

Hierarchy in Gilbert's tessellation: Multifractal behavior

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We investigate the fragment-size distribution produced by a hierarchical generalization of Gilbert's two-dimensional random tessellation. The study is motivated by experimental data on fragmentation processes, obtained by Kooij and coworkers [S. Kooij *et al.*, *Nat. Commun.* **12**, 2521 (2021)], where the reasons for which experimental data can exhibit a scale-free behavior resulting in power distributions (for unstressed materials) or exponential distribution with a characteristic length scale (for stressed material) are analyzed in great detail. We study two tessellation models, Dichotomous Gilbert (DG) and Hierarchical Gilbert (HG), which implement different hierarchical splitting rules. Both models generate fragment-area distributions well described by a truncated power-law function, namely a power multiplied by a stretched exponential function or generalized Gamma function. The Gilbert's classic tessellation model gives rise, on the contrary, to a simple exponential distribution. To further characterize the underlying complexity, we perform a multifractal analysis of the fragment areas. The results demonstrate that Gilbert's hierarchical fragmentation reveals a multifractal scaling.

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I. INTRODUCTION

The world of space tessellation, in any dimension, is truly vast. Voronoi tessellation in two and three dimensions is probably the best-known way of partitioning space into convex polygons or polyhedra without gaps, arranged face to face [1].

Besides Voronoi, another tessellation method attracting considerable interest is Gilbert's tessellation. Many other tessellation patterns exist, and readers can find a fairly comprehensive overview of them in Ref. [2].

However, random spatial tessellations and spatial tessellations in general are, in the words of Reinbach and Jung, "an important class of models in stochastic geometry" [3,4], a branch of mathematics that can be profitably used to study or model intriguing physical, chemical, biological, and engineering processes in which the division of space into polygons (polyhedra) plays a central role. Among the dozens and more of these processes related to tessellations, one of the best known is certainly fragmentation. The latter, in turn, plays a decisive role in many branches of physics, including, among others: astrophysics [5,6], nuclear physics [7], materials science [8–11], geophysics [12], and statistical mechanics [13,14].

The mathematical treatment of the physics of fragmentation processes, both crushing and crack formation, is rather complex.

A pivotal contribution to the theory of sequential (or cascade) fragmentation was made by Brown, whose general equation links the fragment size distribution to a "hazard function" [5]. While Brown's original power-law hazard function

yields a Weibull-type distribution, recent work demonstrates that a more complex hazard function, derived from modeling fragmentation as a two-dimensional Voronoi fractal, gives rise to a generalized gamma distribution $g_3(x)$ [15]. This underscores how geometric models can inform and extend analytical fragmentation theories.

Given the remarkable similarity between the shape of fragments resulting from experiments and the shape of tiles from theoretical tessellations, a purely geometric approach may be a first step towards a deeper understanding of the experiments [16,17].

For instance, the low drying of clay leads to its fragmentation when the evenly distributed stress in space exceeds a certain threshold value. The resulting crack pattern bears a striking resemblance to that generated by a Voronoi tessellation or by a Voronoi fractal [15]. In contrast, the fragmentation caused by an impulsive force on a brittle material resembles a Gilbert tessellation much more.

In the paper by Kooij *et al.* [10], the authors, thanks to rather refined experiments, were able to settle a long-standing question regarding the function of fragment mass distribution after a shattering event. They, studying the explosive fragmentation of glass Prince Rupert's drops and the fragmentation of two-dimensional stressed and stress free glass plates, uncover that the two types of distributions (exponential or power) are strictly connected to the breakup process that can be, respectively, random or hierarchical. The connection between the fragmentation process and the resulting fragment size distribution function prompted us to undertake the (purely geometric) work presented here. In particular, we focus on the planar Gilbert process. Since our approach remains at this stage purely geometric, it does not, by itself, provide a criterion for deciding *a priori* which specific hierarchical rule best matches a given real fragmentation process; identifying such a structural correspondence is left as an open question for future work.

