

Stochastic thermodynamics of nonreciprocally interacting particles and fields

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Nonreciprocal interactions that violate Newton’s law “*actio = reactio*” are ubiquitous in nature and are currently intensively investigated in active matter, chemical reaction networks, population dynamics, and many other fields. An outstanding challenge is the thermodynamically consistent formulation of the underlying stochastic dynamics that obeys local detailed balance and allows for a rigorous analysis of the stochastic thermodynamics of nonreciprocally interacting particles. Here, we present such a framework for a broad class of active systems and derive, by systematic coarse-graining, exact expressions for the macroscopic entropy production. Four independent contributions to the thermodynamic dissipation can be identified, among which the energy flux sustaining vorticity currents manifests the presence of nonreciprocal interactions. Then, Onsager’s nonreciprocal relations, the fluctuation-response relation, the fluctuation relation, and the thermodynamic uncertainty relations for nonreciprocal systems are derived. Finally, we demonstrate that our general framework is applicable to a plethora of active matter systems and chemical reaction networks and opens new paths to understand the stochastic thermodynamics of nonreciprocally interacting many-body systems.

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I. INTRODUCTION

The discovery of fluctuation relations has shed new light on the classical field of thermodynamics [1–19] and established the refined framework of stochastic thermodynamics (ST) [20,21]. Stochastic thermodynamics provides a framework to define the first and second laws of thermodynamics for a single stochastic transition [22]. This is achieved by defining the energy, work, heat, and entropy production for a stochastic transition, so that these thermodynamic quantities have probability distributions. Fluctuations vanish in the thermodynamic limit, where the resulting stochastic thermodynamic quantities converge to the deterministic thermodynamic description, thereby recovering classical thermodynamics. Stochastic thermodynamics applies to small systems with inherent stochastic fluctuations in the presence of externally imposed driving of control parameters. The validity of this framework relies on a key constraint of timescale separation, which results in the Markovian property for the transitions: the states of the stochastic process comprise the slow degrees of freedom, while the collective fast degrees of freedom form the thermal environment (bath). The Local Detailed Balance (LDB) condition establishes the equivalence between the forward and backward transition rates for the system and the thermodynamic transition cost supported by the environment [23,24], ensuring thermodynamic consistency for all transitions.

Newton’s third law of motion states that for every action force, there is an equal and opposite reaction force. However, multiple real-world counterexamples to Newton’s third law exist. Systems that violate Newton’s third law are called nonreciprocal [25]. In particular, a multitude of biological systems provides frontiers for studying and understanding the implications of such violations. For instance, toy models for ecological systems [26–36], stochastic systems [37–39], neural networks [40–46], mixtures of nonreciprocal particles [47], birds with vision cones [48], solids with odd elasticity [49], particles breaking action-reaction symmetry [50,51], chiral active particles [52,53], nonreciprocal flocking models [54–60], nonreciprocal frustration [61], and systems described by nonreciprocal Cahn-Hilliard equations [62–68]. Recent developments in understanding the microscopic dynamics of nonreciprocal systems have revealed more sophisticated behaviors and phases absent in reciprocal systems, such as traveling waves, oscillations, spiral waves, and effervescence [54,55,62–68]. The nonreciprocal phases have been further elucidated by delineating the underlying topological properties [54].

Active matter and nonreciprocal systems are conventionally analyzed using hydrodynamic descriptions [69,70], which successfully recover the phases observed in the microscopic dynamics [55]. Deriving hydrodynamic equations of motion requires identifying the relevant degrees of freedom, termed order parameters, with the help of symmetry arguments: a top-down approach [69,70]. Subsequently, fluctuations of the order parameters can be incorporated using more sophisticated exact coarse-graining techniques [71,72]. This procedure is usually effective and reliable for understanding system dynamics, phases, and phase transitions.

However, for microscopic systems, the hydrodynamic description fails both quantitatively and qualitatively in the small

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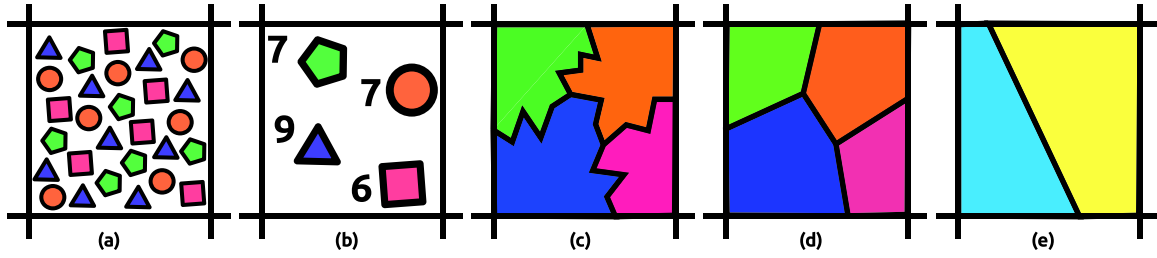


FIG. 1. Scheme of various levels of coarse-graining considered in this paper, microscopic, mesoscopic, macroscopic-fluctuating, macroscopic-deterministic, and hydrodynamic. (a) Microscopic particles are confined to the lattice site (denoted by the black #). Four different types of particles are illustrated: triangle (blue), square (pink), pentagon (green), and circle (orange). (b), (c) The mesoscopic (macroscopic) description is obtained for the fluctuating particle number (density). (d) The straight interface between the density fields represents the suppression of the fluctuation and convergence to the mean-field dynamical description. (e) Cyan and Yellow denote the relevant hydrodynamic order parameter that does not necessarily preserve the microscopic properties.

particle number regime. For example, the onset of the phase transition predicted by the hydrodynamic description differs from that predicted by the microscopic description [73]. Moreover, it does not correctly predict the order of the phase transition, which is usually addressed by introducing *ad hoc* noise corrections [73]. The hydrodynamic description also fails to adequately describe and interpret the thermodynamic implications at the macro- or mesoscale because of coarse-graining. The effective hydrodynamic description does not resolve microscopic dissipation at the hydrodynamic scale. In particular, microscopic states with different thermodynamic properties are coarse-grained into a single order parameter. In addition, the mapping between microscopic and macroscopic or mesoscopic control parameters is less transparent. We aim to construct the macrostate dynamics using a bottom-up approach, implementing the coarse-graining procedure only to an order at which the microscopic dissipation is both qualitatively and quantitatively preserved, thereby avoiding a subsequent coarse-graining to obtain an effective description in terms of relevant order parameters.

The coarse-graining scheme is illustrated in Fig. 1. Microscopic particles with different dynamic and thermodynamic properties exist. Lumping together microstates with identical thermodynamic and physical properties forms a mesostate: the particle number. The macroscopic limit corresponds to taking the limit of a large number of particles per site. It defines the scaling from the intensive density order parameter to the extensive particle number. The densities are fluctuating stochastic fields, indicated by the rough edges. The thermodynamic limit suppresses macroscopic fluctuations, leading to reaction-diffusion dynamics for deterministic density fields. In comparison, the top-down construction of an effective hydrodynamic description fails to track microscopic thermodynamic dissipation both qualitatively and quantitatively.

Despite significant progress in understanding the dynamics of active matter on different scales, its implications for the thermodynamics of nonreciprocal systems remain elusive. For reciprocal active matter systems, stochastic thermodynamics has been formulated for active Brownian particles [74–76] and for flocking models [77], and fluctuation relations have been derived [78,79]. To quantify thermodynamic irreversibility, information-theoretic measures have been utilized, for instance, for the Active Ising model [80] and the nonreciprocal Cahn-Hilliard equation [81–84], which have been proven to

be good statistical measures for computing lower bounds on thermodynamic entropy production [85–91]. However, the information-theoretic definition of irreversibility does not necessarily coincide with thermodynamic entropy production [85–91]. The information-theoretic measure can be defined without the state transitions satisfying local detailed balance (LDB), thus lacking thermodynamic consistency [80–83].

In this paper, we combine the three main motifs mentioned so far: Stochastic Thermodynamics, Coarse-graining, and Nonreciprocal Interacting Particles. We aim to formulate a microscopic Markov jump process description for interacting nonreciprocal particles, which enables a thermodynamically consistent definition of their dynamics and thermodynamics. Subsequently, we aim to obtain a coarse-grained mesoscopic or macroscopic description that preserves microscopic thermodynamic dissipation. We ensure this by properly identifying the LDB condition at the mesoscale or macroscale. This allows us to obtain thermodynamically consistent dynamics for the time evolution of the mesostate or macrostate, ensuring that thermodynamics for nonreciprocal systems is defined at these scales. This procedure establishes the connection between microscopic and macroscopic or mesoscopic control parameters, enabling qualitative and quantitative equivalence between dynamics and thermodynamic quantities. Importantly, this extends the applicability of stochastic thermodynamics to coarse-grained many-body descriptions.

Throughout this work, the definition of a ‘microscopic’ nonreciprocally interacting particle refers to the ‘minimal’ experimentally relevant observation scale, where an ‘effective’ nonreciprocity is observed. This nonreciprocity may arise from asymmetric interactions with the surrounding medium, complex internal dynamics, or the cognitive capabilities of living systems, corresponding to active matter models, chemical reaction networks, and population dynamics models, respectively. For example, two particles of type “1” and “2” of different shapes and materials submerged in a medium interact asymmetrically with the surrounding environment, which generates effective asymmetric nonreciprocal forces between them [50,51]. Although composed of atoms and molecules that may satisfy Newton’s third law at the smallest scales, “1” and “2” are treated as effective particles exhibiting nonreciprocal interactions due to their asymmetric interactions with the surrounding medium. We treat such an

‘effective nonreciprocal particle’ as the fundamental microscopic object of our study. The notion of a microscopic particle refers to such an independent degree of freedom. Throughout this work, the term microscopic means the following: it is not intended as ‘elemental’ in the sense of irreducible, rather as the effective small-scale description of the system’s independent units (degree of freedom). Questions such as how does such nonreciprocity originate, and what is the energetic cost to generate it? are not the aim of this work. Rather, we focus on the open question: given the abundant existence of effective nonreciprocal particles, is it possible to formulate a thermodynamically consistent microscopic theoretical description of their dynamics? We focus on a broad class of lattice models with nonreciprocal interactions, for which we introduce a thermodynamically consistent dynamics. Then it is a well-posed question to derive the stochastic thermodynamics for it, but we intended this as a purely theoretical framework without any concrete experimental realization in mind.

We define a class of thermodynamically consistent lattice models for nonreciprocally interacting microscopic particles coupled to an external driving reservoir [1,14,20]. We achieve this by formulating the LDB condition for the dynamics of microscopic nonreciprocal particles. In stochastic thermodynamics, thermodynamic consistency refers specifically to the LDB condition [23,24,74], the LDB condition for the microscopic system ensures thermodynamic consistency of these models; we adhere to this convention throughout this paper. The LDB for the microscopic system ensures thermodynamic consistency of these models. This allows us to define the microscopic entropy production rate (EPR) for externally driven nonreciprocal particles using a Markov Jump Process (MJP) with dynamics governed by the Master equation [1,14,20]. Furthermore, we define a coarse-grained description (mesoscopic or macroscopic) by lumping microstates with identical dynamical and thermodynamic properties [92]. The mesoscopic (macroscopic) scale is defined for particle number (density) as a coarse-grained state. The coarse-grained description is chosen flexibly depending on system-specific requirements. We employ the Doi-Peliti exact coarse-graining method, which preserves microscopic discreteness in the coarse-grained description [92].

Next, we formulate transitions between coarse-grained states using the LDB in a thermodynamically consistent way. This enables us to quantify and decompose coarse-grained state dissipation using stochastic thermodynamics for any nonreciprocal system. In addition, the dynamic and thermodynamic equivalence of microscopic and coarse-grained systems is ensured. The fluctuating dynamics of coarse-grained states generalize Macroscopic Fluctuation Theory (MFT) for nonreciprocal systems [93]. We obtain a nonquadratic dissipation function, which plays an important role in the exact quantification of the EPR, particularly for far-from-equilibrium systems [94,95]. We decompose the EPR into its linearly independent contributions through an orthogonal decomposition, consisting of four components: relaxation of the reciprocal interaction energy functional, nonreciprocal interaction, coupling to the external reservoir, and external driving work. We show that nonreciprocal phase transitions are equivalent to dynamical phase transitions and are characterized by different

scaling regimes for the nonreciprocal EPR, which physically correspond to sustaining dissipative vorticity currents.

Finally, we demonstrate nonreciprocal and nonequilibrium analogs of reciprocal systems. For instance, irreversible thermodynamic relations: Onsager’s nonreciprocal relations, fluctuation-response relations, higher-order nonlinear responses, and stochastic thermodynamic relations: fluctuation relations, thermodynamic kinetic uncertainty relations, and information thermodynamics, which are generalized to macroscopic systems. In the deterministic macroscopic limit, we formulate irreversible thermodynamics, Arrhenius transition state theory (LDB), and reaction-diffusion systems (dynamics) for interacting nonreciprocal density fields [69,70,96]. We elaborate on the application of our framework using a few prototypical models. Overall, our framework extends the applicability of stochastic thermodynamics for nonreciprocal and externally driven systems across different observation scales.

II. MICROSCOPIC DESCRIPTION

We consider a multiparticle system that comprises several particle types and resides in a discretized lattice cells (the sites of the lattice): a regular lattice with lattice constant l and continuous time. We set $l = 1$ throughout this paper unless we explicitly specify the limit $l \rightarrow 0$. Our definition of lattice cell seems analogous to the ‘subvolume’ defined in Refs. [97,98] or the ‘voxel’ defined in Ref. [99]. We measure the infinitesimal time step in units of the fastest microscopic transition among the set of all possible transitions, so that both reactive and diffusive transitions defined subsequently are relevant. Because of this self-adjusting formulation, our setup incorporates reaction-dominated and diffusion-dominated regimes in a specific limit of observation timescale. This preserves the physical effects associated with all microscopic transitions and avoids any assumption associated with timescale separation between transitions [94,95]. The number of particles of type i at lattice site index $\#$ is denoted by $N_i^\#$. The configuration of the lattice is denoted by $\{N\}$, whose dimension is determined by the lattice size and the number of particle types. The volume of the d -dimensional lattice space is denoted by $\mathcal{V}$ and is equal to the number of lattice sites. The faster degrees of freedom form the environment, which provides the thermodynamic cost for all microstate changes of the system, consisting of the slower degrees of freedom [20,23,24].

A. Microscopic Boltzmann weight

1. Theoretical formulation

The microscopic interaction energy experienced by the type i particle due to the type j particle is denoted by v_{ij} . $v_{ij} < 0$ implies that the type i particle is attracted toward the type j particle, while $v_{ij} > 0$ indicates a repulsive interaction. Analogously, v_{ij} represents the thermodynamic cost, supported by the environment, to place a type i particle in the presence of a type j particle, see Fig. 2(a). The violation of the actio = reactio symmetry implies $v_{ij} \neq v_{ji}$ microscopically. $v_{ij} = v_{ji}$ indicates that the particles are reciprocal and respect the actio = reactio symmetry. To separate and quantify the actio = reactio symmetry-preserving and

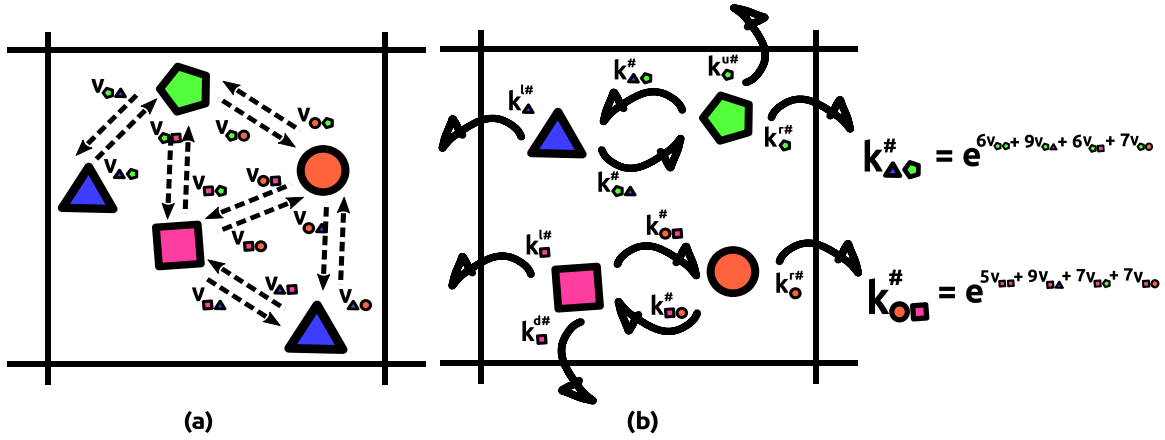


FIG. 2. (a) Illustration of the microscopic interactions between particle types, denoted by v_{ij} , confined to the lattice site $\#$, with $v_{ij} \neq v_{ji}$. (b) All possible transitions are indicated by curved harpoons. The exact expressions for two transition rates $k_{\gamma\gamma'}$ in terms of v_{ij} for the microscopic lattice configuration in Fig. 1(a) are demonstrated. Throughout this work, superscript $\#$ and subscript i denote the lattice site and particle-type index, respectively. For microscopic interaction coefficients between particle types, two indices are needed; the convention ‘left-right’ denotes ‘influenced-influencer’ particle. Reactive or diffusive transitions correspond to changes in particle type or lattice location, respectively. For transitions, the ‘right-to-left’ notation denotes the ‘initial-to-final’ microstate. Latin letters i, j refer to state-space properties requiring the particle type specification, e.g., v_{ij} , v_{ij}^r , v_{ij}^{nr} , $\epsilon_i^\#$, ϵ_i^r , f_i^{nr} . Greek letters γ, γ' refer to transition-space properties specifying particle type changes, e.g., $\Delta_{\gamma\gamma'}^\#$, $f_{\gamma\gamma'}^{ch}$, $k_{\gamma\gamma'}^\#$. This distinction clarifies the difference between state-space and transition-space quantities.

-breaking components, we define the reciprocal $v_{ij}^r = (v_{ij} + v_{ji})/2$ and nonreciprocal $v_{ij}^{nr} = (v_{ij} - v_{ji})/2$ interaction energies. v_{ij}^r and v_{ij}^{nr} correspond to the symmetry-preserving and symmetry-breaking parts, respectively, which fixes ‘the orthogonal reference gauge’ such that $v_{ij}^r = v_{ji}^r$ and $v_{ij}^{nr} = -v_{ji}^{nr}$ by construction.

$\epsilon_i^\#$ is microscopic Boltzmann weight of a type i particle at lattice site $\#$, quantifying the total thermodynamic cost of placing the particle at the site. The reciprocal and nonreciprocal components of the microscopic Boltzmann weight $\epsilon_i^\#$ of the type i particle at $\#$ are denoted as ϵ_i^r and f_i^{nr} , respectively [100], such that

$$\epsilon_i^\# = \epsilon_i^r + f_i^{nr} \quad (1)$$

is decomposed into its reciprocal and nonreciprocal components,

$$\epsilon_i^r = \beta \sum_{j \neq i} v_{ij}^r N_j^\# + \beta v_{ii}^r (N_i^\# - 1), \quad f_i^{nr} = \beta \sum_j v_{ij}^{nr} N_j^\#. \quad (2)$$

We drop the lattice index $\#$ for ϵ_i^r and f_i^{nr} for brevity, but it is understood that these quantities are evaluated with the appropriate lattice $\#$ and type i indexes. Here, $N_i^\# - 1$ accounts for the self-interaction of the particle. β is the inverse temperature, and the Boltzmann constant is set to unity throughout this work. ϵ_i^r and f_i^{nr} correspond to the actio = reactio symmetry preserving and breaking terms, respectively, and quantify the total reciprocal and nonreciprocal interaction energy experienced by the type i particle due to all other particles. ϵ_i^r and f_i^{nr} are local properties of each particle. For noninteracting ideal particles, $\epsilon_i^\# = 0$, since $v_{ij} = 0 \forall j$.

We examine the possibility of representing the locally defined microscopic Boltzmann weight as a global property of the lattice, namely, the total interaction energy ϵ^{int} . This is possible only if $v_{ij} = v_{ji}$, which is satisfied by v_{ij}^r and violated

by v_{ij}^{nr} . Hence, the reciprocal part of the microscopic Boltzmann weight can be expressed as the microscopic interaction energy functional ϵ^{int} of the lattice,

$$\epsilon^{\text{int}} = \frac{1}{2} \beta \sum_{\#, i} v_{ii}^r N_i^\# (N_i^\# - 1) + \beta \sum_{\#, i \neq j} v_{ij}^r N_i^\# N_j^\#. \quad (3)$$

In the absence of nonreciprocity, i.e., $v_{ij}^{nr} = 0 \forall i, j$, the system satisfies the Boltzmann distribution generated by ϵ^{int} . A detailed discussion on thermodynamics follows.

Equation (3) does not necessarily represent the equilibrium interaction energy functional. Consider a decomposition $v_{ij} = v_{ij}^{\text{eq}} + v_{ij}^{\text{neq}}$, where v_{ij}^{eq} is derived from the equilibrium energy functional of the form of Eq. (3) and v_{ij}^{neq} is the remaining microscopic nonequilibrium interaction energy attributed to the violation of Newton’s third law. This gives $v_{ij}^r = v_{ij}^{\text{eq}} + (v_{ij}^{\text{neq}} + v_{ji}^{\text{neq}})/2$ and $v_{ij}^{nr} = (v_{ij}^{\text{neq}} - v_{ji}^{\text{neq}})/2$, so that v_{ij}^r and v_{ij}^{nr} incorporate the symmetric and antisymmetric parts of v_{ij}^{neq} , respectively. In conclusion, even though the microscopic physical interaction rules are governed by v_{ij} , v_{ji} , v_{ij}^{eq} , v_{ij}^{neq} , the underlying geometric and thermodynamic structure is captured by v_{ij}^r , v_{ij}^{nr} , and $|\epsilon^{\text{int}}| \geq |\epsilon^{\text{eq}}|$ gives a tighter bound on the conservative interaction energy functional.

Moreover, this analysis shows that the decomposition of v_{ij} and v_{ji} (gauge fixing) is not necessarily unique, unless the orthogonal reference gauge is imposed. Furthermore, only the orthogonal reference gauge leads to an observation scale-invariant decomposition of actio-reactio symmetry conserving and breaking terms. Consequently, it also yields the physically correct thermodynamic dissipation of the system. The decomposition of $\epsilon_i^\#$ into ϵ_i^r and f_i^{nr} is equivalent to the Helmholtz-Hodge decomposition of $\epsilon_i^\#$ into a conservative gradient force ϵ_i^r derived from ϵ^{int} and a divergence-free non-conservative force f_i^{nr} .

2. Illustration of the model setup

We illustrate the significance of the concept of Boltzmann weights in nonreciprocal systems with a minimal model, namely, the sheep-dog dynamics. Here, v_{sd} represents the energy cost for an external agent to place a sheep on the lattice when a dog is already present. Since the sheep repels the dog, let us assume that five units of energy are required to implement this operation experimentally, implying $v_{sd} = 5$ units. Similarly, v_{ds} is the energy cost for an external agent to place a dog on the lattice when a sheep is already present. Because the dog is attracted to the sheep, this operation releases energy. This calibration is equivalently achieved by performing the inverse operation of removing the dog from the lattice when both sheep and dog are present, which requires $-v_{ds}$ units of energy (note the sign convention due to defining the cost for the inverse operation). Let us assume that three units of energy are needed to remove the dog from the lattice in the presence of the sheep. Therefore, $v_{ds} = -3$ units, signifying an attractive interaction of the dog toward the sheep. All parameters of our theoretical model, namely, the microscopic interaction energies v_{ij} for different particle types, are to be interpreted in this way.

B. Dynamics

1. Single-particle microstate transition rates

The particle can change its type or its lattice location, which are defined as the reactive and diffusive transitions, respectively. A reactive transition from type γ' to γ at the lattice site $\#$ is indicated by $\Delta_{\gamma\gamma'}^\#$, resulting in $\{N\} \rightarrow \{N + \Delta_{\gamma\gamma'}^\#\}$, see Fig. 2(b). Similarly, a diffusive transition of a type i particle from the lattice site $\#$ in the direction $\vec{D}$ is indicated by $\Delta_i^{\vec{D}\#}$, resulting in $\{N\} \rightarrow \{N + \Delta_i^{\vec{D}\#}\}$. The set of forward and backward transitions is consistently defined between the initial $\{N\}$ and final lattice configurations $\{N + \Delta_{\gamma\gamma'}^\#\}$ and $\{N + \Delta_i^{\vec{D}\#}\}$, respectively, for reactive and diffusive transitions. For example, let us consider a single reactive transition of type 1 to type 2 with a single lattice site, then the forward transition has to be consistently defined from the initial lattice configuration $\{N\} = (N_1, N_2)$ to the final lattice configuration $\{N + \Delta_{21}\} = (N_1 - 1, N_2 + 1)$, and the other way around for the backward transition. The transition rates of reactive ($k_{\gamma'\gamma}^\#$) and diffusive ($k_i^{\vec{D}\#}$) jumps are constructed using $\epsilon_i^\#$:

$$k_{\gamma'\gamma}^\# = d_{\gamma\gamma'} e^{\epsilon_\gamma^\#}, \quad k_i^{\vec{D}\#} = d_i^{\vec{D}} e^{\epsilon_i^\#}. \quad (4)$$

The thermodynamically consistent exponential parametrization of $k_{\gamma'\gamma}^\#$ and $k_i^{\vec{D}\#}$ in $\epsilon_i^\#$ allows one to visualize the thermodynamic meaning of $\epsilon_i^\#$. For noninteracting ideal particles, $\epsilon_i^\# = 0$; therefore, $k_{\gamma'\gamma}^\# = d_{\gamma\gamma'}$ and $k_i^{\vec{D}\#} = d_i^{\vec{D}}$ are the equilibrium transition rates for noninteracting ideal particles, which also quantify the strength of the transition rates in Eq. (4). They satisfy $d_{\gamma\gamma'} = d_{\gamma'\gamma}$ and $d_i^{\vec{D}} = d_i^{\vec{D}^{-1}}$.

The LDB condition for the forward and backward microscopic transitions constructed using Eq. (4) reads

$$\frac{k_{\gamma'\gamma}^\#}{k_{\gamma\gamma'}^\#} = e^{\epsilon_\gamma^\# - \epsilon_{\gamma'}^\# + f_{\gamma\gamma'}^{\text{ch}}}, \quad \frac{k_i^{\vec{D}\#}}{k_i^{(\vec{D}\#)^{-1}}} = e^{\epsilon_i^\# - \epsilon_i^{(\vec{D}\#)^{-1}} + \vec{D} \cdot \vec{f}_i^{\text{sp}}}. \quad (5)$$

Equation (5) expresses the microscopic LDB for nonreciprocal particles in a more familiar form compared with its original definition in Eq. (4). However, the time-symmetric part of the transition rates cancels out in Eq. (5) due to the relations $d_{\gamma'\gamma} = d_{\gamma\gamma'}$ and $d_i^{\vec{D}} = d_i^{\vec{D}^{-1}}$.

