

Exact transport theory of dendritic competition and arrest

Beñat Gurrutxaga-Lerma , Matthew Hughes, and Nils Warnken

School of Metallurgy and Materials, University of Birmingham, B15 2TT Edgbaston, Birmingham, United Kingdom



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We solve exactly the transient two-dimensional advection diffusion problem generated by one and two semi-infinite internal Dirichlet lines, representing slender dendrites advancing into a supersaturated liquid while fixing the solute concentration on their surfaces. For a single dendrite, the mixed boundary value problem is reduced to a Wiener-Hopf equation, factorized explicitly, and inverted to the time domain by the Cagniard-de Hoop method, yielding the single-layer density in closed form in terms of the complex error function. For two parallel dendrites separated by a lateral distance h , the interaction enters through the transcendental kernel $(1 + e^{-\kappa h})^{-1}$, whose lower Wiener-Hopf factor is represented by a logarithmic Cauchy integral, and the resulting finite distance feeding field is reduced to a regularized oscillatory integral. The exact two-dendrite solution reveals a structural obstruction to purely local growth laws: the leading inverse square-root tip singularity is universal, with identical amplitudes for both dendrites independently of spacing, even though the neighbor modifies the feeding field at every finite distance behind the tip. Competition therefore cannot be encoded in any local tip amplitude alone. This obstruction forces a nonlocal coupling variable, which we identify as a finite tip zone functional that retains the subleading near tip structure responsible for competitive depletion. The interaction further resolves into a discrete hierarchy of exponentially screened modes with dominant screening length h/π , which sets the natural scale of competitive suppression. These exact transport results provide the analytical backbone for a reduced theory of dendrite arrest and winner selection, and are validated against independent finite-difference computations.

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I. INTRODUCTION

During directional solidification of a binary alloy, the morphological instability of the planar front gives rise to an array of dendritic crystals whose tips advance into the supersaturated liquid. The steady-state tip velocity and radius of an isolated dendrite growing into an infinite melt are determined, in the diffusion controlled regime, by two classical ingredients: the Ivantsov relation between the solutal Péclet number and the far-field supersaturation [1], which fixes the product ρV but not the individual values, and the microscopic solvability condition, which selects a discrete operating point (ρ, V) through the singular perturbation introduced by surface energy anisotropy [2–6]. For a single-needle crystal in an unbounded domain, this theory is well established and the transport field is known exactly through the Ivantsov paraboloidal solution and its generalizations [7].

In a solidification experiment, however, dendrites do not grow in isolation. Neighboring dendrites compete for the same finite pool of solute, and the resulting diffusive interaction governs the selection of primary arm spacing, the elimination of unfavorably oriented grains, and the columnar to equiaxed

transition. The qualitative picture is well understood: each dendrite acts as a solute sink, depleting the concentration field in its vicinity; when two dendrites are sufficiently close, their depletion zones overlap and each receives less solute than it would in isolation; any geometric asymmetry is then amplified by a positive feedback between tip position and solute supply, so that the lagging dendrite is progressively starved and eventually arrests [8–10]. Despite this clear physical picture, a quantitative analytical theory of dendrite competition has remained elusive.

The existing analytical models of primary spacing and competitive growth rely on geometric approximations of the diffusion field rather than on solutions of the transport equation. The classical spacing theories approximate the solutal field around each dendrite by a one-dimensional profile or by the field of an array of hemispheres, paraboloids, or ellipsoids, and extract a spacing criterion by requiring that the tip undercooling or the interdendritic supersaturation remain consistent with the imposed thermal gradient [10–12]. These are scaling arguments, not solutions of the two or three-dimensional diffusion equation with the correct internal boundary conditions. Subsequent refinements have introduced modified Ivantsov functions that account for an imposed far-field gradient or for interaction with nearest neighbors through ad hoc correction factors [13–16], but none solve the coupled transport problem exactly or derive the competitive arrest criterion from the resolved concentration field.

Numerical approaches have been more successful at capturing the full complexity of dendrite competition. Phase-field

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simulations can resolve the solutal field, the interface shape, and the competitive elimination process in detail [17–19], but the computational cost of resolving the thin diffuse interface limits the accessible parameter range, and the results do not yield closed form predictions. The dendritic needle network model provides a computationally efficient multiscale framework in which each dendrite is represented as a thin needle whose tip velocity and radius are determined by a solvability condition, while the solutal field is computed on a coarser grid [20–23]. Within that framework, the flux of solute into each tip is characterized by a flux intensity factor that plays the same role as the stress intensity factor in fracture mechanics, and dendrite interactions enter through the shared diffusion field [20]. However, the outer transport problem in the dendritic needle network model is solved numerically, so the interaction between neighboring dendrites is not available in closed form.

An alternative analytical route to dendrite interactions is provided by slender body theory. In the limit of small tip radius relative to spacing and diffusion length, each dendrite collapses onto a one-dimensional line source of solute, and the outer solutal field satisfies a reduced integral equation for the source strength [24,25]. This construction explicitly identifies the outer transport problem as a line source integral equation and includes array interactions at the outer level, but the resulting equations must be solved numerically, surface energy is not included in the tip selection mechanism (it is replaced by an array-based selection rule), and the connection between the outer solution and the inner capillary tip region is not formalized as a matched asymptotic expansion.

From the standpoint of mathematical technique, the transport problem around a semi-infinite absorbing line in a uniform flow belongs to the class of mixed boundary value problems solvable by the Wiener-Hopf method. The technique was originally developed for wave diffraction by a semi-infinite screen [26] and has since been applied to a wide range of elliptic and hyperbolic problems in acoustics, elastodynamics, and electromagnetic theory. Its application to thermal and diffusion problems is less common but not unprecedented. The heat transfer from a finite heated film on a wall in shear flow has been solved by Wiener-Hopf factorization, yielding exact expressions for the local Nusselt number and the thermal wake [27–29]. More recently, transient thermal mixed boundary value problems on a half space, with the boundary condition switching from Dirichlet to Neumann or Robin at a point, have been solved by combining the Wiener-Hopf technique with the Cagniard-de Hoop method of contour deformation to invert the Laplace transform exactly [30–32]. The latter work demonstrates that the Cagniard-de Hoop method, originally developed for transient wave propagation [33], extends naturally to parabolic equations, producing well conditioned single integral representations for the time domain solution. The same contour deformation principle has been applied to transient electromagnetic diffusion in geophysics [34].

Despite these developments, the present problem appears not to have been solved before. To the author’s knowledge, neither the transient advection-diffusion equation with an internal absorbing half line nor its two dendrite counterpart has previously been treated by Wiener-Hopf methods in solidifica-

tion theory, and the Cagniard-de Hoop inversion has not been used in this setting. The broad analytic architecture is reminiscent of the classical Sommerfeld half plane problem, but the resemblance is only formal. In the present case the operator is parabolic, the imposed flow shifts the branch structure of the kernel into the complex plane, and the unknown carried by the Wiener-Hopf equation is the interfacial uptake density itself. These differences produce a genuinely new factorization and inversion problem, and lead to an exact transport theory that, to the author’s knowledge, has no precedent in the dendrite literature.

Achieving these solutions leads to the definition of a *process zone* at the dendrite tip as a means of studying dendritic competition. This is done in analogy with fracture mechanics. Indeed, the analogy between the tip of a dendrite and the tip of a crack has been noted in the solidification literature, most explicitly in the dendritic needle network model, where the solutal flux intensity factor at the dendrite tip is defined in direct analogy with the stress intensity factor of fracture mechanics [20]. In fracture mechanics, it is well established that the leading inverse square-root singularity of the stress field at a crack tip is universal: its angular and radial dependence are determined by local geometry alone, while the stress intensity factor encodes the influence of remote loading and boundary conditions [35,36]. When two cracks interact, the stress intensity factor of each is modified by the presence of the other, but the form of the singularity is unchanged. The analogous statement for the solutal flux at a dendrite tip has been implicitly assumed in the dendritic needle network model but has not, to our knowledge, been proved from the exact solution of a transport problem with competing internal boundaries.

The purpose of the present paper is to fill this gap. We formulate the solutal transport around one and two semi-infinite dendrites as a mixed boundary value problem for the two-dimensional advection diffusion equation with internal Dirichlet lines, solve the resulting Wiener-Hopf equation exactly in the single-dendrite case, reduce the two-dendrite problem to a scalar Wiener-Hopf equation with a transcendental interaction kernel, and invert the transforms to physical space by the Cagniard-de Hoop technique. The analysis is embedded within a sharp interface framework in which the line problem is identified as the outer limit of the full sharp interface free boundary problem, and the finite tip zone functional that governs competitive arrest is motivated by sharp interface matching and consistent with inner solvability. A companion paper [37] tackles the case of transient dendrites impinging on an oblique wall.

The remainder of the paper is organized as follows. Section II formulates the single-dendrite transport problem, which is then solved in Secs. II E and II F via Wiener-Hopf factorization and an exact Cagniard-de Hoop inversion to closed form. Section III extends the analysis to two competing dendrites and develops the Mittag-Leffler mode decomposition of the two-dendrite problem. Section IV couples the exact transport solution to a growth law, presents the $A_1 = A_2$ universality result as a structural obstruction to the local tip law, and derives the arrest and winner selection criteria from the process zone functional. Section V closes the paper with its conclusions and future directions.

II. SINGLE-DENDRITE PROBLEM

We begin with the reduced outer transport problem for a dendrite. We will represent it as an internal Dirichlet half line in an infinite space. This representation is not intended as a literal representation of a fully resolved dendritic microstructure with side branching, curvature, or three-dimensional tip shape. Rather, it is the leading outer transport problem in the slender regime

$$\frac{\rho_i}{h} \ll 1, \quad \frac{\rho_i}{\ell_D} \ll 1, \quad \ell_D := \frac{D}{v},$$

in which the finite capillary tip is asymptotically small relative to both the lateral spacing and the solutal diffusion length. In this regime, the full free boundary problem separates into an inner tip selection problem and an outer transport problem whose leading order trace is precisely an internal line sink with prescribed interfacial concentration. This is the same slender body philosophy that underlies reduced models such as the dendritic needle network construction [20–23] and the line source theories of Spencer and Huppert [24,25]. The present analysis should therefore be read as an exact analytical benchmark for that canonical outer problem. Effects such as side branching, weak line curvature, unequal tip shapes, or fully three-dimensional needle geometry are expected to enter at higher order through perturbations of the effective line geometry and line source strength, rather than through a change in the leading transport mechanism itself.

We begin with the single-dendrite case, which admits a complete closed form solution and serves as the building block for the two dendrite interaction analysis of Sec. II G.

We consider the two-dimensional advection-diffusion equation for a scalar concentration field $c(x, y, t)$:

$$\frac{\partial c}{\partial t} + V_x \frac{\partial c}{\partial x} + V_y \frac{\partial c}{\partial y} = D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right), \quad (1)$$

where $D > 0$ is the diffusivity and $\mathbf{V} = (V_x, V_y)$ is a constant advection velocity. The domain is all of $\mathbb{R}^2$, with the initial condition $c(x, y, 0) = 0$.

The physical problem is to maintain a prescribed concentration c_{LC} on an internal line segment representing, for instance, the solid-liquid interface of a growing dendrite. Without loss of generality, we orient the line along the y axis:

$$\Gamma = \{(x_0, y) : 0 \leq y \leq L\}. \quad (2)$$

The boundary condition on Γ is of Dirichlet type:

$$c(x_0, y, t) = c_{LC}, \quad 0 \leq y \leq L, \quad t > 0. \quad (3)$$

This is fundamentally different from prescribing a source term $f = c_{LC}\delta(x - x_0)$ in Eq. (1), which would inject mass at a constant rate and produce a concentration that grows as $\sqrt{t}$ at the line, rather than remaining fixed at c_{LC} .

A. Single-layer potential formulation

Since the domain is unbounded and the line Γ is internal (not a boundary of the domain), the natural framework is a potential theory representation (cf. Ref. [38], Chap. 3). Let Φ denote the free space Green's function of Eq. (1)

(q.v. Ref. [39]):

$$\Phi(\mathbf{x}, t; \mathbf{x}', s) = \frac{1}{4\pi D(t-s)} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}' - \mathbf{V}(t-s)|^2}{4D(t-s)}\right), \quad (4)$$

with $t > s$. We represent the concentration as a single-layer potential:

$$c(\mathbf{x}, t) = \int_0^t \int_{\Gamma} \Phi(\mathbf{x}, t; \mathbf{x}', s) \mu(\mathbf{x}', s) d\ell' ds, \quad (5)$$

where $\mu(\mathbf{y}', s)$ is an unknown density (with dimensions of flux) supported on Γ . Physically, μ is the source strength that the line must exert at each point and each instant to maintain the prescribed concentration. It is not known a priori; it is what we solve for.

Enforcing the Dirichlet condition Eq. (3) by evaluating Eq. (5) on Γ itself gives

$$c_{LC} = \int_0^t \int_0^L K(y, y', t-s) \mu(y', s) dy' ds, \quad 0 \leq y \leq L, \quad (6)$$

where $K(y, y', \tau) = \Phi|_{x=x'=x_0}$ is the Green's function evaluated on the line.

Equation (6) is a Volterra integral equation of the first kind for μ (in time). In principle, one could possibly solve it numerically and obtain the layer potential μ from which to generate the full field solution, here we discuss its analytic solution instead via the Wiener-Hopf technique [26] instead.

B. Coordinate reduction

By symmetry, the solution is even in $(x - x_0)$: the two sides of the internal line carry the same concentration. It therefore suffices to work in the half-space $\xi = |x - x_0| > 0$.

For convenience, we shift the origin to the tip of the line segment by introducing

$$\xi = |x - x_0|, \quad \eta = y - L. \quad (7)$$

In these coordinates, the semi-infinite Dirichlet set becomes $\{\xi = 0, \eta < 0\}$. The region $\eta > 0$ (above the tip) carries no boundary condition (the line simply terminates) so the normal flux c_ξ must be continuous (and, by symmetry, zero) there.

We restrict to the case $V_x = 0, V_y = v > 0$ (flow along the line), which is the relevant geometry for a growing dendrite. The PDE in (ξ, η, t) is

$$c_t + vc_\eta = D(c_{\xi\xi} + c_{\eta\eta}), \quad \xi > 0, \quad (8)$$

with the mixed boundary conditions

$$c(0, \eta, t) = c_{LC} \quad (\eta < 0), \quad c_\xi(0, \eta, t) = 0 \quad (\eta > 0). \quad (9)$$

If μ denotes the jump of diffusive normal flux across the line, then we define

$$\mu(\eta, t) = -2Dc_\xi(0^+, \eta, t), \quad (10)$$

so that μ is supported on $\eta < 0$.

Remark on initial conditions

The preceding formulation assumes zero initial data, $c(\xi, \eta, 0) = 0$. This is the physically natural choice when the dendrite appears at $t = 0$: for $t < 0$ the line does not exist, so

the concentration field is undisturbed, and at $t = 0$ the Dirichlet condition switches on and the field builds up from rest. The Wiener-Hopf solution constructed below is the complete solution to this problem; no additional initial condition term is needed.

A different physical scenario arises when a dendrite pre-exists in a stationary medium and advection is switched on at $t = 0$. In that case, for $t < 0$ the dendrite maintains $c = c_{LC}$ on the line with $v = 0$ (pure diffusion), and the concentration field has reached its steady state $c^{ss}(\xi, \eta)$. At $t = 0$ the flow $v > 0$ begins, and the field must evolve from the initial datum $c(\xi, \eta, 0) = c^{ss}(\xi, \eta)$. Writing the total field as $c = c^D + c^{corr}$, where c^D is the zero initial data Wiener-Hopf solution constructed below, the correction c^{corr} satisfies the advection-diffusion equation with *homogeneous* Dirichlet data $c^{corr} = 0$ on $\eta < 0$ (since both c and c^D enforce c_{LC} there) and the initial condition $c^{corr}(\xi, \eta, 0) = c^{ss}(\xi, \eta)$. This is a well-posed initial and boundary-value problem with homogeneous internal boundary data, not a free Cauchy evolution in the full plane. It can be solved by the same Wiener-Hopf machinery with a modified forcing that encodes the initial transient, but the kernel factorization is unchanged. We do not pursue this extension here; in the remainder of the paper we work with zero initial data throughout.

C. Integral transforms

We apply two integral transforms to Eq. (8) in succession. First, we apply a Laplace transform in t with variable p and zero initial data:

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t) dt, \quad \text{Re}(p) > 0. \quad (11)$$

Second, we apply a Fourier transform in η with variable k :

$$\hat{f}(k) = \int_{-\infty}^\infty f(\eta) e^{ik\eta} d\eta. \quad (12)$$

With this convention, $\hat{c}_\eta = -ik\hat{c}$, and functions supported on $\eta < 0$ have Fourier transforms analytic in $\text{Im}(k) < 0$ (hereafter, “− functions”), while those supported on $\eta > 0$ are analytic in $\text{Im}(k) > 0$ (hereafter, “+ functions”).

Applying both transforms to Eq. (8) yields the ODE in ξ :

$$D\hat{c}_{\xi\xi} - (Dk^2 - ivk + p)\hat{c} = 0. \quad (13)$$

The general solution bounded as $\xi \rightarrow +\infty$ is

$$\hat{c}(\xi, k, p) = G(k, p) e^{-\kappa(k, p)\xi}, \quad (14)$$

where $G(k, p)$ is the unknown boundary trace and

$$\kappa(k, p) = \sqrt{k^2 - \frac{iv}{D}k + \frac{p}{D}}. \quad (15)$$

The branch is chosen so that $\text{Re}(\kappa) > 0$ for $\text{Re}(p) > 0$.

