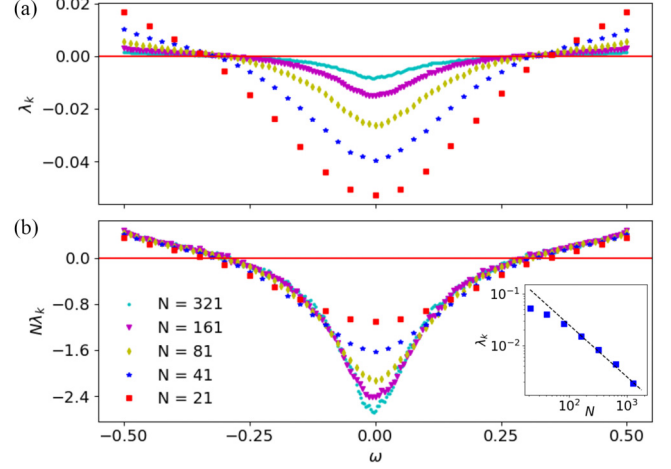


FIG. 4. (a) Dependence of reliability of Kuramoto units on their natural frequency (ω) below the synchronization transition ($\varepsilon = 0.6$, $\varepsilon_c \approx 0.64$) and different ensemble size, $N = 21, 41, 81, 161, 321$. Natural frequencies ω_k are uniformly distributed and equally spaced. (b) The same data as in panel (a) but scaled with N . The inset reports the scaling of the Lyapunov exponent of the unit with $\omega = 0$ (log-log scale, dashed line: $\lambda \propto 1/N$).

itive λ_k peaking near the synchronization transition. This behavior is qualitatively unaffected by the shape of the distribution $g(\omega)$, as illustrated by the two panels in Fig. 3.

For fixed coupling strength ε , reliability depends on the oscillator's natural frequency, as shown in Fig. 4. In particular, reliability is mostly strong (minimum of TLE) for those units the frequency of which lies at the center of the distribution $g(\omega)$ and decreases toward the distribution's tails, where oscillators are antireliable. Comparison of different ensemble sizes shows that the magnitude of the TLE scales as $1/N$ [Fig. 4(b)]. The fraction of reliable and antireliable oscillators in this setup does not depend on the system size, with approximately 30% being antireliable.

Note that the separation of the oscillators into two types resembles the separation of units in chimera states into synchronous and asynchronous subgroups. The difference is that, contrary to the chimera setup where identical units are considered (we will explore reliability properties of such a chimera below in Sec. VID), here the oscillators differ by their natural frequencies, and the reliability property is a function of this frequency.

Above, we explored the case of attractive coupling $\varepsilon > 0$ in the Kuramoto model. In Fig. 5 we report similar results for repulsive coupling $\varepsilon < 0$. Compared with Figs. 3 and 4, at a moderate repulsive coupling, the central oscillators are antireliable while the edge (peripheral) ones are reliable. Furthermore, all oscillators become reliable at strong repulsive coupling.

V. FLUCTUATION-DISSIPATION RELATION FOR THE KURAMOTO MODEL

The expression Eq. (7) for the TLE of the Kuramoto model provides an interesting connection to the fluctuation properties of the dynamics. One can define the pairwise correlations

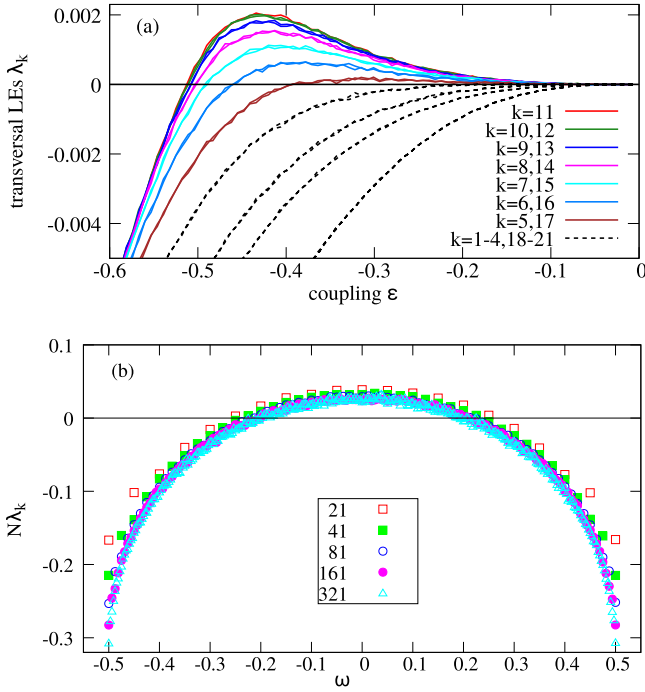


FIG. 5. (a) TLEs for the Kuramoto model with $N = 21$ and uniform distribution of the natural frequencies, for negative values of the coupling parameter ε . Edge units 1–4 and 18–21 have negative TLEs, shown with dashed lines; for the colors of other units, see the panel. (b) TLEs of the Kuramoto model vs their natural frequency (ω) for repulsive coupling $\varepsilon = -0.45$, for different ensemble sizes $N = 21, 41, 81, 161, 321$. Natural frequencies ω_k are uniformly distributed and equally spaced.