The environment supports the thermodynamic cost for the change in the microscopic Boltzmann weight. Compared with Eq. (4), we have modified Eq. (5) by introducing a nonconservative external driving force $f_{\gamma\gamma'}^{\text{ch}}$ and $\vec{f}_i^{\text{sp}}$ “along the transition,” which correspond to chemical and self-propulsion driving. In accordance with the fundamental assumption of stochastic thermodynamics, under the timescale separation between the system’s slow degrees of freedom and the environment’s fast degrees of freedom, the thermodynamic cost of such external constant nonconservative driving of the system is supported by the environment [20,23,24]. Here, the environment is physically or experimentally realized by an externally coupled reservoir of a very large number of particles. External driving refers to particle self-propulsion driven by an external reservoir, like Janus particles that are driven by the Hydrogen Peroxide contained in the surrounding liquid in active colloid experiments. Similarly, the reactive transition between particles are driven by enzymes in the surrounding liquid. This self-propulsion or chemical driving mechanism is generally assumed to be independent of the interactions between and reactions of the system particles. Throughout this work, the convention ‘externally nonconservatively driven’ refers to such a ‘thermodynamic reservoir.’ Therefore, the term external means ‘from outside the system,’ and externally driven means a reaction or diffusive movement fueled by large external reservoirs which are not explicitly modeled, as sketched in Fig. 2(b).

Since an external coupling to a reservoir is a property of a transition, $f_{\gamma\gamma'}^{\text{ch}}$ and $\vec{f}_i^{\text{sp}}$ are introduced in Eq. (5). Nevertheless, they could have been introduced directly in Eq. (4), using exponential tilts $e^{f_{\gamma\gamma'}^{\text{ch}}/2}$ and $e^{-f_{\gamma\gamma'}^{\text{ch}}/2}$ for the forward and backward reactive transitions (or $e^{\vec{D} \cdot \vec{f}_i^{\text{sp}}/2}$ and $e^{-\vec{D} \cdot \vec{f}_i^{\text{sp}}/2}$ for the forward and backward diffusive transitions). Note that the nonreciprocal nonconservative forces f_i^{nr} act in the state space, whereas chemical or self-propulsion nonconservative driving is associated with transitions between states and lies in the transition space. This fundamental physical distinction between the two is clear: nonreciprocity requires the minimal notion of two states to define a dissipative current, whereas driven discrete-state processes require three states and three transitions [1].

We use shorthand notations $\Delta_{\gamma\gamma'}^\#(*) = (*)_{\gamma'}^\# - (*)_\gamma^\#$ and $\Delta_i^{\vec{D}\#}(*) = (*)_i^{\vec{D}\#} - (*)_i^{(\vec{D}\#)^{-1}}$ for the change in $\epsilon_i^\#$, ϵ_i^r , and f_i^{nr} due to a transition. We introduce a shorthand notation Δ to refer to a transition, applied as a diffusive or reactive transition depending on the context. Therefore, using Eqs. (1) and (2), $\Delta_{\gamma\gamma'}^\# \epsilon = \epsilon_\gamma^\# - \epsilon_{\gamma'}^\#$ and $\Delta_i^{\vec{D}\#} \epsilon = \epsilon_i^{\vec{D}\#} - \epsilon_i^{(\vec{D}\#)^{-1}}$. The corresponding reciprocal nonreciprocal decompositions, $\Delta_{\gamma\gamma'}^\# \epsilon = \Delta_{\gamma\gamma'}^\# \epsilon^r + \Delta_{\gamma\gamma'}^\# f^{\text{nr}}$ and $\Delta_i^{\vec{D}\#} \epsilon = \Delta_i^{\vec{D}\#} \epsilon^r + \Delta_i^{\vec{D}\#} f^{\text{nr}}$. The set of all reactive transitions is denoted by $\{\Delta_{\gamma\gamma'}^\#\}$, and the set of all diffusive transitions is denoted by $\{\Delta_i^{\vec{D}\#}\}$. Four such possibilities exist for a two-dimensional square lattice: upward,

downward, leftward, and rightward. Microscopic interactions and transitions are shown in Fig. 2.

To demonstrate the connection of Eq. (4) to the Glauber-dynamics [101] $k_{21}^\# = e^{\epsilon_1^\#}/(e^{\epsilon_1^\#} + e^{\epsilon_2^\#})$ and $k_{12}^\# = e^{\epsilon_2^\#}/(e^{\epsilon_1^\#} + e^{\epsilon_2^\#})$ or the Metropolis-dynamics [102] $k_{12}^\# = \min(1, e^{-\epsilon_1^\# + \epsilon_2^\#})$ and $k_{21}^\# = \min(1, e^{-\epsilon_2^\# + \epsilon_1^\#})$, we consider a reactive transition $\Delta_{12}^\#$ between type 1 and 2 particles with $f_{12}^{\text{ch}} = 0$. Obviously, Glauber-dynamics and Metropolis-dynamics have the exponential form of Eq. (4), but with factors $d_{\gamma\gamma'} = 1/(e^{\epsilon_1^\#} + e^{\epsilon_2^\#})$ and $d_{\gamma\gamma'} = 1/\max(e^{\epsilon_1^\#}, e^{\epsilon_2^\#})$, respectively, that does not represent the rate for noninteracting particles since it depends on $\epsilon_1^\#$ and $\epsilon_2^\#$. The noninteracting transition rates $d_{\gamma\gamma'}$ represent the broad range of timescales for different transitions that are present in generic reaction networks, which requires careful treatment without assumptions; see Refs. [94,95] for details. This is carefully treated with the definition in Eq. (4), which preserves the different timescales associated with different transitions. Note that the Glauber and Metropolis rates are saturating and therefore advantageous for equilibrium Monte Carlo simulations [101,102], but they do not reflect the realistic dynamics of nonequilibrium systems. In any case, the time-symmetric components of the transition rates become irrelevant after taking the ratio of transition rates for the LDB in Eq. (5). The dynamics defined by Eq. (4) can straightforwardly be simulated by the exact Gillespie algorithm [103].

2. Multiparticle microstate transition rates

Particles with identical type and lattice index have the same dynamic and thermodynamic properties. Hence, $k_{\gamma\gamma'}$ are lumped together to give multiparticle transition rates $k_{\gamma\gamma'}^\#(\{N\}) = N_{\gamma'}^\# k_{\gamma\gamma'}^\#$ and $k_i^{\bar{\mathcal{D}}\#}(\{N\}) = N_i^\# k_i^{\bar{\mathcal{D}}\#}$. The LDB for the multiparticle transition rates [reactive $k_{\gamma\gamma'}^\#(\{N\})$ and diffusive $k_i^{\bar{\mathcal{D}}\#}(\{N\})$] is

$$\frac{k_{\gamma\gamma'}^\#(\{N\})}{k_{\gamma\gamma'}^\#(\{N + \Delta_{\gamma\gamma'}^\#\})} = e^{-\Delta_{\gamma\gamma'}^\# s^b + \Delta_{\gamma\gamma'}^\# \epsilon + f_{\gamma\gamma'}^{\text{ch}}}, \quad (6)$$

$$\frac{k_i^{\bar{\mathcal{D}}\#}(\{N\})}{k_i^{(\bar{\mathcal{D}}\#)^{-1}}(\{N + \Delta_i^{\bar{\mathcal{D}}\#}\})} = e^{-\Delta_i^{\bar{\mathcal{D}}\#} s^b + \Delta_i^{\bar{\mathcal{D}}\#} \epsilon + \bar{\mathcal{D}} \cdot \tilde{f}_i^{\text{sp}}}. \quad (6)$$

Here, Δs^b incorporates the change in the microscopic Boltzmann entropy, defined as a global lattice property [104]:

$$s^b = \ln \left[\frac{(\sum_{i,\#} N_i^\#)!}{\prod_{i,\#} N_i^\#!} \right]. \quad (7)$$

It quantifies the statistical degeneracy of the microstate and is a system property defined for the lattice. Importantly, $\Delta_{\gamma\gamma'}^\# \epsilon^r = \Delta_{\gamma\gamma'}^\# \epsilon^{\text{int}}$ and $\Delta_i^{\bar{\mathcal{D}}\#} \epsilon^r = \Delta_i^{\bar{\mathcal{D}}\#} \epsilon^{\text{int}}$, as satisfied by the reciprocal microscopic Boltzmann weight. We introduce the notation $\Delta_{\gamma\gamma'}^\# \epsilon^{\text{int}} = \epsilon^{\text{int}}(\{N\}) - \epsilon^{\text{int}}(\{N + \Delta_{\gamma\gamma'}^\#\})$, which represents a change of a function of the lattice state, a global property; this applies similarly to s^b . In contrast, the change in the nonreciprocal microscopic Boltzmann weight $\Delta_{\gamma\gamma'}^\# f^{\text{nr}}$ cannot be represented as a change of a global function of the lattice state. This results in a fundamental difference between the boundary ($\Delta_{\gamma\gamma'}^\# \epsilon^{\text{int}}$ and $-\Delta_{\gamma\gamma'}^\# s^b$) and the bulk terms ($\Delta_{\gamma\gamma'}^\# f^{\text{nr}}$) of the transition affinity.

3. Master equation

The probability of the multiparticle lattice microstate $\{N\}$ at time t is denoted by $P_{\{N\}}(t)$. The master equation for the dynamic evolution of the probability $P_{\{N\}}(t)$ for the lattice microstate reads [105,106]:

$$\partial_t P_{\{N\}}(t) = - \sum_{\{\Delta_{\gamma\gamma'}^\#\}} j_{\gamma\gamma'}^\#(\{N\}) - \sum_{\{\Delta_i^{\bar{\mathcal{D}}\#}\}} j_i^{\bar{\mathcal{D}}\#}(\{N\}). \quad (8)$$

The reactive and diffusive microscopic transition currents are

$$j_{\gamma\gamma'}^\#(\{N\}) = k_{\gamma\gamma'}^\#(\{N\})P_{\{N\}} - k_{\gamma\gamma'}^\#(\{N + \Delta_{\gamma\gamma'}^\#\})P_{\{N + \Delta_{\gamma\gamma'}^\#\}},$$

$$j_i^{\bar{\mathcal{D}}\#}(\{N\}) = k_i^{\bar{\mathcal{D}}\#}(\{N\})P_{\{N\}} - k_i^{(\bar{\mathcal{D}}\#)^{-1}}(\{N + \Delta_i^{\bar{\mathcal{D}}\#}\})P_{\{N + \Delta_i^{\bar{\mathcal{D}}\#}\}}. \quad (9)$$

C. Thermodynamics

1. Transition affinities

The stochastic state entropy s^{state} of $\{N\}$ is defined as [18]

$$s^{\text{state}} = -\ln(P_{\{N\}}). \quad (10)$$

The total microscopic energy functional ε incorporates energetic and entropic contributions, $\varepsilon = \varepsilon^{\text{int}} - s^b$. The stochastic Massieu potential is

$$\varepsilon^{\text{stoch}} = \varepsilon - s^{\text{state}}. \quad (11)$$

Importantly, ε^{int} , s^b , s^{state} are properties of the lattice; therefore, a cumulative change in them due to multiple stochastic transitions depends only on the initial and final lattice states. This subsequently applies to ε and $\varepsilon^{\text{stoch}}$. In comparison, $\Delta_i^{\bar{\mathcal{D}}\#} f^{\text{nr}}$ and $\Delta_{\gamma\gamma'}^\# f^{\text{nr}}$ are defined locally on the lattice. The stochastic transition entropy is defined as

$$\Delta_{\gamma\gamma'}^\# \sigma = \Delta_{\gamma\gamma'}^\# \varepsilon^{\text{int}} - \Delta_{\gamma\gamma'}^\# s^b - \Delta_{\gamma\gamma'}^\# s^{\text{state}} + \Delta_{\gamma\gamma'}^\# f^{\text{nr}} + f_{\gamma\gamma'}^{\text{ch}},$$

$$\Delta_i^{\bar{\mathcal{D}}\#} \sigma = \Delta_i^{\bar{\mathcal{D}}\#} \varepsilon^{\text{int}} - \Delta_i^{\bar{\mathcal{D}}\#} s^b - \Delta_i^{\bar{\mathcal{D}}\#} s^{\text{state}} + \Delta_i^{\bar{\mathcal{D}}\#} f^{\text{nr}} + \bar{\mathcal{D}} \cdot \tilde{f}_i^{\text{sp}}. \quad (12)$$

Equation (12) quantifies the thermodynamic cost of each microscopic transition supported by the environment. We adhere to the convention ‘forces generate currents’ throughout this paper, where forces refer to the transition affinities, for instance, $\Delta \varepsilon$, Δs^{state} , Δf^{nr} , $\tilde{f}_i^{\text{sp}}$, $f_{\gamma\gamma'}^{\text{ch}}$. Transition affinities are categorized into boundary (conservative) terms ($\Delta \varepsilon$, Δs^{state}) and bulk (nonconservative) terms (Δf^{nr} , $\tilde{f}_i^{\text{sp}}$, $f_{\gamma\gamma'}^{\text{ch}}$), which correspond to relaxation and dissipative currents, respectively [20]. Considering a set of consecutive stochastic transitions over observation time τ , the conservative and nonconservative total entropy production due to all transitions is $O(1)$ and $O(\tau)$, respectively. Importantly, the nonconservative forces fundamentally differ in their origin. Δf^{nr} and $f_{\gamma\gamma'}^{\text{ch}}$ (or $\tilde{f}_i^{\text{sp}}$) lie in the state-space and the transition-space, respectively. Moreover, f_i^{nr} depends on the local particle occupancy, unlike $f_{\gamma\gamma'}^{\text{ch}}$ (or $\tilde{f}_i^{\text{sp}}$), which are constants. Thus, the nonconservative driving due to nonreciprocal interactions depends on the initial $\{N\}$ and final lattice configurations $\{N + \Delta_{\gamma\gamma'}^\#\}$ and $\{N + \Delta_i^{\bar{\mathcal{D}}\#}\}$, respectively, for reactive and diffusive transitions, since f_i^{nr} depends on $\{N\}$, as highlighted earlier in Sec. II B.

The nonconservative driving due to nonreciprocal interactions is more dynamic in comparison to fixed chemical or self-propulsion driving forces.

2. Microscopic EPR

The microscopic reactive and diffusive mean EPR are defined as $\langle \dot{\sigma}^r \rangle$ and $\langle \dot{\sigma}^d \rangle$, respectively [1,20]:

$$\begin{aligned} \langle \dot{\sigma}^r \rangle &= \sum_{\{\Delta_{\gamma\gamma'}^\#\}} j_{\gamma\gamma'}^\#(\{N\}) \ln \left(\frac{k_{\gamma\gamma'}^\#(\{N\}) P_{\{N\}}}{k_{\gamma\gamma'}^\#(\{N + \Delta_{\gamma\gamma'}^\#\}) P_{\{N + \Delta_{\gamma\gamma'}^\#\}}} \right), \\ \langle \dot{\sigma}^d \rangle &= \sum_{\{\Delta_i^{\bar{D}\#}\}} j_i^{\bar{D}\#}(\{N\}) \ln \left(\frac{k_i^{\bar{D}\#}(\{N\}) P_{\{N\}}}{k_i^{(\bar{D}\#)^{-1}}(\{N + \Delta_i^{\bar{D}\#}\}) P_{\{N + \Delta_i^{\bar{D}\#}\}}} \right). \end{aligned} \quad (13)$$

Here, $\sum_{\{\Delta_{\gamma\gamma'}^\#\}}^{[N]}$ and $\sum_{\{\Delta_i^{\bar{D}\#}\}}^{[N]}$ denote the sum over all lattice configurations for the reactive and diffusive transitions, respectively. This satisfies the fundamental definition of mean EPR, namely, force times current: $\langle \dot{\sigma}^r \rangle = \sum_{\{\Delta_{\gamma\gamma'}^\#\}}^{[N]} j_{\gamma\gamma'}^\#(\{N\}) \Delta_{\gamma\gamma'}^\# \sigma$ and $\langle \dot{\sigma}^d \rangle = \sum_{\{\Delta_i^{\bar{D}\#}\}}^{[N]} j_i^{\bar{D}\#}(\{N\}) \Delta_i^{\bar{D}\#} \sigma$, due to the LDB condition Eq. (6) and the definitions Eqs. (10) and (12). $\langle \dot{\sigma}^r \rangle > 0$ and $\langle \dot{\sigma}^d \rangle \geq 0$ due to the inequality $(x - y) \ln(x/y) \geq 0$, which leads to the second law of thermodynamics. The total mean microscopic EPR reads $\langle \dot{\sigma} \rangle = \langle \dot{\sigma}^r \rangle + \langle \dot{\sigma}^d \rangle$.

3. Conservative and nonconservative decomposition of EPR

We reorganize $\langle \dot{\sigma} \rangle$ using Eqs. (8), (9), and (6), obtaining the EPR contributions due to the reciprocal, nonreciprocal, Gibbs entropic, and external nonconservative driving forces.

$$\langle \dot{\sigma} \rangle = -d_t \langle \varepsilon \rangle + d_t s^{\text{gb}} + \langle \dot{\sigma}^{\text{nr}} \rangle + \langle \dot{\sigma}^{\text{sp}} \rangle + \langle \dot{\sigma}^{\text{ch}} \rangle. \quad (14)$$

Here, $-d_t \langle \varepsilon \rangle$ and $d_t s^{\text{gb}}$ are attributed to the rate of change of $\varepsilon^{\text{int}} - s^b$ and s^{state} from Eq. (12). We define the mean microscopic work rate $\langle \dot{w} \rangle = -\sum_{\{N\}} P_{\{N\}} \partial_t \varepsilon^{\text{int}}$ due to the explicit time-dependent driving of the control parameters $\{\lambda\}$ of ε^{int} [19]. Hence, $d_t \langle \varepsilon \rangle = \sum_{\{N\}} \partial_t P_{\{N\}} \varepsilon^{\text{int}} + P_{\{N\}} \partial_t \varepsilon^{\text{int}}$ incorporates the stochastic work. The Gibbs entropy and the Gibbs EPR read [104]

$$s^{\text{gb}} = -\sum_{\{N\}} P_{\{N\}} \ln(P_{\{N\}}), \quad d_t s^{\text{gb}} = -\sum_{\{N\}} d_t P_{\{N\}} \ln(P_{\{N\}}). \quad (15)$$

Such that $s^{\text{gb}} = \langle s^{\text{state}} \rangle$ holds. The mean microscopic nonreciprocal EPR is

$$\langle \dot{\sigma}^{\text{nr}} \rangle = \sum_{\{\Delta_{\gamma\gamma'}^\#\}}^{[N]} j_{\gamma\gamma'}^\#(\{N\}) \Delta_{\gamma\gamma'}^\# f^{\text{nr}} + \sum_{\{\Delta_i^{\bar{D}\#}\}}^{[N]} j_i^{\bar{D}\#}(\{N\}) \Delta_i^{\bar{D}\#} f^{\text{nr}}. \quad (16)$$

The mean microscopic nonconservative EPR due to the chemical and self-propulsion driving is

$$\begin{aligned} \langle \dot{\sigma}^{\text{ch}} \rangle &= \sum_{\{\Delta_{\gamma\gamma'}^\#\}}^{[N]} j_{\gamma\gamma'}^\#(\{N\}) f_{\gamma\gamma'}^{\text{ch}}, \\ \langle \dot{\sigma}^{\text{sp}} \rangle &= \sum_{\{\Delta_i^{\bar{D}\#}\}}^{[N]} j_i^{\bar{D}\#}(\{N\}) \bar{D} \cdot \bar{f}_i^{\text{sp}}. \end{aligned} \quad (17)$$

Equation (14) is the second law of thermodynamics decomposed on the basis of the origin of the forces acting along the transitions. The first two terms are derived from conservative forces. Because their time-integrated EPR is given by the change of a functional defined for the lattice, it depends only on the initial and final states. In contrast, the remaining terms give the EPR due to the nonconservative forces, namely, nonreciprocal, self-propulsion, and chemical driving.

4. Orthogonal decomposition of the EPR

We define the Boltzmann probability distribution $P_{\{N\}}^\varepsilon(t) = e^{-\varepsilon + \psi_\varepsilon}$ with reference energy functional ε . We compute the free energy $\psi_\varepsilon = -\ln(\mathcal{Z}^\varepsilon)$ in terms of the partition function $\mathcal{Z}^\varepsilon = \sum_{\{N\}} e^{-\varepsilon}$. In the absence of nonconservative forces, i.e., $f_i^{\text{nr}}, \bar{f}_i^{\text{sp}}, f_{\gamma\gamma'}^{\text{ch}} = 0$, the system satisfies the Boltzmann distribution. We decompose Eq. (11) using $P_{\{N\}}^\varepsilon(t)$ to [107–109]

$$\varepsilon^{\text{stoch}} = \psi_\varepsilon + \ln \left(\frac{P_{\{N\}}(t)}{P_{\{N\}}^\varepsilon(t)} \right). \quad (18)$$

Comparing Eq. (18) with Eq. (11), analogously to Eq. (10), we define the reference state entropy $s_\varepsilon^{\text{state}}$,

$$s_\varepsilon^{\text{state}} = -\ln \left(\frac{P_{\{N\}}(t)}{P_{\{N\}}^\varepsilon(t)} \right). \quad (19)$$

$s_\varepsilon^{\text{state}}$ quantifies the thermodynamic distance of $P_{\{N\}}(t)$ from the reference Boltzmann probability $P_{\{N\}}^\varepsilon(t)$. The total thermodynamic distance between the probability distributions $P_{\{N\}}(t)$ and $P_{\{N\}}^\varepsilon(t)$ is given by the Kullback-Leibler (KL) divergence [107–114], $D_\varepsilon^{\text{KL}}(t) = \sum_{\{N\}} P_{\{N\}}(t) \ln(P_{\{N\}}(t)/P_{\{N\}}^\varepsilon(t))$, where, $-D_\varepsilon^{\text{KL}}(t) = \langle s_\varepsilon^{\text{state}} \rangle$ holds. Analogously to $d_t s^{\text{gb}}$, the rate of change of $D_\varepsilon^{\text{KL}}(t)$ is defined as

$$d_t D_\varepsilon^{\text{KL}}(t) = \sum_{\{N\}} d_t P_{\{N\}}(t) \ln \left(\frac{P_{\{N\}}(t)}{P_{\{N\}}^\varepsilon(t)} \right). \quad (20)$$

We introduce the orthogonal decomposition of $\langle \dot{\sigma}^{\text{ch}} \rangle$ and $\langle \dot{\sigma}^{\text{sp}} \rangle$,

$$\langle \dot{\sigma}^{\text{ch}} \rangle = \sum_{\{\Delta_{\gamma\gamma'}^\#\}}^{[N]} j_{\gamma\gamma'}^{\text{ch}} f_{\gamma\gamma'}^{\text{ch}}, \quad \langle \dot{\sigma}^{\text{sp}} \rangle = \sum_{\{\Delta_i^{\bar{D}\#}\}}^{[N]} j_i^{\text{sp}} \bar{D} \cdot \bar{f}_i^{\text{sp}}. \quad (21)$$

Here, $j_{\gamma\gamma'}^{\text{ch}}$ is the antisymmetric part of $j_{\gamma\gamma'}^\#(\{N\})$ under the adjoint transformation $f_{\gamma\gamma'}^{\text{ch}} \rightarrow -f_{\gamma\gamma'}^{\text{ch}}$. Similarly, j_i^{sp} is the antisymmetric part of $j_i^{\bar{D}\#}(\{N\})$ under the adjoint transformation $\bar{f}_i^{\text{sp}} \rightarrow -\bar{f}_i^{\text{sp}}$. We have dropped the # and $\{N\}$ index for

brevity. The exact expressions for $j_{\gamma\gamma'}^{\text{ch}}$ and j_i^{sp} read

$$j_{\gamma\gamma'}^{\text{ch}} = d_{\gamma\gamma'} \left(P_{\{N\}} N_{\gamma'}^{\#} e^{\epsilon_{\gamma'}^{\#}} + P_{\{N+\Delta_{\gamma\gamma'}\}} N_{\gamma} e^{\epsilon_{\gamma}^{\#}} \right) \sinh \left(\frac{f_{\gamma\gamma'}}{2} \right),$$

$$j_i^{\text{sp}} = d_i^{\text{D}} \left(P_{\{N\}} N_i^{\#} e^{\epsilon_i^{\#}} + P_{\{N+\Delta_i^{\text{D}\#}\}} N_i^{\text{D}\#} e^{\epsilon_i^{\text{D}\#}} \right) \sinh \left(\frac{\tilde{\text{D}} \cdot \tilde{f}_i^{\text{sp}}}{2} \right). \quad (22)$$

$j_{\gamma\gamma'}^{\text{ch}}$ and j_i^{sp} in Eq. (22) are the dissipative currents generated by nonconservative driving forces $f_{\gamma\gamma'}$ and $\tilde{f}_i^{\text{sp}}$, compared with the total reactive and diffusive currents $j_{\gamma\gamma'}^{\#}(\{N\})$ and $j_i^{\text{D}\#}(\{N\})$ in Eq. (17) that include currents generated by conservative forces. The total bidirectional current, commonly called traffic, is obtained by replacing $-$ with $+$ in Eq. (9), which also equivalently quantifies the scaled fluctuations resulting from the transitions. The form of Eq. (22) reveals that the amplitude of $j_{\gamma\gamma'}^{\text{ch}}$ and j_i^{sp} , and subsequently $\langle \dot{\sigma}^{\text{ch}} \rangle$ and $\langle \dot{\sigma}^{\text{sp}} \rangle$, is proportional to the traffic (scaled fluctuations) in the direction orthogonal to the external nonconservative driving force.

Utilizing Eqs. (18) to (21), the second law of thermodynamics Eq. (14) has the following orthogonal form:

$$\langle \dot{\sigma} \rangle = -d_t \psi_{\varepsilon} - d_t D_{\varepsilon}^{\text{KL}}(t) + \langle \dot{\sigma}^{\text{nr}} \rangle + \langle \dot{\sigma}^{\text{sp}} \rangle + \langle \dot{\sigma}^{\text{ch}} \rangle. \quad (23)$$

In Eq. (23), we have decomposed the EPR into its four linearly independent orthogonal contributions. First, $-d_t \psi_{\varepsilon}$ quantifies the rate of change of the free energy attributed to the external driving work required to change the control parameters $\{\lambda\}$ of ε . Second, $-d_t D_{\varepsilon}^{\text{KL}}(t)$ quantifies the EPR due to the relaxation toward the Boltzmann distribution $P_{\{N\}}^{\varepsilon}$, where $-d_t \psi_{\varepsilon}$ and $-d_t D_{\varepsilon}^{\text{KL}}(t)$ are physically interpreted as the boundary terms in the $\{\lambda\}$ and $P_{\{N\}}$ spaces, respectively. Third, $\langle \dot{\sigma}^{\text{nr}} \rangle$ quantifies the EPR due to the antisymmetric forces (and vorticity currents generated by them) between the nonreciprocal particles. Fourth, $\langle \dot{\sigma}^{\text{sp}} \rangle + \langle \dot{\sigma}^{\text{ch}} \rangle$ quantifies the EPR due to the external nonconservative forces along the transitions, a thermodynamic cost supported by an external chemical or self-propulsion reservoir, which generates the dissipative transition currents [115].

One identifies the nonadiabatic EPR $\langle \dot{\sigma}_{\varepsilon}^{\text{na}} \rangle = -d_t D_{\varepsilon}^{\text{KL}}(t)$ and the remaining terms as the housekeeping EPR $\langle \dot{\sigma}_{\varepsilon}^{\text{hs}} \rangle$. The orthogonal decomposition is more fundamental because it does not require the existence of a steady state because ε is well-defined for any dynamic state. The lower bound on the total entropy production (EP) is obtained using $\langle \dot{\sigma}_{\varepsilon}^{\text{na}} \rangle$ [111,114], and it reads $\langle \dot{\sigma}_{\varepsilon}^{\text{na}} \rangle \geq D_{\varepsilon}^{\text{KL}}(0) - D_{\varepsilon}^{\text{KL}}(\tau) \geq D_{\varepsilon}^{\text{KL}}(P(0)||P(\tau))$. Other notable consequences of the orthogonal decomposition have utilized different reference gauges [116–123]. Choosing $P_{\{N\}}^{\text{ss}}$ as the reference distribution, the nonadiabatic-housekeeping decomposition of $\langle \dot{\sigma} \rangle$ is detailed in Appendix.