D. Factorization of the dispersion relation

To set up the Wiener-Hopf problem we need to factor κ into upper and lower half plane contributions. Completing the square in Eq. (15):

$$\kappa^2 = (k - i\beta)^2 + \lambda^2, \quad (16)$$

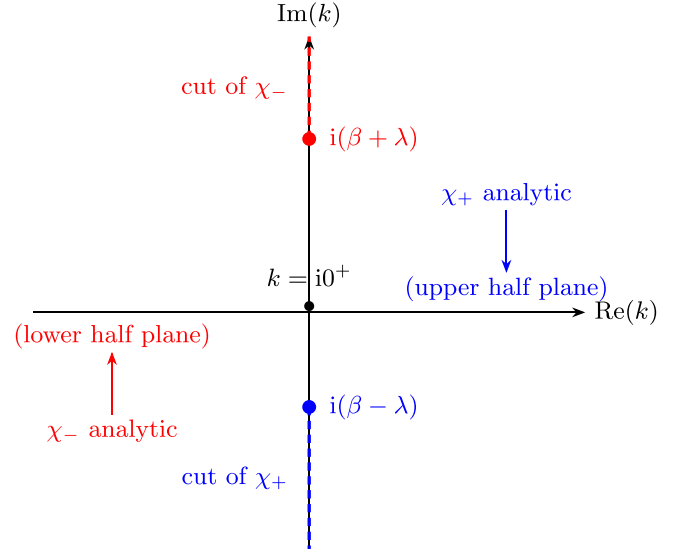


FIG. 1. Analytic structure in the complex k plane for $\text{Re}(p) > 0$. The branch point of χ_+ (blue) lies in the lower half plane with its cut extending downward; the branch point of χ_- (red) lies in the upper half plane with its cut extending upward. The simple pole at $k = i0^+$ arises from the half-line Dirichlet datum.

where we define

$$\beta = \frac{v}{2D}, \quad \lambda = \sqrt{\beta^2 + \frac{p}{D}}, \quad v = \sqrt{\frac{p}{D}}. \quad (17)$$

For $\text{Re}(p) > 0$, we have $\lambda > \beta > 0$ (both real and positive). The dispersion relation factors as

$$\kappa^2 = (k - i(\beta - \lambda))(k - i(\beta + \lambda)). \quad (18)$$

The two branch points are (see Fig. 1) $k = i(\beta - \lambda)$, which lies in the lower half k plane (since $\beta - \lambda < 0$); and $k = i(\beta + \lambda)$, which lies in the upper half k plane.

We define the normalized split factors:

$$\chi_+(k, p) = \sqrt{\frac{k - i(\beta - \lambda)}{i(\lambda - \beta)}}, \quad \chi_-(k, p) = \sqrt{\frac{k - i(\beta + \lambda)}{-i(\beta + \lambda)}}, \quad (19)$$

so that

$$\kappa = v\chi_+\chi_-. \quad (20)$$

By construction χ_+ is analytic in $\text{Im}(k) > -(\lambda - \beta)$, i.e., the upper half plane; χ_- is analytic in $\text{Im}(k) < \beta + \lambda$, i.e., the lower half plane; $\chi_+(0) = \chi_-(0) = 1$ (this normalization is crucial).

E. Wiener-Hopf equation

We now proceed to solve the problem for $G(k, p)$ via the Wiener-Hopf technique.

1. Decomposition of the boundary trace

The Fourier transform of the boundary trace $\bar{c}(0, \eta, p)$ splits into contributions from the two half lines:

$$G(k, p) = G_-(k, p) + G_+(k, p), \quad (21)$$

where G_- is the transform of the known Dirichlet data on $\eta < 0$, and G_+ is the transform of the (unknown) field on $\eta > 0$.

The Dirichlet datum is $\bar{c}(0, \eta, p) = c_{LC}/p$ for $\eta < 0$. Its half range Fourier transform is

$$G_-(k, p) = \frac{c_{LC}}{p} \int_{-\infty}^0 e^{ik\eta} d\eta = \frac{c_{LC}}{ip(k - i0^+)}. \quad (22)$$

This is a $-$ -function (analytic in $\text{Im}(k) < 0$), as it must be, since the datum is supported on $\eta < 0$. It has a simple pole at $k = i0^+$ in the upper half plane. The unknown G_+ is a $+$ -function (analytic in $\text{Im}(k) > 0$).

2. Flux condition

The symmetry condition $c_\xi(0^+, \eta, t) = 0$ for $\eta > 0$ means that the transform of the normal flux,

$$Q(k, p) = -D\kappa(k, p)G(k, p), \quad (23)$$

must be a $-$ -function.

3. Setting up and solving the functional equation

From Eqs. (21) and (23), we write

$$G_- + G_+ = G = -\frac{Q}{D\kappa} = -\frac{Q}{Dv\chi_+\chi_-}. \quad (24)$$

Multiply both sides by χ_+ :

$$\chi_+G_- + \chi_+G_+ = -\frac{Q}{Dv\chi_-}. \quad (25)$$

The right-hand side is a $-$ -function (product of two $-$ -functions). The second term on the left, χ_+G_+ , is a $+$ -function. We need to decompose the first term, χ_+G_- , into its $+$ and $-$ parts.

Additive splitting of χ_+G_- . The product χ_+G_- has two types of singularity: (i) in the upper half plane, the simple pole of G_- at $k = i0^+$; (ii) in the lower half plane, the branch cut of χ_+ . We decompose χ_+G_- into its $+$ and $-$ parts by writing

$$\chi_+G_- = \underbrace{(\chi_+ - 1)G_-}_{+ \text{ function}} + \underbrace{G_-}_{- \text{ function}}. \quad (26)$$

The second piece, G_- , is a $-$ -function: it is analytic in the lower half plane, with its only singularity (the pole at $k = i0^+$) in the upper half plane.

The analyticity of the first piece, $(\chi_+ - 1)G_-$, requires more care. Individually, $(\chi_+ - 1)$ is analytic in the upper half plane (a $+$ -function), while G_- is analytic in the lower half plane (a $-$ -function); in general, the product of a $+$ and a $-$ -function has singularities in both half planes and is neither $+$ nor $-$. However, in the present case, we must perform a direct singularity audit of the product. The factor G_- has a simple pole at $k = i0^+$ in the upper half plane, and no other singularities there. But the factor $(\chi_+ - 1)$ has a simple zero at exactly the same point, since $\chi_+(0) = 1$ by construction. This zero cancels the pole, and the product $(\chi_+ - 1)G_-$ has a removable singularity at $k = i0^+$. Its only surviving singularity is the branch cut of χ_+ in the lower half plane. It is therefore analytic in the upper half plane, i.e., a $+$ -function.

Substituting Eq. (26) into Eq. (25) and grouping $+$ functions on the left and $-$ functions on the right we get

$$\underbrace{\chi_+G_+ + (\chi_+ - 1)G_-}_{+ \text{ function}} = \underbrace{-\frac{Q}{Dv\chi_-}}_{- \text{ function}} - G_-. \quad (27)$$

The left-hand side is analytic in $\text{Im}(k) > 0$; the right-hand side is analytic in $\text{Im}(k) < 0$. Both sides are therefore analytic in the entire complex plane. Moreover, as $|k| \rightarrow \infty$, we have: $\chi_+ \sim k^{1/2}$, G_+ decays at least as k^{-1} , and $(\chi_+ - 1)G_- \sim k^{-1/2}$, so the left-hand side decays to zero. By Liouville's theorem [40], the entire function is identically zero. From the $+$ side:

$$\begin{aligned} \chi_+G_+ &= -(\chi_+ - 1)G_- = (1 - \chi_+)G_- \\ \Rightarrow G_+ &= \left(\frac{1}{\chi_+} - 1\right)G_-. \end{aligned} \quad (28)$$

Hence, the full boundary trace is

$$\begin{aligned} G(k, p) &= G_- + G_+ = G_- + \left(\frac{1}{\chi_+} - 1\right)G_- \\ &= \frac{G_-}{\chi_+(k, p)} = \frac{c_{LC}}{ip(k - i0)\chi_+(k, p)}. \end{aligned} \quad (29)$$

4. Source density $\hat{\mu}$

From Eqs. (10), (14), and (23), we find that the source density is then given by

$$\begin{aligned} \hat{\mu}(k, p) &= 2D\kappa(k, p)G(k, p) = 2Dv\chi_-G_- \\ &= \frac{2c_{LC}\sqrt{D/p}\chi_-(k, p)}{i(k - i0)}. \end{aligned} \quad (30)$$

Since χ_- is analytic in the lower half k plane and the pole at $k = i0$ is in the upper half plane, $\hat{\mu}$ is a $-$ -function, consistent with μ being supported on $\eta < 0$.

F. Exact inversion of $\hat{\mu}$ via Cagniard-de Hoop

We now perform the formal inversion of the line density $\hat{\mu}(k, p)$ Eq. (30) via the Cagniard-de Hoop [30,33,41,42] method.

1. Reduction to a one-sided Laplace transform

We first formally invert the $\hat{\mu}(k, p)$ in space. Since μ is supported on $\eta < 0$, it is natural to introduce the distance behind the tip,

$$s = -\eta = L - y > 0, \quad (31)$$

so that $\mu(-s, t)$ is the density at distance s behind the tip. To recover $\bar{\mu}(-s, p)$ from $\hat{\mu}(k, p)$, we write the formal Fourier inversion:

$$\bar{\mu}(-s, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mu}(k, p) e^{iks} dk. \quad (32)$$

Since $\hat{\mu}$ is analytic in $\text{Im}(k) < 0$ (a $-$ -function), we may deform the real- k contour downward by setting $k = -iq$ with $\text{Re}(q) > 0$. Then $dk = -idq$, $e^{iks} = e^{q s}$, and the contour becomes a vertical line in the q plane running from $c - i\infty$ to

$c + i\infty$. The integral Eq. (32) becomes

$$\begin{aligned}\bar{\mu}(-s, p) &= \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \tilde{\mu}(q, p) e^{qs} (-i) dq \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\mu}(q, p) e^{qs} dq,\end{aligned}\quad (33)$$

where $\tilde{\mu}(q, p) := \hat{\mu}(-iq, p)$. The right-hand side is precisely the inverse Laplace transform $\mathcal{L}_{q \rightarrow s}^{-1}[\tilde{\mu}]$. In other words, the Fourier prefactor $1/(2\pi)$ combines with the $-i$ from the change of variable to produce the Bromwich prefactor $1/(2\pi i)$; no additional factors are generated or lost.

The function $\tilde{\mu}(q, p)$ is also the forward one-sided Laplace transform of $\bar{\mu}(-s, p)$ in s :

$$\tilde{\mu}(q, p) = \int_0^\infty e^{-qs} \bar{\mu}(-s, p) ds. \quad (34)$$

Evaluating χ_- at $k = -iq$:

$$\chi_-(-iq, p) = \sqrt{\frac{-iq - i(\beta + \lambda)}{-i(\beta + \lambda)}} = \sqrt{1 + \frac{q}{\beta + \lambda}}, \quad (35)$$

and the pole factor becomes $i(-iq - i0)^{-1} = 1/q$. Hence,

$$\tilde{\mu}(q, p) = \frac{2c_{\text{LC}}\sqrt{D/p}}{q} \sqrt{1 + \frac{q}{\beta + \lambda}}. \quad (36)$$

This is a known Laplace transform [43]. Using the standard identity

$$\mathcal{L}_{q \rightarrow s}^{-1}\left[\frac{1}{q} \sqrt{1 + \frac{q}{a}}\right] = \text{erf}(\sqrt{as}) + \frac{e^{-as}}{\sqrt{\pi as}}, \quad a > 0, \quad (37)$$

which can be verified by computing the forward transform of the right-hand side using the standard results $\int_0^\infty e^{-qs} \text{erf}(\sqrt{as}) ds = \frac{1}{q} \sqrt{\frac{a}{a+q}}$ and $\int_0^\infty \frac{e^{-(a+q)s}}{\sqrt{\pi as}} ds = \frac{1}{\sqrt{a(a+q)}}$, we obtain the exact Laplace domain density:

$$\bar{\mu}(-s, p) = 2c_{\text{LC}} \sqrt{\frac{D}{p}} \left[\text{erf}(\sqrt{s(\beta + \lambda)}) + \frac{e^{-s(\beta + \lambda)}}{\sqrt{\pi s(\beta + \lambda)}} \right], \quad (38)$$

with $s > 0$.

a. Tip singularity. As $s \downarrow 0$, $\text{erf} \rightarrow 0$ and the exponential term diverges as $s^{-1/2}$:

$$\bar{\mu}(-s, p) \sim \frac{2c_{\text{LC}}}{\sqrt{\pi s}} \sqrt{\frac{D}{p(\beta + \lambda)}}, \quad s \downarrow 0.$$

This inverse square-root singularity at the tip (cf. [1]) emerges automatically from the exact solution; it need not be assumed *a priori*. Its physical origin is the abrupt termination of the Dirichlet constraint at the tip: as one approaches the endpoint $y = L$ from below, the concentration is pinned at c_{LC} on one side of the tip and free on the other, so the flux required to maintain the constraint diverges in the manner familiar from classical potential theory.

The divergence is local in origin, produced by the termination of the pinned concentration and fed from the melt in an arbitrarily small neighborhood of the tip. Diffusion equilibrates that neighborhood faster than any external source can

deplete it, and Sec. IV shows that a second dendrite at finite separation leaves the coefficient of $s^{-1/2}$ at its isolated value.

b. Far-field limit. As $s \rightarrow \infty$, $\text{erf} \rightarrow 1$ and the exponential term vanishes, giving $\bar{\mu} \rightarrow 2c_{\text{LC}}\sqrt{D/p}$, which inverts to $2c_{\text{LC}}\sqrt{D/(\pi t)}$. This is exactly the flux from a one-dimensional (1D) half-space diffusion problem with a fixed concentration boundary, physically correct, since far behind the tip the problem is locally one-dimensional: the tip region is too distant to influence the local flux.

2. Inversion of the Laplace transform in p

The density in physical time is given by the Bromwich integral:

$$\mu(-s, t) = \frac{1}{2\pi i} \int_{\mathcal{B}} e^{pt} \bar{\mu}(-s, p) dp, \quad (39)$$

where $\mathcal{B}$ is the Bromwich contour, vertical to the right of all singularities in the p plane.

3. Singularity structure in the p plane

The function $\bar{\mu}(-s, p)$ contains two sources of non analyticity in p : $\sqrt{D/p}$, which has a branch point at $p = 0$; and $\lambda(p) = \sqrt{\beta^2 + p/D}$, which has a branch point at $p = -D\beta^2$. With the standard branch cut convention, the combined cut runs along the entire negative real axis from $p = 0$ to $-\infty$, encompassing both branch points.

A direct contour deformation in the p plane would require integrating along the full cut, with distinct expressions on the segment $(-D\beta^2, 0)$ (where only $\sqrt{p}$ is discontinuous) and on $(-\infty, -D\beta^2)$ (where both $\sqrt{p}$ and λ are discontinuous). This is correct but awkward. A much cleaner route is to change variables so that the spurious branch point at $p = 0$ disappears.

4. Deformation onto the imaginary axis

The required substitution is

$$v = \sqrt{\frac{p}{D}}, \quad p = Dv^2, \quad dp = 2Dv dv, \quad (40)$$

taken with $\text{Re}(v) > 0$ so that the right half v plane covers the cut p plane once. Three simplifications follow at once. The branch point of $\sqrt{D/p}$ at $p = 0$ is removed, since $\sqrt{D/p} = 1/v$ is meromorphic; this factor then cancels identically against the v in the Jacobian, so no algebraic singularity survives at the origin. What remains is $\lambda(v) = \sqrt{\beta^2 + v^2}$, whose branch points sit at the two isolated points $v = \pm i\beta$ on the imaginary axis. The single awkward cut of the p plane, along which the discontinuity changed character at $p = -D\beta^2$, is thereby replaced by a pair of symmetric cuts with uniform structure.

Under Eq. (40) the Bromwich contour maps to a hyperbola $\mathcal{B}'$ in the right half v plane, opening about the real axis. The integrand is analytic throughout the region between $\mathcal{B}'$ and the imaginary axis, and the connecting arcs at infinity contribute nothing because $e^{Dv^2 t}$ decays in the right half plane. Cauchy's theorem therefore permits $\mathcal{B}'$ to be collapsed onto the imaginary axis, indented around the branch points at $\pm i\beta$. This deformation is exact, not asymptotic, and it is the step

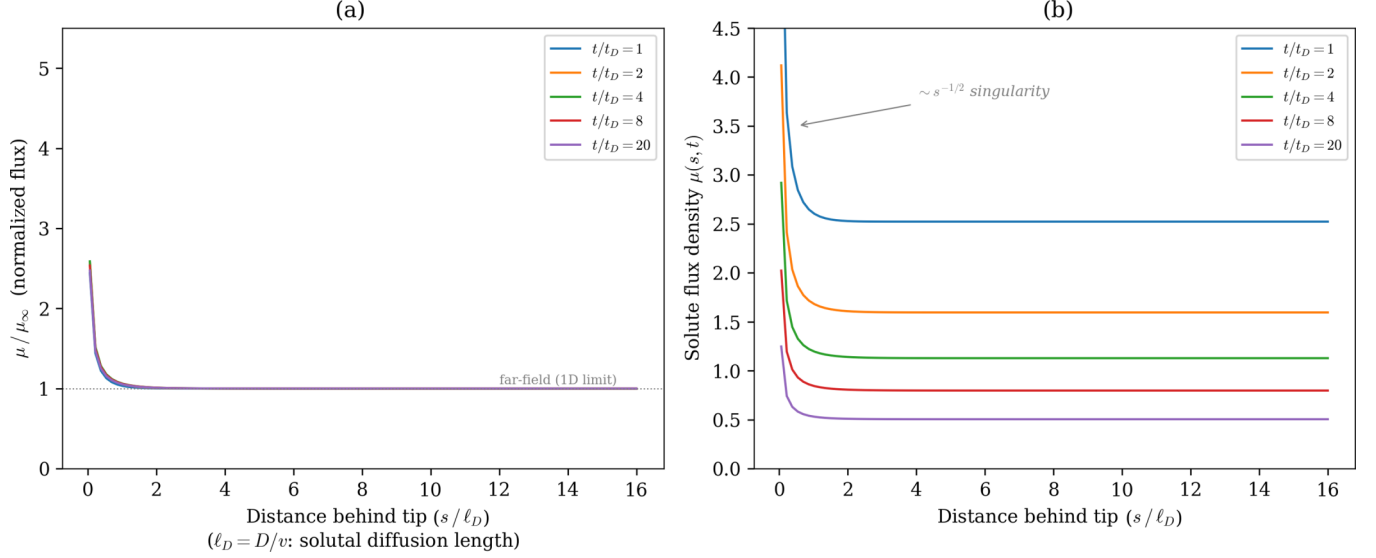


FIG. 2. Single-dendrite line density $\mu(s, t)$ as a function of the distance s/ℓ_D behind the tip, where $\ell_D = D/v$ is the solutal diffusion length, for several dimensionless times t/t_D with $t_D = D/v^2$. (a) Density normalized by the far-field value $\mu_\infty = 2c_{LC}\sqrt{D/(\pi t)}$, showing convergence to the 1D limit $\mu/\mu_\infty = 1$ for large s . The inverse square-root singularity at the tip ($s \downarrow 0$) is evident. (b) Unnormalized density; the overall magnitude decreases as $t^{-1/2}$.

that makes the solution numerically tractable: the resulting integral runs over a single real variable with an integrand that is bounded and oscillation free.