between the phase oscillators as $c_{jk} = c_{kj} = \langle \cos(x_j - x_k) \rangle$ (see Refs. [22–24]). This allows for representing the TLE Eq. (7) as

$$\lambda_k = -\frac{\varepsilon}{N} \sum_{j \neq k} c_{jk}. \quad (8)$$

This equality can be interpreted as a fluctuation-dissipation relation because it connects the properties of fluctuations (correlations) to the stability properties of replica reliability (TLE). We illustrate correlations c_{jk} in the Kuramoto ensemble with a uniform distribution of frequencies in Fig. 6.

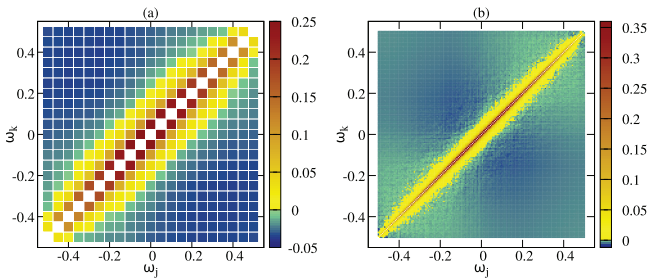


FIG. 6. Correlation matrices c_{jk} for the Kuramoto ensembles with a uniform distribution of natural frequencies and (a) $N = 21$ and (b) $N = 161$, for $\varepsilon = 0.5$. To better contrast positive and negative correlations, a yellow-red palette is used for positive correlations and a green-blue one for negative values.

The oscillators with close frequencies (along the diagonal) are positively correlated, although this positive correlation is small for peripheral units (with frequencies at the edges of the distribution). Examination of different system sizes N from 21 to 321 shows that the number of positively correlated neighbors is around 10–15 and practically N independent, and the values of maximal positive correlations also do not depend on N . In contradistinction, oscillators with significantly different frequencies are anticorrelated ($c_{jk} < 0$), and the values scale as $|c_{jk}| \sim N^{-1}$. Thus, the TLE, which is according to Eq. (8) a sum of the corresponding column or row, is negative for central units and positive for peripheral ones if coupling is attractive $\varepsilon > 0$, in accordance with Fig. 4. The correlation pattern for repulsive coupling is similar to that shown in Fig. 6, but because the correlations are now multiplied by negative ε , central units have positive TLEs while peripheral units have negative TLEs, as in Fig. 5.

Remarkably, the sum of all TLEs can be expressed in another fluctuation-dissipation relation via the variance of the complex Kuramoto order parameter $Z = N^{-1} \sum_k \exp[ix_k]$. Indeed, direct calculation yields, with the account of Eq. (8),

$$\begin{aligned} \langle |Z|^2 \rangle &= N^{-2} \sum_{k,j} \langle e^{i(x_j - x_k)} \rangle \\ &= N^{-2} \left(N + \sum_{k,j \neq k} c_{jk} \right) = N^{-1} \left(1 - \varepsilon^{-1} \sum_k \lambda_k \right). \end{aligned} \quad (9)$$

This relation is exact for any ensemble size N .

VI. OTHER MODELS OF COUPLED OSCILLATORS

In this section, we report internal reliability properties for different models of coupled oscillators that are similar to the globally coupled Kuramoto model [Eq. (6)].

A. Phase oscillators with Winfree coupling

Here we explore a globally coupled ensemble of N phase oscillators with a Winfree-type [25] coupling:

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} S(\varphi_k) \sum_{j \neq k} Q(\varphi_j), \quad (10)$$

where $S(\varphi)$ is a phase sensitivity (phase response) function, and $Q(\varphi)$ is the forcing term. Below, we consider an example with

$$S(\varphi) = \cos \varphi, \quad Q(\varphi) = \sin \varphi, \quad N = 21.$$

To demonstrate that internal reliability can be observed for frequency distributions of different types, we sampled frequencies from a double-hump distribution here. Namely, we first produced uniformly distributed quantities $-1/2 \leq y_i \leq 1/2$ and then calculated $\omega_i = \bar{\omega} + \Delta(0.3y_i + 0.75 \sin \pi y_i)$ with $\bar{\omega} = 10$, $\Delta = 1$. The transversal LEs in the system of Eq. (10) are defined as

$$\lambda_k = - \left\langle \frac{\varepsilon}{N} \sin \varphi_k \sum_{j \neq k} \sin \varphi_j \right\rangle.$$