D. Coarse-graining

We implement the Doi-Peliti coarse-graining (DPCG) procedure to obtain a coarse-grained description [124–137]. The DPCG procedure incorporates the fluctuations (discreteness of the microscopic particle nature) owing to the

second-quantization approach used. The technical details of the coarse-graining procedure and its physical implications are summarized in Ref. [92].

III. FLUCTUATING MACROSCOPIC DESCRIPTION

The set of coarse-grained mesostates is denoted by $\{N_i^{\#}\}$. Here, $\{N_i^{\#}\}$ is defined over the discrete (lattice) space for the mesoscopic description. Physically, it corresponds to lumping the microstates (particles) with the same dynamic and thermodynamic properties into a single extensive coarse-grained mesostate. It achieves the coarse-graining step from Figs. 1(a) to 1(b). We define $\Omega = N_{\text{tot}}/\mathcal{V}$, where N_{tot} and $\mathcal{V}$ are the total number of particles in the whole lattice and the volume of the lattice (number of lattice sites), respectively. Thus, Ω quantifies the mean number of particles per lattice site. $N_i^{\#}$ is $O(\Omega)$ and its fluctuations due to microscopic transitions are $O(1)$ [92,138].

To proceed further with the mesoscopic to macroscopic coarse-graining step, Figs. 1(b) to 1(c), we introduce the mesoscopic to macroscopic scaling $N_i^{\#} = \Omega \rho_i(\vec{x})$ between the macrostate $\rho_i(\vec{x})$ and the particle number mesostate $N_i^{\#}$. Therefore, ρ_i and its fluctuations due to microscopic transitions are $O(1)$ and $O(1/\Omega)$, respectively [92,138]. By construction, $\Omega \geq 1$, as $\Omega < 1$ contradicts the mesoscopic to macroscopic coarse-graining. The space field $\vec{x}$ tracks the lattice index, which we drop in our notations, and its meaning is presumed. Importantly, we use the macrostate convention consistent with the large-deviation theory [92,138], where an intensive variable is defined, and its fluctuations are suppressed with the large-deviation scaling parameter Ω . Thus, the notion of an intensive variable is defined in the context of fluctuations. The macrostate ρ_i does not necessarily coincide with the convention of the density used in hydrodynamics unless $\Omega = \mathcal{V}$ is chosen. This is fulfilled in the hydrodynamic limit, where the average number of particles per lattice site scales with the system volume. Hence, $N_i^{\#} = \mathcal{V} \rho_i$ holds. Importantly, this flexible definition of the macrostate ρ_i allows us to define and study the systems on intermediate observation scales between the mesoscopic and hydrodynamic descriptions. $\Omega = 1$ corresponds to the mesoscopic description that uses the particle number as the coarse-grained state. The thermodynamic limit of $\Omega \rightarrow \infty$ suppresses macrostate fluctuations and recovers the deterministic limit corresponding to the reaction-diffusion systems, Fig. 1(d).

A. Macroscopic Boltzmann weight

1. Energy functional and nonconservative forces

The macroscopic Boltzmann weight μ_i of the macrostate ρ_i is identified using the LDB [92]. μ_i is the macroscopic Boltzmann weight of the macrostate ρ_i , quantifying the total thermodynamic cost of placing a macrostate ρ_i . It consists of an ideal (entropic) part $\mu_i^{\text{id}} = \ln(\rho_i)$ and a reciprocal interaction (energetic) part $\mu_i^{\text{int}} = \sum_j V_{ij} \rho_j$, such that $\mu_i = \mu_i^{\text{id}} + \mu_i^{\text{int}}$. Here, $V_{ij} = \Omega(e^{\beta v_{ij}^{\text{r}} + \beta v_{ij}^{\text{nr}}} - 1)$ quantifies the contribution to μ_i due to the presence of ρ_j arising from the interaction between ρ_i and ρ_j . V_{ij} serves as the second Virial coefficient for the interacting nonreciprocal fields, such that $V_{ij} \neq V_{ji}$. $\mu_i^{\text{int}} < 0$ indicates attractive interactions experienced by

ρ_i , whereas $\mu_i^{\text{int}} > 0$ corresponds to repulsive interactions. Analogously, μ_i can be decomposed into reciprocal and nonreciprocal macroscopic Boltzmann weights:

$$\mu_i^r = \ln \rho_i + \sum_j V_{ij}^r \rho_j, \quad F_i^{\text{nr}} = \sum_j V_{ij}^{\text{nr}} \rho_j, \quad (24)$$

where V_{ij}^r and V_{ij}^{nr} denote the reciprocal and nonreciprocal interaction coefficients for ρ_i due to ρ_j , respectively:

$$V_{ij}^r = \Omega [\cosh(\beta v_{ij}^{\text{nr}}) e^{\beta v_{ij}^r} - 1], \quad V_{ij}^{\text{nr}} = \Omega \sinh(\beta v_{ij}^{\text{nr}}) e^{\beta v_{ij}^r}. \quad (25)$$

Importantly, $V_{ij}^r = V_{ji}^r$ and $V_{ij}^{\text{nr}} = -V_{ji}^{\text{nr}}$, as required by $v_{ij}^r = v_{ji}^r$ and $v_{ij}^{\text{nr}} = -v_{ji}^{\text{nr}}$. This ensures that the microscopic ‘orthogonal gauge fixing’ is scale-invariant, guaranteeing a unique and exact decomposition of reciprocal and nonreciprocal contributions at the coarse-grained macroscopic level. Alternative gauge choices would violate scale invariance and lead to physically incorrect decompositions of the EPR.

The nonlinear dependence of V_{ij}^r and V_{ij}^{nr} arises from incorporating the Poissonian statistics of the microscopic occupancy variables [92]. The reciprocal macroscopic weight μ_i^r is derived from the global macroscopic energy functional E , defined such that $\mu_i^r = \delta E / \delta \rho_i$:

$$E = \int_{\mathcal{V}} \left[\frac{1}{2} \sum_{i,j} V_{ij}^r \rho_i \rho_j + \sum_i \rho_i \ln \left(\frac{\rho_i}{e} \right) \right]. \quad (26)$$

E quantifies the total reciprocal interaction energy E^{int} and the macroscopic Boltzmann entropic term S^b , such that $E = E^{\text{int}} - S^b$. Consequently, $\mu_i = \mu_i^r + F_i^{\text{nr}}$ represents a macroscopic Helmholtz-Hodge decomposition of μ_i into symmetry-preserving (μ_i^r) and symmetry-violating (F_i^{nr}) components. The macroscopic nonconservative driving forces along reactive and diffusive transitions are related to their microscopic counterparts as $F_{\gamma\gamma'}^{\text{ch}} = f_{\gamma\gamma'}^{\text{ch}}$ and $\vec{F}_i^{\text{sp}} = l \vec{f}_i^{\text{sp}}$.

2. Equilibrium energy functional

The functional E is not the equilibrium energy functional E^{eq} . Setting $v_{ij}^{\text{nr}} = 0$, $\vec{f}_i^{\text{sp}} = 0$, and $f_{\gamma\gamma'}^{\text{ch}} = 0$ yields $E^{\text{eq}} = E(V_{ij}^{\text{eq}})$, where $V_{ij}^{\text{eq}} = \Omega[e^{\beta v_{ij}^r} - 1]$ is the second Virial coefficient for equilibrium reciprocal interactions. Notably, E satisfies $|E| \geq |E^{\text{eq}}|$, with equality only if $v_{ij}^{\text{nr}} = 0$ for all i, j . Unlike E^{eq} , the functional E incorporates nonreciprocal interactions and their effect on symmetric macrostate correlations through the $\cosh(\beta v_{ij}^{\text{nr}})$ term in Eq. (25). E is the macroscopic analog of Eq. (3). Analogously highlighted in section II A, E corresponds to the geometric decomposition of reciprocal interactions that generates the conservative reciprocal interactions through the Helmholtz-Hodge decomposition of μ_i ; therefore, E serves as a Lyapunov functional for nonreciprocal systems in the absence of external driving forces and vanishing nonreciprocal vorticity currents. The proof follows trivially from the decomposition of macroscopic mean EPR (36) and (41), the second law of thermodynamics, and vanishing EPR cost associated with external driving forces and vanishing nonreciprocal vorticity currents.

B. Dynamics

1. Local detailed balance

The macroscopic reactive and diffusive transitions are defined as $\Delta_{\gamma\gamma'} : \rho_{\gamma'} \rightarrow \rho_{\gamma}$ and $\Delta_i^{\vec{D}} : \rho_i \rightarrow \rho_i^{\vec{D}}$, respectively, where $\vec{D}$ quantifies the direction vector for the diffusive transition. Using the macroscopic transition probability measure from the DPFT [92], the macroscopic transition rates for the reactive ($\Delta_{\gamma\gamma'}$) and diffusive ($\Delta_i^{\vec{D}}$) transitions are

$$K_{\gamma\gamma'} = d_{\gamma\gamma'} e^{\mu_{\gamma'} + \frac{1}{2} F_{\gamma\gamma'}^{\text{ch}}}, \quad K_i^{\vec{D}} = \tilde{d}_i^{\vec{D}} e^{\mu_i + \frac{1}{2} \vec{D} \cdot \vec{F}_i^{\text{sp}}}. \quad (27)$$

Here, $\tilde{d}_i^{\vec{D}} = d_i l^2 \cosh(\vec{D} \cdot \vec{F}_i^{\text{sp}} / 2)$ is the macroscopic diffusion constant along $\vec{D}$, measured using the microscopic diffusive length scale l . The local detailed balance (LDB) constructed from Eq. (27) for the reactive ($\Delta_{\gamma\gamma'}$) and diffusive ($\Delta_i^{\vec{D}}$) transitions reads [92]:

$$\frac{K_{\gamma\gamma'}}{K_{\gamma'\gamma}} = e^{\mu_{\gamma'} - \mu_{\gamma} + F_{\gamma\gamma'}^{\text{ch}}}, \quad \frac{K_i^{\vec{D}}}{K_i^{(\vec{D})^{-1}}} = e^{\mu_i - \mu_i^{\vec{D}} + \vec{D} \cdot \vec{F}_i^{\text{sp}}}. \quad (28)$$

The LDB constrains the ratio of forward and backward transitions (a dynamic quantity) using μ_i (a thermodynamic quantity). For $\Delta_{\gamma\gamma'}$, we define the macroscopic affinity $A_{\gamma\gamma'} = \mu_{\gamma'} - \mu_{\gamma} + F_{\gamma\gamma'}^{\text{ch}}$ and its symmetric counterpart $U_{\gamma\gamma'} = \mu_{\gamma'} + \mu_{\gamma}$. Macroscopic and microscopic external chemical driving forces are related by $F_{\gamma\gamma'}^{\text{ch}} = f_{\gamma\gamma'}^{\text{ch}}$. Similarly, for $\Delta_i^{\vec{D}}$, we define $A_i^{\vec{D}} = \mu_i - \mu_i^{\vec{D}} + \vec{D} \cdot \vec{F}_i^{\text{sp}}$ and $U_i^{\vec{D}} = 2\mu_i$. Macroscopic and microscopic self-propulsion are related by $\vec{F}_i^{\text{sp}} = l \vec{f}_i^{\text{sp}}$. Equations (27) and (28) are the macroscopic analogs of the microscopic equations (4) and (5), respectively. Importantly, Eq. (27) provides a more fundamental formulation than Eq. (28), as it does not assume a timescale separation between different transition rates. Since Eq. (28) omits the time-symmetric part of the transition rates by taking the ratio of forward and backward transitions, it is analogous to the microscopic counterpart discussed previously.

2. Generalized macroscopic fluctuating dynamics

The deterministic evolution of ρ_i is given by the ‘most likely path’ of the Doi-Peliti action, namely, the ‘Instanton.’ The ‘dominant macrostate fluctuations’ are characterized by the local curvature of the Instanton [92]. The stochastic equation of motion (EOM) for ρ_i is [92]

$$\begin{aligned} \partial_t \rho_i = & -\vec{\nabla} \cdot \vec{J}_i^{\vec{D}} - \sum_{i \in \{\Delta_{\gamma\gamma'}\}} J_{\gamma\gamma'} + \vec{\nabla} \cdot (\sqrt{2B_i^{\vec{D}}} \tilde{\xi}_i^{\vec{D}}) \\ & + \sum_{i \in \{\Delta_{\gamma\gamma'}\}} \sqrt{2B_{\gamma\gamma'}^{\mathcal{R}}} \hat{\xi}_{\gamma\gamma'}^{\mathcal{R}}. \end{aligned} \quad (29)$$

We adopt the convention that outward transition currents from ρ_i ($-J_{\gamma\gamma'}$ and $-\vec{\nabla} \cdot \vec{J}_i^{\vec{D}}$) are represented with a negative sign. $\hat{\xi}_{\gamma\gamma'}^{\mathcal{R}}$ and $\tilde{\xi}_i^{\vec{D}}$ are standard Gaussian white noises with unit variance and zero mean. The set $\{\Delta_{\gamma\gamma'}\}$ denotes all macroscopic reactive transitions. The reactive and diffusive transition

currents are

$$\begin{aligned}\bar{J}_i^{\bar{D}} &= -D_i^{\bar{D}}(\{\rho\})\bar{\nabla}^{\bar{D}}\mu_i + \bar{J}_i^{sp}, \\ J_{\gamma\gamma'} &= 2D_{\gamma\gamma'}(\{\rho\})\sinh\left(\frac{A_{\gamma\gamma'}}{2}\right).\end{aligned}\quad (30)$$

Here, $\bar{\nabla}^{\bar{D}}$ is the gradient along $\bar{D}$, and similarly, the Laplacian along $\bar{D}$ is denoted by $\Delta^{\bar{D}}$. The macroscopic self-propulsion current is $\bar{J}_i^{sp} = 2d_i l e^{\mu_i} \sinh(\bar{D} \cdot \bar{F}_i^{sp}/2)$. Choosing the basis vector $\bar{D} = \{\parallel, \perp\}$ as parallel and perpendicular to $\bar{F}_i^{sp}$, the simplified form reads $-\bar{\nabla} \cdot \bar{J}_i^{\bar{D}} = \tilde{d}_i^{\parallel} \Delta^{\parallel} e^{\mu_i} + \tilde{d}_i^{\perp} \Delta^{\perp} e^{\mu_i} + \bar{\nabla}^{\parallel} \cdot \bar{J}_i^{sp}$. Here, $\tilde{d}_i^{\parallel} = d_i l^2 \cosh(F_i^{sp}/2)$ and $\tilde{d}_i^{\perp} = d_i l^2$, indicating that self-propulsion renormalizes the diffusion coefficients differently in directions parallel and perpendicular to the self-propulsion force. Without loss of generality, we consider the scaled macroscopic diffusion constants $\tilde{d}_i = d_i l^2$ or equivalently fix $l = 1$, using the diffusive length scale as the unit of spatial distance. In the continuum limit $l \rightarrow 0$, $\tilde{d}_i^{\parallel} = \tilde{d}_i^{\perp} = \lim_{l \rightarrow 0} d_i l^2$ and $J_i^{sp} = \tilde{d}_i^{\parallel} e^{\mu_i} f_i^{sp}$ [92], so the macrostates lose the directional dependence of the diffusion coefficients. The anisotropic diffusion coefficients, however, have been shown to generate novel phases [139–143], highlighting their importance.

We define the mobility for diffusive and reactive transitions as

$$D_i^{\bar{D}}(\{\rho\}) = \tilde{d}_i^{\bar{D}} e^{\mu_i}, \quad D_{\gamma\gamma'}(\{\rho\}) = d_{\gamma\gamma'} e^{(U_{\gamma\gamma'}/2)}. \quad (31)$$

Mobilities characterize the strength of macroscopic transition currents generated by the macroscopic transition affinities. They play the same role as diffusive mobility and transport coefficients for $\bar{J}_i^{\bar{D}}$ and $\bar{J}_i^{sp}$, respectively. For repulsive interactions experienced by ρ_i (i.e., $\mu_i > 0$), the amplitude of the transition mobilities increases exponentially, signifying escape from a thermodynamically unfavorable state. Similarly, the transition currents are exponentially suppressed for ρ_i experiencing attractive interactions ($\mu_i < 0$). The variances of the reactive and diffusive currents are [92]

$$\begin{aligned}2B_i^{\bar{D}}(\{\rho\}) &= \frac{2}{\Omega} D_i^{\bar{D}}(\{\rho\}), \\ 2B_{\gamma\gamma'}(\{\rho\}) &= \frac{2}{\Omega} D_{\gamma\gamma'}(\{\rho\}) \cosh\left(\frac{A_{\gamma\gamma'}}{2}\right).\end{aligned}\quad (32)$$

Equation (32) is the Einstein-Smoluchowski relation connecting current fluctuations to transition mobility, where Ω plays a role analogous to the inverse temperature β [92,138]. Importantly, the same mobility controls the mean current in Eq. (30) through a hyperbolic relation, which gives rise to the fluctuation-response relation.

We define the traffic as the sum of the magnitudes of the forward and backward currents [144–148]. For reactive and diffusive transitions, the traffic reads

$$T_{\gamma\gamma'} = 2D_{\gamma\gamma'}(\{\rho\}) \cosh\left(\frac{A_{\gamma\gamma'}}{2}\right), \quad T_i^{\bar{D}} = 2D_i(\{\rho\}). \quad (33)$$

Traffic in Eq. (33) is related to the scaled variance of the transition currents in Eq. (32): $T_{\gamma\gamma'} = 2\Omega B_{\gamma\gamma'}(\{\rho\})$ and $T_i^{\bar{D}} = 2\Omega B_i^{\bar{D}}(\{\rho\})$.

The mean current and traffic [Eqs. (30) and (33)] exhibit $\sinh()$ and $\cosh()$ forms due to their thermodynamically consistent exponential parametrization in the affinity A . However, they have a fundamental thermodynamic meaning independent of exponential parametrization: namely, the mean current and traffic correspond to the time-antisymmetric and time-symmetric components of the nonequilibrium currents [94,95]. Consequently, they characterize the thermodynamic length and the dynamical activity for the transition, which physically correspond to the total directional transition current and the inverse timescale for the transition [94,95]. The shortcomings of the Glauber-dynamics and Metropolis-dynamics discussed in Sec. II B are better understood with Eq. (29), where different transitions contribute with different ‘diffusivity’ or ‘variance,’ quantified by the traffic. This exactly preserves the different timescales associated with different transitions that contribute to the macrostate ρ_i . Therefore, due to the thermodynamically consistent exact coarse-graining procedure [92], Eq. (29) serves as a foundational cornerstone for future studies to investigate macroscopic stochastic thermodynamics without strong assumptions on timescale separation.

Although Eq. (29) is written in the continuum macroscopic limit $l \rightarrow 0$, equivalent analogs exist for systems with finite l , e.g., lattice gas models with $l = 1$. In such cases, the discrete analogs of the Laplacian and gradient operators replace the continuum-space counterparts, similar to reactive transition dynamics [92].

C. Thermodynamics

1. Macroscopic thermodynamics

The total mean macroscopic EPR is $\langle \dot{\Sigma} \rangle = \langle \dot{\Sigma}^{\mathcal{D}} \rangle + \langle \dot{\Sigma}^{\mathcal{R}} \rangle + d_t S^{\text{gb}}$. It is decomposed into the reactive mean macroscopic bulk EPR ($\langle \dot{\Sigma}^{\mathcal{R}} \rangle$), the diffusive mean macroscopic bulk EPR ($\langle \dot{\Sigma}^{\mathcal{D}} \rangle$), and the boundary term, i.e., the Gibbs EPR ($d_t S^{\text{gb}}$):

$$\begin{aligned}\langle \dot{\Sigma}^{\mathcal{D}} \rangle &= \int_{\mathcal{V}} \sum_i \langle \bar{J}_i^{\bar{D}} \cdot (-\bar{\nabla}^{\bar{D}}\mu_i + \bar{F}_i^{sp}) \rangle, \\ \langle \dot{\Sigma}^{\mathcal{R}} \rangle &= \int_{\mathcal{V}} \sum_{\{\Delta_{\gamma\gamma'}\}} \langle J_{\gamma\gamma'}(\mu_{\gamma'} - \mu_{\gamma} + F_{\gamma\gamma'}^{\text{ch}}) \rangle, \\ d_t S^{\text{gb}} &= - \int \mathbb{D}[\{\rho\}] d_t \mathcal{P}[\{\rho\}] \ln(\mathcal{P}[\{\rho\}]).\end{aligned}\quad (34)$$

Here, $\mathbb{D}[\{\rho\}]$ denotes the path integral over the macrostate space, and $\mathcal{P}[\{\rho\}]$ is the probability distribution for the macrostate. The mean EPR in Eq. (34) has the form of a transition current multiplied by the corresponding transition affinity. The transition affinity is obtained using the macroscopic local detailed balance (LDB) condition in Eq. (28). The macroscopic Gibbs entropy is defined as $S^{\text{gb}} = - \int \mathbb{D}[\{\rho\}] \mathcal{P}[\{\rho\}] \ln(\mathcal{P}[\{\rho\}])$, which is related to the macroscopic state entropy S^{state} , defined as [18,20]:

$$S^{\text{state}} = - \ln(\mathcal{P}[\{\rho\}]), \quad (35)$$

so that $\langle S^{\text{state}} \rangle = S^{\text{gb}}$ holds.

2. Conservative and nonconservative decomposition of the total macroscopic EPR

$\langle \dot{\Sigma} \rangle$ is decomposed into four contributions, analogous to Eq. (14):

$$\langle \dot{\Sigma} \rangle = -d_t \langle E \rangle + d_t S^{\text{gb}} + \langle \dot{\Sigma}^{\text{nr}} \rangle + \langle \dot{\Sigma}^{\text{ch}} \rangle + \langle \dot{\Sigma}^{\text{sp}} \rangle. \quad (36)$$

Equation (36) represents the macroscopic second law of thermodynamics. The term $-d_t \langle E \rangle + d_t S^{\text{gb}}$ corresponds to the conservative EPR $\langle \dot{\Sigma}^c \rangle$, so Σ^c depends only on the initial and final states. The exact expressions for $\langle \dot{\Sigma}^{\text{ch}} \rangle$ and $\langle \dot{\Sigma}^{\text{sp}} \rangle$ are

$$\langle \dot{\Sigma}^{\text{ch}} \rangle = \int_{\mathcal{V}} \sum_{\{\Delta_{\gamma\gamma'}\}} \langle J_{\gamma\gamma'} \rangle F_{\gamma\gamma'}^{\text{ch}}, \quad \langle \dot{\Sigma}^{\text{sp}} \rangle = \int_{\mathcal{V}} \sum_i \langle \vec{J}_i^{\text{D}} \rangle \cdot \vec{F}_i^{\text{sp}}. \quad (37)$$

The nonreciprocal EPR $\langle \dot{\Sigma}^{\text{nr}} \rangle$ consists of reactive $\langle \dot{\Sigma}_{\mathcal{R}}^{\text{nr}} \rangle = \int_{\mathcal{V}} \sum_{\{\gamma\gamma'\}} \langle J_{\gamma\gamma'} \rangle (F_{\gamma'}^{\text{nr}} - F_{\gamma}^{\text{nr}})$ and diffusive contributions $\langle \dot{\Sigma}_{\mathcal{D}}^{\text{nr}} \rangle = -\int_{\mathcal{V}} \sum_i \langle \vec{J}_i^{\text{D}} \rangle \cdot \vec{\nabla} F_i^{\text{nr}}$, such that $\dot{\Sigma}^{\text{nr}} = \dot{\Sigma}_{\mathcal{R}}^{\text{nr}} + \dot{\Sigma}_{\mathcal{D}}^{\text{nr}}$. Using the EOM (29) leads to $\dot{\Sigma}^{\text{nr}} = -\sum_i \int_{\mathcal{V}} F_i^{\text{nr}} \partial_t \rho_i$. With $V_{ij}^{\text{nr}} = -V_{ji}^{\text{nr}}$, this further simplifies to

$$\langle \dot{\Sigma}^{\text{nr}} \rangle = \int_{\mathcal{V}} \sum_{i,j} V_{ij}^{\text{nr}} \langle \omega_{ij} \rangle. \quad (38)$$

Here, $\omega_{ij} = \rho_i \partial_t \rho_j - \rho_j \partial_t \rho_i$ defines the macroscopic vorticity between ρ_i and ρ_j . $\langle \Sigma^{\text{nr}} \rangle$ corresponds to the sustained vorticity currents between the macrostates generated by the macroscopic nonreciprocal forces. Importantly, the vorticity currents are defined in the macrostate space, in contrast with the dissipative currents defined in the transition space. Moreover, vorticity currents are defined using two macrostates, unlike chemical reaction networks, which require a transition cycle of three states to formulate a dissipative current [1]. From a fundamental perspective, this reveals a sophisticated mechanism of nonequilibriumness in nonreciprocal systems, with no equivalent counterparts in the known literature.

3. Nonreciprocal phase transitions are dynamical phase transitions

The dynamics of nonreciprocal systems exhibit dynamic phases due to $\mathcal{PT}$ symmetry breaking [54], for instance, traveling waves and temporal oscillations. Here, we focus on nonreciprocal phase transitions that exhibit a transition from a static to a dynamic phase. A static phase is defined as the $\mathcal{PT}$ symmetry-preserving phase; in contrast, the $\mathcal{PT}$ symmetry is broken in the dynamic phase. We consider a wave solution $\rho_i = \rho_i^{\text{avg}} + \phi_i^{\text{ow}} \cos(\theta_i^{\text{ow}} - v^{\text{ow}} t) + \phi_i^{\text{tw}} \cos(x - v^{\text{tw}} t + \theta_i^{\text{tw}})$. Here, ρ_i^{avg} is the time-integrated and space-integrated average value of ρ_i , and $w = \{tw, ow\}$ denotes the traveling wave and the temporal oscillating wave, respectively. The ϕ_i^{ow} and ϕ_i^{tw} characterize the wave amplitudes. Similarly, v^{ow} and v^{tw} are the wave velocities. Integrating Eq. (38) over the periodic space and oscillation time period τ^{ow} , the nonreciprocal EP for the system reads

$$\langle \Sigma^{\text{nr}} \rangle = \mathcal{V} \tau^{\text{ow}} \sum_{i,j,w} V_{ij}^{\text{nr}} v^w \phi_i^w \phi_j^w \sin(\theta_j^w - \theta_i^w). \quad (39)$$

TABLE I. Scaling of macroscopic mean EP as a function of the observation time τ .