On the imaginary axis the two sides of the contour carry boundary values of λ that are complex conjugates of one another. The contributions from $\text{Im}(v) > 0$ and $\text{Im}(v) < 0$ therefore combine into twice the real part of a single integral, and the factors of i cancel. This is the analytic expression of a physical requirement: $\mu(-s, t)$ is a solute uptake per unit

length and must be real. Carrying out the deformation and taking the real part, both of which are set out in Appendix A, gives the exact real integral representation

$$\mu(-s, t) = \frac{4Dc_{LC}}{\pi} \text{Re} \left[\int_0^\infty e^{-D\tau^2 t} F(\beta + \lambda(i\tau)) d\tau \right]. \quad (41)$$

Writing this out explicitly,

$$\mu(y, t) = \frac{4Dc_{LC}}{\pi} \int_0^\infty e^{-D\tau^2 t} \text{Re} \left[\text{erf}(\sqrt{(L-y)(\beta + \lambda(i\tau))}) + \frac{e^{-(L-y)(\beta + \lambda(i\tau))}}{\sqrt{\pi(L-y)(\beta + \lambda(i\tau))}} \right] d\tau, \quad (42)$$

with $\lambda(i\tau)$ given by Eq. (A9). This is the desired real closed form solution to the line density for a single dendrite.

Figure 2 shows the line density $\mu(s, t)$ computed by direct numerical quadrature of Eq. (42) in dimensionless units ($D = 1$, $v = 2$, $c_{LC} = 1$). The exact solution was achieved through simple Gauss-Hermite quadrature. The solution displays the inverse square-root singularity $\mu \sim A^{(1)}(t)/\sqrt{s}$ at the tip, which acts as the solutal analog of the stress intensity factor at a crack tip, and plays a central role in the later competition analysis. Far behind the tip ($s \gg \ell_D$), the density asymptotes to the one-dimensional result $\mu_\infty = 2c_{LC}\sqrt{D/(\pi t)}$: the dendrite locally resembles a semi-infinite absorbing wall in a uniform flow, and the transverse structure becomes irrelevant. The transition between these two regimes occurs on the solutal diffusion length $\ell_D = D/v$, which sets the lateral extent of the depletion wake, in units of the solutal diffusion time $t_D = D/v^2$. Figure 2(a) normalizes the density by the far-field value $\mu_\infty = 2c_{LC}\sqrt{D/(\pi t)}$; the collapse onto unity for large s/ℓ_D confirms the 1D far-field limit derived above. Near the tip ($s \rightarrow 0$), the $s^{-1/2}$ singularity is clearly visible at all times: this is the signature of the abrupt termi-

nation of the Dirichlet constraint at the tip, and it emerges automatically from the exact solution. Figure 2(b) shows the unnormalized density, illustrating how μ decays as $t^{-1/2}$ at long times while the tip singularity remains the dominant feature.

To confirm the exact solution independently, we solve the mixed boundary value problem by explicit finite differences on a regular grid covering the half space $\xi \geq 0$, $\eta \in [\eta_{\min}, \eta_{\max}]$. The Dirichlet condition $c = c_{LC}$ is imposed at $\xi = 0$ for $\eta < 0$, and the symmetry condition $c_\xi = 0$ at $\xi = 0$ for $\eta \geq 0$. The advection diffusion equation is advanced using a forward time, centered space (FTCS) scheme with upwind differencing for the advection term and a time step chosen to satisfy both the diffusive stability condition ($\Delta t < \frac{1}{2} \min(\Delta \xi, \Delta \eta)^2/D$) and the advective CFL condition ($\Delta t < \Delta \eta/v$). The line density is extracted as $\mu(\eta, t) = -2D[c(1, j) - c_{LC}]/\Delta \xi$ for $\eta < 0$.

Figure 3 compares the finite-difference solution against the exact Cagniard-de Hoop quadrature Eq. (42) at four dimensionless times $t/t_D = 1, 2, 4, 8$, using a grid of 120×320 points with $\Delta \xi = \Delta \eta = 0.05$. Figure 3(a) shows the density

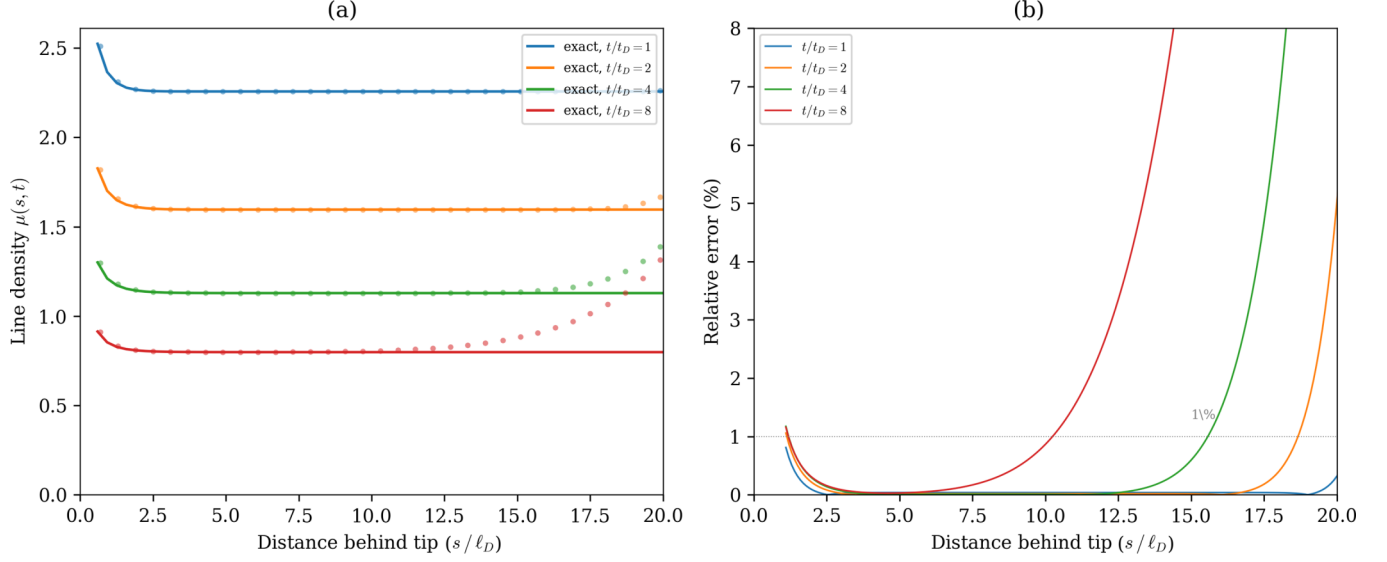


FIG. 3. Numerical verification of the exact single-dendrite solution. (a) Line density $\mu(s, t)$ from the exact Cagniard-de Hoop quadrature (solid lines) and from explicit finite differences on a 120×320 grid (circles), at four dimensionless times. (b) Relative error of the finite-difference solution. In the physical interior ($2 \lesssim s/\ell_D \lesssim 12$), the error is below 1%. Deviations near the tip are due to the unresolved $s^{-1/2}$ singularity; deviations at large s are due to the finite computational domain.

profiles: the exact solution (lines) and the finite-difference result (circles) are visually indistinguishable for $2 \lesssim s/\ell_D \lesssim 12$. Near the tip ($s/\ell_D \lesssim 1$), the finite-difference solution cannot resolve the inverse square-root singularity and underestimates the density, as expected for any regular grid discretization. At the far end of the domain ($s/\ell_D \gtrsim 15$), the finite domain boundary contaminates the solution. Figure 3(b) quantifies the relative error: in the physical interior, the error is below 1% at all times, confirming that the exact solution is correct.

G. Recovering the concentration field

Once $\mu(y', s)$ is known at all times from Eq. (42), the concentration at any point (x, y) in the domain is recovered from the single-layer potential Eq. (5):

$$c(x, y, t) = \int_0^t \int_0^L \Phi(x, y, t; x_0, y', s) \mu(y', s) dy' ds. \quad (43)$$

The integral is nonsingular for $(x, y) \notin \Gamma$ and can be evaluated by standard quadrature, as we do in Sec. IV.

III. TWO-DENDRITE PROBLEM

The features just described are intrinsic to an isolated dendrite. The question now is: What changes when a second dendrite is placed alongside? If the two dendrites were infinitely far apart, then each would produce its own independent depletion wake, and the total field would simply be the superposition of two single-dendrite solutions. But when the lateral spacing h is comparable to ℓ_D , the wakes overlap, the liquid between the two dendrites is depleted by both, and each dendrite finds itself drawing from a field that has already been partially consumed by the other. Because the Dirichlet condition $c = c_{LC}$ must still be maintained on each line, the density μ_i that each dendrite requires to sustain this condition depends on the concentration field produced by both dendrites. The individual densities μ_1 and μ_2 are therefore

not simply superpositions of the single-dendrite solution just achieved; they are the self-consistent response of the coupled two-line system. Computing this coupled response is the purpose of this section.

We thus consider two semi-infinite internal Dirichlet lines, located at

$$x = x_1, \quad x = x_2 = x_1 + h,$$

and extending over $\eta = y - L < 0$. Both are held at the same constant concentration c_{LC} . The geometry is shown schematically in Fig. 4.

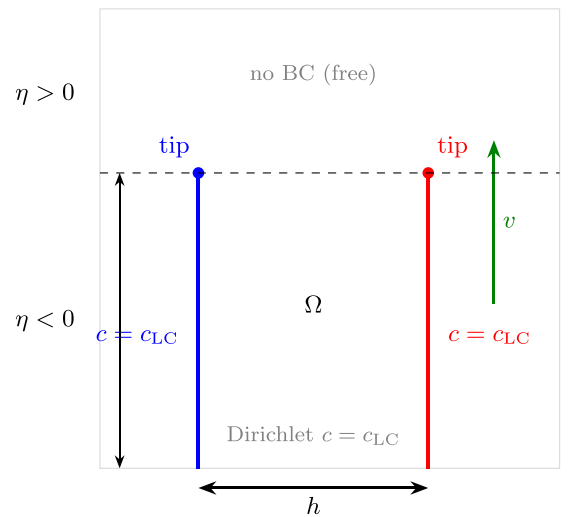


FIG. 4. Schematic of the two dendrite geometry. Two semi-infinite internal Dirichlet lines (blue and red), separated by a lateral distance h , extend over $\eta = y - L < 0$. Both are held at the prescribed concentration c_{LC} . The advection velocity v is directed along the positive y axis. For $\eta > 0$ (above the tips), the lines terminate and no boundary condition is imposed.

The advection-diffusion equation is again

$$c_t + vc_\eta = D(c_{xx} + c_{\eta\eta}),$$

and we use the same Laplace transform in time and Fourier transform in η as before. With the same notation as in the one dendrite problem, we define

$$\kappa(k, p) = \sqrt{k^2 - \frac{iv}{D}k + \frac{p}{D}}, \quad \text{Re}(\kappa) > 0. \quad (44)$$

Then, for each fixed (k, p) , the transformed field satisfies

$$D\partial_{xx}\widehat{c} - D\kappa^2\widehat{c} = 0, \quad x \neq x_1, x_2.$$

To represent the effect of the two dendrites, we introduce transformed line densities $\widehat{\mu}_1(k, p)$ and $\widehat{\mu}_2(k, p)$, one for each line. The transformed field is then given by the superposition of the corresponding single-layer kernels:

$$\widehat{c}(x, k, p) = \frac{1}{2D\kappa} [\widehat{\mu}_1(k, p)e^{-\kappa|x-x_1|} + \widehat{\mu}_2(k, p)e^{-\kappa|x-x_2|}]. \quad (45)$$

Evaluating Eq. (45) on the two lines gives

$$\widehat{c}(x_1, k, p) = \frac{1}{2D\kappa} [\widehat{\mu}_1(k, p) + e^{-\kappa h}\widehat{\mu}_2(k, p)], \quad (46)$$

$$\widehat{c}(x_2, k, p) = \frac{1}{2D\kappa} [e^{-\kappa h}\widehat{\mu}_1(k, p) + \widehat{\mu}_2(k, p)]. \quad (47)$$

Exactly as in the one dendrite problem, the trace on each line is the sum of a prescribed minus part and an unknown plus part. We therefore write

$$G_j(k, p) := \widehat{c}(x_j, k, p) = G_-(k, p) + G_{j,+}(k, p), \quad j = 1, 2, \quad (48)$$

where

$$G_-(k, p) = \frac{c_{LC}}{p} \frac{1}{i(k - i0)}. \quad (49)$$

Here G_- is the Fourier transform of $\frac{c_{LC}}{p}H(-\eta)$, so it is a minus function, while $G_{j,+}$ is analytic in the upper half plane. Likewise, each density $\widehat{\mu}_j$ is a minus function.

Equations (46) and (47) may be written as the matrix relation

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \frac{1}{2D\kappa} \begin{pmatrix} 1 & e^{-\kappa h} \\ e^{-\kappa h} & 1 \end{pmatrix} \begin{pmatrix} \widehat{\mu}_1 \\ \widehat{\mu}_2 \end{pmatrix}. \quad (50)$$

Because the coefficient matrix is symmetric, it is natural to introduce symmetric and antisymmetric combinations:

$$G_s = \frac{G_1 + G_2}{2}, \quad G_a = \frac{G_1 - G_2}{2}, \quad (51)$$

$$\widehat{\mu}_s = \frac{\widehat{\mu}_1 + \widehat{\mu}_2}{2}, \quad \widehat{\mu}_a = \frac{\widehat{\mu}_1 - \widehat{\mu}_2}{2}. \quad (52)$$

Substituting Eqs. (51) and (52) into Eq. (50) diagonalizes the system and yields

$$G_s = \frac{1 + e^{-\kappa h}}{2D\kappa} \widehat{\mu}_s, \quad (53)$$

$$G_a = \frac{1 - e^{-\kappa h}}{2D\kappa} \widehat{\mu}_a. \quad (54)$$

Since both dendrites are held at the same concentration c_{LC} , they have the same prescribed minus part G_- . Thus, the

antisymmetric forcing vanishes, and by symmetry one has

$$G_a = 0, \quad \widehat{\mu}_a = 0. \quad (55)$$

Hence,

$$G_1 = G_2 =: G, \quad \widehat{\mu}_1 = \widehat{\mu}_2 =: \widehat{\mu},$$

and the problem reduces to the symmetric mode alone. Equation (53) becomes

$$G(k, p) = \frac{1 + e^{-\kappa(k, p)h}}{2D\kappa(k, p)} \widehat{\mu}(k, p). \quad (56)$$

Equivalently,

$$\widehat{\mu}(k, p) = K_s(k, p)G(k, p), \quad K_s(k, p) := \frac{2D\kappa(k, p)}{1 + e^{-\kappa(k, p)h}}. \quad (57)$$

Finally, since

$$G(k, p) = G_-(k, p) + G_+(k, p),$$

$$G_+(k, p) := G(k, p) - G_-(k, p) \in H^+,$$

while $\widehat{\mu}(k, p) \in H^-$, the symmetric mode Wiener-Hopf equation is

$$K_s(k, p)[G_-(k, p) + G_+(k, p)] = \widehat{\mu}(k, p), \quad (58)$$

with $\widehat{\mu} \in H^-$, $G_+ \in H^+$. Substituting Eq. (49) gives the scalar Wiener-Hopf equation

$$K_s(k, p)G_+(k, p) = \widehat{\mu}(k, p) - \frac{c_{LC}}{p} \frac{K_s(k, p)}{i(k - i0)}, \quad (59)$$

with kernel

$$K_s(k, p) = \frac{2D\kappa(k, p)}{1 + e^{-\kappa(k, p)h}}. \quad (60)$$

This is the exact symmetric mode Wiener-Hopf equation for the two dendrite geometry. In the limit $h \rightarrow \infty$, one has $e^{-\kappa h} \rightarrow 0$, and Eq. (60) reduces to the one dendrite kernel

$$K_s(k, p) \rightarrow 2D\kappa(k, p),$$

as expected.