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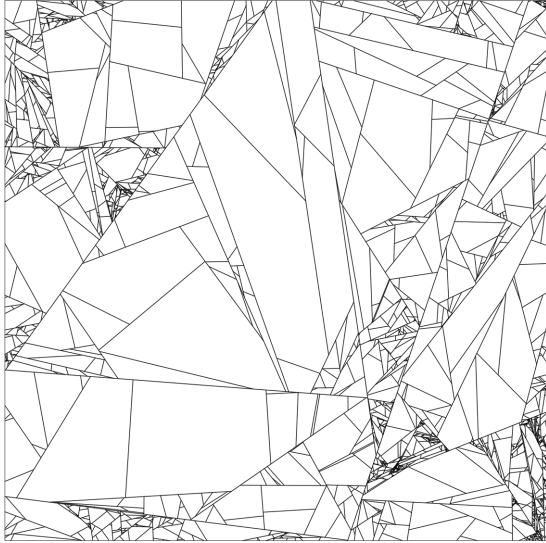


FIG. 1. Spatial structure resulting from a simulation of the DG model obtained on a domain $\mathcal{D} = 1000 \times 1000$ with a final number of polygons $N = 2045$.

It is well known that Gilbert's canonical process gives rise to a tile areas distribution function that is described by a simple exponential function [17]. Here, we focus on two planar Gilbert's tessellations where we introduced a certain degree of hierarchy into the stochastic process. The purpose of our work is to investigate the statistical properties of these two modified versions of Gilbert's canonical tessellation, i.e., to analyze how hierarchy modifies the exponential distribution characteristic of the canonical process. Ultimately, it is a geometric approach to studying the effects of hierarchy on a stationary Poisson process.

To this end, we introduce and focus our interest on two specific types of processes referred to as *Dichotomous Gilbert* (DG) and *Hierarchical Gilbert* (HG).

II. RESULTS AND DISCUSSION

The DG model is an iterative process structured as follows. In the first iteration, starting from an initial domain $\mathcal{D}$, we chose a random point within it and, following the classical Gilbert process, we create a random line passing through the point, splitting $\mathcal{D}$ into two new subdomains. From the second iteration onward, for every subdomain present at the previous iterations, we select a random point within it, and split the subdomains into two new fragments with a random line passing through the point. This process continues for a predetermined number of iterations n and produces, at the end, 2^n domains (Fig. 1).

The HG model introduces a new kind of randomness. While also dealing with an iterative process it incorporates a secondary random selection. The first iteration is the same of DG model but from the second iteration onward we select randomly only one of the specific subdomains and split it into two new fragments with a random line passing through a point randomly chosen within it. This process, continues for a predetermined number of iterations n and produces $n + 1$ domains (Fig. 2). To reproduce the tessellation model introduced

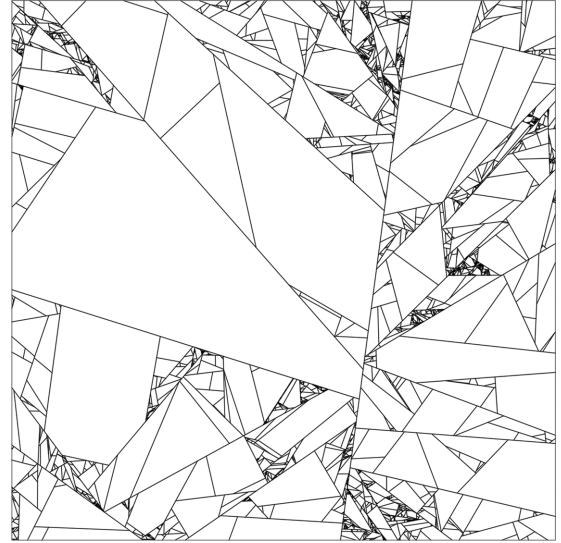


FIG. 2. Spatial structure resulting from a simulation of the HG model obtained on a domain $\mathcal{D} = 1000 \times 1000$ with a final number of polygons $N = 5000$.

above, we developed a Python script that includes Shapely library to create, split and compute geometric properties of the domains. The simulations of DG and HG models were performed on an initial domain of $\mathcal{D} = 1000 \times 1000$ so to minimize, as much as possible, the impact of numerical errors when working with small areas generated in later iterations and reducing the *finite size scale effect*. The tessellation processes were repeated for different values of the number of iterations n . Furthermore, for each fixed n , the simulation was repeated multiple times in order to obtain a total of approximately 250000 polygons ensuring a fully adequate statistical sampling. In addition, as we have already pointed out in a previous article [15], the effects of the limited domain on which the tessellation is performed are not a cause for concern, since what we are simulating here is the fragmentation of a tile which is a finite object.