Phase	Σ^c	Σ^{nr}	Σ^{sp}	Σ^{ch}
Static	$O(1)$	$O(1)$	$O(\tau)$	$O(\tau)$
Dynamic	$O(1)$	$O(\tau)$	$O(\tau)$	$O(\tau)$

Equation (39) quantifies the nonreciprocal EP density over the total volume and time period. It has the physical meaning of the kinetic energy for a wave, where the mass is given by v^w , V_{ij}^{nr} and $\sin(\theta_j^w - \theta_i^w)$, and the velocity is given by the wave amplitudes ϕ_i^w . For $V_{ij}^{\text{nr}} \geq 0$, it physically implies that ρ_i repels ρ_j while ρ_j is attracted to ρ_i , thus $\theta_j^{\text{tw}} \geq \theta_i^{\text{tw}}$ and $\theta_j^{\text{ow}} \geq \theta_i^{\text{ow}}$, which further implies $\langle \Sigma^{\text{nr}} \rangle \geq 0$. Importantly, the out-of-phase state $\theta_j^{\text{tw}} - \theta_i^{\text{tw}} = \pi/2$ maximizes $\langle \Sigma^{\text{nr}} \rangle$, while the in-phase state $\theta_j^{\text{tw}} = \theta_i^{\text{tw}}$ minimizes $\langle \Sigma^{\text{nr}} \rangle$. The in-phase to out-of-phase steady-state transition has been a key motif of nonreciprocal systems [54]. Hence, it signifies that the transition from the in-phase to out-of-phase steady-state is equivalent to switching from the minimum to the maximum nonreciprocal EPR for steady-state selection. Moreover, $\langle \Sigma^{\text{nr}} \rangle$ for out-of-phase oscillation scales with the observation time τ . This signifies a constant thermodynamic dissipation cost required to sustain the vorticity currents between the macrostates. Physically, it connects the dynamical phase transition to an analogy with equilibrium phase transitions, where the EPR for the nonequilibrium phase transition is analogous to the free energy for the equilibrium phase transition [149]. It is characterized by different scaling regimes for $\langle \Sigma^{\text{nr}} \rangle$. The dissipative nature of the vorticity currents is realized only in the dynamic phase. Importantly, nonreciprocal phase transitions are equivalent to dynamical phase transitions and are characterized by nonanalytical behavior of $\langle \Sigma^{\text{nr}} \rangle$.

The scaling of different contributions of Σ with the observation time τ is summarized in Table I. The self-propulsion and chemical driving EP follow scaling $O(\tau)$ for dissipative currents [94,95]. However, a discontinuity (or a kink) in the self-propulsion or chemically driven EPR can be observed and is associated with a dynamical phase transition [149–152]. The nonreciprocal phase transitions are characterized by different scaling regimes for Σ^{nr} , which correspond to sustaining temporal or spatial vorticity currents.

4. Orthogonal decomposition of the EPR

We define the macroscopic relative state entropy:

$$S_E^{\text{state}} = -\ln \left(\frac{\mathcal{P}[\{\rho\}]}{\mathcal{P}^E[\{\rho\}]} \right). \quad (40)$$

Using the definition in Eq. (40), $-d_t \langle E \rangle + d_t S^{\text{gb}}$ in Eq. (36) is further decomposed into the orthogonal form [109]:

$$\langle \dot{\Sigma} \rangle = -d_t \psi_E - d_t D_E^{\text{KL}}(\mathcal{P}[\{\rho\}]) + \langle \dot{\Sigma}^{\text{nr}} \rangle + \langle \dot{\Sigma}^{\text{ch}} \rangle + \langle \dot{\Sigma}^{\text{sp}} \rangle. \quad (41)$$

ψ_E is the macroscopic free energy defined as $\psi_E = -\ln(\mathcal{Z}^E)$. Here, $\mathcal{Z}^E = \int \mathbb{D}[\{\rho\}] e^{-E}$ is the partition function, and the macroscopic Boltzmann distribution is $\mathcal{P}^E[\{\rho\}] = e^{-E + \psi_E}$. It holds that $\langle S_E^{\text{state}} \rangle = -D_E^{\text{KL}}(\mathcal{P}[\{\rho\}])$. Using the symmetry

of the nonconservative external driving forces $F_{\gamma\gamma'}$ and $\vec{F}_i^{sp}$, namely, the orthogonal decomposition of the transition affinities, Eq. (37) is reduced to:

$$\langle \dot{\Sigma}^{\text{ch}} \rangle = \int_{\mathcal{V}} \sum_{\{\Delta_{\gamma\gamma'}\}} \langle J_{\gamma\gamma'}^{\text{ch}} \rangle F_{\gamma\gamma'}^{\text{ch}}, \quad \langle \dot{\Sigma}^{sp} \rangle = \int_{\mathcal{V}} \sum_i \langle \vec{J}_i^{sp} \rangle \cdot \vec{F}_i^{sp}. \quad (42)$$

Here, $\vec{J}_i^{sp} = 2d_i e^{\mu_i} \sinh(\vec{F}_i^{sp}/2) = T_i^{\perp} \sinh(\vec{F}_i^{sp}/2)$ is the antisymmetric part of $\vec{J}_i^{\vec{D}}$ under the adjoint transformation $\vec{F}_i^{sp} \rightarrow -\vec{F}_i^{sp}$. Similarly, $J_{\gamma\gamma'}^{\text{ch}} = d_{\gamma\gamma'}(e^{\mu_{\gamma}} + e^{\mu_{\gamma'}}) \sinh(F_{\gamma\gamma'}/2) = T_{\gamma\gamma'}^{\perp} \sinh(F_{\gamma\gamma'}/2)$ is the antisymmetric part of $J_{\gamma\gamma'}$ under the adjoint transformation $F_{\gamma\gamma'}^{\text{ch}} \rightarrow -F_{\gamma\gamma'}^{\text{ch}}$. Importantly, $T_i^{\perp}$ and $T_{\gamma\gamma'}^{\perp}$ are the scaled (by Ω) variances of currents [Eq. (32)] in the direction orthogonal to the external driving, which are obtained by plugging in $\vec{F}_i^{sp} = 0$ and $F_{\gamma\gamma'}^{\text{ch}} = 0$, respectively. Thus, in Eq. (42), the nonconservative EPR depends on a nonlinear function of the external driving force and the current variance in the direction orthogonal to the driving. Importantly, $\langle \dot{\Sigma}^{\text{ch}} \rangle$ and $\langle \dot{\Sigma}^{sp} \rangle$ depend on e^{μ_i} and are proportional to the macrostate mobility.

Equation (41) formulates the orthogonal decomposition of $\langle \dot{\Sigma} \rangle$. It decomposes $\langle \dot{\Sigma} \rangle$ into four linearly independent components. First, $-d_i \psi_E$ quantifies the rate of change of the free energy due to the external work needed to drive E through the change in control parameters. Second, $-d_i D_E^{\text{KL}}(\mathcal{P}[\{\rho\}])$ quantifies the EPR due to the relaxation of E . Third, $\langle \dot{\Sigma}^{\text{nr}} \rangle$ is the nonreciprocal EPR, whose case-specific simplifications for the $\mathcal{PT}$ preserving-breaking phases have been discussed before. Fourth, $\langle \dot{\Sigma}^{\text{ch}} \rangle + \langle \dot{\Sigma}^{sp} \rangle$ quantifies the EPR due to the nonconservative forces along the transitions, namely, the self-propulsion and chemical driving for the diffusive and reactive systems, respectively.

The orthogonal decomposition for dissipation functions that are quadratic in the driving affinity, and thus a linear relationship between the current and the affinity, has been proven and rigorously studied [93,153–155]. In contrast, the nonlinear relation between $J_{\gamma\gamma'}^{\text{ch}}$ and $F_{\gamma\gamma'}^{\text{ch}}$ (or $\vec{J}_i^{sp}$ and $\vec{F}_i^{sp}$) leads to nonquadratic dissipation functions in Eq. (42). It gives exact and tighter bounds on the EPR. A more rigorous proof of the orthogonal decomposition for nonquadratic dissipation functions has been derived in Refs. [156–161], and its implications have been studied in Refs. [162–169]. The proof relies on the dynamical large deviation approach [117,170,171]. In Ref. [92], we show that the large deviation functional of the nonreciprocal systems is the same as the one used in Refs. [156–161]. This rigorously justifies the orthogonal decomposition for nonreciprocal systems. In contrast with previous works, the novelty of our approach lies in the proposal of orthogonal decomposition in state space (μ_i and $\epsilon_i^{\#}$) and transition-space ($F_{\gamma\gamma'}^{\text{ch}}$, $\vec{F}_i^{sp}$ and $f_{\gamma\gamma'}^{\text{ch}}$, $\vec{f}_i^{sp}$) for both microscopic and macroscopic systems. The antisymmetric part in state space and transition space gives $\langle \dot{\Sigma}^{\text{nr}} \rangle$ and $\langle \dot{\Sigma}^{\text{ch}} \rangle$, $\langle \dot{\Sigma}^{sp} \rangle$, respectively. The symmetric part gives the rate of change of the macroscopic stochastic Massieu potential $-d_i \psi_E - d_i D_E^{\text{KL}}(\mathcal{P}[\{\rho\}])$. Our formulation reveals similarities and differences in the underlying thermodynamic geometrical structure between the nonreciprocal and reciprocal systems. Importantly, the orthogonal decomposition of the

thermodynamic cost in Eq. (41) is completely determined by physically measurable quantities, in particular, external driving forces and the corresponding current means and variances.

Importantly, the macroscopic self-propulsion EPR with $l = 1$, $\langle \dot{\Sigma}_i^{sp} \rangle = 2d_i^{\parallel} F_i^{sp} \tanh(|F_i^{sp}|/2)$, is bounded below by the continuum-space macroscopic self-propulsion EPR $\dot{\Sigma}_i^{sp} = \vec{d}_i^{\parallel} (f_i^{sp})^2$ in the continuum limit $l \rightarrow 0$ [92]. This highlights the importance of the observation length scale for the coarse-grained macroscopic description. The macroscopic continuum description oversimplifies the coarse-grained description beyond the inherent diffusive length scale of the system, resulting in an underestimation of thermodynamic dissipation due to nonconservative self-propulsion forces [74–76]. This underestimation arises from the quadratic dissipation function in the continuum limit. Correctly identifying the discreteness or finiteness of the diffusive length scale restores the exact microscopic dissipation at the macroscale.

5. Temporal cross-correlations between macrostates and relaxation EP

We define the temporal correlation $C_{ij}(t, \Delta t) = \rho_i(t + \Delta t) \rho_j(t)$ between macrostates and their antisymmetric $C_{ij}^a(t, \Delta t) = C_{ji}(t, \Delta t) - C_{ij}(t, \Delta t)$ and symmetric $C_{ij}^s(t, \Delta t) = C_{ji}(t, \Delta t) + C_{ij}(t, \Delta t)$ decomposition [172]. It can be trivially verified that $C_{ij}^a(t, \Delta t) = \Delta t \omega_{ij}$ for small Δt . Integrating Eq. (38) from an initial time t_i to a final time t_f , with $\tau = t_f - t_i$, leads to the following expression for the relaxation process:

$$\begin{aligned} \langle \Sigma^{\text{nr}} \rangle &= \int_{\mathcal{V}} \sum_{\{i>j\}} V_{ij}^{\text{nr}} (\langle C_{ij}^a(t_i, \tau) \rangle - \langle C_{ij}^a(t_i, 0) \rangle), \\ \langle E^{\text{int}} \rangle &= \frac{1}{2} \int_{\mathcal{V}} \sum_{\{i,j\}} V_{ij}^r (\langle C_{ij}^s(t_i, \tau) \rangle - \langle C_{ij}^s(t_i, 0) \rangle). \end{aligned} \quad (43)$$

Equation (43) relates the EPR due to reciprocal $\langle E^{\text{int}} \rangle$ and nonreciprocal $\langle \dot{\Sigma}^{\text{nr}} \rangle$ interactions between the macrostates with the symmetric and antisymmetric temporal correlations between the initial and final states. $\langle C_{ij}^s(t_i, \tau) \rangle$ and $\langle C_{ij}^a(t_i, \tau) \rangle$ are convenient to obtain experimentally. Using Eq. (43) in Eq. (41) yields a tighter bound on $\langle \dot{\Sigma} \rangle$ by using the relaxation of $C_{ij}^s(t_i, \tau)$ and $C_{ij}^a(t_i, \tau)$ from the initial state to the final state. This bound is similar to the thermodynamic uncertainty relation (TUR) but is obtained using macrostate correlations instead of the precision of the transition currents [173]. Hence, it should be compared with Refs. [174–180]. $-\langle \dot{E}^{\text{int}} \rangle > 0$ ensures that attractive (repulsive) reciprocal interactions $V_{ij}^r < 0$ ($V_{ij}^r > 0$) increase (decrease) the symmetric temporal macrostate correlation $\langle C_{ij}^s(t_i, \tau) \rangle$ during the relaxation process. Similarly, $\langle \dot{\Sigma}^{\text{nr}} \rangle > 0$ implies that $\langle C_{ij}^a(t_i, \tau) \rangle$ increases for $V_{ij}^{\text{nr}} > 0$.

Importantly, this highlights the physical correctness of ‘the orthogonal gauge’. The choice of any other gauge would incorrectly assign a part of the symmetric macrostate correlations to Σ^{nr} , which is physically contradictory. Consequently, E^{int} and Σ^{nr} are not linearly independent for other gauges, and a subsequent redefinition of the linearly independent contributions to the EPR leads to a formulation equivalent to the orthogonal gauge fixing.

TABLE II. The summary of the impact and importance of fluctuations across different coarse-grained descriptions.

Level of description	Mesoscopic $\Omega = 1$	Macroscopic $\Omega = \mathcal{V} \gg 1$	Deterministic $\Omega = \mathcal{V} = \infty$
State	Fluctuating mean particle number	Fluctuating mean particle density	Deterministic mean particle density
Occupancy noise	Poissonian Eq. (25) [92]	Gaussian corrections around the mean field	Vanishes, recovering the mean-field limit
Transition noise	Poissonian and $O(1)$ [94,95]	Gaussian and $O(1/\Omega)$	Vanishes

D. Mesoscopic, macroscopic, and deterministic limits

The parameter Ω dictates the scale of the coarse-grained description of the system. It consists of three important limits: the mesoscopic, macroscopic, and deterministic limits, characterized respectively by Ω being one, large, and infinite. Importantly, the intensive scaling of the microscopic Boltzmann weight $\epsilon_i^\#$ requires the constraint $v_{ij}^r, v_{ij}^{nr} \propto 1/\Omega$. A Taylor-series expansion of Eq. (25) in v_{ij}^r, v_{ij}^{nr} (or equivalently in $1/\Omega$) leads to the macroscopic interaction coefficients $V_{ij}^r = \beta\Omega v_{ij}^r + \frac{1}{2}\Omega\beta^2[(v_{ij}^r)^2 + (v_{ij}^{nr})^2] + O(\Omega\beta^3 v_{ij}^3)$ and $V_{ij}^{nr} = \beta\Omega v_{ij}^{nr} + \Omega\beta^2 v_{ij}^{nr} v_{ij}^r + O(\Omega\beta^3 v_{ij}^3)$. The macroscopic interaction coefficients satisfy the gauge-fixing conditions $V_{ij}^r = V_{ji}^r$ and $V_{ij}^{nr} = -V_{ji}^{nr}$. Taking the limit $\Omega \rightarrow \infty$ yields the deterministic limit, $\bar{V}_{ij}^r = \lim_{\Omega \rightarrow \infty} \beta\Omega v_{ij}^r$ and $\bar{V}_{ij}^{nr} = \lim_{\Omega \rightarrow \infty} \beta\Omega v_{ij}^{nr}$. Throughout this paper, we focus on the macroscopic coarse-grained description, although the system-specific scale Ω should be imposed depending on the physical context. In particular, when the mean number of particles per lattice site is small, a mesoscopic description is a better alternative for coarse-grained physical analysis.

Table II summarizes the implications of fluctuations at different observation scales: the mesoscopic, macroscopic, and deterministic scales. The relevant coarse-grained states are the number of particles at a lattice site, the particle density at a lattice point, and the particle density at a lattice point, respectively. The suitable physical models corresponding to these scales are the Lattice Gas Models (LGMs), Macroscopic Fluctuation Theory (MFT), and Chemical Reaction Networks (CRNs). CRNs are a special case where $\Omega = \mathcal{V}$. While structurally similar, these descriptions differ significantly in other physical aspects, particularly concerning the nature of the noise effects.

Noise plays a key role through two mechanisms: particle occupancy and the transitions between them. Equation (25) highlights that Poissonian occupancy noise renormalizes the mesoscopic V_{ij} , leading to a nonlinear dependence on v_{ij} . It is crucial for correctly predicting the microscopic phase diagram using coarse-grained mesoscopic EOMs [92]. Moreover, the mean EPR correctly incorporates the microscopic noise effects of occupancy. In comparison, the Gaussian-Langevin approximation (29) of the mesoscopic Poissonian transition noise is sufficient due to the van Kampen closure approximation and the correct identification of the fluctuation-response relation, as discussed in the subsequent section. The nonquadratic dissipation function leads to a nonquadratic Hamilton-Jacobi equation [181–185], and Eq. (29) incorpo-

rates this mesoscopic effect despite the Gaussian-Langevin formulation. A more systematic analysis of mesoscopic Poissonian transition fluctuations is detailed in Refs. [94,95]. Importantly, we have not obtained the macroscopic EPR using the Langevin equation, which avoids the underestimation of the macroscopic EPR near equilibrium [94,95]. In particular, the correct identification of effective transition affinities using the mean and fluctuations of the transition currents resolves this issue [94,95].

IV. THERMODYNAMIC RELATIONS

Here, we outline different thermodynamic relations using the macroscopic description. However, one could also use the microscopic description, which is more directly relatable in Stochastic Thermodynamics. We deliberately utilize the macroscopic description to extend and exhibit the applicability of Stochastic Thermodynamics to interacting many-body systems.

A. Nonreciprocal Onsager relations

In this section, we demonstrate the nonreciprocal Onsager relation [186,187]. Assuming vanishing external driving, $\bar{F}_i^{sp} = 0$ and $F_{\gamma\gamma'}^{\text{ch}} = 0$, $J_i^{\bar{\mathcal{D}}} = 0$ in Eq. (30) reads

$$\bar{J}_i^{\bar{\mathcal{D}}} = - \sum_j (D_{ij}^r(\{\rho\}) + D_{ij}^{nr}(\{\rho\})) \bar{\nabla}^{\bar{\mathcal{D}}} \rho_j, \quad (44)$$

where $D_{ij}^r(\{\rho\}) = D_i^{\bar{\mathcal{D}}}(\{\rho\})(\beta V_{ij}^r + \delta_{ij}/\rho_i)$ and $D_{ij}^{nr}(\{\rho\}) = \beta D_i(\{\rho\})V_{ij}^{nr}$. The term $D_{ij}^r(\{\rho\})$ satisfies Onsager's reciprocal relation [186,187], since $V_{ij}^r = \partial^2 E / \partial \rho_j \partial \rho_i = \partial^2 E / \partial \rho_i \partial \rho_j$. In contrast, $D_{ij}^{nr}(\{\rho\})$ satisfies Onsager's antireciprocal relation, due to $V_{ij}^{nr} = -V_{ji}^{nr}$, or equivalently $\partial F_i^{nr} / \partial \rho_j = -\partial F_j^{nr} / \partial \rho_i$. We introduce the mobility, entropic, symmetric, and antisymmetric interaction matrices $\mathbb{D}$, $\mathbb{S}$, $\mathbb{D}^r$, and $\mathbb{D}^{nr}$, with the ij th element of the matrices being $\delta_{ij} D_i(\{\rho\})$, δ_{ij}/ρ_i , D_{ij}^r , and D_{ij}^{nr} , respectively, and let $\nabla \rho$ denote the column vector of gradients.

Thus, $\mathbf{J}^{\bar{\mathcal{D}}} = (\mathbb{D}^r + \mathbb{D}^{nr}) \nabla \rho = \mathbb{D}[\mathbb{S} + \beta(\mathbb{V}^r + \mathbb{V}^{nr})] \nabla \rho$ with the corresponding $\mathbf{F}^{\bar{\mathcal{D}}} = (\mathbb{F}^r + \mathbb{F}^{nr}) \nabla \rho = \mathbb{D}^{-1}(\mathbb{D}^r + \mathbb{D}^{nr}) \nabla \rho$. Hence, $\langle \dot{\Sigma}^{\bar{\mathcal{D}}} \rangle = \langle \mathbf{F}^{\bar{\mathcal{D}}} \cdot \mathbf{J}^{\bar{\mathcal{D}}} \rangle = \nabla \rho^T (\mathbb{D}^r + \mathbb{D}^{nr})^T \mathbb{D}^{-1} (\mathbb{D}^r + \mathbb{D}^{nr}) \nabla \rho$. The square root of $\mathbb{D}^{-1}$, $\mathbb{D}^{-\frac{1}{2}}$, is also a diagonal matrix satisfying $\mathbb{D}^{-\frac{1}{2}} = \mathbb{D}^{-\frac{1}{2}T}$, which reduces $\langle \dot{\Sigma}^{\bar{\mathcal{D}}} \rangle = \|\mathbb{D}^r + \mathbb{D}^{nr}\|^2$ to the norm obtained using $\mathbb{D}^{-1}$. In addition, the symmetric

and skew-symmetric matrices satisfy $\mathbb{D}^{rT} = \mathbb{D}^r$ and $\mathbb{D}^{nrT} = -\mathbb{D}^{nr}$, such that $\langle \dot{\Sigma}^{\mathcal{D}} \rangle = \|\mathbb{D}^r\|^2 + \|\mathbb{D}^{nr}\|^2$. Thus, $\langle \dot{\Sigma}^{\mathcal{D}} \rangle = \langle \nabla \rho^T (\mathbb{D}^{rT} \mathbb{D}^{-1} \mathbb{D}^r + \mathbb{D}^{nrT} \mathbb{D}^{-1} \mathbb{D}^{nr}) \nabla \rho \rangle$, or equivalently, $\langle \dot{\Sigma}^{\mathcal{D}} \rangle = \langle \nabla \rho^T (\mathbb{F}^{rT} \mathbb{D}^T \mathbb{F}^r + \mathbb{F}^{nrT} \mathbb{D}^T \mathbb{F}^{nr}) \nabla \rho \rangle$. The term $\nabla \rho^T \mathbb{D}^{rT} \mathbb{D}^{-1} \mathbb{D}^r \nabla \rho$ relates the symmetric response coefficients of $\mathbf{J}^{\bar{\mathcal{D}}}$ to the mean EPR, analogous to Onsager's reciprocal relation near equilibrium [186–188], while $\nabla \rho^T \mathbb{D}^{nrT} \mathbb{D}^{-1} \mathbb{D}^{nr} \nabla \rho$ relates the antisymmetric response coefficients to the mean EPR, representing the Onsager nonreciprocal relation. By construction, the norm is positive, leading to two independently positive terms for $\langle \dot{\Sigma}^{\mathcal{D}} \rangle$, interpretable as an orthogonal decomposition for nonreciprocal systems.

Using $\rho = \rho^{ss} + \delta \rho$ and/or $\mathbf{J}^{\bar{\mathcal{D}}} = \mathbf{J}_{ss}^{\bar{\mathcal{D}}} + \delta \mathbf{J}^{\bar{\mathcal{D}}}$, and exploiting the symmetry of fluctuations around the steady-state profile ρ^{ss} or $\mathbf{J}_{ss}^{\bar{\mathcal{D}}}$, one can further decompose $\langle \dot{\Sigma}^{\mathcal{D}} \rangle$ into nonadiabatic and housekeeping contributions. Similarly, $\langle \dot{\Sigma}^{\mathcal{R}} \rangle$ can be incorporated by linearizing the reactive currents around equilibrium ρ^{eq} ($\mathbf{J}_{eq}^{\bar{\mathcal{R}}}$) or steady-state ρ^{ss} ($\mathbf{J}_{ss}^{\bar{\mathcal{R}}}$), requiring identification of analogous symmetric and antisymmetric couplings $\mathbb{D}^r$ and $\mathbb{D}^{nr}$ in the discrete transition space [188].

B. Fluctuation-response relation, higher-order current cumulants, and responses

In statistical physics, the fluctuation-response relation (FRR) is a fundamental principle that connects equilibrium fluctuations with the linear response of the system [182,189–198]. The nonequilibrium analog of FRR has been postulated [145–148,153,182,199–211]. We examine the FRR for nonreciprocal and driven systems in this section. We define the n th scaled cumulant $\langle^n J_{\gamma\gamma'} \rangle_C$ of $J_{\gamma\gamma'}$. By construction, $\langle^1 J_{\gamma\gamma'} \rangle_C = J_{\gamma\gamma'}$ and $\langle^2 J_{\gamma\gamma'} \rangle_C = T_{\gamma\gamma'} = 2\Omega B_{\gamma\gamma'}$. Here, Ω defines the scaling between the scaled cumulant and the cumulant, for example, the traffic $T_{\gamma\gamma'}$ and the variance $B_{\gamma\gamma'}(\{\rho\})$. It satisfies the following hierarchical relationship [92]:

$$\begin{aligned} \langle^n J_{\gamma\gamma'} \rangle_C &= \langle^{n-2} J_{\gamma\gamma'} \rangle_C, \\ \langle^n J_{\gamma\gamma'} \rangle_C &= J_{\gamma\gamma'}, \quad n \text{ is odd}, \\ \langle^n J_{\gamma\gamma'} \rangle_C &= T_{\gamma\gamma'}, \quad n \text{ is even}. \end{aligned} \quad (45)$$

Equation (45) reveals the recursive structure of the current cumulants. In particular, only the first and second cumulants are independent. This ensures that the van Kampen moments closure expansion is truncated up to the second order for the transition dynamics [94,95,105,212,213]. Thus, Eq. (45) highlights the validity of the van Kampen closure for far-from-equilibrium, nonreciprocal, and externally driven systems. Physically, this important result enables the study of the macroscopic dynamics of nonreciprocal systems using the first two moments of the transition currents. A similar motif has previously been observed for MFT [200]. This ensures the correctness of the Langevin-Gaussian approximation of the macrostate stochastic dynamics formulated in the dynamics section.

We define the response function $^A \zeta_{\gamma\gamma'} = \partial J_{\gamma\gamma'} / \partial A_{\gamma\gamma'}$ and $^A \bar{\zeta}_i^{\bar{\mathcal{D}}} = \partial \bar{J}_i^{\bar{\mathcal{D}}} / \partial \bar{A}_i^{\bar{\mathcal{D}}}$. The response function satisfies the FRR:

$$^A \zeta_{\gamma\gamma'} = \Omega B_{\gamma\gamma'}, \quad ^A \bar{\zeta}_i^{\bar{\mathcal{D}}} = \Omega B_i^{\bar{\mathcal{D}}}. \quad (46)$$

Here, Ω plays an analogous role to the inverse temperature for the macroscopic stochastic dynamics [181]. We use the convention of evaluating the response at the reference affinity $A_{\gamma\gamma'}^*$. Hence, it characterizes the reference state and the probability distribution around which the response is evaluated, for instance, a steady state or an equilibrium distribution. We define the response function for the currents with respect to the change in symmetric reactive transition affinity, $^U \zeta_{\gamma\gamma'} = \partial J_{\gamma\gamma'} / \partial U_{\gamma\gamma'}$ and $^U \bar{\zeta}_i^{\bar{\mathcal{D}}} = \partial \bar{J}_i^{\bar{\mathcal{D}}} / \partial \bar{U}_i^{\bar{\mathcal{D}}}$:

$$^U \zeta_{\gamma\gamma'} = \frac{1}{2} J_{\gamma\gamma'}, \quad ^U \bar{\zeta}_i^{\bar{\mathcal{D}}} = \frac{1}{2} \bar{J}_i^{\bar{\mathcal{D}}}. \quad (47)$$

Equations (46) and (47) satisfy the generic linear-response relations between current and traffic [144,209].

We further delineate the underlying generic structure for the higher-order response function. The n th-order response function for the m th current cumulant is defined as $^A \zeta_{\gamma\gamma'}^m = \partial^n \langle^m J_{\gamma\gamma'} \rangle_C / \partial A_{\gamma\gamma'}^n$ and $^U \zeta_{\gamma\gamma'}^m = \partial^n \langle^m J_{\gamma\gamma'} \rangle_C / \partial U_{\gamma\gamma'}^n$:

$$^A \zeta_{\gamma\gamma'}^m = \frac{1}{2^n} \langle^{m+n} J_{\gamma\gamma'} \rangle_C, \quad ^U \zeta_{\gamma\gamma'}^m = \frac{1}{2^n} \langle^m J_{\gamma\gamma'} \rangle_C. \quad (48)$$

Equation (48) is the higher-order FRR that relates the non-linear response of any higher-order current cumulant to other current cumulants:

$$^A \zeta_{\gamma\gamma'}^{m_1} = \delta(n_1 + m_1 - n_2 - m_2) 2^{n_2 - n_1} ^A \zeta_{\gamma\gamma'}^{m_2}, \quad (49)$$

where $\delta(n_1 - n_2)$ is the Kronecker delta function. Equations (49) and (48) generalizes the far-from-equilibrium higher-order FRR for nonreciprocal and driven systems [214–216].