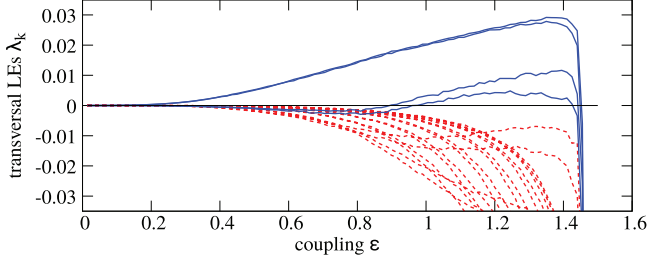


FIG. 7. Transversal LEs for the Winfree model [Eq. (10)]. Only the four oscillators with peripheral frequencies (the two smallest ones and the two largest ones) can become antireliable, and these TLEs are shown with bold blue lines; other oscillators (dashed lines) are always reliable.

Figure 7 shows these Lyapunov exponents as functions of the coupling constant ε .

B. Rotators described by second-order equations

A popular system, especially in modeling power grids, is that of coupled rotators [26] (sometimes called “oscillators with inertia”):

$$\mu \ddot{\phi}_k + \dot{\phi}_k = \omega_k + \frac{\varepsilon}{N} \sum_j \sin(\phi_j - \phi_k). \quad (11)$$

In the linear approximation, a small difference between the m th rotator and its replica $\delta_m = \phi'_m - \phi_m$, $v_m = \dot{\phi}'_m - \dot{\phi}_m$ evolves according to equations

$$\begin{aligned} \dot{\delta}_m &= v_m, \\ \mu \dot{v}_m &= -v_m - \delta_m \frac{\varepsilon}{N} \sum_{j \neq m} \cos(\phi_j - \phi_m). \end{aligned}$$

To calculate the transversal Lyapunov exponent for the m th oscillator, one has to calculate

$$\lambda_m \approx \frac{1}{T} \ln \frac{\Delta_m(T)}{\Delta_m(0)}, \quad \Delta_m = (\delta_m^2 + v_m^2)^{1/2},$$

for a large enough T .

These LEs are shown in Fig. 8 for $\varepsilon = 0.6$ and $\mu = 1$, and different N . Here a uniform distribution of frequencies $\omega_k = -0.5 + (k-1)/(N-1)$ is adopted. One can see a good overlap at $N = 50$ and 100 . For these parameters, approximately

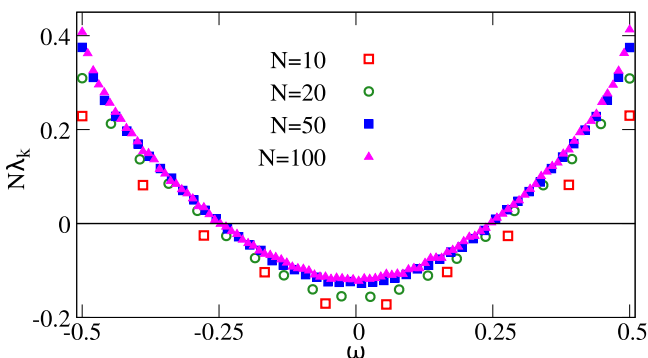


FIG. 8. Normalized Lyapunov exponents for coupled rotators [Eq. (11)] as functions of ω_m for different ensemble sizes N .

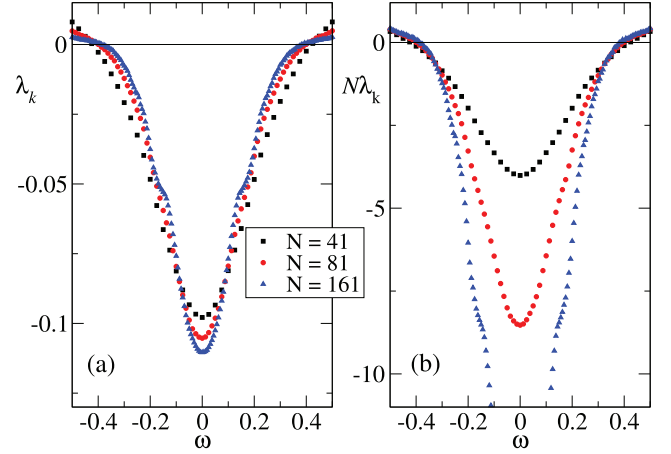


FIG. 9. (a) Transversal LEs for an ensemble of Stuart-Landau oscillators in Eq. (12) with $\varepsilon = 0.53$ for different sizes N and uniform frequency distribution. (b) Same data as in panel (a) vertically scaled by N .

half of all LEs are positive, meaning that approximately half of all units are reliable.