This allows us to investigate the behavior of the statistical geometrical properties at different levels of the hierarchy. In particular, for DG model, we performed simulations for $n \in [6, 11]$, which corresponds to a number of polygons equal to $N_{DG} = [64, 128, 256, 512, 1024, 2048]$, while, for HG model, we directly specified the total number of polygons, repeating the process in order to obtain $N_{HG} = [100, 250, 500, 1000, 2500, 5000]$.

To investigate the effect of hierarchy, we focus our attention on the distribution function of the tile sizes (area). To do this and ensure a statistical reliability we introduce a data rejection criteria excluding all the data points associated with a probability density function (pdf), which exhibits a relative error greater than 10%. We emphasize that such relative errors, however, only occur around small area values. Figure 3 presents the resulting pdfs for two extremal values of the number of polygons, associated with the DG model, while Fig. 4 reports the results associated with the HG model.

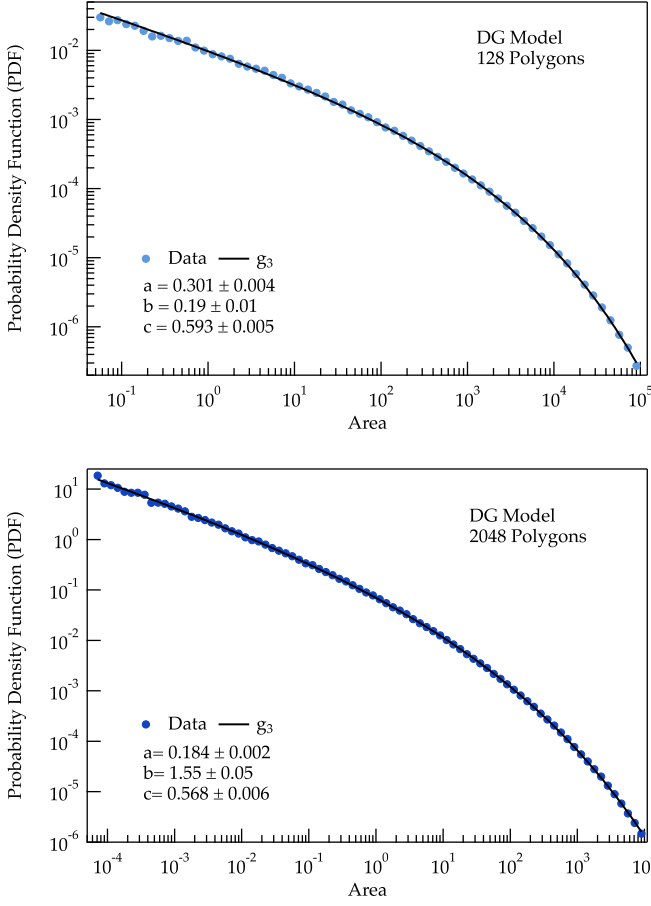


FIG. 3. Probability density function (pdf) of a fragment areas as a function of the area A for the DG model, shown for systems with $N = 128$ and $N = 2048$ polygons. The black solid lines represent the best-fit curves obtained using Eq. (1).

The pdfs obtained from simulations are extremely well described by the generalized Gamma distribution:

$$g_3(x) = \frac{ab^{c/a}}{\Gamma(\frac{c}{a})} x^{c-1} e^{-bx^a}, \quad (1)$$

where x represents the area of the fragments; the subscripted number 3 recalls that Eq. (1) has three parameters, (a, b, c) that characterize the distribution. The generalized Gamma function behaves as a power law, described by the term x^{c-1} , modified by a stretched exponential cut-off, e^{-bx^a} . The presence of the power law component suggests scale invariance within a certain range, also indicated by the absence of a characteristic scale. At the same time, the cutoff emerges as due to finite-size effect, which is dominant at scales comparable to the simulation domain. The fact that the system is well-described by Eq. (1) quantifies the empirical observation that the spatial distribution of fragments induced by the hierarchical process produces strong inhomogeneities in the resulting pattern. Figure 5 shows the dependence of the fitted parameters as a function of N for both models. It is interesting to note that while both models share the same functional form, they are characterized by a distinctly different behaviors and values of the parameters. However, in both models, the c parameter, which governs the power-law regime, remains