A. Solution of the two dendrite Wiener-Hopf equation

For two identical semi-infinite dendrites located at

$$x = x_1, \quad x = x_2 = x_1 + h,$$

and extending over $y < L$, the symmetric mode reduction derived above gives

$$\widehat{\mu}_1(k, p) = \widehat{\mu}_2(k, p) =: \widehat{\mu}(k, p),$$

and the scalar Wiener-Hopf equation

$$K_s(k, p)[G_-(k, p) + G_+(k, p)] = \widehat{\mu}(k, p), \quad (61)$$

with $\widehat{\mu} \in H^-$, $G_+ \in H^+$, and

$$K_s(k, p) = \frac{2D\kappa(k, p)}{1 + e^{-\kappa(k, p)h}}, \quad \kappa(k, p) = \sqrt{k^2 - \frac{iv}{D}k + \frac{p}{D}}, \quad (62)$$

and

$$G_-(k, p) = \frac{c_{LC}}{p} \frac{1}{i(k - i0)}. \quad (63)$$

As in the one dendrite problem, we write

$$\beta = \frac{v}{2D}, \quad v = \sqrt{\frac{p}{D}}, \quad \lambda = \sqrt{\beta^2 + v^2},$$

so that

$$\kappa(k, p) = v\chi_+(k, p)\chi_-(k, p),$$

with χ_+ analytic in the upper half plane, χ_- analytic in the lower half plane, and $\chi_+(0, p) = 1$. The new feature is the interaction factor

$$\frac{1}{1 + e^{-\kappa(k, p)h}}. \quad (64)$$

We introduce its formal Wiener-Hopf factorization

$$\frac{1}{1 + e^{-\kappa(k, p)h}} = M_+(k, p)M_-(k, p), \quad (65)$$

where M_+ is analytic and nonzero in the upper half plane, M_- is analytic and nonzero in the lower half plane, and the normalization

$$M_+(0, p) = 1 \quad (66)$$

is imposed. Formally, this gives the factorization

$$K_s(k, p) = K_{s,+}(k, p)K_{s,-}(k, p), \quad (67)$$

with

$$\begin{aligned} K_{s,+}(k, p) &= \chi_+(k, p)M_+(k, p), \\ K_{s,-}(k, p) &= 2Dv\chi_-(k, p)M_-(k, p). \end{aligned} \quad (68)$$

By construction,

$$K_{s,+}(0, p) = 1,$$

$$K_{s,-}(k, p)F(k, p) = \widehat{\mu}(k, p) - K_{s,-}(k, p)G_-(k, p). \quad (69)$$

The scalar equation (61) is solved by the same argument as in the single-dendrite case. The normalization $K_{s,+}(0, p) = 1$ cancels the pole of G_- at $k = 0$, so that the function $K_{s,+}[G_- + G_+] - G_-$ is analytic in the upper half plane while $\widehat{\mu} - K_{s,-}G_-$ is analytic in the lower half plane. The two agree on the common strip and are bounded at infinity, so by Liouville's theorem both vanish identically. The construction is given in Appendix B. This yields

$$\widehat{\mu}(k, p) = K_{s,-}(k, p)G_-(k, p). \quad (70)$$

Substituting Eqs. (68) and (63) gives

$$\widehat{\mu}(k, p) = 2c_{\text{LC}}\sqrt{\frac{D}{p}}\frac{\chi_-(k, p)M_-(k, p)}{i(k - i0)}. \quad (71)$$

Since the forcing is symmetric, this is the transformed density on each dendrite:

$$\widehat{\mu}_1(k, p) = \widehat{\mu}_2(k, p) = \widehat{\mu}(k, p). \quad (72)$$

We now invert in the tangential variable. As before, write

$$s = L - y > 0,$$

and regard $\widehat{\mu}(-s, p)$ as a function of $s > 0$. Setting $k = -iq$, with $\text{Re}(q) > 0$, and defining

$$m_-(q, p; h) := M_-(-iq, p), \quad (73)$$

one obtains from Eq. (71)

$$\begin{aligned} \widehat{\mu}(q, p) &:= \widehat{\mu}(-iq, p) \\ &= 2c_{\text{LC}}\sqrt{\frac{D}{p}}\frac{m_-(q, p; h)}{q}\sqrt{1 + \frac{q}{\beta + \lambda(p)}}. \end{aligned} \quad (74)$$

This is the one-sided Laplace transform in s . We therefore define

$$H(s, p; h) := \mathcal{L}_{q \rightarrow s}^{-1}\left[\frac{m_-(q, p; h)}{q}\sqrt{1 + \frac{q}{\beta + \lambda(p)}}\right]. \quad (75)$$

Then

$$\widehat{\mu}(-s, p) = 2c_{\text{LC}}\sqrt{\frac{D}{p}}H(s, p; h). \quad (76)$$

In particular,

$$\widehat{\mu}_1(-s, p) = \widehat{\mu}_2(-s, p) = \widehat{\mu}(-s, p). \quad (77)$$

Equation (75) is the exact formal inversion in the tangential variable. If $h \rightarrow \infty$, then $M_- \rightarrow 1$, so $m_- \rightarrow 1$, and Eq. (75) reduces to the one dendrite formula

$$H(s, p; \infty) = \text{erf}(\sqrt{s(\beta + \lambda)}) + \frac{e^{-s(\beta + \lambda)}}{\sqrt{\pi s(\beta + \lambda)}}.$$

The Laplace transform in p is inverted by the same substitution $v = \sqrt{p/D}$ as in the single-dendrite problem, the factor $\sqrt{D/p}$ canceling against the Jacobian $dp = 2Dv dv$; a detailed derivation is offered in Appendix B, which gives

$$\mu(-s, t) = \frac{2Dc_{\text{LC}}}{\pi i} \int_{\mathcal{B}'} e^{Dv^2 t} H(s, Dv^2; h) dv. \quad (78)$$

The contour $\mathcal{B}'$ is the same hyperbolic image of the Bromwich contour as in the one dendrite problem. However, the analytic structure of the integrand in the v plane is richer, because the interaction factor $H(s, Dv^2; h)$ inherits the singularities of $m_-(q, p; h)$, which in turn are determined by the poles and branch structure of $1/(1 + e^{-\kappa h})$. At $k = 0$ (which dominates the far-field contribution), $\kappa = v$, and the poles of $1/(1 + e^{-v h})$ lie at $v = i(2n - 1)\pi/h$, $n = 1, 2, \dots$. These are the interaction poles introduced by the second dendrite, and they lie on the imaginary v axis above $i\beta$ (for h not too large) and between the branch points of $\lambda(v)$; see Fig. 5.

Deforming $\mathcal{B}'$ onto the imaginary axis, exactly as in Appendix A 1, gives

$$\mu(-s, t) = \frac{2Dc_{\text{LC}}}{\pi} \int_{-\infty}^{\infty} e^{-D\tau^2 t} H(s, D(i\tau + 0^+)^2; h) d\tau. \quad (79)$$

The boundary values on the upper and lower halves of the deformed contour are complex conjugates, and therefore

$$\mu(-s, t) = \frac{4Dc_{\text{LC}}}{\pi} \text{Re} \left[\int_0^{\infty} e^{-D\tau^2 t} H(s, D(i\tau + 0^+)^2; h) d\tau \right]. \quad (80)$$

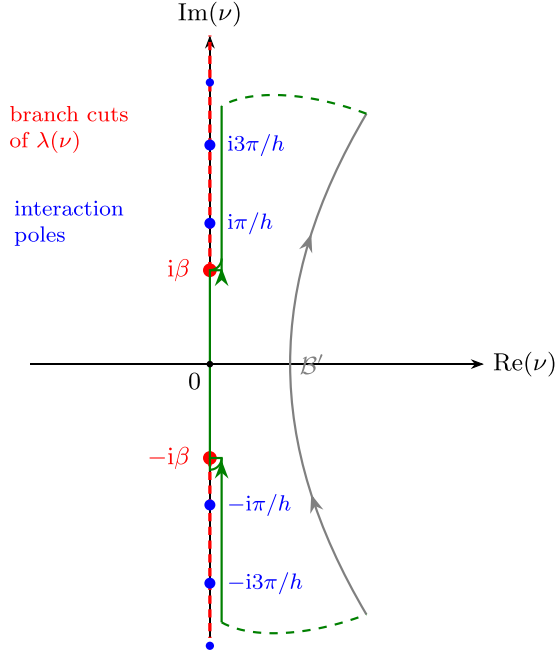


FIG. 5. Analytic structure in the ν plane for the two-dendrite problem. The branch cuts of $\lambda(\nu) = \sqrt{\beta^2 + \nu^2}$ (red dashed, as in the single-dendrite case) run from $\pm i\beta$ to $\pm i\infty$. The interaction factor $1/(1 + e^{-\kappa h})$ introduces an infinite sequence of simple poles on the imaginary axis (blue dots) at $\nu = \pm i(2n-1)\pi/h$, $n = 1, 2, \dots$. The original hyperbolic contour $\mathcal{B}'$ (gray) is deformed onto the imaginary axis (green), passing to the right of all singularities. The Mittag-Leffler truncation at order N retains only the first N pairs of interaction poles.

Returning to the physical coordinate $y = L - s$, the densities on the two dendrites are therefore

$$\mu_1(y, t) = \mu_2(y, t) = \frac{4Dc_{LC}}{\pi} \text{Re} \left[\int_0^\infty e^{-D\tau^2 t} H(L - y, D(i\tau + 0^+)^2; h) d\tau \right] \quad (81)$$

for $y < L$, where H is defined by Eq. (75).

For completeness, the interaction factor M_- may be written formally in logarithmic Cauchy integral form as

$$\log M_-(k, p) = -\frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\log(1 + e^{-\kappa(\xi, p)h})}{\xi - k} d\xi, \quad (82)$$

with $\text{Im}(k) < 0$, and the branch chosen so that $M_-(k, p) \rightarrow 1$ as $|k| \rightarrow \infty$ in the lower half plane. The integral in Eq. (82) is along the real ξ axis, with integrand $\log(1 + e^{-\kappa(\xi, p)h})/(\xi - k)$. For real ξ , the function $\kappa(\xi, p)$ is smooth and positive (assuming $\text{Re}(p) > 0$), and the logarithm decays exponentially as $|\xi| \rightarrow \infty$ since $\kappa \rightarrow |\xi|$ for large $|\xi|$. The integral is therefore absolutely convergent and well-suited to standard adaptive quadrature, e.g., Gauss-Legendre on a suitably truncated interval $[-\Xi, \Xi]$ with $\Xi \sim 5\sqrt{|p|/D}$ together with analytic tail corrections. For a given value of k in the lower half plane, the Cauchy kernel $1/(\xi - k)$ is bounded and smooth, so no special treatment of the singularity is needed. Equations (71), (75), and (81) therefore constitute the formal

Wiener-Hopf and Cagniard-de Hoop solution for the two dendrite geometry.

B. Mittag-Leffler expansion of the competition factor

Equation (81) is exact within the reduced transport problem and is already suitable for numerical quadrature. Its drawback is not correctness but opacity. The interaction between the two dendrites is hidden inside the transcendental Wiener-Hopf factor M_- , or equivalently inside the function $m_-(q, p; h)$ appearing in Eq. (75). As a result, one cannot read off directly how the lateral spacing h controls the strength, range, and modal structure of the competitive suppression. The purpose of the present subsection is therefore not to replace the exact solution, but to extract from it a more explicit representation of the competition mechanism.

As a way of resolving this difficulty, here we expand the line densities via Mittag-Leffler series [44], by replacing the transcendental interaction factor with an explicit hierarchy of rational terms, which as will be seen each correspond to a specific screened interaction mode between the two dendrites.

The exact symmetric mode density [see Eq. (81)] may be written in the form

$$\bar{\mu}(-s, p) = 2c_{LC} \sqrt{\frac{D}{p}} H(s, p; h), \quad (83)$$

where H is defined through the lower Wiener-Hopf factor $m_-(q, p; h)$ by

$$H(s, p; h) = \mathcal{L}_{q \rightarrow s}^{-1} \left[\frac{m_-(q, p; h)}{q} \sqrt{1 + \frac{q}{\beta + \lambda}} \right], \quad (84)$$

for

$$\lambda = \sqrt{\beta^2 + \frac{p}{D}}, \quad \beta = \frac{v}{2D}.$$

The obstruction to an explicit closed formula is therefore not the inversion in s itself, but the factor m_- . This factor is inherited from the transcendental interaction term $(1 + e^{-\kappa h})^{-1}$, with

$$\kappa(k, p) = \sqrt{(k - i\beta)^2 + \lambda^2}.$$

Unlike the one dendrite problem, the kernel now contains both the branch structure of κ and the infinite discrete set of zeros determined by

$$1 + e^{-\kappa h} = 0 \iff \kappa h = i(2n-1)\pi, \quad n = 1, 2, \dots$$

In this sense the exact function $H(s, p; h)$ is irreducible within the present Wiener-Hopf framework: unless one can split m_- in closed form, the exact density remains tied to the abstract factorization.

A more explicit and, for the present purpose, more informative route is to expand the competition factor itself before attempting any further factorization. The relevant identity is the Mittag-Leffler expansion [40]

$$\frac{1}{1 + e^{-z}} = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{z}{2}\right) = \frac{1}{2} + 2z \sum_{n=1}^\infty \frac{1}{z^2 + a_n^2}, \quad (85)$$

with $a_n = (2n-1)\pi$, which converges locally uniformly away from the poles $z = ia_n$. Applying Eq. (85) with

$z = h\kappa(k, p)$ gives

$$\frac{1}{1 + e^{-\kappa h}} = \frac{1}{2} + 2h\kappa \sum_{n=1}^{\infty} \frac{1}{h^2 \kappa^2 + a_n^2}. \quad (86)$$

The symmetric kernel

$$K_s(k, p) = \frac{2D\kappa(k, p)}{1 + e^{-\kappa(k, p)h}}$$

therefore becomes

$$K_s(k, p) = D\kappa(k, p) + 4Dh\kappa(k, p)^2 \sum_{n=1}^{\infty} \frac{1}{h^2 \kappa(k, p)^2 + a_n^2}. \quad (87)$$

This representation is far more transparent than the formal factor m_- . Writing

$$\gamma_n(p) = \sqrt{\lambda^2 + \frac{a_n^2}{h^2}}, \quad (88)$$

one has

$$h^2 \kappa(k, p)^2 + a_n^2 = h^2 ((k - i\beta)^2 + \gamma_n(p)^2). \quad (89)$$

Hence, each term in Eq. (87) contributes a pair of simple poles at

$$k = i\beta \pm i\gamma_n(p). \quad (90)$$

These poles control how the influence of the second dendrite decays along the length of the first. When the Bromwich inversion in q is performed to recover the tangential density $\mu(s, p)$, each pole at $k = i\beta + i\gamma_n$ contributes a residue that decays as $e^{-(\beta + \gamma_n)s}$ in the distance s behind the tip. The n th mode therefore has a characteristic penetration depth

$$\ell_n = \frac{h}{a_n} = \frac{h}{(2n - 1)\pi}, \quad n = 1, 2, \dots, \quad (91)$$

beyond which its contribution is exponentially small. We call ℓ_n the *screening length* of the n th competitive mode: it is the distance behind the tip over which the presence of the neighboring dendrite materially alters the solute density on the focal dendrite. The first mode, with $\ell_1 = h/\pi$, is the longest ranged interaction and therefore the leading correction to the isolated dendrite. Higher modes decay on progressively shorter scales and encode increasingly local competition between the two lines. This is precisely the physical content that is hidden when everything is left inside the factor m_- .

For practical use, we truncate Eq. (86) after N terms and define

$$\Theta_N(z) = \frac{1}{2} + 2z \sum_{n=1}^N \frac{1}{z^2 + a_n^2}, \quad (92)$$

so that

$$K_s^{(N)}(k, p) = 2D\kappa(k, p)\Theta_N(h\kappa(k, p)). \quad (93)$$

The corresponding approximate density is defined by replacing the exact interaction factor by Θ_N in the transformed solution. Thus,

$$H_N(s, p; h) := \mathcal{L}_{q \rightarrow s}^{-1} \left[\Theta_N(h\sqrt{\lambda^2 - (q + \beta)^2}) \frac{1}{q} \sqrt{1 + \frac{q}{\beta + \lambda}} \right]. \quad (94)$$

The approximation to the Laplace domain line density is then

$$\bar{\mu}_N(-s, p) = 2c_{\text{LC}} \sqrt{\frac{D}{p}} H_N(s, p; h). \quad (95)$$

Since the two dendrites are identical and equally forced, one has

$$\bar{\mu}_{1,N}(-s, p) = \bar{\mu}_{2,N}(-s, p) = \bar{\mu}_N(-s, p). \quad (96)$$

The quantity H_N is no longer irreducible. Indeed, for each fixed N , the factor $\Theta_N(h\sqrt{\lambda^2 - (q + \beta)^2})$ has only finitely many singularities in the q plane, namely the poles generated by

$$q = -\beta \pm \gamma_n(p), \quad n = 1, \dots, N. \quad (97)$$

Thus, the Bromwich inversion in Eq. (94) may be evaluated by the usual residue calculus together with the same square-root branch already present in the one dendrite problem. In particular, H_N is a finite sum of the isolated dendrite profile

$$H_\infty(s, p) = \text{erf}(\sqrt{s(\beta + \lambda)}) + \frac{e^{-s(\beta + \lambda)}}{\sqrt{\pi s(\beta + \lambda)}}, \quad (98)$$

together with finitely many exponentially screened competitive corrections generated by the poles Eq. (97). Symbolically,

$$H_N(s, p; h) = C_0^{(N)}(p; h)H_\infty(s, p) + \sum_{n=1}^N C_n^{(N)}(p; h)\mathcal{K}_n(s, p; h), \quad (99)$$

where each $\mathcal{K}_n$ is obtained explicitly from the residue at $q = -\beta - \gamma_n(p)$, and therefore decays like $e^{-(\beta + \gamma_n)s}$ for large s . The coefficients $C_n^{(N)}$ are completely explicit residue coefficients determined by Θ_N . Their exact closed forms are lengthy but elementary.