C. Ensemble of Stuart-Landau oscillators

We now discuss an example of oscillators displaying both amplitude and phase dynamics. We consider a network of N globally coupled Stuart-Landau oscillators with diffusive coupling:

$$\dot{z}_k = -|z_k|^2 z_k + (1 + i\omega_k)z_k + \frac{\varepsilon}{N} \sum_{j \neq k} (z_j - z_k), \quad (12)$$

where z_k are complex state variables. In the uncoupled limit $\varepsilon = 0$, each oscillator k relaxes on a stable limit cycle with unit radius and frequency ω_k . Analogously to rotators, a uniform distribution of frequency $\omega_k = -0.5 + (k-1)/(N-1)$ is here adopted.

A small (complex) perturbation δ_k between the state of oscillator k and its replica evolves according to

$$\dot{\delta}_k = -2|z_k|^2 \delta_k - z_k^2 \delta_k^* + (1 + i\omega_k)\delta_k - \varepsilon \frac{N}{N-1} \delta_k, \quad (13)$$

and the corresponding TLE λ_k is obtained as

$$\lambda_k \approx \frac{1}{T} \ln \frac{|\delta_k(T)|}{|\delta_k(0)|} \quad (14)$$

for large T . The transition to the synchronized state was found for $\varepsilon \simeq 0.58$; here we choose $\varepsilon = 0.53$ as a point representative of the unsynchronized regime close to the transition.

In Fig. 9(a), we show the corresponding spectrum of TLEs. For positive exponents of antireliable oscillators the scaling $1/N$ holds [these values nearly overlap in Fig. 9(b)], while negative exponents only weakly vary with N . We interpret this behavior as follows. Positive TLEs are dominated by the phase dynamics, which implies the same scaling as in the Kuramoto model or in coupled rotators. Conversely, negative TLEs are (possibly) dominated by the amplitude dynamics, damping of which does not depend on the number of units. We found that the fraction of “amplitude-dominated” oscillators decreases

upon decreasing ϵ in favor of “phase-dominated” ones (data not shown).

D. Internal reliability in a chimera state

Here, we would like to discuss possible connections to other setups in networks of oscillators, where identical units are explored. Symmetry breaking in such networks can lead to so-called chimera states, where some groups of oscillators are synchronized and build a cluster of identical states, while other groups remain desynchronized [27–30]. Stability or instability of clusters of identical units in chimeras is governed by an evaporation Lyapunov exponent [17]. This exponent is very close to the transversal LE introduced in this paper to characterize internal reliability. The difference lies in the fact that, in the chimera setup, both identical units affect other oscillators within the network, whereas in the reliability setup, only the prototype unit acts on other oscillators. If one neglects this difference, then it is natural to expect that synchronized units in a chimera state are reliable and desynchronized units are antireliable.

To confirm this, we applied the approach of internal reliability to the chimera setup proposed in Ref. [27]. Here, one takes two identical subpopulations φ_i, θ_i , $i = 1, \dots, N$ of phase oscillators, and couples them with a common phase shift α and different coupling strengths inside and outside the subpopulations:

$$\begin{aligned}\dot{\varphi}_i &= \frac{\mu}{N} \sum_{j=1, j \neq i}^N \sin(\varphi_j - \varphi_i - \alpha) \\ &\quad + \frac{\nu}{N} \sum_{j=1}^N \sin(\theta_j - \varphi_i - \alpha), \\ \dot{\theta}_i &= \frac{\mu}{N} \sum_{j=1, j \neq i}^N \sin(\theta_j - \theta_i - \alpha) \\ &\quad + \frac{\nu}{N} \sum_{j=1}^N \sin(\varphi_j - \theta_i - \alpha).\end{aligned}\quad (15)$$

In the chimera state, oscillators in subpopulation φ are synchronized, $\varphi_1 = \varphi_2 = \dots = \varphi_N$, while oscillators in subpopulation θ remain different. To such a state with $N = 32$, we applied the replica Φ_k to a prototype φ_k , and the replica Θ_k to a prototype θ_k . Equations for these variables read

$$\begin{aligned}\dot{\Phi}_k &= \frac{\mu}{N} \sum_{j=1, j \neq k}^N \sin(\varphi_j - \Phi_k - \alpha) \\ &\quad + \frac{\nu}{N} \sum_{j=1}^N \sin(\theta_j - \Phi_k - \alpha), \\ \dot{\Theta}_k &= \frac{\mu}{N} \sum_{j=1, j \neq k}^N \sin(\theta_j - \Theta_k - \alpha) \\ &\quad + \frac{\nu}{N} \sum_{j=1}^N \sin(\varphi_j - \Theta_k - \alpha).\end{aligned}\quad (16)$$