We use the FRR to infer the transition affinity A_{Δ}^* using the mean current $\langle J_{\Delta} \rangle$ and the current variance $2B_{\Delta}$ of the transition Δ [94,95]:

$$A_{\Delta}^* = 2 \tanh^{-1} \left(\frac{J_{\Delta}}{2\Omega B_{\Delta}} \right). \quad (50)$$

Using Eq. (50), we formulate an inference-based inverse problem. The affinity $A_{\Delta^o}^*$ of an observable transition Δ_o is inferred using the observable mean current J_{Δ^o} and its variance $2B_{\Delta^o}$ [94,95]:

$$A_{\Delta^o}^* = 2 \tanh^{-1} \left(\frac{J_{\Delta^o}}{2\Omega B_{\Delta^o}} \right). \quad (51)$$

$A_{\Delta^o}^*$ is the effective driving force corresponding to the observable current J_{Δ^o} . This concludes the formulation of the FRR for the macroscopic fluctuating dynamics of nonreciprocal systems.

C. Fluctuation relations

The fluctuation relations (FRs) stand as fundamental principles that illuminate the behavior of far-from-equilibrium fluctuating systems [2–5,7,8,11–13,16,217,218]. They offer profound insight into the nature of fluctuations, shedding light on the asymmetry (symmetry) between the forward and backward physical processes at the microscopic level [6,11,13,14,16,17]. By examining the statistical properties of systems undergoing nonequilibrium stochastic dynamics, the FR unveil universal laws about the irreversible nature of thermodynamic processes. This bridges the gap between

macroscopic irreversibility and the underlying microscopic dynamics, paving the way for a deeper understanding of the interplay between order and fluctuations in physical systems. The FR generalize the fluctuation-dissipation relation (FRR) and Onsager's regression hypothesis for systems operating far from equilibrium [219–223].

Here, we focus on a unified formalism of the FR based on measure theory [224–227] combined with the large-deviation principle [138]. In measure theory, the Radon-Nikodym derivative (RND) is defined as the transition-probability measure between the process and the corresponding reference process. The LDB condition establishes the connection between the RND as a mathematical property; that is, a transition-probability measure, and its physical interpretation as a stochastic transition EP. The measure-theoretical formalism of the FR based on the RND utilizes the contraction of the rate functional for the empirical microscopic transition currents to the corresponding rate functional for the EP [94,95]. The orthogonal decomposition of the stochastic EP ensures that the operation and the reference operation chosen to evaluate the RND are spanned by operations that commute with each other. From the large-deviation approach, this is equivalent to choosing the linearly independent empirical observable EP [95,228–232]. The orthogonality condition delineates the linearly independent symmetry operations that act on the system [95,228–232].

We define a scaled-intensive observable $\hat{O} = \frac{1}{\Omega_O} \int_0^\tau O dt$ with a scaling factor Ω_O . We consider the most general observable $\vec{O} = \{\Delta_0^\tau \hat{S}_E^{\text{state}}, \Delta_i^\tau \psi_E, \hat{W}, \{\hat{\omega}_{ij}\}, \{\hat{J}_{\gamma\gamma'}^{\text{ch}}\}, \{\hat{J}_i^{\text{sp}}\}\}$ in vector representation with $\vec{\Omega}_O = \{\ln(\Omega), \Omega, \Omega, \{\Omega\tau\}, \{\Omega\tau\}, \{\Omega\tau\}\}$ and $\vec{\Omega}_O^{-1} = \{[\ln(\Omega)]^{-1}, \Omega^{-1}, \Omega^{-1}, (\Omega\tau)^{-1}, (\Omega\tau)^{-1}, (\Omega\tau)^{-1}\}$. Here, we have utilized the counting observable for the transition currents. The cumulant generating function $G_{\vec{O}}(\vec{\chi}_O)$ for $\vec{O}$ reads

$$G_{\vec{O}}(\vec{\chi}_O) = \ln \langle e^{\vec{\chi}_O \cdot \vec{\Omega}_O \odot \vec{O}} \rangle, \quad (52)$$

with the observable conjugate vector $\vec{\chi}_{\vec{O}} = \{\chi_{\Delta_0^\tau \hat{S}_E^{\text{state}}}, \chi_{\Delta_i^\tau \psi_E}, \chi_{\hat{W}}, \{\chi_{\hat{\omega}_{ij}}\}, \{\chi_{\hat{J}_{\gamma\gamma'}^{\text{ch}}}\}, \{\chi_{\hat{J}_i^{\text{sp}}}\}\}$. The scaling of $G_{\vec{O}}(\vec{\chi}_O)$ is employed using $\max\{\vec{\Omega}_O\}$, which dominates the scaled $G_{\vec{O}}(\vec{\chi}_O)$ (SCGF). The probability density measure $\mathcal{P}(\vec{O})$ satisfies the following symmetry:

$$\ln \left(\frac{\mathcal{P}(\vec{O})}{\mathcal{P}(-\vec{O})} \right) = \vec{A}_{\vec{O}} \cdot \vec{\Omega}_O \odot \vec{O}, \quad G_{\vec{O}}(\vec{\chi}_O) = G_{\vec{O}}(-\vec{A}_{\vec{O}} - \vec{\chi}_O). \quad (53)$$

Defined precisely, $\mathcal{P}(\vec{O}) = \lim_{\delta\vec{O} \rightarrow 0} \mathcal{P}(\vec{O} < \vec{O} < \vec{O} + \delta\vec{O})$ and $\odot$ is the component-wise Hadamard product defined between two vectors. Equation (53) is the finite- $\vec{\Omega}_O$ Gallavotti-Cohen fluctuation relation symmetry, which requires identifying $\vec{A}_{\vec{O}} = \{1, -1, 1, \{V_{ij}^{\text{nr}}\}, \{F_{\gamma\gamma'}^{\text{ch}}\}, \{\bar{F}_i^{\text{sp}}\}\}$, the forces conjugate to $\vec{O}$ [6,215,216,221,222,227,233–240].

We aim to identify the effective macroscopic symmetry of the coarse-grained observable denoted by $\vec{O}^\parallel$, such that $\vec{O}$ is decomposed into $\vec{O} = \vec{O}^\parallel \oplus \vec{O}^\perp$. Using Bayes' theorem and

Eq. (53), the FR for $\vec{O}^\parallel$ reads [228–232]

$$\frac{\mathcal{P}(\vec{O}^\parallel)}{\mathcal{P}(-\vec{O}^\parallel)} = \frac{e^{\vec{A}_{\vec{O}^\parallel} \cdot \vec{\Omega}_{\vec{O}^\parallel}}}{\langle e^{-\vec{O}^\perp} | \vec{O}^\parallel \rangle}, \quad (54)$$

where $\langle e^{-\vec{O}^\perp} | \vec{O}^\parallel \rangle = \int \mathbb{D}\vec{O}^\perp \mathcal{P}(\vec{O}^\perp | \vec{O}^\parallel) e^{-\vec{A}_{\vec{O}^\perp} \cdot \vec{\Omega}_{\vec{O}^\perp} \odot \vec{O}^\perp}$ quantifies the projection of the conditional probability measure of $\vec{O}^\perp$ into the $\vec{O}^\parallel$ space. $\langle e^{-\vec{O}^\perp} | \vec{O}^\parallel \rangle = 1$ for linearly independent observables $\vec{O}^\parallel$ and $\vec{O}^\perp$, which recovers the FR for $\vec{O}^\parallel$. The proof follows trivially by the Taylor-series expansion of $e^{-\vec{O}^\perp}$ and the orthogonality relation $\langle \vec{O}^\perp | \vec{O}^\parallel \rangle = 0$. This emphasizes the importance of choosing linearly independent orthogonal observables. Physically, this generates a hierarchy of detailed fluctuation relations and delineates independent underlying symmetries of the EP.

By choosing $\vec{O}^\parallel = \vec{\Sigma} = \{\hat{W}, \Delta_0^\tau \hat{S}_E^{\text{state}}, \hat{\Sigma}^{\text{nr}}, \hat{\Sigma}^{\text{ch}}, \hat{\Sigma}^{\text{sp}}\}$ and $\vec{O}^\perp = \Delta_0^\tau \psi_E$, we obtain the detailed master FR for nonreciprocal systems:

$$\frac{\mathcal{P}(W, \Delta_0^\tau S_E^{\text{state}}, \Sigma^{\text{nr}}, \Sigma^{\text{ch}}, \Sigma^{\text{sp}})}{\mathcal{P}(-W, -\Delta_0^\tau S_E^{\text{state}}, -\Sigma^{\text{nr}}, -\Sigma^{\text{ch}}, -\Sigma^{\text{sp}})} = e^{W - \Delta_0^\tau \psi_E + \Delta_0^\tau S_E^{\text{state}} + \Sigma^{\text{nr}} + \Sigma^{\text{ch}} + \Sigma^{\text{sp}}}. \quad (55)$$

We implemented the contraction from the linearly independent empirical current observable [Eq. (53)] to the linearly independent EP [Eq. (55)] [94,95]. Equation (55) satisfies the Lebowitz-Spohn symmetry, with the generating function satisfying $G_{\vec{\Sigma}}(\vec{\chi}_{\vec{\Sigma}}) = G_{\vec{\Sigma}}(-1 - \vec{\chi}_{\vec{\Sigma}})$ [11,13,14]. The stochastic work W is not a linearly independent variable; rather, the irreversible work $W - \Delta_0^\tau \psi_E$ is a linearly independent variable [231,232]. This is effectively realized by the factor of $W - \Delta_0^\tau \psi_E$ on the left-hand side of Eq. (55). Hence, $\Delta_0^\tau \psi_E$ drops out, and its projection onto the work gives the factor $W - \Delta_0^\tau \psi_E$ on the right-hand side of Eq. (55).

Assuming the specific case of $F_{\gamma\gamma'}^{\text{ch}} = 0$ and $\bar{F}_i^{\text{sp}} = 0$, which corresponds to nonreciprocity being the only source of the system's nonequilibriumness, Eq. (55) is reduced to the Crooks-Tasaki relation for nonreciprocal systems [12,15,16]:

$$\frac{\mathcal{P}(W, \Delta_0^\tau S_E^{\text{state}}, \Sigma^{\text{nr}})}{\mathcal{P}(-W, -\Delta_0^\tau S_E^{\text{state}}, -\Sigma^{\text{nr}})} = e^{W - \Delta_0^\tau \psi_E + \Delta_0^\tau S_E^{\text{state}} + \Sigma^{\text{nr}}}. \quad (56)$$

Assuming that the external driving control parameter changes instantly relaxes the probability distribution to the reference Boltzmann distribution (no error during driving, i.e., a quasistatic driving process [231,232,241]), we have $\Delta_0^\tau S_E^{\text{state}} = 0$. Equation (56) is then reduced to a familiar form of the Crooks-Tasaki fluctuation theorem, but defined here for nonreciprocal systems. By integrating Eq. (56), we obtain the Jarzynski fluctuation relation for nonreciprocal systems:

$$\langle e^{-W - \Delta_0^\tau S_E^{\text{state}} - \Sigma^{\text{nr}}} \rangle = e^{-\Delta \psi_E}. \quad (57)$$

Using Eq. (43), we relate the external driving work needed with the symmetric and antisymmetric correlations between the initial and final macrostates. Using Jensen's inequality, the second law of thermodynamics (approximate law), Eq. (41), is recovered. However, using $C_{ij}^a(t, \tau)$ and the fluctuation theorem (exact law) Eq. (57) gives a tighter bound on the mean work. The work done on nonreciprocal systems requires extra

thermodynamic cost associated with changing the antisymmetric temporal cross-correlations between the macrostates. The validity of the nonreciprocal Crooks-Tasaki and Jarzynski relations (56) and (57) is subject to not crossing the $\mathcal{PT}$ symmetry-breaking bifurcation manifold in the space of control parameters. Driving through a $\mathcal{PT}$ transition from a static to a dynamic phase undergoes divergent fluctuations, which could lead to nontrivial FR symmetries and exponents. A more systematic study is needed for such physical problems.

By choosing $\Delta_0^\tau S_E^{\text{state}} \neq 0$, the stochastic work incorporates the irreversible work contribution due to the nonquasistatic driving of the control parameters [85,86,231,232,241–244]. Physically, it quantifies the mismatch between the instantaneous nonequilibrium probability distribution and the corresponding instantaneous Boltzmann probability distribution specified by the instantaneous control parameters of E . In the absence of external work ($W = 0$), $\Delta_0^\tau S_E^{\text{state}} \neq 0$ quantifies the relaxation EP and restores the finite-time FR.

Assuming $W = 0$ and combining the boundary term $\Delta\psi_E$ with the nonconservative EP $\Sigma^{\text{nr}} + \Sigma^{\text{sp}} + \Sigma^{\text{ch}}$, Eq. (55) is reduced to the following equation:

$$\frac{\mathcal{P}(\Delta_0^\tau S_E^{\text{state}}, \Sigma_E^{\text{hk}})}{\mathcal{P}(-\Delta_0^\tau S_E^{\text{state}}, -\Sigma_E^{\text{hk}})} = e^{\Delta_0^\tau S_E^{\text{state}} + \Sigma_E^{\text{hk}}}. \quad (58)$$

Equation (58) represents generic detailed fluctuation relations for the nonadiabatic, housekeeping, and total EP, applicable to nonreciprocal systems [116]. It is the effective version of the underlying FR symmetry Eq. (55). The detailed FR for the nonadiabatic, housekeeping, and total EP are obtained by using $E = ss$ [116,242,245], the Hatano-Sasa relation for the excess EP for a transition from an initial to a final steady state [116,246], or the Speck-Seifert relation for the housekeeping EP for a transition from an initial to a final steady state [247].

The steady-state condition is not required to formulate the orthogonal decomposition, making it applicable in a broader context, with a suitable gauge-fixing choice of E according to the required physical or experimental constraints [95,116,117,119–123,162–169].

D. Thermodynamic-kinetic uncertainty relation

The thermodynamic uncertainty relation (TUR) delineates the trade-off between the precision of an observable current and the minimum EP required to sustain it [173,248–250]. Here, the precision is the ratio between the square of the mean observable transition current and the variance of the transition current. The TUR gives a tighter bound on the EP than the second law of thermodynamics. In other words, the precision of an observable current comes at a minimum EP required to sustain it. The bounds on thermodynamic dissipation can be improved by choosing other appropriate physical observables and constraints [210,251–262] and by their unification into the thermodynamic-kinetic uncertainty relation (TKUR) [263]. The TKUR's connection to Speed Limits (SLs) [263–271] and to the FR symmetry [272–276] has been established. Here, we demonstrate the TKUR for nonreciprocal systems, while a more fundamental analysis of the TKUR and nonquadratic speed limits is outlined in Refs. [94,95].

1. Coarse-grained observable transition currents

We consider a set of macroscopic observable transition currents denoted by $O^J = \{\Delta^o\}$ and the corresponding mean inferred EPR $\langle \dot{\Sigma}_{O^J} \rangle$. The mapping $\{\{\Delta_{\gamma\gamma'}\}, \{\Delta_i^{\tilde{D}}\}\} = \{\Delta\} \rightarrow \{\Delta^o\}$ is considered to be many-to-one, so that each microscopic transition contributes to one macroscopic observable transition current. This mapping is represented using an observable matrix $\mathbb{O}^J$, with the row and column indices corresponding to the observable and microscopic currents, respectively. Mathematically, the mapping $\{\{\Delta_{\gamma\gamma'}\}, \{\Delta_i^{\tilde{D}}\}\} \rightarrow \{\Delta^o\}$ implies $\mathbb{O}_{o'j}^J = 1 \Rightarrow \mathbb{O}_{o''j}^J = 0 \forall o'' \in \{\Delta^o\} - o'$. The observable current and traffic vectors satisfy $\tilde{J}_{\Delta^o} = \mathbb{O}^J \tilde{J}_\Delta$ and $\tilde{T}_{\Delta^o} = \mathbb{O}^J \tilde{T}_\Delta$. In particular, using Eqs. (30) and (32), one can equivalently obtain the same expressions for the mean and variance of the observable current. Then, the mean inferred EPR $\langle \dot{\Sigma}_{O^J} \rangle$ reads [94,95]

$$\langle \dot{\Sigma}_{O^J} \rangle = \sum_{\{\Delta^o\}} 2J_{\Delta^o} \tanh^{-1} \left(\frac{J_{\Delta^o}}{T_{\Delta^o}} \right). \quad (59)$$

The inequality $\langle \dot{\Sigma} \rangle \geq \langle \dot{\Sigma}_{O^J} \rangle$ is proven using the generalized log-sum inequality and the definitions $\tilde{J}_{\Delta^o} = \mathbb{O}^J \tilde{J}_\Delta$ and $\tilde{T}_{\Delta^o} = \mathbb{O}^J \tilde{T}_\Delta$ [94,95,277]. The nonquadratic formulation gives a tighter bound on the EP than the traditional quadratic TUR [270] and is closely related to [263,268,271,278]. The mean observable inferred EP $\langle \Sigma_{O^J} \rangle$ satisfies the inequality [94,95]

$$\langle \Sigma_{O^J} \rangle \geq \sum_{\{\Delta^o\}} 2J_o^\tau \tanh^{-1} \left(\frac{J_o^\tau}{T_o^\tau} \right), \quad (60)$$

where $J_o^\tau = \int_0^\tau J_o dt$ and $T_o^\tau = \int_0^\tau T_o dt$ are the time-integrated current and traffic. We have utilized Jensen's inequality to obtain the inequality in Eq. (60) from the equality in Eq. (59). The nonquadratic dissipation function gives a tighter bound than the quadratic TKUR, and its detailed physical and technical analysis is summarized in Refs. [94,95].

2. Coarse-grained vorticities and state-correlations

Here, we aim to examine the EPR inference using state-space observables. We choose a specific subcase of the set of vorticity currents between the observable macrostate $\rho_o = \mathbb{O}^\rho \tilde{\rho}$, where $\mathbb{O}^\rho$ quantifies the many-to-one mapping from $\{\rho_i\} \rightarrow \{\rho_o\}$. $O^\omega = \{\omega_{oo'}\}$ is quantified by the observable vorticity operator $\mathbb{O}^\omega$, which connects microscopic vorticities $\{\omega_{ij}\}$ to observable vorticities through $\tilde{\omega}_O = \mathbb{O}^\omega \tilde{\omega}$. The antisymmetric (vorticity) and symmetric (traffic) components are denoted by $\langle \omega_{oo'} \rangle = \langle \rho_o \partial_t \rho_{o'} - \rho_{o'} \partial_t \rho_o \rangle$ and $\langle \omega_{oo'}^s \rangle = \langle \rho_o \partial_t \rho_{o'} + \rho_{o'} \partial_t \rho_o \rangle$, respectively. The mean inferred EPR using the observable vorticities reads [94,95]

$$\langle \dot{\Sigma}_{O^\omega} \rangle = \sum_{\{\omega_{oo'}\}} 2\langle \omega_{oo'} \rangle \tanh^{-1} \left(\frac{\langle \omega_{oo'} \rangle}{|\langle \omega_{oo'}^s \rangle|} \right). \quad (61)$$

The inferred mean EP $\langle \dot{\Sigma}_{O^\omega} \rangle$ provides a lower bound on $\langle \dot{\Sigma} \rangle$. Integrating Eq. (61) and using Jensen's inequality leads to a bound on the inferred EP using the state correlations:

$$\langle \Sigma_{O^\omega} \rangle \geq \sum_{\{\omega_{oo'}\}} 2\langle \Delta_0^\tau C_{oo'}^a \rangle \tanh^{-1} \left(\frac{\langle \Delta_0^\tau C_{oo'}^a \rangle}{|\langle \Delta_0^\tau C_{oo'}^s \rangle|} \right). \quad (62)$$

Here, $C_{\sigma\sigma'}^a(t, \Delta t) = \rho_\sigma(t + \Delta t)\rho_{\sigma'}(t) - \rho_{\sigma'}(t + \Delta t)\rho_\sigma(t)$ and $C_{\sigma\sigma'}^s(t, \Delta t) = \rho_\sigma(t + \Delta t)\rho_{\sigma'}(t) + \rho_{\sigma'}(t + \Delta t)\rho_\sigma(t)$ characterize the antisymmetric and symmetric parts of the observable state correlation. Furthermore, $\Delta_0^\tau C_{\sigma\sigma'}^a = C_{\sigma\sigma'}^a(t_i, \tau) - C_{\sigma\sigma'}^a(t_i, 0)$ and $\Delta_0^\tau C_{\sigma\sigma'}^s = C_{\sigma\sigma'}^s(t_i, \tau) - C_{\sigma\sigma'}^s(t_i, 0)$ denote the differences between the initial and final times.

Using $\{\omega_{\sigma\sigma'}\} = \{\omega_{ij}\}$, the tightest possible bound on $\langle \dot{\Sigma} \rangle$ is obtained using state-space correlations. Physically, this signifies that considering the temporal cross-correlations between all linearly independent macrostates yields the tightest possible bound on the EP using the state-space TKUR. The set of Eqs. (61) and (62) provides the formulation of the short-time and finite-time TKUR using temporal correlations between the observable states [94,95]. This should be compared with the quadratic [178] and nonquadratic [175,179] counterparts.

E. Information thermodynamics

The framework of information thermodynamics takes into account the thermodynamic implications of statistical information. Statistical information on a system's state is encoded in Gibbs's entropy [Eqs. (34) and (35)]. An information gain can be achieved through a feedback-controlled operation or measurement [244,279–282]. This information gain reduces the statistical uncertainty associated with the system, resulting in a lower EP, signifying the reduced statistical uncertainty.

We define the hidden variable as Y , which does not affect the transition dynamics (bulk term) of the relevant macrostates $\{\rho\}$. Then, the stochastic mutual information $I_{\{\rho\},Y}$ between them is defined as

$$I_{\{\rho\},Y} = -\ln \left(\frac{\mathcal{P}_{\{\rho\},Y}}{\mathcal{P}_{\{\rho\}} \mathcal{P}_Y} \right) = -\ln(\mathcal{P}_{\{\rho\}|Y}) + \ln(\mathcal{P}_{\{\rho\}}). \quad (63)$$

Here, Bayes' theorem is used to reach the second equality in Eq. (63). Mutual information quantifies the statistical correlation (or dependence) between $\{\rho\}$ and the hidden variable Y . It vanishes if the macrostate of the system is statistically independent of the hidden variable; that is, $\mathcal{P}_{\{\rho\}|Y} = \mathcal{P}_{\{\rho\}} \forall Y \in \mathcal{V}_Y$, $\{\rho\} \in \mathcal{V}_{\{\rho\}}$. Here, $\mathcal{V}_Y$ and $\mathcal{V}_{\{\rho\}}$ denote the probability measure spaces for the hidden variable Y and macrostates $\{\rho\}$, respectively. The mean mutual information is defined as $\langle I_{\{\rho\},Y} \rangle = \int_{\mathcal{V}_Y} \int_{\mathcal{V}_{\{\rho\}}} I_{\{\rho\},Y} \mathcal{P}_{\{\rho\},Y}$. Here, $\int \mathbb{D}[\{\rho\}] \equiv \int_{\mathcal{V}_{\{\rho\}}}$ is the shorthand notation used for summation over the probability measure space of the macrostate $\{\rho\}$. Then, the mutual information EPR is defined as $\langle \dot{I}_{\{\rho\},Y} \rangle = \int_{\mathcal{V}_Y} \int_{\mathcal{V}_{\{\rho\}}} \dot{I}_{\{\rho\},Y} d\mathcal{P}_{\{\rho\},Y}$, whose exact expressions for the mutual information EP and EPR are

$$\begin{aligned} \langle I_{\{\rho\},Y} \rangle &= S^{\text{gb}}(\mathcal{P}_{\{\rho\}|Y}) - S^{\text{gb}}(\mathcal{P}_{\{\rho\}}), \\ \langle \dot{I}_{\{\rho\},Y} \rangle &= d_t S^{\text{gb}}(\mathcal{P}_{\{\rho\}|Y}) - d_t S^{\text{gb}}(\mathcal{P}_{\{\rho\}}). \end{aligned} \quad (64)$$

Incorporating the mutual information EP and EPR, the second law of thermodynamics Eq. (41) is modified. Thus, the Gibbs EPR is replaced by the conditional Gibbs EPR. The conditional Gibbs entropy satisfies the inequality $S^{\text{gb}}(\mathcal{P}_{\{\rho\}}) \geq S^{\text{gb}}(\mathcal{P}_{\{\rho\}|Y})$. Physically, this corresponds to the information gained due to the hidden variable Y , which reduces the statistical ignorance (Gibbs entropy) of the macrostate $\{\rho\}$ by encapsulating its correlations with the hidden variable Y . $\langle \dot{I}_{\{\rho\},Y} \rangle \leq 0$ holds, which gives the sign convention for the in-

formation gain. Equation (64) satisfies the equality, $\langle \dot{I}_{\{\rho\},Y} \rangle = d_t S^{\text{gb}}(\mathcal{P}_{\{\rho\}|Y}) - d_t S^{\text{gb}}(\mathcal{P}_{\{\rho\}}) - d_t S^{\text{gb}}(\mathcal{P}_Y)$. Similarly, the fluctuation theorem follows trivially by modifying Eq. (55) to incorporate stochastic mutual information Eq. (63), or equivalently, by replacing the reference state entropy with the reference conditional state entropy $S_E^{\text{state}} \rightarrow S_E^{\text{state}|Y} = -\ln(\mathcal{P}_{\{\rho\}|Y}/\mathcal{P}_{\{\rho\}}^E)$. This is a manifestation of the physical symmetry of measurement (certainty) and uncertainty. We have briefly highlighted the procedure to incorporate information thermodynamics for nonreciprocal systems. The rest is rather trivial.

V. ILLUSTRATIVE EXAMPLES

A. Diffusive systems

We consider systems without reactive transitions, for which $\{\gamma\gamma'\} = \emptyset$ is an empty set.

1. Generalized Kawasaki-Dean equation

Considering $\vec{F}_{sp} = 0$, Eq. (29) is reduced to [92]

$$\partial_t \rho_i = \tilde{d}_i \vec{\nabla} \cdot (e^{\mu_i} \vec{\nabla} \mu_i) + \frac{1}{\sqrt{\Omega}} \vec{\nabla} \cdot (\sqrt{2\tilde{d}_i} e^{\mu_i} \vec{\xi}_i^{\mathcal{D}}). \quad (65)$$

Equation (65) generalizes the Kawasaki-Dean equation for interacting systems [71,72], valid in the regime of small interaction parameters. In contrast, Eq. (65) encapsulates the effect of the interaction on the amplitude of the noise. For an ideal noninteracting reciprocal particle, $\mu_i = \ln \rho_i$, one recovers the Kawasaki-Dean equation for the ideal particles.