The first truncation already has a clear interpretation. With $N = 1$,

$$\Theta_1(z) = \frac{1}{2} + \frac{2z}{z^2 + \pi^2}, \quad (100)$$

so that the two dendrite interaction is approximated by a single competitive mode with screening length h/π . The corresponding γ_1 is

$$\gamma_1(p) = \sqrt{\lambda^2 + \frac{\pi^2}{h^2}}. \quad (101)$$

Thus, the first correction to the isolated dendrite is exponentially damped on the scale $1/(\beta + \gamma_1)$ along the tangential coordinate, and its strength is controlled by the lateral spacing h . This provides a direct quantitative picture of dendrite competition: decreasing h shifts γ_1 upward, strengthens the competitive correction, and compresses the interaction zone.

Retaining the first term of the expansion isolates the dominant competitive mode and yields a closed form that exposes its range and sign. The kernel splits as

$$H_1(s, p; h) = \frac{1}{2}H_\infty(s, p) + \Delta H_1(s, p; h), \quad (102)$$

where the first term is one half of the isolated dendrite profile, the $z \rightarrow 0$ limit in which both dendrites draw on a single shared diffusion field, and ΔH_1 is the competitive correction proper.

Its Laplace inversion is carried out in Appendix B. The interaction contributes two simple poles at $q = -\beta \pm \gamma_1$, with γ_1 given by Eq. (101), together with the branch cut inherited

from the isolated problem. Collecting the residues and the cut contribution,

$$\Delta H_1(s, p; h) = R_+ e^{(-\beta+\gamma_1)s} + R_- e^{-(\beta+\gamma_1)s} + \Delta H_1^{\text{cut}}(s, p; h), \quad (103)$$

with $a := \beta + \lambda$ and

$$R_+ = \frac{-i(\gamma_1 + \lambda)\sqrt{\gamma_1 - \lambda}}{h\gamma_1\sqrt{a}(\gamma_1 - \beta)}, \quad (104)$$

$$R_- = \frac{(\lambda - \gamma_1)\sqrt{\lambda + \gamma_1}}{h\gamma_1\sqrt{a}(\beta + \gamma_1)}. \quad (105)$$

The structure of Eq. (103) is the physical content of the truncation. The branch cut term decays as $e^{-(\beta+\lambda)s}$ and therefore shares the exponential envelope of the isolated dendrite. The residue R_- decays as $e^{-(\beta+\gamma_1)s}$ with $\gamma_1 > \lambda$, so the competitive correction is shorter ranged than the isolated profile and is localized behind the tip. Since $\gamma_1^2 - \lambda^2 = \pi^2/h^2$, the excess decay rate is set by the lateral spacing alone and the associated range is h/π . This is the analytical origin of the mesoscopic competition scale we will use in Sec. IV, and it is the precise sense in which the interaction is felt at finite distance behind the tip rather than at the tip itself. Note also that $\lambda < \gamma_1$ renders R_- negative: the retained mode reduces the uptake density on each dendrite, as expected for mutual screening.

The residue R_+ grows as $e^{(\gamma_1-\beta)s}$ and is spurious. The exact factor m_- is analytic in the right q half plane, so no such pole exists in the untruncated solution. It is subdominant near the tip, is partially canceled by the cut contribution, and becomes appreciable only for $s \gg h/\pi$. The $N = 1$ approximation is therefore quantitatively reliable precisely on the interval $\ell \lesssim h/\pi$ over which the process zone functional is evaluated, which is the regime of interest throughout Sec. IV.

The complete $N = 1$ Laplace domain density on each dendrite, derived explicitly in Appendix B, is

$$\begin{aligned} \bar{\mu}_1(-s, p) &= \bar{\mu}_2(-s, p) \\ &= 2c_{\text{LC}} \sqrt{\frac{D}{p}} \left[\frac{1}{2} H_\infty(s, p) + \Delta H_1(s, p; h) \right], \end{aligned} \quad (106)$$

with $s > 0$ and H_∞ given by Eq. (98). The truncation error of the general N -term expansion is bounded in Appendix B: it grows linearly in the interaction argument $z = h\kappa$, so the expansion is reliable for $z \lesssim N\pi$. Because the Cagniard-de Hoop quadrature samples z only up to $z_{\text{eff}} \sim h/\sqrt{D\tau}$, an order $N \gtrsim h/(\pi\sqrt{D\tau})$ suffices, which for typical solidification parameters is $N \gtrsim 10$.

Once H_N has been constructed, the time-dependent densities follow by the same Laplace inversion as before:

$$\mu_{1,N}(y, t) = \mu_{2,N}(y, t) = \frac{1}{2\pi i} \int_B e^{pt} \bar{\mu}_N(-(L-y), p) dp. \quad (107)$$

Equivalently, after the same Cagniard-de Hoop deformation used in the one dendrite problem,

$$\begin{aligned} \mu_{1,N}(y, t) &= \mu_{2,N}(y, t) = \frac{4Dc_{\text{LC}}}{\pi} \text{Re} \\ &\times \left[\int_0^\infty e^{-D\tau^2 t} H_N(L-y, D(i\tau + 0^+)^2; h) d\tau \right], \end{aligned} \quad (108)$$

with $y < L$.

Thus, the exact function $H(s, p; h)$ is tied to a transcendental Wiener-Hopf factor and is not naturally expressible in a finite closed form. The Mittag-Leffler expansion Eq. (85) replaces that irreducible object by an explicit hierarchy of competitive modes. The approximation H_N is then both analytically simpler and physically more revealing: each retained mode corresponds to a specific screened channel through which one dendrite suppresses the uptake of the other. In that sense, the Mittag-Leffler representation is not merely a technical approximation. It is the natural mode decomposition of dendrite competition.

Figure 6 illustrates the quality and limitations of the Mittag-Leffler approximation. Figure 6(a) compares the truncated expansions $\Theta_N(z)$ for $N = 1, 2, 5, 10, 50$ against the exact interaction factor $1/(1 + e^{-z})$. A structural feature of the expansion must be noted: every term $2z/(z^2 + a_n^2) \rightarrow 0$ as $z \rightarrow \infty$, so any finite truncation $\Theta_N(z) \rightarrow 1/2$ for large z , whereas the exact function tends to 1. The convergence of the series is locally uniform—on any bounded interval $[0, Z]$, higher N gives a better approximation—but it is not uniform on $[0, \infty)$. The crossover between good approximation and degradation occurs at $z \sim N\pi$: for $z \ll N\pi$, Θ_N tracks the exact curve closely; for $z \gg N\pi$, it collapses back toward $1/2$. Thus, $N = 10$ is accurate for $z \lesssim 30$ and $N = 50$ for $z \lesssim 150$, but no finite N captures the saturation to unity at arbitrarily large z . Physically, the argument $z = h\kappa$ is the dimensionless interaction strength: it measures how many diffusion lengths separate the two dendrites at a given tangential wave number. When $z \gg 1$ (widely spaced dendrites, or high wave number), the interaction factor tends to unity and the dendrites do not interact; when $z \rightarrow 0$ (close spacing), both dendrites draw from the same diffusion field and the effective supply to each is halved. For the Cagniard-de Hoop quadrature, the Gaussian weight $e^{-D\tau^2 t}$ ensures that only values of κ up to $O(1/\sqrt{D\tau})$ contribute significantly, so the effective range of z is bounded by $h/\sqrt{D\tau}$; the required N is therefore set by the ratio $h/(\pi\sqrt{D\tau})$. Figure 6(b) shows the Laplace-domain density profile $\bar{\mu}(-s, p)$ for a single dendrite (black) and for two dendrites at lateral spacings $h = 2, 4, 10$ times the solutal diffusion length $\ell_D = D/v$. For each spacing, the solid line is the exact far-field result, the dashed line the $N = 10$ Mittag-Leffler approximation, and the dotted line the $N = 1$ approximation. At $h = 10\ell_D$ the three curves are nearly indistinguishable (the neighbor is too far away to matter), while at $h = 2\ell_D$ the suppression is substantial and the $N = 1$ approximation deviates visibly from the exact.

IV. COMPETITIVE GROWTH, ARREST, AND WINNER SELECTION

The preceding sections solved the concentration field exactly for two dendrites in a fixed geometry and showed that, within this exact solution, the leading tip singularity is the same for both dendrites:

$$A_1(t) = A_2(t) = A^{(1)}(t),$$

independently of the lateral spacing h . Any growth law that couples the tip velocity to the local singular amplitude alone therefore predicts no competition. The central issue of the present section is to identify the correct nonlocal feeding

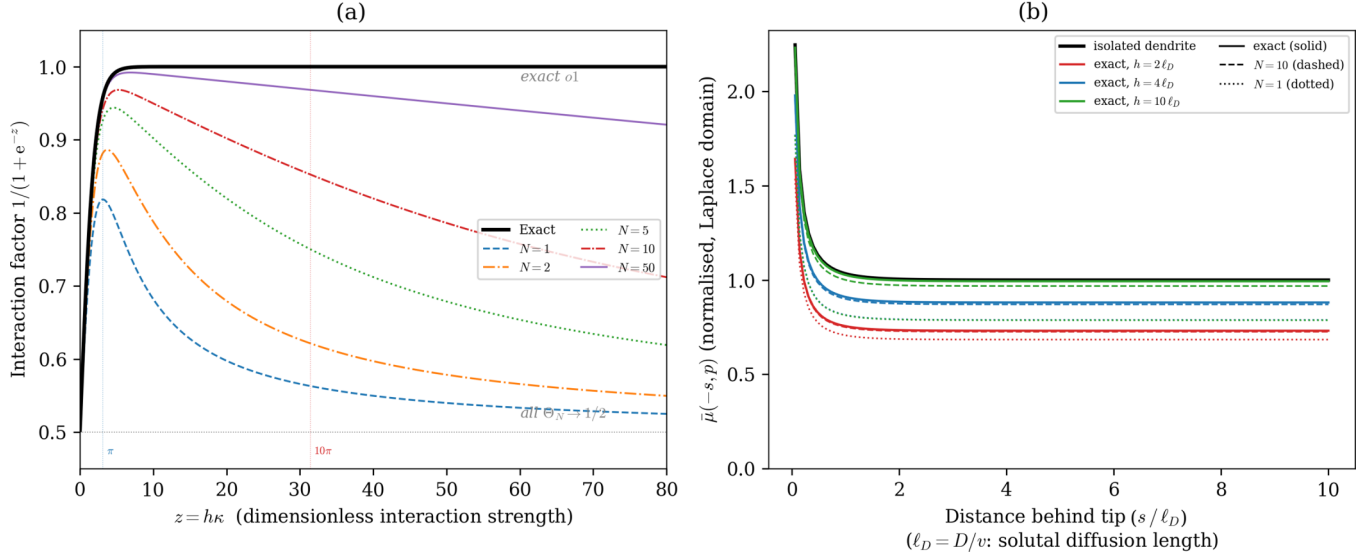


FIG. 6. Mittag-Leffler approximation of the two dendrite interaction. (a) Convergence of the truncated expansion $\Theta_N(z)$ to the exact interaction factor $1/(1+e^{-z})$, for $N = 1, 2, 5, 10, 50$. The argument $z = h\kappa$ is the dimensionless interaction strength; $z \gg 1$ corresponds to isolated dendrites. Note that every finite truncation satisfies $\Theta_N(z) \rightarrow 1/2$ as $z \rightarrow \infty$, whereas the exact function tends to 1; the approximation is accurate for $z \lesssim N\pi$ and degrades beyond. (b) Laplace-domain density profile at $p = D(v/2D)^2$ for a single dendrite (black) and for two dendrites at three lateral spacings $h/\ell_D = 2, 4, 10$ where $\ell_D = D/v$ is the solutal diffusion length. For each spacing: solid = exact, dashed = $N = 10$, dotted = $N = 1$. The distance behind the tip is measured in units of ℓ_D .

variable, formulate a reduced growth law in terms of it, and show how arrest and winner selection emerge once the exact transport field is coupled back to the tip motion.

The resolution is not another local tip quantity, but a finite process zone functional $\Psi_i(\ell, t)$. Because Ψ_i integrates the line density over a mesoscopic interval behind the tip, it retains the subleading coefficients $B_i, C_i, \dots$ in which the competitive interaction is encoded. The argument proceeds in three stages. First, we show why every purely local tip law fails, both physically and mathematically. Second, we define the process zone and derive its exact representation from the Wiener-Hopf solution. Third, we formulate the reduced growth law in terms of Ψ_i , explain the arrest mechanism, and justify the choice of the process zone length ℓ .

A. Process zone

For a two dendrite geometry with tips at L_1 and L_2 , the exact Wiener Hopf solution gives the line density $\mu_i(L_i(t) - s, t)$ on each dendrite as a function of the distance $s > 0$ behind the tip. The failure of a purely local law is already suggested by the sharp interface structure of the problem. The physical dendrite tip has a finite capillary and kinetic core of radius ρ_i , whereas the outer transport field varies on much larger scales such as the lateral spacing h , the convective diffusion length D/v , and the distance to other macroscopic features. We are therefore in the slender regime

$$\varepsilon_i := \frac{\rho_i}{\ell_{\text{out},i}} \ll 1, \quad (109)$$

where $\ell_{\text{out},i}$ denotes a characteristic outer length on which the liquid field varies appreciably. In this regime the rounded tip is seen from the outer scale only through the endpoint of an

internal Dirichlet line, and the detailed inner geometry can enter only through an effective coupling law.

Introduce the distance behind tip i ,

$$s := L_i(t) - y > 0. \quad (110)$$

Near the tip, the exact outer density has the generic endpoint expansion

$$\mu_i(L_i(t) - s, t) = \frac{A_i(t)}{\sqrt{s}} + B_i(t) + C_i(t)\sqrt{s} + O(s), \quad s \downarrow 0. \quad (111)$$

Equation (111) is the standard endpoint structure of the reduced outer problem. The coefficients A_i, B_i, C_i are not local constants, but global quantities determined by the full transport field.

The coefficient

$$A_i(t) := \lim_{s \downarrow 0} \sqrt{s} \mu_i(L_i(t) - s, t) \quad (112)$$

is the local tip feeding amplitude. It is the solutal analog of the stress intensity factor at a crack tip. Since A_i is the dominant local quantity, the most obvious growth law for the tip dynamics is to let the tip speed depend on A_i alone:

$$\dot{L}_i = \mathcal{V}(A_i), \quad i = 1, 2, \quad (113)$$

where $\mathcal{V}$ is monotone increasing, or in the simplest threshold free case $\dot{L}_i = M A_i^+$ with $M > 0$. This is the direct analog of the local tip law used in the dendritic needle network model, where a flux intensity factor extracted from the outer field closes the solvability condition [45].

The exact two-dendrite solution, however, shows that this local tip growth law cannot detect competition. As proved earlier, the lower Wiener Hopf factor $m_-(q, p; h)$ satisfies $m_- \rightarrow 1$ as $q \rightarrow \infty$, so the large q asymptotics of the

transformed density are independent of h . More precisely,

$$\tilde{\mu}(q, p) \sim 2c_{\text{LC}} \sqrt{\frac{D}{p(\beta + \lambda)}} q^{-1/2}, \quad q \rightarrow \infty.$$

By the Abelian theorem for the one-sided Laplace transform [46], this large q behavior controls the small s singularity of $\tilde{\mu}(-s, p)$ and yields

$$A_1(t) = A_2(t) = A^{(1)}(t), \quad (114)$$

independently of the lateral spacing. The neighbor modifies μ_i at every finite distance behind the tip, but the endpoint singularity itself is blind to the interaction.

Thus, under any local growth law of the form (113), both dendrites receive the same velocity at all times, regardless of their spacing. No arrest occurs. Two identical dendrites growing side by side would then be, from the standpoint of their tip fields, indistinguishable from two isolated dendrites infinitely far apart. This is not an artefact of a particular kinetic law. It is a structural property of the exact outer field.

The obstruction is therefore twofold. First, the outer density is singular at $s = 0$, so the inner tip problem cannot couple to a point value of μ_i there. Second, even the singular coefficient A_i extracted from the local asymptotics is universal in the two-dendrite problem and is therefore blind to competitive shielding. Any admissible competition variable must therefore look beyond the leading singularity and must sample a finite interval behind the tip.

This contrasts with two interacting cracks. In that problem, the remote load reaches each tip through the elastic medium, and the second crack perturbs that transmission, so each stress intensity factor depends on the separation. The dendrite pins the concentration at c_ℓ along its whole length, and the depletion field of the neighbor is absorbed by that constraint at every point behind the tip. What remains to be transmitted to the tip itself is drawn from a neighborhood that shrinks to zero, within which diffusion equilibrates arbitrarily fast, so the neighbor is felt there only as a uniform offset. The offset enters the regular terms of the uptake and leaves the coefficient of $s^{-1/2}$ at its isolated value.

The dendrite therefore screens its own tip. Solute intercepted along the absorbing length never reaches the endpoint, and the record of competition accumulates behind the tip over the region where the two depletion wakes overlap. No quantity evaluated at the tip reproduces the role of K , and the feeding variable must sample the melt over a finite distance behind it.

This concerns the exact symmetric configuration, in which the reflection relating the two dendrites forces $A_1 = A_2$. Tips at different positions or advancing at different speeds are not related by that symmetry, and flux intensity factors extracted numerically from such configurations can differ between them. The difference then records the asymmetry of the configuration rather than screening between the tips.