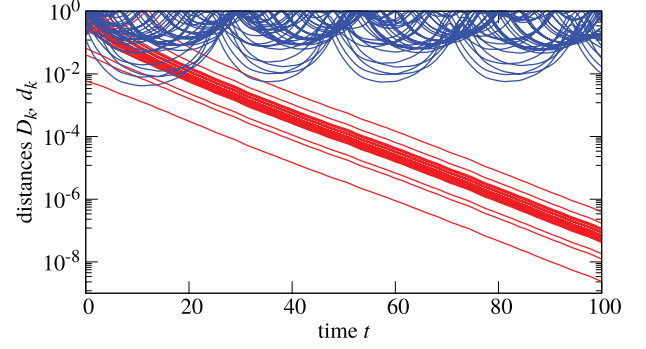


FIG. 10. Distances D_k for synchronized oscillators in a chimera state (red lines), and d_k for nonsynchronized oscillators (blue lines) vs time for $N = 32$. Parameters of the model are the same as in Ref. [27]: $\alpha = \pi/2 - 0.1$, $\mu = 0.6$, $\nu = 0.4$.

The initial states of the replicas were random. In Fig. 10 we show the evolution of the replica-prototype distances $d_k = |\sin((\Theta_k - \theta_k)/2)|$ and $D_k = |\sin((\Phi_k - \varphi_k)/2)|$ with time. One can see that for the synchronized subpopulation, all distances D_k decay exponentially with time, indicating reliability. In contrast, for the nonsynchronous subgroup, the distances d_k oscillate, indicating antireliability.

E. Disordered Kuramoto-type network

Furthermore, we explored several examples of random Kuramoto networks $\dot{x}_k = \omega_k + \epsilon \sum_{j \neq k} A_{kj} \sin(x_j - x_k)$, with a uniform distribution of natural frequencies and randomly sampled A_{jk} . For both symmetric and asymmetric random matrices, the typical picture is that peripheral units (i.e. units with natural frequencies in the tails of the distribution) are antireliable, and the central units (i.e. with the natural frequencies close to the mean frequency) are reliable. However, some central units become antireliable in a certain range of coupling strengths ϵ . Quantitative characterization of the internal reliability of random networks requires extensive statistical evaluations, which exceed the scope of this paper.

VII. IMPERFECT REPLICAS

Here, we discuss the properties of the replica behavior if its dynamical equations are not identical to those of the prototype. In particular, we take the Kuramoto ensemble [Eq. (6)], and replicate the unit k with a detuning $\delta\omega$ in the frequency:

$$\begin{aligned}\dot{x}_j &= \omega_j + \frac{\epsilon}{N} \sum_{i \neq j} \sin(x_i - x_j), \quad j = 1, \dots, N, \\ \dot{y}_k &= \omega_k + \delta\omega_k + \frac{\epsilon}{N} \sum_{i \neq k} \sin(x_i - y_k).\end{aligned}\quad (17)$$

To characterize the properties of such a replica, we calculate the time average phase difference to the prototype unit $d = \langle |\sin((x_k - y_k)/2)| \rangle$ and the difference in the observed frequencies $\Delta\Omega = \langle \dot{y}_k - \dot{x}_k \rangle$. We plot these measures as functions of the detuning $\delta\omega$ for a reliable oscillator with the central frequency $\omega_k = 0$ and for an antireliable oscillator with $\omega_k = -0.5$ in Fig. 11. The behavior of the phase difference d is not surprising: for the antireliable oscillator, it is

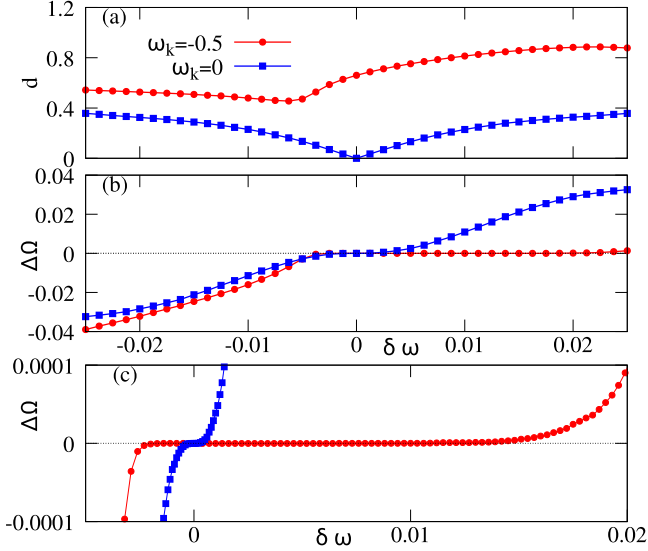


FIG. 11. Difference between the phases d (a) and the frequency difference $\Delta\Omega$ (b) vs $\delta\omega$ for reliable ($\omega_k = 0$, blue squares) and antireliable ($\omega_k = -0.5$, red circles) units. Panel (c) shows a zoomed region of panel (b) with small values of $|\Delta\Omega|$. Parameters $\varepsilon = 0.6$, $N = 21$.