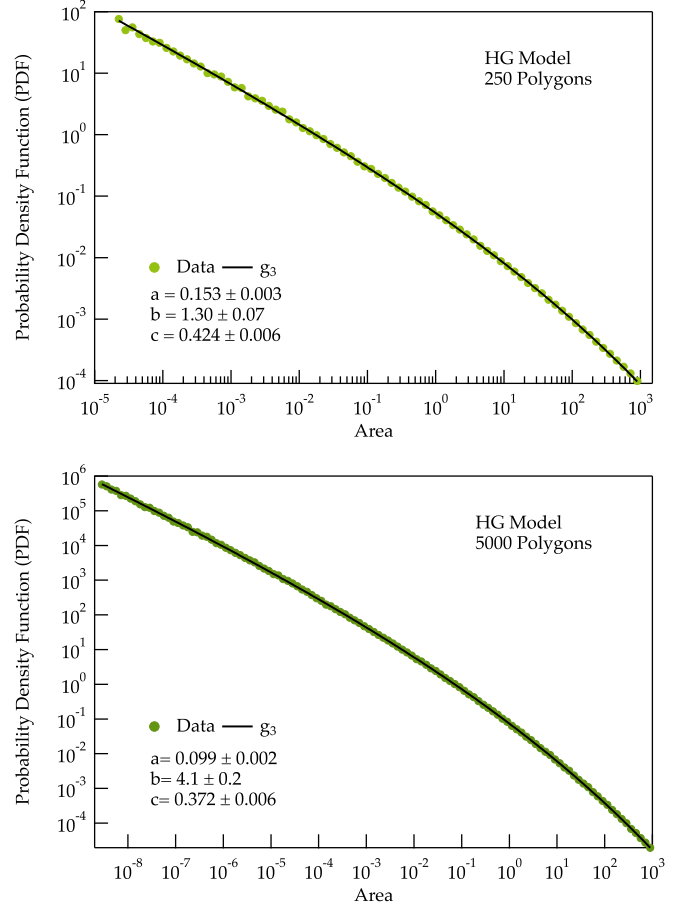


FIG. 4. Probability density function (pdf) of a fragment areas as a function of the area A for the HG model, shown for systems with $N = 250$ and $N = 5000$ polygons. The black solid lines represent the best-fit curves obtained using Eq. (1).

approximately constant across scales. In contrast, parameters a and b , controlling the stretched exponential term, display the same behavior as a function of N : a decreases while b increases. To assess the quality of the fit, we performed a χ^2 statistical test on all datasets with a 95% confidence level. Importantly, all tests performed have been successfully passed, supporting the idea of generalized Gamma distribution [Eq. (1)] as an appropriate candidate for describing our data.

The generalized-gamma fit already distinguishes DG from HG through its parameter values, but, being a statistic aggregated over the full ensemble of fragments, it cannot resolve how the splitting statistics vary across generations or branches of the fragmentation process. Going further, the concept of a hierarchical process characterized by an inhomogeneous probability measure suggests a complementary way to reinterpret the system. For this reason, we decided to perform a multifractal analysis [18–21]. The core idea is to introduce a probability measure μ_i defined as $\mu_i = \mathcal{D}_i / \mathcal{D}$ where $\mathcal{D}_i$ represents the area of the i -th fragment and $\mathcal{D}$ the initial one. Because μ_i depends only on the normalized fragment area and not on its spatial position, the underlying probabilistic measure is effectively defined on a one-dimensional support, which is the unit interval $[0,1]$, with $\sum_{i=1}^N \mu_i = 1$.