1. Process functional at the dendritic tip

The exact two-dendrite solution leaves the competition problem in a striking but incomplete form. On the one hand, the neighboring dendrite modifies the feeding field at every finite distance behind the tip. On the other hand, the leading inverse square-root singularity remains unchanged, Eq. (114),

so any purely local tip law is blind to that interaction. The purpose of the present subsection is to resolve this paradox in full. We first explain why the finite inner structure of the tip already forces a regularized, nonpointwise coupling to the outer field. We then show that, in the competition problem, the relevant averaging scale cannot remain microscopic, because the universal local singularity still dominates there. This leads naturally to a mesoscopic process zone functional $\Psi_i(\ell, t)$, which retains the subleading near tip structure responsible for competitive shielding. Finally, we derive the exact Wiener-Hopf representation of that functional which will serve as the transport input for the reduced growth law below.

This failure of locality is already implicit in the sharp interface structure of the problem. The physical dendrite tip has a finite capillary and kinetic core of radius ρ_i , whereas the reduced outer field is singular at the mathematical endpoint. The inner tip problem therefore cannot couple to a pointwise value of μ_i at $s = 0$. It can only sample the incoming field through an average over a finite interval behind the tip. If that interval is taken on the microscopic scale $O(\rho_i)$ inherited from the inner structure, then the resulting quantity regularizes the singularity and recovers the local amplitude A_i at leading order. For an isolated dendrite, this would already provide the natural outer to inner coupling variable. In the present competition problem, however, such a microscopic average remains too local. Since $A_1 = A_2$ and $\rho_i \ll h$ in the slender regime, it is dominated by the universal local singular field and is therefore essentially blind to the depletion induced by the neighboring dendrite.

The decisive interaction enters only through the finite distance structure of the feeding profile behind the tip, namely through the subleading coefficients $B_i, C_i, \dots$ in the endpoint expansion (111). To detect that interaction one must sample a larger interval behind the tip, still small on the outer transport scale, but no longer asymptotically tied to the microscopic tip radius. This is the origin of the process zone, which we borrow from fracture mechanics.

The structural analogy with fracture mechanics is thus clear. At the tip of a crack, the elastic field has a universal inverse square-root singularity whose amplitude is the stress intensity factor K . When that local singular field is insufficient to characterize the effective driving force, one passes to a nonlocal quantity such as the Rice J integral, which retains information from a finite region surrounding the tip [47,48]. The present transport problem exhibits the same transition. Here the coefficient A_i plays the role of a solutal flux intensity factor, but in the exact symmetric two-dendrite problem it is identical for both competitors and therefore cannot distinguish them.

The comparison with the dendritic needle network model is equally instructive. In the formulation of Tournet and Karma [22,45], a local flux intensity factor F is extracted from the numerically computed outer field and enters the solvability tip law through the law $\rho^2 V \propto F^2$, or equivalently $\rho V^2 \propto F^2$ after elimination by the Ivantsov relation [1]. In that setting, the local intensity factor does depend on neighbor geometry because the outer field is solved numerically in the full evolving configuration. The present problem has a different structure. Here the outer field is solved exactly in the reduced two line geometry, and that exact solution shows that the

local singular amplitude is the same for both dendrites on the symmetric branch. A DNN style contour extraction [45] of the local intensity factor would therefore return the same value for both tips. If the tip speed were governed by that quantity alone, then both dendrites would advance at the same speed regardless of their spacing, and no arrest would ever occur. Yet arrest is observed in experiments and phase-field simulations of dendritic competition. The resolution is that the competitive interaction enters not through the leading singularity, but through the finite distance depletion behind the tip.

This motivates, in direct analogy to fracture mechanics [49,50], the introduction of the *process zone* as the finite region behind the dendrite tip from which the advancing tip effectively draws solute. The relevant averaging length is therefore not the microscopic scale $O(\rho_i)$ inherited from the inner tip structure, but a mesoscopic scale ℓ that remains small relative to the outer transport scales and yet is large enough to resolve the competitive depletion produced by the neighboring dendrite. Associated with this process zone, we define the *process zone functional*

$$\Psi_i(\ell, t) := \frac{1}{2\sqrt{\ell}} \int_0^\ell \mu_i(L_i(t) - s, t) ds, \quad (115)$$

which measures the mean feeding of dendrite i over a finite tip zone of length ℓ .

The normalization is chosen so that the universal singular field contributes at order one:

$$\frac{1}{2\sqrt{\ell}} \int_0^\ell \frac{A_i(t)}{\sqrt{s}} ds = A_i(t).$$

Substituting the endpoint expansion (111) into Eq. (115) gives

$$\begin{aligned} \Psi_i(\ell, t) &= \frac{1}{2\sqrt{\ell}} \int_0^\ell \left(\frac{A_i(t)}{\sqrt{s}} + B_i(t) + C_i(t)\sqrt{s} + O(s) \right) ds \\ &= \frac{1}{2\sqrt{\ell}} \left(2A_i(t)\sqrt{\ell} + B_i(t)\ell + \frac{2}{3}C_i(t)\ell^{3/2} + O(\ell^2) \right) \\ &= A_i(t) + \frac{1}{2}B_i(t)\sqrt{\ell} + \frac{1}{3}C_i(t)\ell + O(\ell^{3/2}). \end{aligned} \quad (116)$$

Thus, Ψ_i recovers the local singular amplitude at leading order while retaining the first nonuniversal corrections. For an isolated dendrite, Ψ_i reduces to A_i at leading order. For two competing dendrites, by contrast, the neighbor enters precisely through the subleading coefficients $B_i, C_i, \dots$, so the competitive depletion becomes visible only after averaging over a finite interval behind the tip. This is why the relevant variable for arrest is the finite process zone functional Ψ_i , not the local tip intensity alone.

a. Exact mathematical form of the process zone functional. The functional (115) can be evaluated directly from the exact two-dendrite transport solution. In Laplace space, write

$$\mathcal{J}(\ell, p; h) := \int_0^\ell H(s, p; h) ds, \quad (117)$$

so that Ψ_i is recovered after inverse transformation. This quantity should not be confused with the fracture J integral; here $\mathcal{J}$ denotes the finite zone uptake of the exact density kernel along the dendrite axis.

The Mittag-Leffler expansion provides a computable rational approximation to the two dendrite interaction, but it is not the irreducible exact form. It converges only locally uniformly and degrades once the scaled variable enters the range $z \gtrsim N\pi$. The exact Cagniard de Hoop representation derived earlier from the Wiener Hopf solution remains available and is free of any truncation. We now reduce that exact representation to the simplest regular integral form.

Starting from the exact two-dendrite solution, one finds

$$\begin{aligned} \mathcal{J}(\ell, p; h) &:= \int_0^\ell H(s, p; h) ds \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{q\ell}}{q^2} F(q, p; h) dq, \end{aligned} \quad (118)$$

where

$$F(q, p; h) := m_-(q, p; h) \sqrt{1 + \frac{q}{\beta + \lambda(p)}}, \quad (119)$$

and $m_-(q, p; h) = M_-(-iq, p)$ with M_- the lower Wiener Hopf factor of the interaction kernel $(1 + e^{-\kappa h})^{-1}$.

The factor F is analytic in the open right half q plane. Indeed, $M_-(k, p)$ is analytic for $\text{Im}(k) < 0$, so $m_-(q, p; h) = M_-(-iq, p)$ is analytic for $\text{Re}(q) > 0$, and the square-root factor has its branch point at $q = -(\beta + \lambda)$, which lies strictly in the left half plane when $\text{Re}(p) > 0$. The only singularity on the boundary $\text{Re}(q) = 0$ is therefore the double pole at $q = 0$ contributed by the factor $1/q^2$.

To deform the Bromwich contour from $\text{Re}(q) = \gamma > 0$ onto the imaginary axis, expand F near $q = 0$:

$$F(q, p; h) = F_0(p; h) + F_1(p; h)q + O(q^2), \quad (120)$$

with

$$\begin{aligned} F_0(p; h) &= m_-(0, p; h), \\ F_1(p; h) &= m'_-(0, p; h) + \frac{m_-(0, p; h)}{2(\beta + \lambda)}. \end{aligned} \quad (121)$$

The value of the leading coefficient follows exactly from the Wiener Hopf normalization. Since $M_+(0, p) = 1$ and

$$M_+(0, p)M_-(0, p) = \frac{1}{1 + e^{-hv}}, \quad v = \sqrt{p/D},$$

one obtains

$$m_-(0, p; h) = \frac{1}{1 + e^{-h\sqrt{p/D}}}. \quad (122)$$

The derivative $m'_-(0, p; h)$ is obtained by differentiating the logarithmic Cauchy integral representation of M_- and evaluating at $k = 0$.

Subtracting the first two Taylor terms and deforming the contour to the imaginary axis $q = i\omega$, with the residue of the double pole contributing the polynomial part, yields

$$\begin{aligned} \mathcal{J}(\ell, p; h) &= F_0\ell + F_1 \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\ell} \frac{M_-(\omega^-, p) \sqrt{1 + \frac{i\omega}{\beta + \lambda}} - F_0 - i\omega F_1}{\omega^2} d\omega, \end{aligned} \quad (123)$$

where

$$M_-(\omega^-, p) := \lim_{\varepsilon \downarrow 0} M_-(\omega - i\varepsilon, p)$$

is the boundary value of the lower Wiener Hopf factor from below the real axis. The subtraction of $F_0 + i\omega F_1$ regularizes the origin: the numerator is now $O(\omega^2)$ as $\omega \rightarrow 0$, so the integrand is $O(1)$ rather than $O(\omega^{-1})$ or $O(\omega^{-2})$, and the integral converges absolutely near $\omega = 0$. At large $|\omega|$, the oscillatory factor $e^{i\omega\ell}$ ensures convergence by the Riemann Lebesgue mechanism.

The boundary value $M_-(\omega^-, p)$ is obtained from the Plemelj Sokhotski formula applied to the logarithmic Cauchy representation of the Wiener Hopf factor:

$$\begin{aligned} \log M_-(\omega^-, p) = & -\frac{1}{2\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{\log(1 + e^{-h\kappa(\xi, p)})}{\xi - \omega} d\xi \\ & + \frac{1}{2} \log(1 + e^{-h\kappa(\omega, p)}), \end{aligned} \quad (124)$$

where

$$\kappa(\xi, p) = \sqrt{\xi^2 - i\nu\xi/D + p/D}.$$

The principal value integral is a Hilbert transform of the smooth, rapidly decaying function $\log(1 + e^{-h\kappa})$ and is therefore well suited to adaptive Gauss Legendre quadrature on a truncated interval $[-\Xi, \Xi]$ with $\Xi \sim 5\sqrt{|p|/D}$.

On the symmetric branch, the exact process zone feeding functional is therefore

$$\Psi(\ell, t; h) = \frac{2D_{\text{CLC}}}{\pi\sqrt{\ell}} \int_0^{\infty} e^{-D\tau^2 t} \text{Re}[\mathcal{J}(\ell, D(i\tau + 0^+)^2; h)] d\tau, \quad (125)$$

with $\mathcal{J}$ given by Eq. (123) and $M_-(\omega^-)$ by Eq. (124). This is a double integral in (τ, ω) with a precomputed Hilbert transform for the Wiener Hopf factor. The τ integral converges because of the Gaussian weight, the ω integral converges because of the regularization at the origin and the oscillatory decay at infinity, and no exponential blow up appears anywhere in the representation.

The Mittag-Leffler approximation enters here directly. Equation (85) amounts to replacing the exact boundary value $M_-(\omega^-, p)$ in Eq. (123) by an explicit rational approximation. That replacement avoids the Hilbert transform and yields a computable modal decomposition, but it introduces the nonuniform convergence discussed earlier. Equations (123) and (125) are therefore the irreducible exact representation of the two dendrite process zone functional within the present Wiener Hopf framework. They reduce the original formal solution, which involves the Bromwich inversion of a transcendental Wiener Hopf factor, to a regularized oscillatory Fourier integral on the real axis, with the factor evaluated on its natural boundary by the Plemelj formula. This is the maximal contour simplification available in the two dendrite geometry: unlike the single-dendrite problem, the factor M_- cannot be analytically continued across the real k axis in a way that would permit further collapse of the contour onto a branch cut. The numerical evaluation of these formulas is described in Appendix C.

Equations (115) and (125) complete the transition from the exact transport solution to the reduced competition theory. The role of the leading singular amplitude is now clarified: it fixes the universal local part of the tip feeding but cannot distinguish the competitors. The process zone functional Ψ_i is the first finite zone quantity that both regularizes the singular tip field and retains the subleading structure through which neighboring dendrites interact. In the sequel, this exact transport quantity will be evaluated at the instantaneous tip geometry and used as the sole transport input in the reduced growth law.

B. Finite process zone feeding and the reduced growth law

Once the correct feeding variable has been identified, the reduced tip dynamics can be closed in terms of the process zone functional. The physical idea is simple. A dendrite tip can advance only if the solute supply into the finite region from which it draws material is sufficient to sustain the constitutional undercooling required by the capillary and kinetic balance at the tip. If that feeding falls below the minimum required level, then the tip decelerates and ultimately arrests. Since the exact transport field enters the growth problem only through the finite zone uptake, the reduced growth law should depend monotonically on $\Psi_i(\ell, t)$ rather than on A_i .

The simplest such growth law is

$$\dot{L}_i(t) = M(\Psi_i(\ell, t; L_1, L_2, h) - \Psi_c)_+, \quad i = 1, 2, \quad (126)$$

where $M > 0$ is a kinetic mobility, $\Psi_c \geq 0$ is the critical process zone feeding required to maintain growth, and $r_+ := \max\{r, 0\}$ enforces the fact that the tip does not retreat. In the threshold free idealization $\Psi_c = 0$, the tip advances whenever $\Psi_i > 0$ and arrests when the finite zone feeding vanishes.

In the physical setting of binary alloy solidification, the mobility M is related to the interface kinetic coefficient, while the threshold Ψ_c encodes the minimum solutal driving required to offset the Gibbs Thomson penalty at the tip. The detailed sharp interface growth tip law, obtained by combining the Ivantsov relation with a solvability law, gives a more refined selected velocity map in which M and Ψ_c are absorbed into a nonlinear response function. For the qualitative analysis of arrest and winner selection, however, the precise functional form is secondary. What matters is that $\dot{L}_i$ is a monotone increasing function of the finite process zone feeding and vanishes once that feeding falls below a critical level.

The transport input entering Eq. (126) is taken directly from the exact two-dendrite solution for the instantaneous geometry. In full generality one may write

$$\begin{aligned} \Psi_i(\ell, t; L_1, L_2, h) = & \frac{2D_{\text{CLC}}}{\pi\sqrt{\ell}} \int_0^{\infty} e^{-D\tau^2 t} \\ & \times \text{Re} \left[\int_0^{\ell} H_i(s, D(i\tau + 0^+)^2; L_1, L_2, h) ds \right] d\tau, \end{aligned} \quad (127)$$

where H_i is the exact Wiener Hopf kernel derived earlier. The modeling content of Eq. (126) lies solely in the decision to evaluate this exact transport quantity at the current geometry and couple it to the tip motion through a monotone threshold law.

It is useful to emphasize the distinction between exact transport and reduced dynamics. For any prescribed pair of tip positions L_1, L_2 , the quantity Ψ_i given by Eq. (127) is exact within the present transport model. The approximation enters only when that exact feeding is inserted into an autonomous law for the tip motion. The tip law is therefore local in geometry, not in transport. What is neglected is the dependence of the concentration field on the past motion of the interfaces. This history free reduction is plausible when the process zone Péclet number

$$\text{Pe}_\ell := \frac{\dot{L} \ell}{D}$$

remains small, and becomes increasingly questionable once $\text{Pe}_\ell = O(1)$. The reduced growth law and its implications for dendrite arrest are explored numerically in Sec. IV G in Fig. 9.

C. Dendrite arrest mechanism

To make the consequences of the exact transport solution explicit, we now specialize the reduced dynamics to the minimal monotone growth law

$$\dot{L}_i(t) = M(\Psi_i(\ell, t; L_1, L_2, h) - \Psi_c)_+, \quad i = 1, 2, \quad (128)$$

where $M > 0$ is a kinetic mobility and $\Psi_c \geq 0$ is the critical process zone feeding required to sustain advance. This law should be viewed as a coarse-grained version of the classical interfacial solute balance

$$V_n(1 - k)C^* = -D \partial_n C,$$

as given in, e.g., Ref. [51], where V_n is the local normal velocity of the solid liquid interface and k is the equilibrium partition coefficient, whereas $-D \partial_n C$ denotes the diffusive solute flux away from the interface in the normal direction. In the full sharp interface problem, the tip velocity is determined by the local normal diffusive flux together with the interfacial concentration, partition condition, and capillary selection. In the present reduced line description, however, the mathematical tip field is singular and its leading local amplitude is universal, so a pointwise flux cannot serve as the competition variable. The process zone functional Ψ_i therefore plays the role of an effective finite zone solute uptake, while the parameters M and Ψ_c absorb the geometric, partition, and capillary factors required to convert that uptake into a tip speed. This law provides the simplest reduced kinetics through which the exact process zone feeding can be converted into an arrest prediction.

Said arrest mechanism is kinetic rather than purely geometric. At early times, the two dendrites' depletion zones have not yet overlapped, and both receive nearly the same solute supply as an isolated dendrite: $\Psi_1 \approx \Psi_2 \approx \Psi^{(1)}$. Both tips therefore advance at comparable speeds. As time progresses, the wakes expand laterally on the diffusive scale $\sqrt{Dt}$, and once $\sqrt{Dt} \sim h$ the depletion zones begin to merge. From this point onward, each dendrite is drawing from a liquid that has already been partially depleted by the other, and the process zone feeding Ψ_i falls below the isolated dendrite value. If the two dendrites are perfectly symmetric, then both are equally shielded and both experience the same reduced feeding. But if one tip is even slightly ahead, then the asymmetry is amplified: The

leading tip extends further into undepleted liquid and receives more solute, while the lagging tip sits more deeply inside the neighbor's wake and receives less. The leader accelerates, the laggard decelerates, and the gap widens. This effect is explored numerically later in Fig. 9.