large in the whole range of detunings, while for the reliable unit, it vanishes at $\delta\omega = 0$ and grows linearly with $\delta\omega$ for small detunings. However, the behavior of the frequencies is quite remarkable. For the reliable unit, the frequencies become different already for small detuning [we found that $\Delta\Omega \sim (\delta\omega)^3$]. In contradistinction, for the antireliable unit, there is a large interval of detunings where $\Delta\Omega \approx 0$ [see Fig. 11(c)]. One can say that in this region, the replica is frequency entrained, while the phases of the replica and the prototype remain different.

VIII. RELIABILITY OF SUBNETWORKS

Here, we extend the notion of internal reliability to the subnetworks. Instead of replicating one unit like in Eq. (3), we replicate all units from a subset $\mathcal{S}$ (the nonreplicated units belong to the complementary subset $\bar{\mathcal{S}}$):

$$\dot{\mathbf{y}}_m = \mathbf{F}_m(\mathbf{y}_m) + \mathbf{H}_m(\mathbf{y}_m; \mathbf{x}_l \in \bar{\mathcal{S}}, \mathbf{y}_j \in \mathcal{S}, j \neq m), \quad m \in \mathcal{S}. \quad (18)$$

The reliability is now defined via the maximal transversal Lyapunov exponents for the whole replicated subnetwork, obtained from the linear system

$$\dot{\mathbf{v}}_m = \left(\frac{\partial \mathbf{F}_m}{\partial \mathbf{x}_m} + \frac{\partial \mathbf{H}_m}{\partial \mathbf{x}_m} \right) \mathbf{v}_m + \sum_{j \in \mathcal{S}, j \neq m} \frac{\partial \mathbf{H}_m}{\partial \mathbf{x}_j} \mathbf{v}_j, \quad m \in \mathcal{S}, \quad (19)$$

which generalizes Eq. (4).

Let us first consider the Kuramoto model and different subnetworks of two replicas. Here, one can look at all combinations of two selected units for replication; the results are presented in Fig. 12. One can see that, in practically all cases, a pair is reliable if both units are individually reliable. This rule applies only as a “rule of thumb” for larger replicated subnetworks. The number of possible larger subnetworks is too large to explore all of them; thus, we performed a statistical

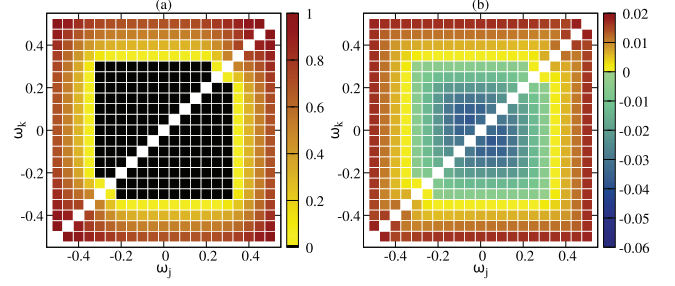


FIG. 12. Kuramoto model with a uniform distribution of frequencies, $N = 21$, $\varepsilon = 0.6$. (a) Distance $d = \langle \sin^2[(x_k - y_k)/2] + \sin^2[(x_j - y_j)/2] \rangle^{1/2}$ between the two prototypes having indices k, j , and the corresponding replicas. The values $d < 0.01$ are depicted in black; otherwise, the color palette is used. (b) TLEs for the pair of replicas, calculated according to Eq. (19). Here, two palettes are used for positive and negative values to improve contrast.

analysis and found the transversal LEs for a random sample of 1000 subnetworks of sizes 4, $\dots$, 10 out of $N = 21$, for the uniform distribution of natural frequencies. The cumulative distributions of the TLEs are presented in Fig. 13(a). One can see that already for subnetworks of size 4, only 10% of all cases are reliable, and starting from size 7, all tested subnetworks are antireliable.

In the following example, we explore the reliability of subnetworks in the recurrent neural network [Eq. (5)]. As mentioned, all individual elements are reliable here with the trivial TLE equal to -1 . Nevertheless, the reliability of a subnetwork is nontrivial, because positive feedbacks due to the interaction of many units in the replicated subnetwork may induce instability. To investigate how the TLE depends on the size of the replicated subnetwork, we have chosen a neural network of size $N = 100$ operating in a chaotic regime and explored 1000 randomly sampled subnetworks for sizes