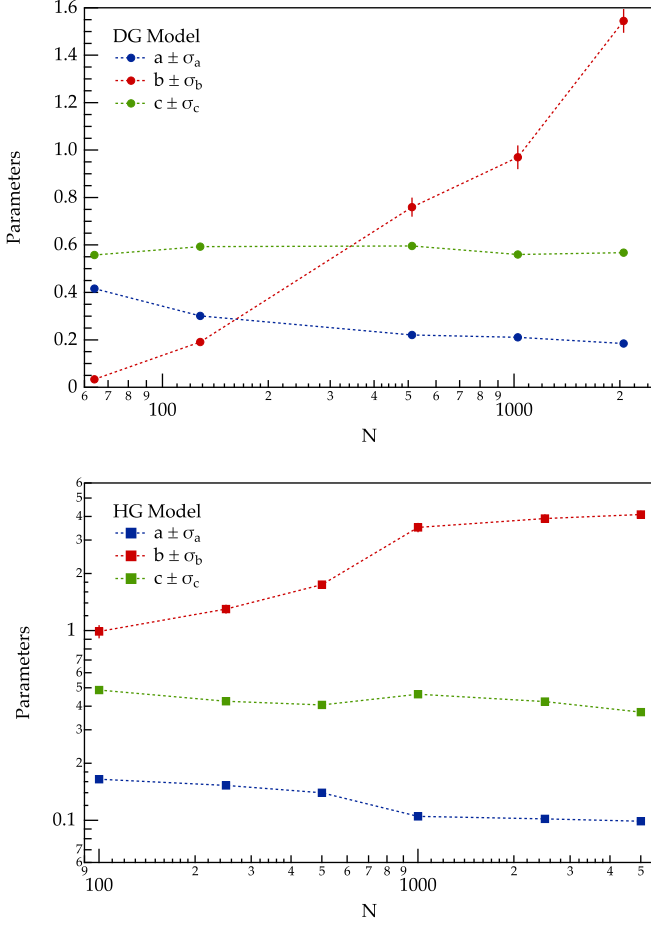


FIG. 5. Scaling of the fitted parameters from Eq. (1) with the number of polygons for the DG and HG model.

We then evaluate the partition function, $Z(q, N) = \sum_{i=1}^N \mu_i^q$, knowing that in the limit of large N , this scales as $Z(q, N) \sim N^{-\tau(q)}$ where $\tau(q)$ is the mass exponent. Unlike the standard box-counting approach, where the spatial scale is set by an externally imposed bin size l uniformly covering the domain, here the natural scale parameter emerges directly from the hierarchical cascade itself: at each generation, the fragmentation process produces, by construction, a new and finer partition of the system, and the tree structure of this partition induces its own natural metric, i.e., an ultrametric, in which distance between fragments is set by the generation at which their lineages diverge rather than by their physical separation. Thus, the number of fragments N (equivalently, the generation index) naturally plays the role of the scale parameter with respect to this ultrametric, in the same formal sense that l does with respect to the Euclidean metric of the domain [22,23]. The latter allows us to estimate the Rényi generalized dimensions $D(q)$, defined as $D(q) = \tau(q)/(q - 1)$ for $q \neq 1$. The multifractal nature of the system is then characterized by a nonlinear behavior of $D(q)$. Equivalently, it can be described by the multifractal spectrum $f(\alpha)$, (α is the local scaling exponent), which is related to $\tau(q)$ via a Legendre transformation.

To perform a statistically significant multifractal analysis, we used an ensemble of 100 independent realizations for

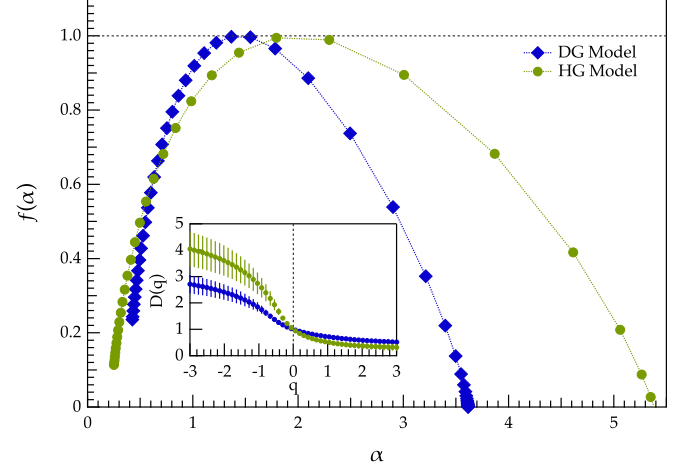


FIG. 6. Multifractal spectrum $f(\alpha)$ as a function of α , and Rényi generalized dimensions $D(q)$ as a function of q for the DG and HG model

each model. To resolve the scaling behavior across multiple scales, we performed the simulations on a larger domain $D = 100000 \times 100000$. The number of fragments, for a single simulation, was varied between 2^9 and 2^{14} , so as to cover at least two orders of magnitude for investigating scaling properties. For each realization, we computed the partition function $Z(q, N)$ and determined $\tau(q)$ over the range $q \in [-3, 3]$ by performing a linear regression. We then used the ensemble average $\langle \tau(q) \rangle$ to estimate the values of the Rényi dimension $D(q)$ and subsequently $f(\alpha)$. In Fig. 6 are shown the multifractal spectrum $f(\alpha)$ and the Rényi dimensions $D(q)$ for the DG and HG model, respectively.