2. Macroscopic fluctuation theory

We consider systems without chemical reaction changes between the particle types. In this case, the generalized macroscopic fluctuation theory EOM for the macrostate reads

$$\begin{aligned} \partial_t \rho_i &= \vec{\nabla}^{\vec{\mathcal{D}}} \cdot \{D_i^{\vec{\mathcal{D}}} (\vec{\nabla}^{\vec{\mathcal{D}}} \mu_i + \vec{\nabla}^{\vec{\mathcal{D}}} F_i^{\text{nr}} - \vec{F}_i^{\text{sp}})\} \\ &+ \frac{1}{\sqrt{\Omega}} \vec{\nabla}^{\vec{\mathcal{D}}} \cdot (\sqrt{2D_i^{\vec{\mathcal{D}}}} \vec{\xi}_i). \end{aligned} \quad (66)$$

Equation (66) is the MFT formulation for nonreciprocal systems [93]. In contrast with Ref. [93], we have systematically derived the coarse-grained EOM with the density-dependent mobility $D_i^{\vec{\mathcal{D}}} = d_i^{\vec{\mathcal{D}}} e^{\mu_i + F_i^{\text{nr}}}$ [92]. The second law of thermodynamics for this system is $\langle \dot{\Sigma} \rangle = -d_t D_E^{\text{KL}}(\mathcal{P}[\{\rho\}]) + \langle \dot{\Sigma}^{\text{nr}} \rangle + \langle \dot{\Sigma}^{\text{sp}} \rangle$. Here, $\langle \dot{\Sigma}^{\text{sp}} \rangle$ is a nonquadratic dissipation function, in contrast with the quadratic one obtained in MFT [93], therefore avoiding the underestimation of the dissipation for far-from-equilibrium systems [94,95].

B. Nonreciprocal two species

1. Chemo-sensing bacteria

We consider a prototypical system of bacteria (b) attracted toward a chemical (c). Thus, the microscopic interaction rules for this physical system read $v_{bc} = -1$, $v_{cb} = 0$, $v_{cc} = 0$, $v_{bb} = 0$. This leads to $v_{bc}^r = -\frac{1}{2}$ and $v_{cb}^{\text{nr}} = -v_{bc}^{\text{nr}} = \frac{1}{2}$. We choose this model with a minimal motif of nonreciprocity and its implications for diffusive dynamics; therefore, the self-propulsion forces vanish, $\tilde{f}_c^{\text{sp}} = \tilde{f}_b^{\text{sp}} = 0$. The macroscopic

energy functional E for the density of the bacteria (ρ_b) and the chemical (ρ_c) reads

$$E = \int_V \left[V_{bc} \rho_b \rho_c + \rho_b \ln \left(\frac{\rho_b}{e} \right) + \rho_c \ln \left(\frac{\rho_c}{e} \right) \right], \quad (67)$$

where $V_{bc}^r = \beta(-\frac{1}{2} + \frac{\beta}{4\Omega})$ and $V_{cb}^{nr} = \beta(\frac{1}{2} - \frac{\beta}{4\Omega})$. Using $\mu_b^r = \ln(\rho_b) + \rho_c V_{bc}^r$ and $\mu_c^r = \ln(\rho_c) + \rho_b V_{cb}^r$, $F_b^{nr} = -V_{cb}^{nr} \rho_c$ and $F_c^{nr} = \frac{1}{2} V_{bc}^{nr} \rho_c$. The mobility of the bacterial density and chemical density are $D_\rho = \tilde{d}_\rho \rho_b e^{\rho_c(V_{bc}^r + V_{cb}^{nr})} = \tilde{d}_\rho \rho_b e^{-\beta \rho_c(1 - \frac{\beta}{2\Omega})}$ and $D_c = \tilde{d}_c \rho_c$, respectively. Thus, $\langle \dot{\Sigma}^{sp} \rangle = \langle \dot{\Sigma}^{ch} \rangle = 0$. The expression for $\langle \dot{\Sigma}^{nr} \rangle$, which quantifies the thermodynamic cost of maintaining the vorticity currents between the bacteria and the chemical, reads

$$\langle \dot{\Sigma}^{nr} \rangle = V_{cb}^{nr} \int_V \langle \rho_c \partial_t \rho_b - \rho_b \partial_t \rho_c \rangle. \quad (68)$$

The second law of thermodynamics states $\langle \dot{\Sigma} \rangle = -d_t \langle E \rangle + d_t S^{gb} + \langle \dot{\Sigma}^{nr} \rangle$. Or equivalently, $\langle \dot{\Sigma} \rangle = -d_t D_E^{KL}(\mathcal{P}[\{\rho\}]) + \langle \dot{\Sigma}^{nr} \rangle$. Here, $d_t D_E^{KL}(\mathcal{P}[\{\rho\}])$ quantifies the EPR due to the relaxation of E , which equivalently quantifies the relaxation of the symmetric macrostate correlations. $\langle W \rangle = 0$, since the control parameters of the system are fixed.

2. Predator-prey model

We consider a predator-prey system with a minimalist attraction-repulsion mechanism, such as dogs-sheep, without any self-propulsion. Thus, the microscopic interaction rules are $v_{ds} = -1$, $v_{sd} = 1$, $v_{dd} = 0$, $v_{ss} = -1$, which implies $v_{ds}^r = -1$, $v_{sd}^r = 1$, $v_{dd}^r = 0$, $v_{ss}^r = -1$. This leads to the macroscopic interaction rules $V_{sd}^r = \frac{\beta^2}{2\Omega}$, $V_{sd}^{nr} = \beta$, $V_{ds}^{nr} = -\beta$, and $V_{ss}^r = -\beta + \frac{\beta^2}{2\Omega}$. We have intentionally chosen $v_{ss} = -1$, which models an attractive interaction between the sheep, physically corresponding to the flocking of the sheep herd. In addition, $f_{ds}^{ch} = \tilde{f}_s^{sp} = \tilde{f}_d^{sp} = 0$. Then, the macroscopic coarse-grained energy functional for this model obtained using the density of the sheep (ρ_s) and the dogs (ρ_d) reads

$$E = \int_V \left[\frac{1}{2} V_{ss}^r \rho_s^2 + V_{sd}^r \rho_s \rho_d + \rho_d \ln \left(\frac{\rho_d}{e} \right) + \rho_s \ln \left(\frac{\rho_s}{e} \right) \right], \quad (69)$$

where the reciprocal macroscopic Boltzmann weights are $\mu_d^r = \ln \rho_d + V_{sd}^r \rho_s$ and $\mu_s^r = \ln \rho_s + V_{sd}^r \rho_d + V_{ss}^r \rho_s$, and the nonreciprocal macroscopic Boltzmann weights are $F_d^{nr} = V_{ds}^{nr} \rho_s$, $F_s^{nr} = V_{sd}^{nr} \rho_d$. The diffusive mobilities are then $D_d = \tilde{d}_d \rho_d e^{(V_{ds}^r + V_{ds}^{nr}) \rho_s}$ and $D_s = \tilde{d}_s \rho_s e^{[V_{ss}^r \rho_s + (V_{sd}^r + V_{sd}^{nr}) \rho_d]}$. In addition, $\langle \dot{\Sigma}^{sp} \rangle = \langle \dot{\Sigma}_R^{nr} \rangle = \langle \dot{\Sigma}^{ch} \rangle = 0$, since $F_{ds}^{ch} = \tilde{F}_s^{sp} = \tilde{F}_d^{sp} = 0$. The expression for $\langle \dot{\Sigma}^{nr} \rangle$ reads

$$\langle \dot{\Sigma}^{nr} \rangle = V_{sd}^{nr} \int_V \langle \rho_s \partial_t \rho_d - \rho_d \partial_t \rho_s \rangle. \quad (70)$$

Assuming $\langle W \rangle = 0$, the second law of thermodynamics for this model is $\langle \dot{\Sigma} \rangle = -d_t \langle E \rangle + d_t S^{gb} + \langle \dot{\Sigma}^{nr} \rangle$. Or, equivalently, $\langle \dot{\Sigma} \rangle = -d_t D_E^{KL}(\mathcal{P}[\{\rho\}]) + \langle \dot{\Sigma}^{nr} \rangle$.

The change in the symmetric part of the macrostate correlations is $O(1/\Omega)$, as $V_{sd}^{nr} \propto O(1/\Omega)$, compared with the chemo-sensing bacteria in Sec. VB1, where $V_{sd}^{nr} \propto O(1)$, which highlights a very important difference between the

two models. Importantly, despite the similarity of $\langle \dot{\Sigma}^{nr} \rangle$ for both models, the symmetric part of the macrostate correlations distinguishes between the underlying different physical mechanisms that could generate similar vorticity currents. In the chemo-sensing bacteria section, it is one-way attraction, and here it is a mutual attraction-repulsion mechanism between the dogs and sheep that leads to nonreciprocal vorticity currents.

C. Thermodynamically consistent active Ising model

1. Single species

Here, we consider the thermodynamically consistent Active Ising Model (AIM). The AIM consists of two different types of self-propelled particles, positive and negative [73]. Particles of the same type attract each other, otherwise they repel. Hence, the microscopic interaction rules are $v_{++} = -1$, $v_{--} = -1$, and $v_{+-} = 1$. This leads to the macroscopic interaction coefficients $V_{--}^r = V_{++}^r = -1 + \frac{\beta}{2\Omega}$ and $V_{+-}^r = 1 + \frac{\beta}{2\Omega}$. The macroscopic reciprocal Boltzmann weights are $\mu_+^r = \ln \rho_+ + \beta(V_{++}^r \rho_+ + V_{+-}^r \rho_-)$ and $\mu_-^r = \ln \rho_- + \beta(V_{--}^r \rho_- + V_{+-}^r \rho_+)$. The diffusive mobility for the density fields ρ_+ and ρ_- reads $D_+ = \tilde{d} e^{\mu_+^r}$ and $D_- = \tilde{d} e^{\mu_-^r}$, respectively. We report a detailed study in Refs. [77,92] and briefly outline the thermodynamic consequences here.

The macroscopic energy functional E is trivially obtained using Eq. (26) and V_{ij}^r . The only nonconservative force for the thermodynamically consistent AIM is the self-propulsion. Thus, $\langle \dot{\Sigma}^{nr} \rangle = \langle \dot{\Sigma}^{ch} \rangle = 0$. The macroscopic mean EPR due to the self-propulsion reads

$$\langle \dot{\Sigma}^{sp} \rangle = \int_V 2 \langle (D_+ + D_-) \rangle F^{sp} \sinh \left(\frac{F^{sp}}{2} \right). \quad (71)$$

The macroscopic second law of thermodynamics reads $\langle \dot{\Sigma} \rangle = -d_t \langle E \rangle + \langle \dot{\Sigma}^{sp} \rangle + d_t S^{gb}$, or equivalently, $\langle \dot{\Sigma} \rangle = -d_t D_E^{KL}(\mathcal{P}[\{\rho\}]) + \langle \dot{\Sigma}^{sp} \rangle$. We report other important implications for the correctness of the macroscopic phase diagram in Ref. [92].

2. Nonreciprocal two species

We aim to study a more sophisticated model that combines both nonreciprocity and self-propulsion simultaneously. The bird-flocking phenomena for a predator-prey bird species with two different preferred flying directions are denoted by $+$ and $-$. The predator (prey) is a Falcon (Starling). The microscopic interaction parameters for this model are $v_{s+s+} = v_{s-s-} = v_{f+f+} = v_{f-f-} = -1$ and $v_{s+s-} = v_{f+f-} = 1$ and $v_{s+f+} = v_{s-f-} = v_{s+f-} = v_{s-f+} = 1$ and $v_{f+s+} = v_{f-s-} = v_{f+s-} = v_{f-s+} = -1$. It leads to the macroscopic interaction coefficients $V_{s+s+}^r = V_{s-s-}^r = V_{f+f+}^r = V_{f-f-}^r = -\beta + \frac{\beta^2}{2\Omega}$, $V_{s+s-}^r = V_{f+f-}^r = \beta + \frac{\beta^2}{2\Omega}$, $V_{s+f+}^r = V_{s-f-}^r = V_{s+f-}^r = V_{s-f+}^r = V_{f+s-}^r = V_{f-s+}^r = -\beta + \frac{\beta^2}{2\Omega}$, $V_{s+f-}^r = V_{f+s+}^r = \beta + \frac{\beta^2}{2\Omega}$. The macroscopic interaction energy functional E is trivially obtained using Eq. (26) and V_{ij}^r . The macroscopic self-propulsion

EPR reads

$$\langle \dot{\Sigma}^{sp} \rangle = \int_{\mathcal{V}} 2 \left[F_s^{sp} \sinh \left(\frac{F_s^{sp}}{2} \right) (D_{s^+} + D_{s^-}) + F_f^{sp} \sinh \left(\frac{F_f^{sp}}{2} \right) (D_{f^+} + D_{f^-}) \right], \quad (72)$$

and the macroscopic nonreciprocal EPR reads

$$\langle \dot{\Sigma}^{nr} \rangle = V_{s^+f^+}^{nr} \int_{\mathcal{V}} \langle \rho_s \partial_t \rho_f - \rho_f \partial_t \rho_s \rangle. \quad (73)$$

Here, $\rho_s = \rho_{s^-} + \rho_{s^+}$ and $\rho_f = \rho_{f^-} + \rho_{f^+}$ quantify the total density of Starlings and Falcons, irrespective of their flying direction. Equations (72) and (73) gives the orthogonal decomposition of the macroscopic second law of thermodynamics, $\langle \dot{\Sigma} \rangle = -d_t \langle E \rangle + \langle \dot{\Sigma}^{nr} \rangle + \langle \dot{\Sigma}^{sp} \rangle + d_t S^{\text{gb}}$. Or equivalently, $\langle \dot{\Sigma} \rangle = -d_t D_E^{\text{KL}}(\mathcal{P}[\{\rho\}]) + \langle \dot{\Sigma}^{nr} \rangle + \langle \dot{\Sigma}^{sp} \rangle$, in the absence of the external driving of E , the driving work is $\langle W \rangle = 0$.

VI. CONCLUSION AND OUTLOOK

We have formulated a generic novel framework of stochastic thermodynamics for nonreciprocal systems that relies on systematic thermodynamically consistent coarse-graining approaches. The novel formulation of the Local Detailed Balance condition for nonreciprocal systems provides a thermodynamically consistent formulation across the scales. Hence, our framework broadens the applicability of stochastic thermodynamics to different observation scales of the system. We further decompose the mean EPR into four orthogonal contributions, namely, the conservative, the non-reciprocal, the external chemical or self-propulsion driving, and the rate of change in the free energy (driving work). They correspond to the entropy-production cost associated with the relaxation (toward the reference Boltzmann distribution), sustaining the vorticity currents (due to nonreciprocity), sustaining the dissipative transition currents, and the quasistatic mean stochastic work, respectively. Importantly, the systematic coarse-graining using the Doi-Peliti field theory ensures the equivalence of the systems' dynamics and thermodynamics across the observation scale.

We compute the dynamic equations of motion for the macrostate using the Langevin approximation, which exhibits the multiplicative demographic noise. This generalizes the Macroscopic Fluctuation Theory to nonreciprocal and externally driven systems. We demonstrated that the microscopic nonreciprocal interactions lead to the manifestation of Onsager's nonreciprocal relations on the macroscale. We formulate the fluctuation-response relation and its generalizations, namely, the higher-order current cumulant response relations. In the spirit of stochastic thermodynamics, we formulate the generic fluctuation relations for nonreciprocal systems. We obtain the tightest Thermodynamic Kinetic Uncertainty Relation for the nonreciprocal systems. It relies on using the observable transition currents and the observable vorticity that lies in the transition and state space, respectively. We briefly highlight how to incorporate

the information thermodynamics for nonreciprocal systems. Our framework opens up a plethora of model-specific extensions of nonreciprocal systems across a different observable scale. Different aspects of this work are summarized in Table III.

Multibody microscopic interactions. One can explore the consequences of the nonlinear dependence of ϵ_i on N_i , which exhibits a richer phenomenology [64]. The structure of our framework (both dynamic and thermodynamic) is robust to such a modification to the nonlinear dependence of the Boltzmann weights on the particle number. Hence, a straightforward generalization is obtained [92]. Importantly, the higher-order multibody interaction plays a key role in ensuring that the interaction energy functional is bounded from below.

Linear cyclic CRN. A recent development in the interacting (nonideal) CRN has led to the applicability of the methodology developed for the noninteracting CRN [1,237,239,283–286] to the interacting CRN under certain conditions [287]. We intentionally avoided the interplay of the interacting systems and the topological properties of the CRN encapsulating the underlying reactive transitions. Nevertheless, the interacting CRNs are fundamentally similar to the noninteracting CRNs, provided that certain constraints on the topological properties of the CRN are met. An important implication is the vanishing of diffusive currents [287]. However, the interplay of diffusive and reactive currents has been the key mechanism of pattern formation [288]. Therefore, this opens up an interesting avenue to explore, under a more generic condition of a spatially extended interacting CRN that does not satisfy the constraints in Refs. [287]. The study of the interplay between diffusive fluxes and the reactive fluxes modeled by interacting CRNs and their thermodynamic and dynamic implications for pattern formation remains an open problem. We expect that the spatially extended nonreciprocal systems might exhibit a richer phenomenology.

Nonlinear CRN. Our framework can be straightforwardly incorporated into the nonlinear deterministic CRN [165,289] by incorporating the topological structure of the CRN. Moreover, the stochastic nonlinear CRN exhibits more sophisticated effects arising due to the interplay of nonlinearity and stochasticity [290–295]. It is an interesting avenue to explore the nonlinear stochastic CRN by incorporating non-reciprocal interactions. Since our framework systematically incorporates the microscopic occupancy and transition fluctuations on the macroscale, it is an inherently better starting point than the deterministic CRN to study such effects.

The role of demographic noise. Demographic noise plays a key role in understanding the physics of ecological models [296–308]. For instance, the steady-state selection, the stability of the attractors for the deterministic and stochastic dynamics [297–308], and the mismatch between the deterministic and stochastic dynamics [150,309–311]. It is an interesting avenue to explore the model-specific implications of demographic noise and potentially richer novel phenomena using the exact Eq. (29) for their thermodynamically consistent dynamics. Importantly, our framework enables us to compute the thermodynamic dissipation cost associated with the different phases as well.

TABLE III. Coarse-graining diagram: where $\mathcal{F}_{\text{GL}}$ and $\mathcal{F}_{\text{GL}}^{\text{id}}$ are the corresponding Ginzburg-Landau energy functional derived using phenomenological arguments and the ideal particle counterpart that incorporates the Boltzmann entropic contributions only.

Level	Dynamics			Thermodynamics				
	State	Equations of Motion	LDB	Gibbs entropic	Boltzmann entropic	Reciprocal interaction	Nonreciprocal interaction	External driving
Microscopic description	Multiparticle state probability $P_{\{N\}}$	Master equation Eq. (8)	Eq. (6)	s_{state} in Eq. (10)	s_b in Eq. (7)	ϵ_i^r in Eq. (2)	f_i^{nr} in Eq. (2)	$f_{\gamma\gamma'}^{\text{ch}}$ and $\tilde{f}_i^{\text{sp}}$ in Eq. (6)
Macroscopic fluctuating description	Fluctuating macrostate density ρ_i	Generalized macroscopic fluctuation theory Eq. (29)	Eq. (28)	S_{state} in Eq. (35)	$\mu_i^{\text{id}} = \ln(\rho_i)$	μ_i^r in Eq. (24)	F_i^{nr} in Eq. (24)	$F_{\gamma\gamma'}^{\text{ch}}$ and $\tilde{F}_i^{\text{sp}}$ in Eq. (28)
Hydrodynamic order parameters [69,70]	Phenomenological order parameters $\phi = g(\rho_i)$	Models A, B, etc.		Absent	$\mu_{\phi_i}^{\text{id}} = \frac{\delta \mathcal{F}_{\text{GL}}^{\text{id}}}{\delta \phi_i}$	$\mu_{\phi_i} = \frac{\delta \mathcal{F}_{\text{GL}}}{\delta \phi_i}$	Usually absent but considered in Refs. [62–68,81–84]	Usually absent but considered in Refs. [93,153,154]

The finite-time optimal control of nonreciprocal systems. We have omitted the time-dependent change of the control parameters of the nonreciprocal models, except for the slow driving of the control parameters of the conservative energy functional E that quantifies the quasistatic work. However, a novel framework for the finite-time optimal control of any control parameters of far-from-equilibrium systems has been recently developed in Ref. [312]. It has revealed the importance of the exact nonquadratic dissipation functions and the discontinuous endpoint jumps in optimal driving protocols, which have been shown to be of paramount importance for far-from-equilibrium systems [94,95]. It is an interesting avenue to explore model-specific phenomena for the optimal control of nonreciprocal systems in a finite driving time.

DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the authors.

APPENDIX: MICROSCOPIC EXCESS AND HOUSE-KEEPING DECOMPOSITION OF THE EPR

The total mean microscopic bulk EPR can be further divided using another physical aspect of the system. For instance, a state variable is often easier to track compared with the underlying thermodynamic energy functional. One particular case arises when the temporal dynamics of the microstate lead to a steady-state probability distribution $P_{\{N\}}^{\text{ss}}$ for the microstate $\{N\}$. The mean bulk EPR ($\langle \dot{\sigma}^{\mathcal{R}} \rangle + \langle \dot{\sigma}^{\mathcal{D}} \rangle$) can be decomposed into two components, called excess and housekeeping EPR, using the steady state of the dynamics [20,112,113,115,245–247]. This decomposition is commonly referred to as the Hatano-Sasa (HS) decomposition of the total mean EPR. The mean excess HS EPR is denoted by $\langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle$,

with the exact expression given by

$$\langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle = \sum_{\{\Delta_{\gamma\gamma'}^{\#}\}}^{\{N\}} j_{\gamma\gamma'}^{\#}(\{N\}) \ln \left(\frac{P_{\{N+\Delta_{\gamma\gamma'}^{\#}\}}^{\text{ss}}(t)}{P_{\{N\}}^{\text{ss}}(t)} \right) + \sum_{\{\Delta_i^{\mathcal{D}\#}\}}^{\{N\}} j_i^{\mathcal{D}\#}(\{N\}) \ln \left(\frac{P_{\{N+\Delta_i^{\mathcal{D}\#}\}}^{\text{ss}}(t)}{P_{\{N\}}^{\text{ss}}(t)} \right). \quad (\text{A1})$$

Using the master equation (8), the mean microscopic excess HS EPR satisfies $\langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle = \sum_{\{N\}} d_t P_{\{N\}}(t) \ln [P_{\{N\}}^{\text{ss}}(t)]$. The time dependence of the steady state indicates the validity of the definition in Eq. (A1) for nonautonomous dynamics implemented through thermodynamic work.

The exact expression for the microscopic housekeeping EPR reads

$$\langle \dot{\sigma}_{\text{hs}}^{\text{hk}} \rangle = \langle \dot{\sigma} \rangle - \langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle - d_t s^{\text{gb}}. \quad (\text{A2})$$

Positivity of excess and housekeeping EPR has been demonstrated [112,113,245–247]. By combining the rate of change of the Gibbs entropy term $d_t s^{\text{gb}}$ with the mean excess HS EPR $\langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle$, one defines the nonadiabatic mean EPR $\langle \dot{\sigma}^{\text{na}} \rangle$. Its closed-form expression $\langle \dot{\sigma}^{\text{na}} \rangle = \langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle + d_t s^{\text{gb}}$ reads

$$\langle \dot{\sigma}^{\text{na}} \rangle = - \sum_{\{N\}} d_t P_{\{N\}}(t) \ln \left(\frac{P_{\{N\}}(t)}{P_{\{N\}}^{\text{ss}}(t)} \right). \quad (\text{A3})$$

The nonadiabatic mean EPR $\langle \dot{\sigma}^{\text{na}} \rangle$ quantifies microscopic EPR due to a mismatch between the instantaneous probability distribution $\{P_{\{N\}}(t)\}$ and the steady-state distribution $\{P_{\{N\}}^{\text{ss}}(t)\}$. In steady state, the instantaneous probability distribution satisfies $\{P_{\{N\}}(t)\} = \{P_{\{N\}}^{\text{ss}}(t)\}$, so that the nonadiabatic mean EPR $\langle \dot{\sigma}^{\text{na}} \rangle$ vanishes. Consequently, in steady state, the total mean microscopic EPR $\langle \dot{\sigma} \rangle$ is completely determined by the mean housekeeping EPR, $\langle \dot{\sigma}_{\text{hs}}^{\text{hk}} \rangle = \langle \dot{\sigma} \rangle$.

Reorganizing the definitions of the mean microscopic excess HS EPR and the microscopic nonadiabatic EPR shows that they are related to the rate of change of the stochastic

Massieu potential. In particular, $\langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle = d_t \langle \ln [P_{\{N\}}^{\text{ss}}(t)] \rangle$ and $\langle \dot{\sigma}^{\text{na}} \rangle = -d_t \langle \ln [P_{\{N\}}(t)] / [P_{\{N\}}^{\text{ss}}(t)] \rangle$, where the time derivative is defined as $d_t \langle * \rangle = \sum_{\{N\}} \langle * \rangle d_t P_{\{N\}}$. The same convention was previously used to define the second law of thermodynamics. Thus, $-\ln [P_{\{N\}}^{\text{ss}}(t)]$ and $\ln [P_{\{N\}}(t)] / [P_{\{N\}}^{\text{ss}}(t)]$ are stochastic Massieu potentials whose temporal variations are $\langle \dot{\sigma}_{\text{hs}}^{\text{ex}} \rangle$ and $\langle \dot{\sigma}^{\text{na}} \rangle$, respectively.

The Kullback-Leibler divergence is defined as $D_{\text{ss}}^{\text{KL}}(P(t) || P^{\text{ss}}(t)) = \sum_{\{N\}} P_{\{N\}}(t) \ln [P_{\{N\}}(t) / P_{\{N\}}^{\text{ss}}(t)]$ [107–114]. We introduce a shorthand notation $D_{\text{ss}}^{\text{KL}}(t) = D^{\text{KL}}(P(t) || P^{\text{ss}}(t))$, where the superscript indicates the reference distribution. Thus, $\langle \dot{\sigma}^{\text{na}} \rangle = -d_t D_{\text{ss}}^{\text{KL}}(P(t))$. Physically, this relates the nonadiabatic EPR to the relaxation of the Kullback-Leibler divergence between the instantaneous and steady-state distributions. Plugging this into Eq. (A2)

gives $\langle \dot{\sigma}_{\text{hs}}^{\text{hk}} \rangle = \langle \dot{\sigma} \rangle |_{P^{\text{ss}}}$, showing that the housekeeping EPR equals the total microscopic EPR in steady state, while the microscopic adiabatic EPR vanishes. For a system without external nonconservative or nonreciprocal forces, $\langle \dot{\sigma}_{\text{hs}}^{\text{hk}} \rangle = 0$ trivially holds. The microscopic EPR due to relaxation is fully quantified by the rate of change of the Kullback-Leibler divergence, $\langle \dot{\sigma} \rangle = -d_t D_{\text{ss}}^{\text{KL}}(t)$.

A stronger bound on the microscopic entropy production is obtained by exploiting the positivity of $\langle \dot{\sigma}^{\text{na}} \rangle$ and its Kullback-Leibler representation [114]. In particular, $\langle \sigma^{\text{na}} \rangle = D_{\text{ss}}^{\text{KL}}(0) - D_{\text{ss}}^{\text{KL}}(\tau)$, where τ is the relaxation time. Using the triangle inequality [111,114], this can be further simplified as $\langle \sigma^{\text{na}} \rangle \geq D^{\text{KL}}(P(0) || P(\tau))$, which is a tighter bound than the second law of thermodynamics due to the non-negativity of the KL divergence.