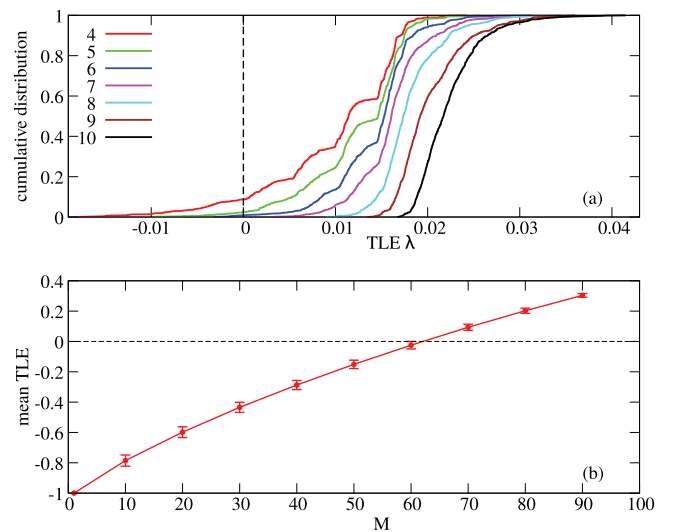


FIG. 13. (a) Cumulative distributions of the TLEs for the Kuramoto model of $N = 21$ units [Eq. (6)] and $\varepsilon = 0.6$, for different sizes of the subnetworks. (b) Mean values and standard deviations (depicted as error bars) of the TLEs for randomly sampled subnetworks of the recurrent neural network [Eq. (5)].

$M = 10, 20, \dots, 90$. The distributions of the calculated TLEs were nearly Gaussian, and we plot in Fig. 13(b) dependencies of the mean and of the standard deviation on the subnetwork size. One can see that the transition is at $M \approx 62$: larger subnetworks are practically all antireliable, while smaller subnetworks are all reliable. Note that the limit $M = N$ corresponds to making a replica of the entire system; the transversal LE is now just the largest LE of the original system (in some sense, a chaotic system as a whole is “antireliable”).

IX. CONCLUSIONS

Summarizing, we explored the internal reliability properties of dynamical networks by determining the stability of particular units and subnetworks when replicated. The standard Kuramoto model of globally coupled phase oscillators exhibits nontrivial reliability properties, featuring for attractive coupling antireliable peripheral (frequencies at the edges of the distribution) oscillators and reliable central (frequencies close to the mean one) oscillators; for repulsive coupling, the situation is opposite. In this model, these properties are well defined for finite networks but disappear in the thermodynamic limit, where the acting fields are constant or periodic. We have demonstrated that other oscillatory networks (phase oscillators with Winfree-type coupling, Stuart-Landau oscillators, rotators described by second-order equations for the angle variables) show similar properties: in all the cases, for attractive coupling, peripheral oscillators are antireliable while central oscillators are reliable. For the recurrent neural network model, which is chaotic in the thermodynamic limit, replicas of small subnetworks are reliable. Still, starting from a certain size, the replicated subnetworks become antireliable.

While the concept applies to generic, regular, and random dynamical networks, for the globally coupled Kuramoto model, we additionally established a fluctuation-dissipation relation. This relation connects the stability exponent of internal reliability of a particular unit to its pairwise phase correlations with others. The sum of all exponents can be expressed in another fluctuation-dissipation relation via the variance of the complex Kuramoto order parameter.

In this paper, we have proven the concept of internal reliability for relatively simple setups of networks with regular

or disordered global coupling, and with “homogeneous” behavior (except for the chimera setup, in other examples, all oscillators were chaotic and nonsynchronized). It would be interesting to extend the study to more complex situations involving multilayer networks (possibly with delay couplings), which demonstrate different clustering patterns [31].

The reliability property is crucial for identifying the parameters of the network elements. As we demonstrated, the difference between states of the replica and the prototype, if they have the same parameters, vanishes only in the case of reliability. In contrast, it remains finite in the case of antireliability.

Finally, we discuss possible experimental realizations of the internal reliability setup. Such a realization is hardly possible if one cannot control connections between the units of a network. However, there are fields such as electronics and electrochemical oscillators, where incoming and outgoing couplings can be easily controlled. In such a situation, preparing a replica of one of the oscillators and driving it with the same input as the prototype is driven is experimentally feasible (similar to an experimental realization of the auxiliary system approach for electronic chaotic oscillators in Ref. [13]).