As is clearly evident, both systems exhibit a strongly nonlinear behavior, confirming the multifractal nature of the processes. Furthermore, the two spectra are distinctly different, unveiling the different stochastic nature of the two processes.

The DG model produces a relatively narrower spectrum, indicating a less heterogeneous character due to the uniform depth of the induced fragmentation process. In contrast, the HG model yields a broader spectrum, reflecting strong heterogeneity and uneven concentration of fragment areas. This leads, in both cases, to localized regions of high fragmentation intensity, which are more evident in the HG model compared to the DG model.

The asymmetry of the two spectra rules out a continuous log-normal cascade model, in favor of a discrete, binomial-type hierarchical branching process, consistent with the two-way splitting rule underlying the DG and HG models.

As mentioned at the beginning, the multiple fragmentation of a solid and its evolution through a cascade are treated mathematically for the first time in Brown's article [5]. The energy initially imparted to the “parent” object is sufficient to fragment it into a set of “children” that may still possess sufficient energy to continue fragmenting into smaller pieces. The process proceeds until the energy possessed by the fragments is sufficient to fragment them further. This cascade mode differs markedly from the cascade process that describes turbulence [18,20], in which a flow of energy from large to

small scales occurs. In contrast, in the disintegration of a solid, energy is initially stored by the parent and the offspring receives it in proportion to its size after each disintegration: no flow of energy from one scale to another.

The weak point in Brown's theory is the hazard function (a function of the failure rate) that must be introduced *a priori* to determine the final fragment size distribution function. Brown chooses a power law that gives rise to a Weibull distribution. Our geometric model returns a size distribution function (or pdf), which is very well described by the generalized gamma function. Incidentally the same type of pdf found in the case of Voronoi fractal tessellation [15]. To the $g_3(x)$ function corresponds the following hazard function:

$$h(x) = \frac{a b^{c/a} \exp(-bx^a)}{\Gamma(c/a, bx^a)} x^{c-1}, \quad (2)$$

or $h(x) = \kappa(x)x^{c-1}$, where $\kappa(x) = \frac{a b^{c/a} \exp(-bx^a)}{\Gamma(c/a, bx^a)}$ is the truncation function. Equation (2) is a truncated power law, the latter being a type of hazard function, often found in the literature, that generates pdf's to which experimental shattering data often fit [24–26]. Actually, the ideal physical theory would associate the hazard function with the type of fragmentation, but unfortunately, at present, no such correspondence exists and the hazard function is determined *a posteriori* by fitting the experimental data. In fact, the link between the hazard distribution and the pdf distribution is given by the following differential equation [instead Eq. (8) in Fanfoni *et al.* [15] is a simple integro-differential equation for determining $p(x)$ from $h(x)$]:

$$\frac{d}{dx} \ln \left[\frac{p(x)}{h(x)} \right] + h(x) = 0. \quad (3)$$

In fact, it is easy to verify that if $h(x) = k x^{\alpha-1}$, $p(x)$ is a Weibull function as it must be, and it is even easier to verify that constant hazard corresponds to a simple exponential function.

III. CONCLUSIONS

In summary, we have introduced two types of hierarchies in a classical Gilbert tessellation. The cascading process that follows gives rise, in both cases, to a tile area distribution function that fits surprisingly well with a generalized gamma function. The latter is a truncated power function which, in turn, in the spirit of Brown's sequential fragmentation theory, is derived from a stretched exponential hazard function that is also truncated. This theoretical finding finds a natural counterpart in the experimental observations of Kooij *et al.* [10], according to which the fragmentation of a brittle sample follows two distinct regimes: a random breakup that gives rise to a simple exponential distribution of mass fragments (canonical Gilbert), whereas a hierarchical crushing leads to a power-law distribution (hierarchical Gilbert, in fact $g_3(x) \sim x^{c-1}$ for $x \rightarrow 0$). Our study demonstrates that the DG and HG cascade processes display a multifractal nature. Both models produce a heterogeneous distribution with different characteristics, quantitatively described by two different multifractal spectra. A direct, quantitative comparison between our geometric models and specific experimental datasets remains, at present, an open challenge and would require fragmentation experiments capable of resolving a sufficiently wide range of fragment sizes to reliably probe the hierarchy-induced effect.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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