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- [1] J. Schnakenberg, Network theory of microscopic and macroscopic behavior of master equation systems, *Rev. Mod. Phys.* **48**, 571 (1976).
 - [2] G. N. Bochkov and I. E. Kuzovlev, General theory of thermal fluctuations in nonlinear systems, *Zh. Eksp. Teor. Fiz.* **72**, 238 (1977) [*Sov. Phys. JETP* **45**, 125 (1977)].
 - [3] G. N. Bochkov and Y. E. Kuzovlev, Fluctuation-dissipation relations for nonequilibrium processes in open systems, *Zh. Eksp. Teor. Fiz.* **76**, 1071 (1979) [*Sov. Phys. JETP* **49**, 543 (1979)].
 - [4] D. J. Evans, E. G. D. Cohen, and G. P. Morriss, Probability of second law violations in shearing steady states, *Phys. Rev. Lett.* **71**, 2401 (1993).
 - [5] D. J. Evans and D. J. Searles, Equilibrium microstates which generate second law violating steady states, *Phys. Rev. E* **50**, 1645 (1994).
 - [6] G. Gallavotti and E. G. D. Cohen, Dynamical ensembles in nonequilibrium statistical mechanics, *Phys. Rev. Lett.* **74**, 2694 (1995).
 - [7] C. Jarzynski, Nonequilibrium equality for free energy differences, *Phys. Rev. Lett.* **78**, 2690 (1997).
 - [8] C. Jarzynski, Equilibrium free-energy differences from nonequilibrium measurements: A master-equation approach, *Phys. Rev. E* **56**, 5018 (1997).
 - [9] K. Sekimoto, Kinetic characterization of heat bath and the energetics of thermal Ratchet models, *J. Phys. Soc. Jpn.* **66**, 1234 (1997).
 - [10] K. Sekimoto, Langevin equation and thermodynamics, *Prog. Theor. Phys. Suppl.* **130**, 17 (1998).
 - [11] J. Kurchan, Fluctuation theorem for stochastic dynamics, *J. Phys. A: Math. Gen.* **31**, 3719 (1998).
 - [12] G. E. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences, *Phys. Rev. E* **60**, 2721 (1999).
 - [13] C. Maes, The fluctuation theorem as a Gibbs property, *J. Stat. Phys.* **95**, 367 (1999).
 - [14] J. L. Lebowitz and H. Spohn, A Gallavotti–Cohen-type symmetry in the large deviation functional for stochastic dynamics, *J. Stat. Phys.* **95**, 333 (1999).
 - [15] H. Tasaki, Jarzynski relations for quantum systems and some applications, [arXiv:cond-mat/0009244](https://arxiv.org/abs/cond-mat/0009244).
 - [16] G. E. Crooks, Path-ensemble averages in systems driven far from equilibrium, *Phys. Rev. E* **61**, 2361 (2000).
 - [17] C. Maes and K. Netočný, Time-reversal and entropy, *J. Stat. Phys.* **110**, 269 (2003).
 - [18] U. Seifert, Entropy production along a stochastic trajectory and an integral fluctuation theorem, *Phys. Rev. Lett.* **95**, 040602 (2005).
 - [19] K. Sekimoto, *Stochastic Energetics* (Springer, Berlin, Heidelberg, 2010).
 - [20] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, *Rep. Prog. Phys.* **75**, 126001 (2012).
 - [21] N. Shiraishi, *An Introduction to Stochastic Thermodynamics: From Basic to Advanced* (Springer Nature, Singapore, 2023), pp. 31–47.
 - [22] We adhere to the convention of using a stochastic transition instead of a stochastic trajectory, since a stochastic trajectory can be visualized as a collection of sequential stochastic transitions. The Markovian property of the transitions ensures this equivalence, provided that the initial probability distributions (boundary terms) are the same. We clarify this notion to avoid confusion with the trajectory-based approach used to compute the information-theoretic definition of entropy production. The information-theoretic approach is not necessarily thermodynamically consistent if each microscopic transition is not properly resolved or if the transition rates do not satisfy the local detailed balance condition.
 - [23] S. Katz, J. L. Lebowitz, and H. Spohn, Phase transitions in stationary nonequilibrium states of model lattice systems, *Phys. Rev. B* **28**, 1655 (1983).
 - [24] S. Katz, J. L. Lebowitz, and H. Spohn, Nonequilibrium steady states of stochastic lattice gas models of fast ionic conductors, *J. Stat. Phys.* **34**, 497 (1984).
 - [25] A. V. Ivlev, J. Bartnick, M. Heinen, C.-R. Du, V. Nosenko, and H. Löwen, Statistical mechanics where Newton’s third law is broken, *Phys. Rev. X* **5**, 011035 (2015).
 - [26] N. S. Goel, S. C. Maitra, and E. W. Montroll, On the volterra and other nonlinear models of interacting populations, *Rev. Mod. Phys.* **43**, 231 (1971).
 - [27] H. Rieger, Solvable model of a complex ecosystem with randomly interacting species, *J. Phys. A: Math. Gen.* **22**, 3447 (1989).

- [28] T. Reichenbach, M. Mobilia, and E. Frey, Noise and correlations in a spatial population model with cyclic competition, *Phys. Rev. Lett.* **99**, 238105 (2007).
- [29] J. Knebel, T. Krüger, M. F. Weber, and E. Frey, Coexistence and survival in conservative Lotka-Volterra networks, *Phys. Rev. Lett.* **110**, 168106 (2013).
- [30] J. Knebel, M. F. Weber, T. Krüger, and E. Frey, Evolutionary games of condensates in coupled birth–death processes, *Nat. Commun.* **6**, 6977 (2015).
- [31] A. Amir, N. Hatano, and D. R. Nelson, Non-Hermitian localization in biological networks, *Phys. Rev. E* **93**, 042310 (2016).
- [32] G. Bunin, Ecological communities with Lotka-Volterra dynamics, *Phys. Rev. E* **95**, 042414 (2017).
- [33] P. M. Geiger, J. Knebel, and E. Frey, Topologically robust zero-sum games and Pfaffian orientation: How network topology determines the long-time dynamics of the antisymmetric Lotka-Volterra equation, *Phys. Rev. E* **98**, 062316 (2018).
- [34] J. Knebel, P. M. Geiger, and E. Frey, Topological phase transition in coupled rock-paper-scissors cycles, *Phys. Rev. Lett.* **125**, 258301 (2020).
- [35] T. Yoshida, T. Mizoguchi, and Y. Hatsugai, Non-Hermitian topology in rock–paper–scissors games, *Sci. Rep.* **12**, 560 (2022).
- [36] T. Yoshida, T. Mizoguchi, and Y. Hatsugai, Chiral edge modes in evolutionary game theory: A kagome network of rock-paper-scissors cycles, *Phys. Rev. E* **104**, 025003 (2021).
- [37] E. Tang, J. Agudo-Canalejo, and R. Golestanian, Topology protects chiral edge currents in stochastic systems, *Phys. Rev. X* **11**, 031015 (2021).
- [38] B. Mahault, E. Tang, and R. Golestanian, A topological fluctuation theorem, *Nat. Commun.* **13**, 3036 (2022).
- [39] A. Nelson and E. Tang, Nonreciprocity is necessary for robust dimensional reduction and strong responses in stochastic topological systems, *Phys. Rev. B* **110**, 155116 (2024).
- [40] H. Sompolinsky and I. Kanter, Temporal association in asymmetric neural networks, *Phys. Rev. Lett.* **57**, 2861 (1986).
- [41] R. Kree and A. Zippelius, Continuous-time dynamics of asymmetrically diluted neural networks, *Phys. Rev. A* **36**, 4421 (1987).
- [42] A. Crisanti and H. Sompolinsky, Dynamics of spin systems with randomly asymmetric bonds: Ising spins and Glauber dynamics, *Phys. Rev. A* **37**, 4865 (1988).
- [43] H. Sompolinsky, A. Crisanti, and H. J. Sommers, Chaos in random neural networks, *Phys. Rev. Lett.* **61**, 259 (1988).
- [44] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, Spectrum of large random asymmetric matrices, *Phys. Rev. Lett.* **60**, 1895 (1988).
- [45] H. Rieger, M. Schreckenberg, and J. Zittartz, Glauber dynamics of neural network models, *J. Phys. A: Math. Gen.* **21**, L263 (1988).
- [46] H. Rieger, M. Schreckenberg, and J. Zittartz, Glauber dynamics of the asymmetric SK-model, *Z. Phys. B: Condens. Matter* **74**, 527 (1989).
- [47] S. A. M. Loos and S. H. L. Klapp, Irreversibility, heat and information flows induced by non-reciprocal interactions, *New J. Phys.* **22**, 123051 (2020).
- [48] M. Durve, A. Saha, and A. Sayeed, Active particle condensation by non-reciprocal and time-delayed interactions, *Eur. Phys. J. E* **41**, 49 (2018).
- [49] C. Scheibner, A. Souslov, D. Banerjee, P. Surówka, W. T. M. Irvine, and V. Vitelli, Odd elasticity, *Nat. Phys.* **16**, 475 (2020).
- [50] R. Soto and R. Golestanian, Self-assembly of catalytically active colloidal molecules: Tailoring activity through surface chemistry, *Phys. Rev. Lett.* **112**, 068301 (2014).
- [51] J. Agudo-Canalejo and R. Golestanian, Active phase separation in mixtures of chemically interacting particles, *Phys. Rev. Lett.* **123**, 018101 (2019).
- [52] K. L. Kreienkamp and S. H. L. Klapp, Clustering and flocking of repulsive chiral active particles with non-reciprocal couplings, *New J. Phys.* **24**, 123009 (2022).
- [53] M. Knežević, T. Welker, and H. Stark, Collective motion of active particles exhibiting non-reciprocal orientational interactions, *Sci. Rep.* **12**, 19437 (2022).
- [54] M. Fruchart, R. Hanai, P. B. Littlewood, and V. Vitelli, Non-reciprocal phase transitions, *Nature (London)* **592**, 363 (2021).
- [55] D. Martin, D. Seara, Y. Avni, M. Fruchart, and V. Vitelli, Transition to collective motion in nonreciprocal active matter: Coarse graining agent-based models into fluctuating hydrodynamics, *Phys. Rev. X* **15**, 041015 (2025).
- [56] M. Mangeat, S. Chatterjee, J. D. Noh, and H. Rieger, Emergent complex phases in a discrete flocking model with reciprocal and non-reciprocal interactions, *Commun. Phys.* **8**, 186 (2025).
- [57] J. Choi, J. D. Noh, and H. Rieger, Flocking with random nonreciprocal interactions, *Phys. Rev. Res.* **7**, L042059 (2025).
- [58] C.-U. Woo, H. Rieger, and J. D. Noh, Collective behavior in the nonreciprocal multi-species Vicsek model, [arXiv:2512.18756](https://arxiv.org/abs/2512.18756).
- [59] C.-U. Woo, H. Rieger, and J. D. Noh, Nonreciprocal yet symmetric multi-species active matter: Emergence of chirality and species separation, [arXiv:2512.18749](https://arxiv.org/abs/2512.18749).
- [60] C.-U. Woo, J. D. Noh, and H. Rieger, Extensive spatio-temporal chaos in non-reciprocal flocking, [arXiv:2604.06462](https://arxiv.org/abs/2604.06462).
- [61] R. Hanai, Nonreciprocal frustration: Time crystalline order-by-disorder phenomenon and a spin-glass-like state, *Phys. Rev. X* **14**, 011029 (2024).
- [62] Z. You, A. Baskaran, and M. C. Marchetti, Nonreciprocity as a generic route to traveling states, *Proc. Natl. Acad. Sci. USA* **117**, 19767 (2020).
- [63] S. Saha, J. Agudo-Canalejo, and R. Golestanian, Scalar active mixtures: The nonreciprocal Cahn-Hilliard model, *Phys. Rev. X* **10**, 041009 (2020).
- [64] S. Saha and R. Golestanian, Effervescence in a binary mixture with nonlinear non-reciprocal interactions, *Nat. Commun.* **16**, 7310 (2025).
- [65] T. Frohoff-Hülsmann and U. Thiele, Nonreciprocal Cahn-Hilliard model emerges as a universal amplitude equation, *Phys. Rev. Lett.* **131**, 107201 (2023).
- [66] T. Frohoff-Hülsmann and U. Thiele, Localized states in coupled Cahn-Hilliard equations, *IMA J. Appl. Math.* **86**, 924 (2021).
- [67] T. Frohoff-Hülsmann, J. Wrembel, and U. Thiele, Suppression of coarsening and emergence of oscillatory behavior in a Cahn-Hilliard model with nonvariational coupling, *Phys. Rev. E* **103**, 042602 (2021).
- [68] F. Brauns and M. C. Marchetti, Nonreciprocal pattern formation of conserved fields, *Phys. Rev. X* **14**, 021014 (2024).
- [69] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena, *Rev. Mod. Phys.* **49**, 435 (1977).

- [70] M. C. Cross and P. C. Hohenberg, Pattern formation outside of equilibrium, *Rev. Mod. Phys.* **65**, 851 (1993).
- [71] K. Kawasaki, Stochastic model of slow dynamics in supercooled liquids and dense colloidal suspensions, *Phys. A (Amsterdam, Neth.)* **208**, 35 (1994).
- [72] D. S. Dean, Langevin equation for the density of a system of interacting Langevin processes, *J. Phys. A: Math. Gen.* **29**, L613 (1996).
- [73] A. P. Solon and J. Tailleur, Revisiting the flocking transition using active spins, *Phys. Rev. Lett.* **111**, 078101 (2013).
- [74] P. Pietzonka and U. Seifert, Entropy production of active particles and for particles in active baths, *J. Phys. A: Math. Theor.* **51**, 01LT01 (2018).
- [75] T. Speck, Efficiency of isothermal active matter engines: Strong driving beats weak driving, *Phys. Rev. E* **105**, L012601 (2022).
- [76] R. Bebon, J. F. Robinson, and T. Speck, Thermodynamics of active matter: Tracking dissipation across scales, *Phys. Rev. X* **15**, 021050 (2025).
- [77] A. T. Mohite and H. Rieger, Thermodynamics of flocking models (unpublished).
- [78] D. Mandal, K. Klymko, and M. R. DeWeese, Entropy production and fluctuation theorems for active matter, *Phys. Rev. Lett.* **119**, 258001 (2017).
- [79] L. Dabelow, S. Bo, and R. Eichhorn, Irreversibility in active matter systems: Fluctuation theorem and mutual information, *Phys. Rev. X* **9**, 021009 (2019).
- [80] Q. Yu and Y. Tu, Energy cost for flocking of active spins: The cusped dissipation maximum at the flocking transition, *Phys. Rev. Lett.* **129**, 278001 (2022).
- [81] T. Suchanek, K. Kroy, and S. A. M. Loos, Irreversible mesoscale fluctuations herald the emergence of dynamical phases, *Phys. Rev. Lett.* **131**, 258302 (2023).
- [82] H. Alston, L. Cocconi, and T. Bertrand, Irreversibility across a nonreciprocal $\mathcal{PT}$ -symmetry-breaking phase transition, *Phys. Rev. Lett.* **131**, 258301 (2023).
- [83] T. Suchanek, K. Kroy, and S. A. M. Loos, Entropy production in the nonreciprocal Cahn-Hilliard model, *Phys. Rev. E* **108**, 064610 (2023).
- [84] T. Suchanek, K. Kroy, and S. A. M. Loos, Time-reversal and parity-time symmetry breaking in non-Hermitian field theories, *Phys. Rev. E* **108**, 064123 (2023).
- [85] R. Kawai, J. M. R. Parrondo, and C. V. den Broeck, Dissipation: The phase-space perspective, *Phys. Rev. Lett.* **98**, 080602 (2007).
- [86] J. Horowitz and C. Jarzynski, Illustrative example of the relationship between dissipation and relative entropy, *Phys. Rev. E* **79**, 021106 (2009).
- [87] A. C. Barato, D. Hartich, and U. Seifert, Information-theoretic versus thermodynamic entropy production in autonomous sensory networks, *Phys. Rev. E* **87**, 042104 (2013).
- [88] B. Gaveau, L. Granger, M. Moreau, and L. S. Schulman, Dissipation, interaction, and relative entropy, *Phys. Rev. E* **89**, 032107 (2014).
- [89] B. Gaveau, L. Granger, M. Moreau, and L. S. Schulman, Relative entropy, interaction energy and the nature of dissipation, *Entropy* **16**, 3173 (2014).
- [90] I. A. Martínez, G. Bisker, J. M. Horowitz, and J. M. R. Parrondo, Inferring broken detailed balance in the absence of observable currents, *Nat. Commun.* **10**, 3542 (2019).
- [91] E. Roldan, J. Barral, P. Martin, J. M. R. Parrondo, and F. Jülicher, Quantifying entropy production in active fluctuations of the hair-cell bundle from time irreversibility and uncertainty relations, *New J. Phys.* **23**, 083013 (2021).
- [92] A. T. Mohite and H. Rieger, Thermodynamically consistent coarse-graining: From interacting particles to fields via second quantization, [arXiv:2508.11430](https://arxiv.org/abs/2508.11430).
- [93] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, *Rev. Mod. Phys.* **87**, 593 (2015).
- [94] A. T. Mohite and H. Rieger, Minimum action principle for entropy production rate of far-from-equilibrium systems, [arXiv:2511.00967](https://arxiv.org/abs/2511.00967).
- [95] A. T. Mohite and H. Rieger, Thermodynamic length in stochastic thermodynamics of far-from-equilibrium systems: Unification of fluctuation relation and thermodynamic uncertainty relation, *Phys. Rev. E* (2026), doi:10.1103/fmlt-px9y.
- [96] A. M. Turing, The chemical basis of morphogenesis, *Philos. Trans. R Soc. Lond. B Biol. Sci.* **237**, 37 (1952).
- [97] J. Hattne, D. Fange, and J. Elf, Stochastic reaction-diffusion simulation with mesord, *Bioinformatics* **21**, 2923 (2005).
- [98] J. Elf and M. Ehrenberg, Spontaneous separation of bi-stable biochemical systems into spatial domains of opposite phases, *IEE Proc. Syst. Bio.* **1**, 230 (2004).
- [99] D. T. Gillespie, L. R. Petzold, and E. Seitaridou, Validity conditions for stochastic chemical kinetics in diffusion-limited systems, *J. Chem. Phys.* **140**, 054111 (2014).
- [100] Despite the equivalence of $\epsilon_i^\#$ to a chemical potential, we avoid calling it so. Rather, it appears in Eq. (4) as the argument in the exponential of the transition rates for a single particle. If all particles are reciprocal, the usual physical meaning of the chemical potential is recovered. However, in principle, what we refer to as the ‘Boltzmann weight’ throughout this work is mathematically and physically equivalent to the ‘chemical potential,’ but defined for a non-reciprocal and far-from-equilibrium system. This remark clarifies the nomenclature.
- [101] R. J. Glauber, Time-dependent statistics of the Ising model, *J. Math. Phys.* **4**, 294 (1963).
- [102] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, *J. Chem. Phys.* **21**, 1087 (1953).
- [103] D. T. Gillespie, Exact stochastic simulation of coupled chemical reactions, *J. Phys. Chem.* **81**, 2340 (1977).
- [104] R. Spinney and I. Ford, Fluctuation relations: A pedagogical overview, *Nonequilibrium Statistical Physics of Small Systems* (John Wiley & Sons, Ltd, Weinheim, Germany, 2013), Chap. 1, pp. 3–56.
- [105] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, North-Holland, 1981).
- [106] C. W. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences* (Springer, 2009).
- [107] P. G. Bergmann and J. L. Lebowitz, New approach to nonequilibrium processes, *Phys. Rev.* **99**, 578 (1955).
- [108] J. L. Lebowitz and P. G. Bergmann, Irreversible Gibbsian ensembles, *Ann. Phys. (NY)* **1**, 1 (1957).
- [109] H. Qian, Relative entropy: Free energy associated with equilibrium fluctuations and nonequilibrium deviations, *Phys. Rev. E* **63**, 042103 (2001).
- [110] F. Schlögl, On stability of steady states, *Z. Phys.* **243**, 303 (1971).

- [111] S. Amari and H. Nagaoka, *Methods of Information Geometry, Translations of Mathematical Monographs* (American Mathematical Society, 2000).
- [112] H. Ge, Extended forms of the second law for general time-dependent stochastic processes, *Phys. Rev. E* **80**, 021137 (2009).
- [113] H. Ge and H. Qian, Physical origins of entropy production, free energy dissipation, and their mathematical representations, *Phys. Rev. E* **81**, 051133 (2010).
- [114] N. Shiraishi and K. Saito, Information-theoretical bound of the irreversibility in thermal relaxation processes, *Phys. Rev. Lett.* **123**, 110603 (2019).
- [115] Y. Oono and M. Paniconi, Steady state thermodynamics, *Prog. Theor. Phys. Suppl.* **130**, 29 (1998).
- [116] C. Pérez-Espigares, A. B. Kolton, and J. Kurchan, Infinite family of second-law-like inequalities, *Phys. Rev. E* **85**, 031135 (2012).
- [117] C. Maes and K. Netočný, Minimum entropy production principle from a dynamical fluctuation law, *J. Math. Phys.* **48**, 053306 (2007).
- [118] C. Maes and K. Netočný, A nonequilibrium extension of the Clausius heat theorem, *J. Stat. Phys.* **154**, 188 (2014).
- [119] A. Dechant, S.-I. Sasa, and S. Ito, Geometric decomposition of entropy production in out-of-equilibrium systems, *Phys. Rev. Res.* **4**, L012034 (2022).
- [120] M. Ding, F. Liu, and X. Xing, Unified theory of thermodynamics and stochastic thermodynamics for nonlinear Langevin systems driven by non-conservative forces, *Phys. Rev. Res.* **4**, 043125 (2022).
- [121] A. Dechant, S.-i. Sasa, and S. Ito, Geometric decomposition of entropy production into excess, housekeeping, and coupling parts, *Phys. Rev. E* **106**, 024125 (2022).
- [122] K. Yoshimura, A. Kolchinsky, A. Dechant, and S. Ito, Housekeeping and excess entropy production for general nonlinear dynamics, *Phys. Rev. Res.* **5**, 013017 (2023).
- [123] R. Nagayama, K. Yoshimura, A. Kolchinsky, and S. Ito, Geometric thermodynamics of reaction-diffusion systems: Thermodynamic trade-off relations and optimal transport for pattern formation, *Phys. Rev. Res.* **7**, 033011 (2025).
- [124] M. Doi, Second quantization representation for classical many-particle system, *J. Phys. A: Math. Gen.* **9**, 1465 (1976).
- [125] M. Doi, Stochastic theory of diffusion-controlled reaction, *J. Phys. A: Math. Gen.* **9**, 1479 (1976).
- [126] L. Peliti, Path integral approach to birth-death processes on a lattice, *J. Phys. France* **46**, 1469 (1985).
- [127] L. Peliti, Renormalisation of fluctuation effects in the $A + A$ to A reaction, *J. Phys. A: Math. Gen.* **19**, L365 (1986).
- [128] P. Grassberger and M. Scheunert, Fock-space methods for identical classical objects, *Fortschr. Phys.* **28**, 547 (1980).
- [129] H. A. Rose, Renormalized kinetic theory of nonequilibrium many-particle classical systems, *J. Stat. Phys.* **20**, 415 (1979).
- [130] A. Mikhailov, Path integrals in chemical kinetics I, *Phys. Lett. A* **85**, 214 (1981).
- [131] A. Mikhailov, Path integrals in chemical kinetics II, *Phys. Lett. A* **85**, 427 (1981).
- [132] A. S. Mikhailov and V. V. Yashin, Quantum field methods in the theory of diffusion-controlled reactions, *J. Stat. Phys.* **38**, 347 (1985).
- [133] J. Cardy, G. Falkovich, and K. Gawedzki, in *Non-equilibrium Statistical Mechanics and Turbulence*, edited by S. Nazarenko and O. V. Zaboronski, London Mathematical Society Lecture Note Series (Cambridge University Press, 2008), pp. 108–161.
- [134] M. Howard and J. Cardy, Fluctuation effects and multiscaling of the reaction-diffusion front for $A + B$ to OE , *J. Phys. A: Math. Gen.* **28**, 3599 (1995).
- [135] K. J. Wiese, Coherent-state path integral versus coarse-grained effective stochastic equation of motion: From reaction diffusion to stochastic sandpiles, *Phys. Rev. E* **93**, 042117 (2016).
- [136] U. C. Täuber, *Critical Dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior* (Cambridge University Press, 2014).
- [137] M. F. Weber and E. Frey, Master equations and the theory of stochastic path integrals, *Rep. Prog. Phys.* **80**, 046601 (2017).
- [138] H. Touchette, The large deviation approach to statistical mechanics, *Phys. Rep.* **478**, 1 (2009).
- [139] S. Chatterjee, M. Mangeat, R. Paul, and H. Rieger, Flocking and reorientation transition in the 4-state active Potts model, *EPL* **130**, 66001 (2020).
- [140] M. Mangeat, S. Chatterjee, R. Paul, and H. Rieger, Flocking with a q -fold discrete symmetry: Band-to-lane transition in the active Potts model, *Phys. Rev. E* **102**, 042601 (2020).
- [141] S. Chatterjee, M. Mangeat, and H. Rieger, Polar flocks with discretized directions: The active clock model approaching the Vicsek model, *Europhys. Lett.* **138**, 41001 (2022).
- [142] M. Karmakar, S. Chatterjee, M. Mangeat, H. Rieger, and R. Paul, Jamming and flocking in the restricted active Potts model, *Phys. Rev. E* **108**, 014604 (2023).
- [143] C.-U. Woo and J. D. Noh, Motility-induced pinning in flocking system with discrete symmetry, *Phys. Rev. Lett.* **133**, 188301 (2024).
- [144] C. Maes, Frenesy: Time-symmetric dynamical activity in nonequilibria, *Phys. Rep.* **850**, 1 (2020).
- [145] M. O. Vlad and J. Ross, Thermodynamic approach to nonequilibrium chemical fluctuations, *J. Chem. Phys.* **100**, 7295 (1994).
- [146] M. O. Vlad and J. Ross, Random paths and fluctuation–dissipation dynamics for one-variable chemical systems far from equilibrium, *J. Chem. Phys.* **100**, 7279 (1994).
- [147] M. O. Vlad and J. Ross, Fluctuation–dissipation relations for chemical systems far from equilibrium, *J. Chem. Phys.* **100**, 7268 (1994).
- [148] J. Ross and M. O. Vlad, Nonlinear kinetics and new approaches to complex reaction mechanisms, *Annu. Rev. Phys. Chem.* **50**, 51 (1999).
- [149] H. Ge and H. Qian, Thermodynamic limit of a nonequilibrium steady state: Maxwell-type construction for a bistable biochemical system, *Phys. Rev. Lett.* **103**, 148103 (2009).
- [150] M. Vellela and H. Qian, Stochastic dynamics and nonequilibrium thermodynamics of a bistable chemical system: The Schlögl model revisited, *J. R. Soc. Interface* **6**, 925 (2009).
- [151] H. Ge and H. Qian, Non-equilibrium phase transition in mesoscopic biochemical systems: From stochastic to nonlinear dynamics and beyond, *J. R. Soc. Interface* **8**, 107 (2011).
- [152] B. Nguyen, U. Seifert, and A. C. Barato, Phase transition in thermodynamically consistent biochemical oscillators, *J. Chem. Phys.* **149**, 045101 (2018).
- [153] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory for stationary nonequilibrium states, *J. Stat. Phys.* **107**, 635 (2002).