Within the growth law (126), dendrite i arrests at time $t = t_*$ if

$$\Psi_i(\ell, t_*; L_1, L_2, h) = \Psi_c \quad (129)$$

and $\Psi_i(\ell, t) \leq \Psi_c$ for all $t \geq t_*$. Physically, this means that the solute supply into the laggard's process zone has fallen below the minimum needed to sustain growth against capillary retardation. Once arrested, the laggard's tip is frozen. Its Dirichlet line continues to act as a solute sink, but the tip no longer advances. The winning dendrite is the one whose process zone feeding remains above threshold. Its feeding also decreases relative to the isolated dendrite benchmark, because the arrested neighbor still depletes the shared liquid, but provided Ψ_{winner} remains above Ψ_c the winner continues to grow.

Arrest therefore emerges from the coupled dynamics: The exact transport solution determines Ψ_i at each instant, the reduced kinetic law converts feeding into tip speed, and the positive feedback between tip position and solute supply drives the laggard's feeding below threshold. The transport dependence enters entirely through the exact kernel H_i , while the kinetic response is parametrized by M and Ψ_c . Everything else, including the time of arrest, the identity of the winner, and the subsequent acceleration of the surviving dendrite, is a prediction of the coupled system.

D. Choice of the process zone length ℓ

The process zone length ℓ is determined by the outer interaction structure, not by the inner tip scales. As detailed in Appendix D, the microscopic tip averaging scale $\delta_i = \Lambda \rho_i$ inherited from the sharp interface matching is distinct from, and much smaller than, the mesoscopic process zone ℓ needed to detect competitive shielding. The role of the microscopic scale is to regularise the singular endpoint field and recover the inner coupling variable at leading order. The role of the mesoscopic scale is different, namely to sample enough of the feeding profile behind the tip for the neighbor induced depletion to become visible.

For the canonical two dendrite kernel $(1 + e^{-\kappa h})^{-1}$, the Mittag-Leffler analysis shows that the dominant competitive mode has screening length h/π : higher order modes are all invariably local to the tip. This identifies the natural outer scale on which the neighbor modifies the feeding field, and thus, the natural choice for the process zone length,

$$\ell = \frac{h}{\pi}. \quad (130)$$

as the lengthscale of the first interaction mode capable of producing an appreciable difference between the two finite zone feedings. In this sense, $\ell = h/\pi$ is not an arbitrary fitting parameter but the natural mesoscopic scale selected by the exact interaction kernel itself. More generally, one expects the qualitative competition criterion to be robust throughout the

mesoscopic window

$$\rho_i \ll \ell \lesssim h,$$

because in that range the average already suppresses the universal microscopic singularity while still remaining local relative to the full outer transport field. In the numerical results below we therefore adopt $\ell = h/\pi$ as the canonical choice.

E. Symmetry breaking and winner selection

The arrest mechanism described above operates even when both dendrites are equally shielded: If the symmetric feeding $\Psi_*(\ell, \bar{L}, h)$ falls below Ψ_c , then both dendrites arrest together. But this is not how competition typically plays out. In practice, one dendrite arrests while the other survives, and the question is what selects the winner.

If the two dendrites are perfectly identical and start from equal tip positions, $L_1(0) = L_2(0)$, then symmetry enforces $\Psi_1(\ell, t) = \Psi_2(\ell, t)$ for all time. Both tips advance at the same speed, both experience the same shielding, and no deterministic winner can emerge. This is not a deficiency of the model; it is the correct prediction for a perfectly symmetric configuration. Winner selection requires a symmetry breaking perturbation: a small initial tip offset, a slight difference in nucleation time, a local fluctuation in undercooling, or any other source of asymmetry that gives one dendrite a transient advantage over the other.

To analyze the fate of such a perturbation, write

$$\bar{L} = \frac{L_1 + L_2}{2}, \quad \delta = \frac{L_1 - L_2}{2},$$

so that $\bar{L}$ tracks the mean tip position and δ measures the half-offset between the two tips. When $|\delta|$ is small, the process zone feeding of each dendrite can be expanded about the symmetric branch:

$$\Psi_1 = \Psi_*(\ell, \bar{L}, h) + \Gamma_\ell(\bar{L}, h) \delta + O(\delta^2), \quad (131)$$

$$\Psi_2 = \Psi_* - \Gamma_\ell \delta + O(\delta^2), \quad (132)$$

where $\Psi_* := \Psi_i|_{\delta=0}$ is the symmetric feeding amplitude and Γ_ℓ is the linear sensitivity of the feeding to a tip offset. The coefficient Γ_ℓ is a property of the exact transport field; it measures how strongly a small geometric asymmetry translates into an asymmetry in the solute supply.

Substituting Eq. (131) into the growth law (126) and subtracting gives, while $\Psi_* > \Psi_c$ (so that both dendrites are still growing),

$$\dot{\delta} = M \Gamma_\ell(\bar{L}, h) \delta + O(\delta^2). \quad (133)$$

This is a linear ODE for the tip offset, and its behavior is controlled entirely by the sign of Γ_ℓ .

If $\Gamma_\ell > 0$, then the offset grows exponentially: A dendrite that is slightly ahead receives more solute ($\Psi_1 > \Psi_*$), advances faster, and extends its lead, while the lagging dendrite receives less solute ($\Psi_2 < \Psi_*$), decelerates, and falls further behind. The feedback is positive and self-reinforcing. The offset grows until the laggard's feeding drops to threshold, at which point it arrests. This is the mathematical mechanism of competitive exclusion: an arbitrarily small initial advantage

is amplified by the asymmetric solute field into a finite and eventually fatal competitive gap.

If $\Gamma_\ell < 0$, then the opposite occurs: The offset decays, the two dendrites self-correct, and both survive. The symmetric state is then linearly stable, and no winner is selected by small perturbations.

Finally, if $\Gamma_\ell = 0$, then the leading-order feedback vanishes and higher-order terms control the dynamics; we do not consider this degenerate case.

In practice, then two dendrites nucleate at nearly the same time and begin to grow into the supersaturated liquid. Initially their depletion zones are independent and both advance as isolated dendrites. After a diffusive delay of order $t \sim h^2/D$, the wakes overlap and mutual shielding sets in. If one tip has even a slight positional advantage $\delta_0 > 0$, and if $\Gamma_\ell > 0$, then the asymmetry is amplified on a timescale $1/(M\Gamma_\ell)$. The lagging dendrite's feeding decreases monotonically until it crosses the arrest threshold Ψ_c , at which point it stops growing. The leading dendrite continues, now partially relieved of the competitor's shielding, and accelerates away. The entire selection process, from the onset of shielding to the arrest of the loser, is determined by three quantities: the exact process zone feeding Ψ_i (from the Wiener-Hopf solution), the sensitivity Γ_ℓ (from the antisymmetric mode), and the threshold Ψ_c (from the capillary penalty).

1. Sensitivity coefficient Γ_ℓ from the antisymmetric mode

The coefficient Γ_ℓ can be computed from the antisymmetric Wiener-Hopf mode, which was set to zero in the perfectly symmetric analysis. When the two tips are offset by δ , the antisymmetric component of the Dirichlet forcing acquires a nonzero contribution of order δ . To first order in the tip offset, the antisymmetric forcing is formally reduced to

$$G_{a,-}(k, p) = \frac{c_{LC} \delta}{p} e^{ik\bar{L}} + O(\delta^2), \quad (134)$$

where the passage from the exact half-line transform $(c_{LC}/p)e^{ikL_i}/(i(k-i0))$ to the expression above uses $k/(k-i0) \approx 1$ away from $k=0$. This is an entire function of k (no pole at $k=0$, because the antisymmetric datum is localized near the tips rather than extending over the entire half-line). The formal antisymmetric Wiener-Hopf equation with kernel

$$K_a(k, p) = \frac{2D\kappa(k, p)}{1 - e^{-\kappa(k, p)h}} \quad (135)$$

is solved by the same multiplicative factorization and Liouville argument as the symmetric case, yielding the antisymmetric density $\hat{\mu}_a = K_{a,-} G_{a,-} + O(\delta^2)$. The sensitivity coefficient is then

$$\Gamma_\ell(\bar{L}, h) = \frac{2Dc_{LC}}{\pi\sqrt{\ell}} \operatorname{Re} \left[\int_0^\infty e^{-D\tau^2 t} \times \int_0^\ell \frac{\partial H_a}{\partial \delta} \Big|_{\delta=0} (s, D(i\tau + 0^+)^2; \bar{L}, h) ds d\tau \right]. \quad (136)$$

The antisymmetric kernel K_a admits the same Mittag-Leffler expansion as the symmetric one, but with poles at $\kappa h = i2n\pi$

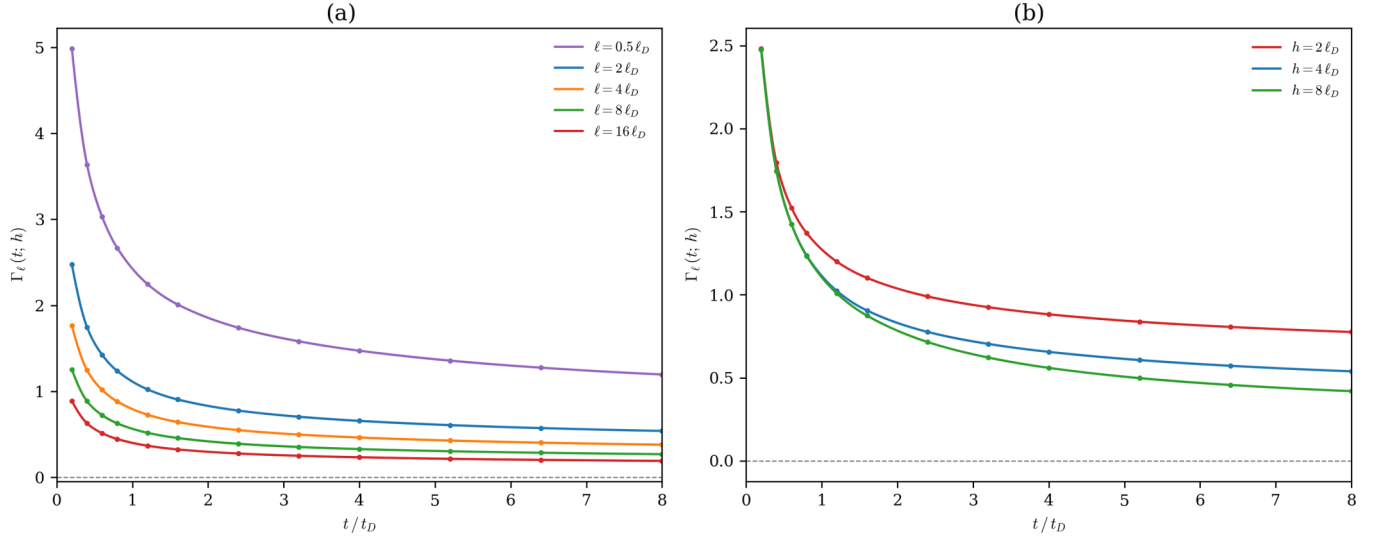


FIG. 7. Antisymmetric sensitivity coefficient $\Gamma_\ell(t; h)$ computed from the exact representation (136) via the Plemelj–Talbot pipeline. The dashed gray line is $\Gamma_\ell = 0$. (a) Fixed spacing $h = 4\ell_D$, with process zone length varying over $\ell/\ell_D = 0.5$ (purple), 2 (blue), 4 (orange), 8 (green), 16 (red). Shorter process zones give larger Γ_ℓ but the sign is always positive. (b) Fixed process zone $\ell = 2\ell_D$, with spacing $h/\ell_D = 2$ (red), 4 (blue), 8 (green). At early times the curves coincide (isolated-dendrite limit); at late times, closer spacing gives larger Γ_ℓ because the overlapping depletion zones amplify the feeding asymmetry. In every case $\Gamma_\ell > 0$: the symmetric state is unstable and any tip offset is amplified.

rather than $i(2n - 1)\pi$. The leading antisymmetric screening length is therefore $h/(2\pi)$, half the symmetric screening length h/π . This is physically natural: the antisymmetric mode, in which one dendrite gains what the other loses, is shorter-ranged than the symmetric mode in which both are suppressed equally.

The sign of Γ_ℓ determines whether the symmetric state is stable or unstable. A proof of the unconditional positivity of the sensitivity coefficient Γ_ℓ is given in Appendix E. Here, we have evaluated $\Gamma_\ell(t; h)$ numerically from the exact representation (136), using the same Plemelj–Talbot scheme described in Appendix C but with the antisymmetric Wiener–Hopf factor $M_{a,-}$ [computed from the logarithmic Cauchy integral with $\log(1 - e^{-h\kappa})$ in place of $\log(1 + e^{-h\kappa})$]. Figure 7 shows the result for three lateral spacings $h/\ell_D = 2, 4, 8$ with $\ell = 2\ell_D$. As expected, the sensitivity is strictly positive at every time, every spacing, and every process zone length tested. The smallest value in the entire survey is $\Gamma_\ell \approx 0.025$ (at $\ell = 16\ell_D$, $t = 8t_D$), and no curve shows any tendency to approach zero. The sign is therefore positive across the full physically relevant parameter range.

Figure 7(a) reveals the dependence on process zone length. Shorter process zones give larger Γ_ℓ , because the normalization $1/(2\sqrt{\ell})$ amplifies the near-tip asymmetry where the subleading competitive correction is strongest. As ℓ increases, Γ_ℓ decreases monotonically but remains positive: the finite-zone average dilutes the near-tip signal with the increasingly uniform far-field wake, but cannot reverse its sign. This confirms that the instability of the symmetric state is not an artifact of any particular choice of ℓ .

Figure 7(b) shows the dependence on lateral spacing at fixed ℓ . At early times ($t \ll h^2/D$), the depletion zones have not yet overlapped and all three curves coincide: Γ_ℓ reduces to the isolated-dendrite sensitivity, which measures how much extra solute a dendrite receives per unit advance of its tip. As

time progresses and the wakes merge, the curves separate, and closer spacing gives *larger* Γ_ℓ . This is the quantitative expression of the positive feedback loop: diffusive shielding not only reduces the total feeding (lowering Ψ_*) but simultaneously amplifies the response to any geometric asymmetry (raising Γ_ℓ).

The symmetric state is therefore unstable for the entire range of spacings, process zone lengths, and times examined. Competitive exclusion is the generic outcome: any initial tip offset, however small, is amplified by the transport field until the lagging dendrite’s feeding falls to threshold.

F. Critical coexistence spacing

The exact two-dendrite transport problem treats the lateral spacing h as a prescribed geometric parameter. It therefore does not select a unique equilibrium spacing. What it does provide, once coupled to the reduced process zone growth law, is a sharply defined *coexistence threshold* h_c , defined as the minimum spacing required for two identical dendrites to sustain simultaneous growth on the symmetric branch.

Consider the reduced dynamics

$$\dot{L}_i(t) = M(\Psi_i(\ell, t; L_1, L_2, h) - \Psi_c)_+, \quad i = 1, 2, \quad (137)$$

with mobility $M > 0$ and process zone threshold $\Psi_c \geq 0$. On the exactly symmetric branch,

$$L_1(t) = L_2(t) =: \Lambda(t), \quad (138)$$

the two dendrites have identical feeding,

$$\Psi_1(\ell, t; \Lambda, \Lambda, h) = \Psi_2(\ell, t; \Lambda, \Lambda, h) =: \Psi_*(\ell, t; h). \quad (139)$$

By common translation invariance, Ψ_* depends on h , ℓ , and t , but not on the absolute tip position Λ .