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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- [1] E. Ott, *Chaos in Dynamical Systems*, 2nd ed. (Cambridge University, Cambridge, England, 2002).
 - [2] Z. F. Mainen and T. J. Sejnowski, Reliability of spike timing in neocortical neurons, *Science* **268**, 1503 (1995).
 - [3] D. S. Goldobin and A. Pikovsky, Antireliability of noise-driven neurons, *Phys. Rev. E* **73**, 061906 (2006).
 - [4] G. B. Ermentrout, R. F. Galán, and N. N. Urban, Reliability, synchrony and noise, *Trends Neurosci.* **31**, 428 (2008).
 - [5] K. K. Lin, E. Shea-Brown, and L.-S. Young, Reliability of coupled oscillators, *J. Nonlinear Sci.* **19**, 497 (2009).
 - [6] A. S. Pikovsky, Synchronization and stochastization of the ensemble of autogenerators by external noise, *Radiophys. Quantum Electron.* **27**, 390 (1984).
 - [7] A. S. Pikovsky, Statistics of trajectory separation in noisy dynamical systems, *Phys. Lett. A* **165**, 33 (1992).
 - [8] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University, Cambridge, England, 2001).
 - [9] J. N. Teramae and D. Tanaka, Robustness of the noise-induced phase synchronization in a general class of limit cycle oscillators, *Phys. Rev. Lett.* **93**, 204103 (2004).
 - [10] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Generalized synchronization of chaos in directionally coupled chaotic systems, *Phys. Rev. E* **51**, 980 (1995).
 - [11] L. M. Pecora and T. L. Carroll, Synchronization in chaotic systems, *Phys. Rev. Lett.* **64**, 821 (1990).

- [12] L. Kocarev and U. Parlitz, Generalized synchronization, predictability, and equivalence of unidirectionally coupled dynamical systems, *Phys. Rev. Lett.* **76**, 1816 (1996).
- [13] H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, Generalized synchronization of chaos: The auxiliary system approach, *Phys. Rev. E* **53**, 4528 (1996).
- [14] C. Letellier, I. Sendiña-Nadal, I. Leyva, and J.-P. Barbot, Generalized synchronization mediated by a flat coupling between structurally nonequivalent chaotic systems, *Chaos* **33**, 093117 (2023).
- [15] C. Letellier, L. Minati, I. Sendina-Nadal, and I. Leyva, A taxonomy for generalized synchronization between flat-coupled systems, [arXiv:2401.11561](https://arxiv.org/abs/2401.11561).
- [16] H. Sompolinsky, A. Crisanti, and H. J. Sommers, Chaos in random neural networks, *Phys. Rev. Lett.* **61**, 259 (1988).
- [17] A. Pikovsky and A. Politi, *Lyapunov Exponents: A Tool to Explore Complex Dynamics* (Cambridge University, Cambridge, England, 2016).
- [18] O. V. Popovych, Y. L. Maistrenko, and P. A. Tass, Phase chaos in coupled oscillators, *Phys. Rev. E* **71**, 065201(R) (2005).
- [19] M. Carlu, F. Ginelli, and A. Politi, Origin and scaling of chaos in weakly coupled phase oscillators, *Phys. Rev. E* **97**, 012203 (2018).
- [20] H. Hong, H. Chaté, L.-H. Tang, and H. Park, Finite-size scaling, dynamic fluctuations, and hyperscaling relation in the Kuramoto model, *Phys. Rev. E* **92**, 022122 (2015).
- [21] F. Peter and A. Pikovsky, Transition to collective oscillations in finite Kuramoto ensembles, *Phys. Rev. E* **97**, 032310 (2018).
- [22] J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Paths to synchronization on complex networks, *Phys. Rev. Lett.* **98**, 034101 (2007).
- [23] M. Schröder, M. Timme, and D. Witthaut, A universal order parameter for synchrony in networks of limit cycle oscillators, *Chaos* **27**, 073119 (2017).
- [24] A. Pikovsky and F. Bagnoli, Dynamics of oscillator populations with disorder in the coupling phase shifts, *New J. Phys.* **26**, 023054 (2024).
- [25] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theor. Biol.* **16**, 15 (1967).
- [26] H. A. Tanaka, A. J. Lichtenberg, and S. Oishi, First order phase transition resulting from finite inertia in coupled oscillator systems, *Phys. Rev. Lett.* **78**, 2104 (1997).
- [27] D. M. Abrams, R. Mirollo, S. H. Strogatz, and D. A. Wiley, Solvable model for chimera states of coupled oscillators, *Phys. Rev. Lett.* **101**, 084103 (2008).
- [28] A. Yeldesbay, A. Pikovsky, and M. Rosenblum, Chimeralike states in an ensemble of globally coupled oscillators, *Phys. Rev. Lett.* **112**, 144103 (2014).
- [29] G. C. Sethia and A. Sen, Chimera states: The existence criteria revisited, *Phys. Rev. Lett.* **112**, 144101 (2014).
- [30] K. Höhlein, F. P. Kemeth, and K. Krischer, Lyapunov spectra and collective modes of chimera states in globally coupled Stuart-Landau oscillators, *Phys. Rev. E* **100**, 022217 (2019).
- [31] M. Lodi, F. D. Rossa, F. Sorrentino, and M. Storace, Analyzing synchronized clusters in neuron networks, *Sci. Reports* **10**, 16336 (2020).