- [154] L. Bertini, A. D. Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Lagrangian phase transitions in nonequilibrium thermodynamic systems, *J. Stat. Mech.* (2010) L11001.
- [155] H. Qian, Y.-C. Cheng, and Y.-J. Yang, Kinematic basis of emergent energetics of complex dynamics, *EPL* **131**, 50002 (2020).
- [156] A. Mielke, R. I. A. Patterson, M. A. Peletier, and D. R. M. Renger, Non-equilibrium thermodynamical principles for chemical reactions with mass-action kinetics, *SIAM J. Appl. Math.* **77**, 1562 (2017).
- [157] M. Kaiser, R. L. Jack, and J. Zimmer, Canonical structure and orthogonality of forces and currents in irreversible Markov chains, *J. Stat. Phys.* **170**, 1019 (2018).
- [158] M. A. Peletier, R. Rossi, G. Savaré, and O. Tse, Jump processes as generalized gradient flows, *Calc. Var.* **61**, 33 (2022).
- [159] D. R. M. Renger and J. Zimmer, Orthogonality of fluxes in general nonlinear reaction networks, *Discrete Contin. Dyn. Syst.* **14**, 205 (2021).
- [160] R. I. A. Patterson, D. R. M. Renger, and U. Sharma, Variational structures beyond gradient flows: A macroscopic fluctuation-theory perspective, *J. Stat. Phys.* **191**, 18 (2024).
- [161] M. A. Peletier and A. Schlichting, Cosh gradient systems and tilting, *Nonlinear Anal.* **231**, 113094 (2023).
- [162] T. J. Kobayashi, D. Loutchko, A. Kamimura, and Y. Sughiyama, Hessian geometry of nonequilibrium chemical reaction networks and entropy production decompositions, *Phys. Rev. Res.* **4**, 033208 (2022).
- [163] T. J. Kobayashi, D. Loutchko, A. Kamimura, and Y. Sughiyama, Kinetic derivation of the Hessian geometric structure in chemical reaction networks, *Phys. Rev. Res.* **4**, 033066 (2022).
- [164] Y. Sughiyama, D. Loutchko, A. Kamimura, and T. J. Kobayashi, Hessian geometric structure of chemical thermodynamic systems with stoichiometric constraints, *Phys. Rev. Res.* **4**, 033065 (2022).
- [165] T. J. Kobayashi, D. Loutchko, A. Kamimura, S. A. Horiguchi, and Y. Sughiyama, Information geometry of dynamics on graphs and hypergraphs, *Inf. Geom.* **7**, 97 (2024).
- [166] D. Loutchko, Y. Sughiyama, and T. J. Kobayashi, The geometry of thermodynamic uncertainty relations in chemical reaction networks [arXiv:2308.04806](#).
- [167] D. R. M. Renger and U. Sharma, Untangling dissipative and Hamiltonian effects in bulk and boundary-driven systems, *Phys. Rev. E* **108**, 054123 (2023).
- [168] M. H. Duong and J. Zimmer, On decompositions of non-reversible processes, *J. Phys.: Conf. Ser.* **2514**, 012007 (2023).
- [169] T. Mizohata, T. J. Kobayashi, L.-S. Bouchard, and H. Miyahara, Information geometric bound on general chemical reaction networks, *Phys. Rev. E* **109**, 044308 (2024).
- [170] C. Maes and K. Netočný, Canonical structure of dynamical fluctuations in mesoscopic nonequilibrium steady states, *Europhys. Lett.* **82**, 30003 (2008).
- [171] C. Maes, K. Netočný, and B. Wynants, Steady state statistics of driven diffusions, *Phys. A (Amsterdam, Neth.)* **387**, 2675 (2008).
- [172] K. Tomita and H. Tomita, Irreversible circulation of fluctuation, *Prog. Theor. Phys.* **51**, 1731 (1974).
- [173] J. M. Horowitz and T. R. Gingrich, Thermodynamic uncertainty relations constrain non-equilibrium fluctuations, *Nat. Phys.* **16**, 15 (2020).
- [174] N. Ohga, S. Ito, and A. Kolchinsky, Thermodynamic bound on the asymmetry of cross-correlations, *Phys. Rev. Lett.* **131**, 077101 (2023).
- [175] A. Kolchinsky, N. Ohga, and S. Ito, Thermodynamic bound on spectral perturbations, with applications to oscillations and relaxation dynamics, *Phys. Rev. Res.* **6**, 013082 (2024).
- [176] A. Dechant, J. Garnier-Brun, and S.-I. Sasa, Thermodynamic bounds on correlation times, *Phys. Rev. Lett.* **131**, 167101 (2023).
- [177] A. Dechant, Thermodynamic constraints on the power spectral density in and out of equilibrium, [arXiv:2306.00417](#).
- [178] N. Shiraishi, Entropy production limits all fluctuation oscillations, *Phys. Rev. E* **108**, L042103 (2023).
- [179] S. Liang and S. Pigolotti, Thermodynamic bounds on time-reversal asymmetry, *Phys. Rev. E* **108**, L062101 (2023).
- [180] T. Van Vu, V. T. Vo, and K. Saito, Dissipation, quantum coherence, and asymmetry of finite-time cross-correlations, *Phys. Rev. Res.* **6**, 013273 (2024).
- [181] M. Freidlin, J. Szücs, and A. Wentzell, *Random Perturbations of Dynamical Systems*, Grundlehren der mathematischen Wissenschaften (Springer, Berlin, Heidelberg, 2012).
- [182] R. Kubo, K. Matsuo, and K. Kitahara, Fluctuation and relaxation of macrovariables, *J. Stat. Phys.* **9**, 51 (1973).
- [183] R. Graham and T. Tél, Existence of a potential for dissipative dynamical systems, *Phys. Rev. Lett.* **52**, 9 (1984).
- [184] R. Graham, Covariant formulation of non-equilibrium statistical thermodynamics, *Z. Phys. B* **26**, 397 (1977).
- [185] H. Gang, Stationary solution of master equations in the large-system-size limit, *Phys. Rev. A* **36**, 5782 (1987).
- [186] L. Onsager, Reciprocal relations in irreversible processes. I., *Phys. Rev.* **37**, 405 (1931).
- [187] L. Onsager, Reciprocal relations in irreversible processes. II., *Phys. Rev.* **38**, 2265 (1931).
- [188] E. T. Jaynes, The minimum entropy production principle, *Annu. Rev. Phys. Chem.* **31**, 579 (1980).
- [189] A. Einstein, *Investigations on the Theory of the Brownian Movement* (Dover, New York, 1905).
- [190] W. Sutherland, LXXV. A dynamical theory of diffusion for non-electrolytes and the molecular mass of albumin, *Philos. Mag.* **9**, 781 (1905).
- [191] H. B. Callen and T. A. Welton, Irreversibility and generalized noise, *Phys. Rev.* **83**, 34 (1951).
- [192] M. S. Green, Markoff random processes and the statistical mechanics of time-dependent phenomena, *J. Chem. Phys.* **20**, 1281 (1952).
- [193] M. S. Green, Markoff random processes and the statistical mechanics of time-dependent phenomena. II. Irreversible processes in fluids, *J. Chem. Phys.* **22**, 398 (1954).
- [194] R. Kubo, Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems, *J. Phys. Soc. Jpn.* **12**, 570 (1957).
- [195] T. Yamamoto, Quantum statistical mechanical theory of the rate of exchange chemical reactions in the gas phase, *J. Chem. Phys.* **33**, 281 (1960).
- [196] R. Zwanzig, Time-correlation functions and transport coefficients in statistical mechanics, *Annu. Rev. Phys. Chem.* **16**, 67 (1965).
- [197] R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.* **29**, 255 (1966).

- [198] E. Helfand, Transport coefficients from dissipation in a canonical ensemble, *Phys. Rev.* **119**, 1 (1960).
- [199] L. F. Cugliandolo and J. Kurchan, Analytical solution of the off-equilibrium dynamics of a long-range spin-glass model, *Phys. Rev. Lett.* **71**, 173 (1993).
- [200] T. Bodineau and B. Derrida, Current fluctuations in nonequilibrium diffusive systems: An additivity principle, *Phys. Rev. Lett.* **92**, 180601 (2004).
- [201] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Fluctuations in stationary nonequilibrium states of irreversible processes, *Phys. Rev. Lett.* **87**, 040601 (2001).
- [202] E. Lippiello, F. Corberi, and M. Zannetti, Off-equilibrium generalization of the fluctuation dissipation theorem for Ising spins and measurement of the linear response function, *Phys. Rev. E* **71**, 036104 (2005).
- [203] T. Speck and U. Seifert, Restoring a fluctuation-dissipation theorem in a nonequilibrium steady state, *Europhys. Lett.* **74**, 391 (2006).
- [204] V. Blickle, T. Speck, C. Lutz, U. Seifert, and C. Bechinger, Einstein relation generalized to nonequilibrium, *Phys. Rev. Lett.* **98**, 210601 (2007).
- [205] M. Baiesi, C. Maes, and B. Wynants, Fluctuations and response of nonequilibrium states, *Phys. Rev. Lett.* **103**, 010602 (2009).
- [206] T. Speck and U. Seifert, Extended fluctuation-dissipation theorem for soft matter in stationary flow, *Phys. Rev. E* **79**, 040102(R) (2009).
- [207] F. Corberi, E. Lippiello, A. Sarracino, and M. Zannetti, Fluctuation-dissipation relations and field-free algorithms for the computation of response functions, *Phys. Rev. E* **81**, 011124 (2010).
- [208] A. Dechant and S.-i. Sasa, Fluctuation–response inequality out of equilibrium, *Proc. Natl. Acad. Sci. USA* **117**, 6430 (2020).
- [209] C. Maes, Response theory: A trajectory-based approach, *Front. Phys.* **8**, 229 (2020).
- [210] N. Shiraishi, Optimal thermodynamic uncertainty relation in Markov jump processes, *J. Stat. Phys.* **185**, 19 (2021).
- [211] Y.-J. Yang and H. Qian, Nonequilibrium stationary process and fluctuation-dissipation relations, *arXiv:2106.13374*.
- [212] L. Onsager and S. Machlup, Fluctuations and irreversible processes, *Phys. Rev.* **91**, 1505 (1953).
- [213] S. Machlup and L. Onsager, Fluctuations and irreversible process. II. Systems with kinetic energy, *Phys. Rev.* **91**, 1512 (1953).
- [214] D. Andrieux and P. Gaspard, A fluctuation theorem for currents and non-linear response coefficients, *J. Stat. Mech.* (2007) P02006.
- [215] K. Saito and Y. Utsumi, Symmetry in full counting statistics, fluctuation theorem, and relations among nonlinear transport coefficients in the presence of a magnetic field, *Phys. Rev. B* **78**, 115429 (2008).
- [216] P. Gaspard, Multivariate fluctuation relations for currents, *New J. Phys.* **15**, 115014 (2013).
- [217] D. M. Carberry, J. C. Reid, G. M. Wang, E. M. Sevick, D. J. Searles, and D. J. Evans, Fluctuations and irreversibility: An experimental demonstration of a second-law-like theorem using a colloidal particle held in an optical trap, *Phys. Rev. Lett.* **92**, 140601 (2004).
- [218] G. M. Wang, J. C. Reid, D. M. Carberry, D. R. M. Williams, E. M. Sevick, and D. J. Evans, Experimental study of the fluctuation theorem in a nonequilibrium steady state, *Phys. Rev. E* **71**, 046142 (2005).
- [219] G. Bochkov and Y. Kuzovlev, Nonlinear fluctuation-dissipation relations and stochastic models in nonequilibrium thermodynamics: I. Generalized fluctuation-dissipation theorem, *Phys. A (Amsterdam, Neth.)* **106**, 443 (1981).
- [220] G. N. Bochkov and Y. E. Kuzovlev, Nonlinear fluctuation-dissipation relations and stochastic models in nonequilibrium thermodynamics: II. Kinetic potential and variational principles for nonlinear irreversible processes, *Phys. A (Amsterdam, Neth.)* **106**, 480 (1981).
- [221] G. Gallavotti, Extension of Onsager’s reciprocity to large fields and the chaotic hypothesis, *Phys. Rev. Lett.* **77**, 4334 (1996).
- [222] D. Andrieux and P. Gaspard, Fluctuation theorem and Onsager reciprocity relations, *J. Chem. Phys.* **121**, 6167 (2004).
- [223] R. Chetrite and S. Gupta, Two refreshing views of fluctuation theorems through kinematics elements and exponential martingale, *J. Stat. Phys.* **143**, 543 (2011).
- [224] Y.-J. Yang and H. Qian, Unified formalism for entropy production and fluctuation relations, *Phys. Rev. E* **101**, 022129 (2020).
- [225] L. Hong, H. Qian, and L. F. Thompson, Representations and divergences in the space of probability measures and stochastic thermodynamics, *J. Comp. App. Math.* **376**, 112842 (2020).
- [226] H. Qian, Y.-C. Cheng, and L. F. Thompson, Ternary representation of stochastic change and the origin of entropy and its fluctuations, *arXiv:1902.09536*.
- [227] D.-Q. Jiang, M. Qian, and F.-X. Zhang, Entropy production fluctuations of finite Markov chains, *J. Math. Phys.* **44**, 4176 (2003).
- [228] P. Talkner, M. Campisi, and P. Hänggi, Fluctuation theorems in driven open quantum systems, *J. Stat. Mech.* (2009) P02025.
- [229] R. García-García, D. Domínguez, V. Lecomte, and A. B. Kolton, Unifying approach for fluctuation theorems from joint probability distributions, *Phys. Rev. E* **82**, 030104(R) (2010).
- [230] R. García-García, V. Lecomte, A. B. Kolton, and D. Domínguez, Joint probability distributions and fluctuation theorems, *J. Stat. Mech.* (2012) P02009.
- [231] S. Lahiri and A. M. Jayannavar, Derivation of not-so-common fluctuation theorems, *Indian J. Phys.* **89**, 515 (2015).
- [232] R. García-García, S. Lahiri, and D. Lacoste, Thermodynamic inference based on coarse-grained data or noisy measurements, *Phys. Rev. E* **93**, 032103 (2016).
- [233] P. Gaspard, Fluctuation theorem for nonequilibrium reactions, *J. Chem. Phys.* **120**, 8898 (2004).
- [234] C. Jarzynski and D. K. Wójcik, Classical and quantum fluctuation theorems for heat exchange, *Phys. Rev. Lett.* **92**, 230602 (2004).
- [235] P. Gaspard, Time-reversed dynamical entropy and irreversibility in Markovian random processes, *J. Stat. Phys.* **117**, 599 (2004).
- [236] D. Andrieux and P. Gaspard, Fluctuation theorem for transport in mesoscopic systems, *J. Stat. Mech.* (2006) P01011.
- [237] D. Andrieux and P. Gaspard, Fluctuation theorem for currents and Schnakenberg network theory, *J. Stat. Phys.* **127**, 107 (2007).
- [238] D. Andrieux and P. Gaspard, Network and thermodynamic conditions for a single macroscopic current fluctuation theorem, *C. R. Phys.* **8**, 579 (2007).

- [239] A. Faggionato and D. DiPietro, Gallavotti–Cohen-type symmetry related to cycle decompositions for Markov chains and biochemical applications, *J. Stat. Phys.* **143**, 11 (2011).
- [240] P. I. Hurtado, C. Pérez-Espigares, J. J. del Pozo, and P. L. Garrido, Symmetries in fluctuations far from equilibrium, *Proc. Natl. Acad. Sci. USA* **108**, 7704 (2011).
- [241] P. Sartori and S. Pigolotti, Thermodynamics of error correction, *Phys. Rev. X* **5**, 041039 (2015).
- [242] A. Saha, S. Lahiri, and A. M. Jayannavar, Entropy production theorems and some consequences, *Phys. Rev. E* **80**, 011117 (2009).
- [243] S. Deffner and E. Lutz, Nonequilibrium entropy production for open quantum systems, *Phys. Rev. Lett.* **107**, 140404 (2011).
- [244] S. Lahiri, S. Rana, and A. M. Jayannavar, Fluctuation theorems in the presence of information gain and feedback, *J. Phys. A: Math. Theor.* **45**, 065002 (2012).
- [245] M. Esposito and C. Van den Broeck, Three detailed fluctuation theorems, *Phys. Rev. Lett.* **104**, 090601 (2010).
- [246] T. Hatano and S.-i. Sasa, Steady-state thermodynamics of Langevin systems, *Phys. Rev. Lett.* **86**, 3463 (2001).
- [247] T. Speck and U. Seifert, Integral fluctuation theorem for the housekeeping heat, *J. Phys. A: Math. Gen.* **38**, L581 (2005).
- [248] A. C. Barato and U. Seifert, Thermodynamic uncertainty relation for biomolecular processes, *Phys. Rev. Lett.* **114**, 158101 (2015).
- [249] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Dissipation bounds all steady-state current fluctuations, *Phys. Rev. Lett.* **116**, 120601 (2016).
- [250] J. M. Horowitz and T. R. Gingrich, Proof of the finite-time thermodynamic uncertainty relation for steady-state currents, *Phys. Rev. E* **96**, 020103(R) (2017).
- [251] T. R. Gingrich and J. M. Horowitz, Fundamental bounds on first passage time fluctuations for currents, *Phys. Rev. Lett.* **119**, 170601 (2017).
- [252] J. P. Garrahan, Simple bounds on fluctuations and uncertainty relations for first-passage times of counting observables, *Phys. Rev. E* **95**, 032134 (2017).
- [253] D. J. Skinner and J. Dunkel, Estimating entropy production from waiting time distributions, *Phys. Rev. Lett.* **127**, 198101 (2021).
- [254] D. J. Skinner and J. Dunkel, Improved bounds on entropy production in living systems, *Proc. Natl. Acad. Sci. USA* **118**, e2024300118 (2021).
- [255] S. K. Manikandan, D. Gupta, and S. Krishnamurthy, Inferring entropy production from short experiments, *Phys. Rev. Lett.* **124**, 120603 (2020).
- [256] S. Otsubo, S. Ito, A. Dechant, and T. Sagawa, Estimating entropy production by machine learning of short-time fluctuating currents, *Phys. Rev. E* **101**, 062106 (2020).
- [257] T. Van Vu, V. T. Vo, and Y. Hasegawa, Entropy production estimation with optimal current, *Phys. Rev. E* **101**, 042138 (2020).
- [258] D.-K. Kim, Y. Bae, S. Lee, and H. Jeong, Learning entropy production via neural networks, *Phys. Rev. Lett.* **125**, 140604 (2020).
- [259] S. Otsubo, S. K. Manikandan, T. Sagawa, and S. Krishnamurthy, Estimating time-dependent entropy production from non-equilibrium trajectories, *Commun. Phys.* **5**, 11 (2022).
- [260] A. Dechant and S.-i. Sasa, Improving thermodynamic bounds using correlations, *Phys. Rev. X* **11**, 041061 (2021).
- [261] I. D. Terlizzi and M. Baiesi, Kinetic uncertainty relation, *J. Phys. A: Math. Theor.* **52**, 02LT03 (2019).
- [262] C. Dieball and A. Godec, Direct route to thermodynamic uncertainty relations and their saturation, *Phys. Rev. Lett.* **130**, 087101 (2023).
- [263] V. T. Vo, T. V. Vu, and Y. Hasegawa, Unified thermodynamic–kinetic uncertainty relation, *J. Phys. A: Math. Theor.* **55**, 405004 (2022).
- [264] N. Shiraishi, K. Funo, and K. Saito, Speed limit for classical stochastic processes, *Phys. Rev. Lett.* **121**, 070601 (2018).
- [265] S. Ito, Stochastic thermodynamic interpretation of information geometry, *Phys. Rev. Lett.* **121**, 030605 (2018).
- [266] S. B. Nicholson, A. del Campo, and J. R. Green, Nonequilibrium uncertainty principle from information geometry, *Phys. Rev. E* **98**, 032106 (2018).
- [267] S. B. Nicholson, L. P. García-Pintos, A. del Campo, and J. R. Green, Time–information uncertainty relations in thermodynamics, *Nat. Phys.* **16**, 1211 (2020).
- [268] V. T. Vo, T. Van Vu, and Y. Hasegawa, Unified approach to classical speed limit and thermodynamic uncertainty relation, *Phys. Rev. E* **102**, 062132 (2020).
- [269] S. Ito and A. Dechant, Stochastic time evolution, information geometry, and the Cramér–Rao bound, *Phys. Rev. X* **10**, 021056 (2020).
- [270] K. Yoshimura and S. Ito, Thermodynamic uncertainty relation and thermodynamic speed limit in deterministic chemical reaction networks, *Phys. Rev. Lett.* **127**, 160601 (2021).
- [271] T. V. Vu and K. Saito, Thermodynamic unification of optimal transport: Thermodynamic uncertainty relation, minimum dissipation, and thermodynamic speed limits, *Phys. Rev. X* **13**, 011013 (2023).
- [272] N. Merhav and Y. Kafri, Statistical properties of entropy production derived from fluctuation theorems, *J. Stat. Mech.* (2010) P12022.
- [273] A. M. Timpanaro, G. Guarnieri, J. Goold, and G. T. Landi, Thermodynamic uncertainty relations from exchange fluctuation theorems, *Phys. Rev. Lett.* **123**, 090604 (2019).
- [274] Y. Hasegawa and T. Van Vu, Fluctuation theorem uncertainty relation, *Phys. Rev. Lett.* **123**, 110602 (2019).
- [275] G. Francica, Fluctuation theorems and thermodynamic uncertainty relations, *Phys. Rev. E* **105**, 014129 (2022).
- [276] G. Francica and L. Dell’Anna, Fluctuation theorems and expected utility hypothesis, *Phys. Rev. E* **109**, 014112 (2024).
- [277] F. M. Dannan, P. Neff, and C. Thiel, On the sum of squared logarithms inequality and related inequalities, *arXiv:1411.1290*.
- [278] J. S. Lee, S. Lee, H. Kwon, and H. Park, Speed limit for a highly irreversible process and tight finite-time Landauer’s bound, *Phys. Rev. Lett.* **129**, 120603 (2022).
- [279] T. Sagawa and M. Ueda, Generalized Jarzynski equality under nonequilibrium feedback control, *Phys. Rev. Lett.* **104**, 090602 (2010).
- [280] T. Sagawa and M. Ueda, Nonequilibrium thermodynamics of feedback control, *Phys. Rev. E* **85**, 021104 (2012).

- [281] P. Sartori, L. Granger, C. F. Lee, and J. M. Horowitz, Thermodynamic costs of information processing in sensory adaptation, *PLoS Comput. Biol.* **10**, e1003974 (2014).
- [282] J. M. R. Parrondo, J. M. Horowitz, and T. Sagawa, Thermodynamics of information, *Nat. Phys.* **11**, 131 (2015).
- [283] D.-Q. Jiang, M. Qian, and M.-P. Qian, *Mathematical Theory of Nonequilibrium Steady States: On the Frontier of Probability and Dynamical Systems* (Springer Science & Business Media, 2004).
- [284] S. L. Kalpazidou, *Cycle Representations of Markov Processes* (Springer Science & Business Media, 2007), Vol. 28.
- [285] B. Altaner, S. Grosskinsky, S. Herminghaus, L. Kathän, M. Timme, and J. Vollmer, Network representations of nonequilibrium steady states: Cycle decompositions, symmetries, and dominant paths, *Phys. Rev. E* **85**, 041133 (2012).
- [286] H. Ge and H. Qian, Nonequilibrium thermodynamic formalism of nonlinear chemical reaction systems with Waage–Guldberg’s law of mass action, *Chem. Phys.* **472**, 241 (2016).
- [287] A. M. Miangolarra and M. Castellana, On non-ideal chemical-reaction networks and phase separation, *J. Stat. Phys.* **190**, 23 (2023).
- [288] F. Brauns, J. Halatek, and E. Frey, Phase-space geometry of mass-conserving reaction-diffusion dynamics, *Phys. Rev. X* **10**, 041036 (2020).
- [289] S. Dal Cengio, V. Lecomte, and M. Poletini, Geometry of nonequilibrium reaction networks, *Phys. Rev. X* **13**, 021040 (2023).
- [290] M. Feinberg, Complex balancing in general kinetic systems, *Arch. Ration. Mech. Anal.* **49**, 187 (1972).
- [291] F. Horn, Necessary and sufficient conditions for complex balancing in chemical kinetics, *Arch. Ration. Mech. Anal.* **49**, 172 (1972).
- [292] M. Feinberg, The existence and uniqueness of steady states for a class of chemical reaction networks, *Arch. Ration. Mech. Anal.* **132**, 311 (1995).
- [293] J. Gunawardena, Chemical reaction network theory for in-silico biologists, Notes available for download at <http://vcp.med.harvard.edu/papers/crnt.pdf> (2003).
- [294] D. F. Anderson, G. Craciun, and T. G. Kurtz, Product-form stationary distributions for deficiency zero chemical reaction networks, *Bull. Math. Biol.* **72**, 1947 (2010).
- [295] D. F. Anderson and S. L. Cotter, Product-form stationary distributions for deficiency zero networks with non-mass action kinetics, *Bull. Math. Biol.* **78**, 2390 (2016).
- [296] F. Baras, M. M. Mansour, and J. E. Pearson, Microscopic simulation of chemical bistability in homogeneous systems, *J. Chem. Phys.* **105**, 8257 (1996).
- [297] G. Domokos and I. Scheuring, Discrete and continuous state population models in a noisy world, *J. Theor. Biol.* **227**, 535 (2004).
- [298] A. Traulsen, J. C. Claussen, and C. Hauert, Coevolutionary dynamics: From finite to infinite populations, *Phys. Rev. Lett.* **95**, 238701 (2005).
- [299] T. Galla, Intrinsic noise in game dynamical learning, *Phys. Rev. Lett.* **103**, 198702 (2009).
- [300] M. Berr, T. Reichenbach, M. Schottenloher, and E. Frey, Zero-one survival behavior of cyclically competing species, *Phys. Rev. Lett.* **102**, 048102 (2009).
- [301] A. Melbinger, J. Cremer, and E. Frey, Evolutionary game theory in growing populations, *Phys. Rev. Lett.* **105**, 178101 (2010).
- [302] T. Rogers, A. J. McKane, and A. G. Rossberg, Demographic noise can lead to the spontaneous formation of species, *Europhys. Lett.* **97**, 40008 (2012).
- [303] S. Pigolotti, S. Bernhardsson, J. Juul, G. Galster, and P. Vivo, Equilibrium strategy and population-size effects in lowest unique bid auctions, *Phys. Rev. Lett.* **108**, 088701 (2012).
- [304] D. A. Kessler and N. M. Shnerb, Generalized model of island biodiversity, *Phys. Rev. E* **91**, 042705 (2015).
- [305] G. Biroli, G. Bunin, and C. Cammarota, Marginally stable equilibria in critical ecosystems, *New J. Phys.* **20**, 083051 (2018).
- [306] H. Weissmann, N. M. Shnerb, and D. A. Kessler, Simulation of spatial systems with demographic noise, *Phys. Rev. E* **98**, 022131 (2018).
- [307] R. West and M. Mobilia, Fixation properties of rock-paper-scissors games in fluctuating populations, *J. Theor. Biol.* **491**, 110135 (2020).
- [308] A. Altieri, F. Roy, C. Cammarota, and G. Biroli, Properties of equilibria and glassy phases of the random Lotka-Volterra model with demographic noise, *Phys. Rev. Lett.* **126**, 258301 (2021).
- [309] T. G. Kurtz, Limit theorems for sequences of jump Markov processes approximating ordinary differential processes, *J. Appl. Probab.* **8**, 344 (1971).
- [310] T. G. Kurtz, The relationship between stochastic and deterministic models for chemical reactions, *J. Chem. Phys.* **57**, 2976 (1972).
- [311] M. Vellela and H. Qian, A quasistationary analysis of a stochastic chemical reaction: Keizer’s paradox, *Bull. Math. Biol.* **69**, 1727 (2007).
- [312] A. T. Mohite and H. Rieger, Generalized finite-time optimal control framework in stochastic thermodynamics, [arXiv:2511.00974](https://arxiv.org/abs/2511.00974).