Physically, $\Psi_*(\ell, t; h)$ represents the solute supply available to each dendrite when both tips are level. If $\Psi_*(\ell, t; h)$

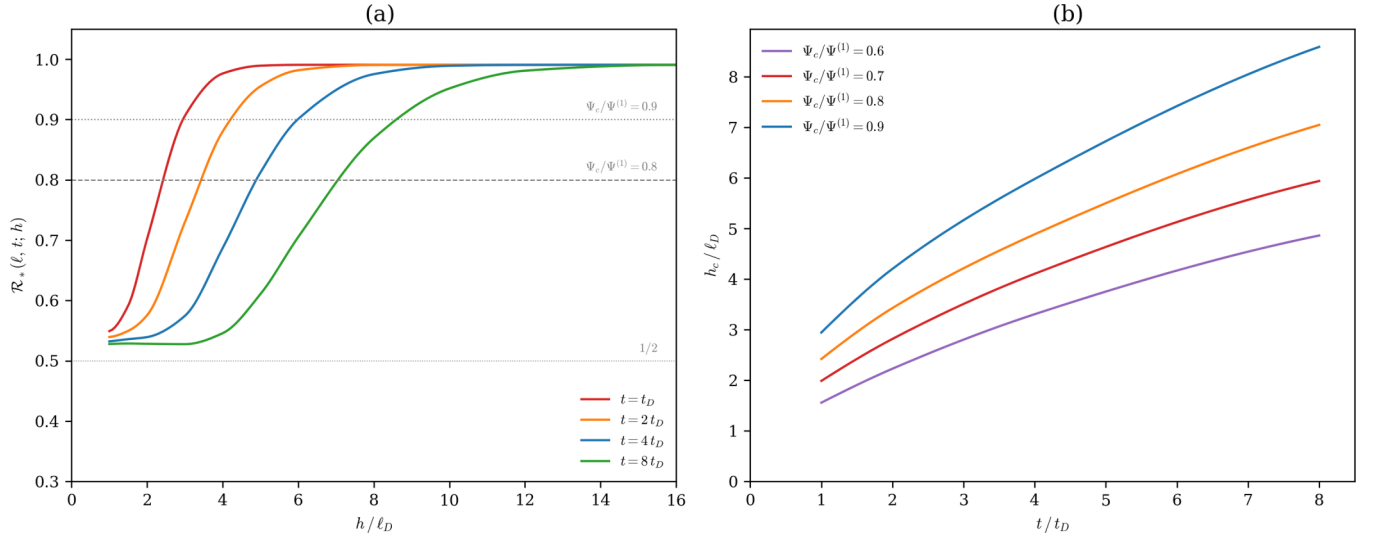


FIG. 8. Critical coexistence spacing for the symmetric two-dendrite problem at $\ell = 2\ell_D$. (a) The symmetric competition ratio $\mathcal{R}_*(\ell, t; h) = \Psi_*(\ell, t; h)/\Psi^{(1)}(\ell, t)$ increases monotonically with spacing and decreases in time as the diffusive wakes overlap. (b) The corresponding coexistence threshold $h_c(t)$, defined by $\mathcal{R}_*(\ell, t; h_c) = \Psi_c/\Psi^{(1)}$, increases both with time and with the threshold level.

lies above threshold, then the pair can continue to grow together. If it falls below threshold, then symmetric coexistence is impossible and both dendrites arrest. This motivates the definition of the critical coexistence spacing

$$h_c(\ell, t; \Psi_c) := \inf\{h > 0 : \Psi_*(\ell, t; h) > \Psi_c\}. \quad (140)$$

When $h \mapsto \Psi_*(\ell, t; h)$ is continuous and strictly increasing, this is equivalently the unique solution of

$$\Psi_*(\ell, t; h_c) = \Psi_c. \quad (141)$$

Let us now normalize by the isolated-dendrite feeding $\Psi^{(1)}(\ell, t)$. Writing

$$\mathcal{R}_*(\ell, t; h) := \frac{\Psi_*(\ell, t; h)}{\Psi^{(1)}(\ell, t)}, \quad (142)$$

the coexistence threshold becomes

$$\mathcal{R}_*(\ell, t; h_c) = \frac{\Psi_c}{\Psi^{(1)}(\ell, t)}. \quad (143)$$

The ratio $\mathcal{R}_*$ is the fraction of the isolated solute supply that remains available to each dendrite when a neighbor is present at spacing h .

Figure 8 shows the resulting structure for the canonical choice $\ell = 2\ell_D$. In it, Fig. 8(a) plots $\mathcal{R}_*(\ell, t; h)$ as a function of spacing at four times. From it, $\mathcal{R}_*$ increases monotonically with h , as expected physically: the further apart the dendrites are, the less strongly their depletion zones overlap. Second, for fixed spacing, $\mathcal{R}_*$ decreases with time. This reflects the gradual build up and overlap of the diffusive wakes. At early times the competition field is still localized near each dendrite and even modest spacing is sufficient to maintain near-isolated feeding. At later times the wake extends further into the liquid, so the same spacing produces a stronger reduction in supply.

Figure 8(b) shows the induced critical spacing $h_c(t)$ for several threshold levels $\Psi_c/\Psi^{(1)}$. The curves are monotone increasing in time: as the transport field develops, a larger separation is required to keep the symmetric pair above threshold.

They are also ordered by threshold in the obvious way: a more demanding feeding requirement produces a larger coexistence spacing. Thus, h_c is not a universal primary spacing, but a time-dependent transport threshold separating symmetric joint growth from symmetric arrest in the reduced two dendrite model.

The present solutions suggest that rather than have a dendritic array select a single spacing, there exists a spacing below which two neighboring dendrites cannot both remain viable at the prescribed process zone threshold. In that sense h_c is the exact transport input for a coexistence criterion, not a stand alone spacing selection law.

G. Computational results

Having derived the exact transport theory, the process zone functional, and the corresponding arrest criterion, we now evaluate the resulting dynamics numerically. The aim of this subsection is twofold. First, we show how the exact feeding functional $\Psi_i(\ell, t; h)$ translates into competitive growth and arrest once inserted into the reduced law Eq. (126). Second, we verify independently that the transport quantity driving that reduced dynamics is itself quantitatively correct, by comparison with direct finite-difference solutions of the full two-dimensional advection diffusion equation.

1. Competitive dynamics

We begin with the dynamical consequences of the theory. Figure 9(a) shows the time-dependent process zone feeding for an isolated dendrite and for a dendrite in the presence of a neighbor at three lateral spacings. At early times, all curves are essentially indistinguishable. This is the transient regime in which the depletion wake generated by each dendrite has not yet spread far enough laterally to interact with the other. In particular, the process zone near either tip does not yet “know” that a competitor is present, and the exact two dendrite feeding remains asymptotic to the isolated value. The onset of competition is therefore delayed, not instantaneous, and

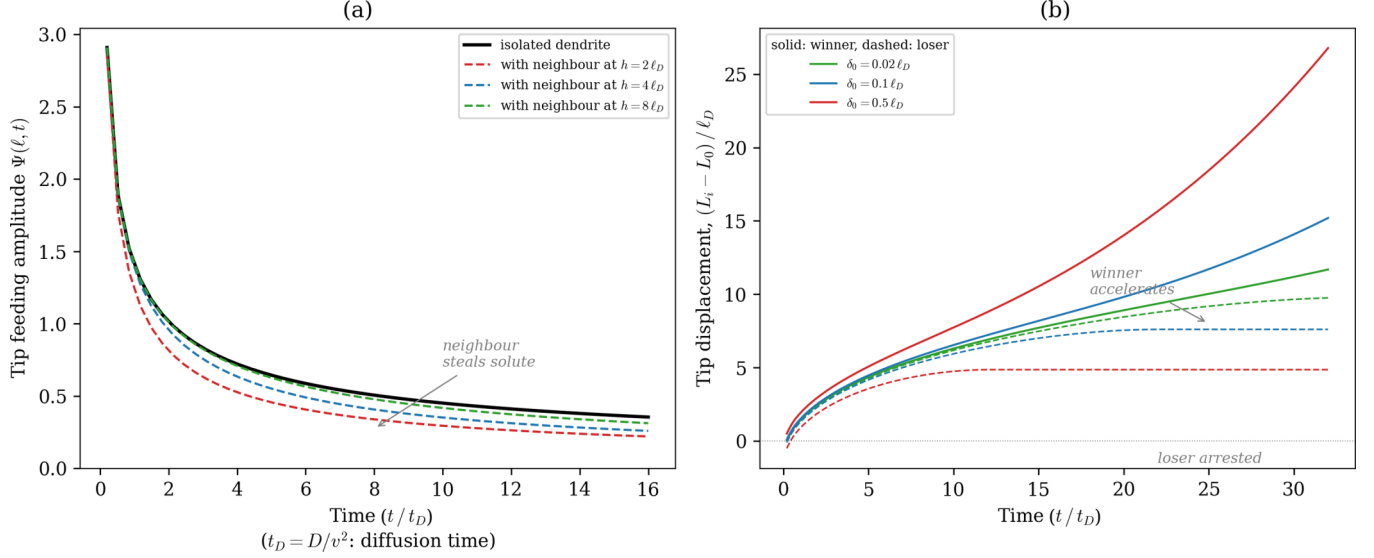


FIG. 9. Competitive growth dynamics. (a) Time-dependent process zone feeding $\Psi(\ell, t)$ for an isolated dendrite (black) and for a dendrite with a neighbor at three lateral spacings $h/\ell_D = 2, 4, 8$ (dashed), where $\ell_D = D/v$ is the solutal diffusion length and $t_D = D/v^2$ the diffusion time. At early times the curves coincide, reflecting the pre overlap regime in which the depletion wakes have not yet interacted. As the wakes merge, the two dendrite feeding falls below the isolated value; the suppression is earlier and stronger at smaller spacing. (b) Tip trajectories under the reduced growth law Eq. (126) at $h = 4\ell_D$, for three initial offsets δ_0 . Solid: winner; dashed: loser. The exact transport asymmetry is converted into dynamical competition: the leader accelerates into less depleted liquid, while the laggard decelerates and arrests.

occurs only once the diffusive wakes have expanded over a distance of order h . In-dimensional terms this is the natural timescale $t \sim h^2/D$ already predicted by the exact transport theory.

Once this overlap regime is reached, the two dendrite feeding curves separate decisively from the isolated benchmark. The feeding is always reduced relative to the isolated dendrite, and the suppression is stronger and earlier for smaller spacing. This is the first direct manifestation of competitive shielding in the exact transport solution. The neighbor does not alter the universal singular amplitude at the mathematical tip, but it does reduce the finite zone solute supply available behind the tip, precisely as encoded by the process zone functional. Figure 9(a) is therefore not merely a numerical plot of Ψ : it is the transport signature of the mechanism developed analytically in Sec. IV. The gap between the isolated and interacting curves is the finite distance competitive depletion that a purely local tip law would miss.

Figure 9(b) then shows what this transport deficit does dynamically once coupled to the reduced growth law. For a fixed spacing $h = 4\ell_D$, three different growth offsets are shown. In every case the same sequence unfolds. The dendrite that is initially slightly ahead samples less depleted liquid, its process zone feeding remains larger, and it therefore advances more rapidly. The lagging dendrite, by contrast, sits more deeply in the shared wake, receives less solute, and decelerates until its feeding falls below threshold and arrest occurs. The winner then continues to grow, still subject to some residual shielding, while the loser remains frozen as a passive sink. What Fig. 9(b) establishes is that the exact transport input, when inserted into even the simplest admissible growth law, is already sufficient to generate the familiar phenomenology of dendritic competition: amplification of a small asymmetry, arrest of the laggard, and survival of a single winner. In this

sense the figure is the numerical realization of the arrest and winner selection criteria derived above.

Note that what is validated independently is the transport input, namely the exact process zone functional $\Psi_i(\ell, t; h)$. The competitive dynamics in Fig. 9(b) are then model predictions obtained by evaluating that validated transport quantity on the instantaneous geometry and coupling it to the reduced law Eq. (126). They are therefore not direct moving boundary simulations, but reduced dynamical consequences of an exact and independently checked transport theory.

2. Two-dendrite transport validation

We now turn from the reduced dynamics back to the transport problem itself. The question is whether the exact Wiener-Hopf theory predicts the correct finite zone feeding when compared against an independent numerical solution of the underlying PDE. To answer this, we compute the exact process zone functional $\Psi(\ell, t; h)$ from the Wiener-Hopf representation and compare it directly with a finite-difference solution of the full two-dimensional advection diffusion equation.

The exact Ψ is evaluated numerically by first computing $\tilde{\Psi}(\ell, p; h)$ in the Laplace domain for $\text{Re}(p) > 0$, using the regularized oscillatory ω -integral Eq. (123) with $M_-(\omega^-)$ obtained from the Plemelj boundary evaluation Eq. (124), and then inverting to the time domain by the Talbot contour method [52,53]. This route stays entirely in the half plane $\text{Re}(p) > 0$ and therefore avoids the interaction poles of the Mittag-Leffler representation, which lie on the imaginary p axis. The finite-difference computation uses the symmetry of the two dendrite geometry: the problem for dendrites at $x = \pm h/2$ is reduced to the half domain $x \geq 0$, with a

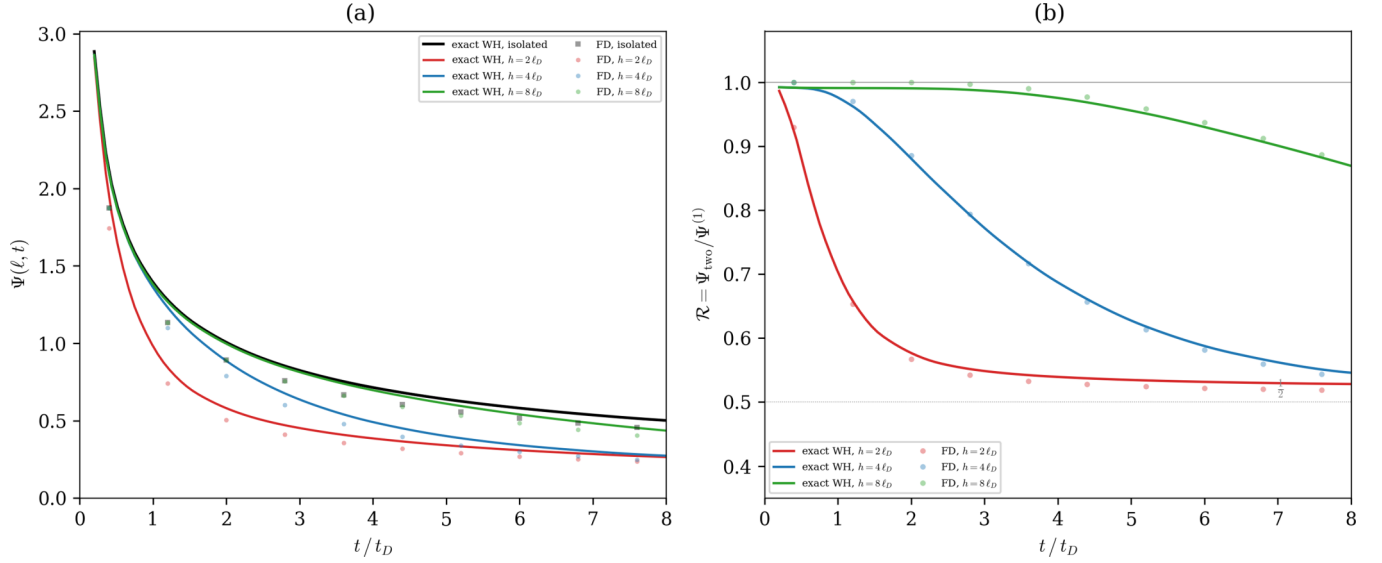


FIG. 10. Two-dendrite transport validation. (a) Process zone feeding $\Psi(\ell, t)$ with $\ell = 2\ell_D$: exact Wiener-Hopf solution evaluated via Talbot inversion (solid lines) and finite-difference data (points). Black: isolated dendrite. Red, blue, green: two dendrites at $h/\ell_D = 2, 4, 8$ respectively. The exact theory captures both the onset and the magnitude of competitive suppression. (b) Competition ratio $\mathcal{R} = \Psi_{\text{two}}/\Psi^{(1)}$: exact theory (lines) and finite-difference data (points). The decay from $\mathcal{R} \approx 1$ marks the onset of wake overlap; smaller spacing produces faster and stronger suppression.

Neumann condition at $x = 0$ and an internal Dirichlet condition at $x = h/2$ for $y < 0$.

Figure 10 shows the comparison for three spacings $h/\ell_D = 2, 4, 8$, with process zone length $\ell = 2\ell_D$. Figure 10(a) compares the time-dependent feeding amplitudes themselves. The isolated dendrite benchmark is reproduced by both methods, confirming that the normalization of the process zone functional is consistent. More importantly, the interacting cases show that the exact theory captures not merely the existence of suppression, but its full temporal structure. For $h = 2\ell_D$ the feeding drops earliest and most strongly, because the depletion wakes overlap almost immediately and the shared liquid between the two dendrites is rapidly exhausted. For $h = 8\ell_D$ the suppression is weaker and delayed, because the wakes remain effectively independent for much longer. The finite-difference data track these exact curves closely over the full time window. The small discrepancy at the earliest times is physically understood: on a regular finite-difference grid, the inverse square-root tip singularity is unresolved, and this produces a slight offset in the extracted finite zone amplitude.

Figure 10(b) recasts the same comparison in terms of the competition ratio Eq. (143),

$$\mathcal{R} = \frac{\Psi_{\text{two}}}{\Psi^{(1)}},$$

which measures directly how much of the isolated dendrite feeding survives in the presence of a neighbor. In this normalization, the transport meaning of competition becomes especially transparent. The ratio begins near unity because the two dendrites are initially unaware of each other, then decays monotonically as the wakes expand and overlap. The stronger the overlap, the lower the ratio. For the closest spacing, the ratio falls rapidly toward 1/2, showing that each dendrite is effectively forced to share the same depleted supply. For the widest spacing shown, the ratio remains close to one

throughout the computed interval, confirming that competitive suppression is then weak. Figure 10 therefore verifies the central transport content of the theory: The interaction kernel $(1 + e^{-\kappa h})^{-1}$ and its Wiener-Hopf factorization produce the correct magnitude and timing of the finite zone competitive depletion.

These comparisons are, to our knowledge, the first direct numerical verification of the exact two dendrite Wiener-Hopf transport solution against an independent PDE computation. Their significance is not merely numerical agreement. They show that the analytically derived process zone functional is the correct object through which the shared diffusion field feeds the tips, and that the competitive suppression predicted by the exact kernel is quantitatively realized in the underlying transport problem.

The transport origin of this suppression is shown directly in Fig. 11, which plots the concentration field at a fixed time. For an isolated dendrite, Fig. 11(a), the depletion wake is narrow and focused, with lateral extent of order ℓ_D . The dendrite draws solute from a relatively undisturbed region on either side, and the wake simply trails behind the advancing tip. Once a second dendrite is introduced, however, the geometry of the wake is qualitatively altered. At $h = 4\ell_D$, Fig. 11(b), the two wakes remain distinguishable but already interact strongly in the region between the dendrites. The liquid there has been depleted by both sinks, and the concentration gradients feeding the inner flanks of the two dendrites are visibly weaker than those on the outer flanks. At $h = 2\ell_D$, Fig. 11(c), this shielding becomes extreme: the interdendritic liquid is almost fully depleted, and the two individual wakes have merged into a single shared depletion zone.

This is the spatial content of the process zone theory. The competition is not localized at the mathematical tip, where the singular field is universal, but is encoded in the finite distance concentration field behind the tip, where the neighbor changes

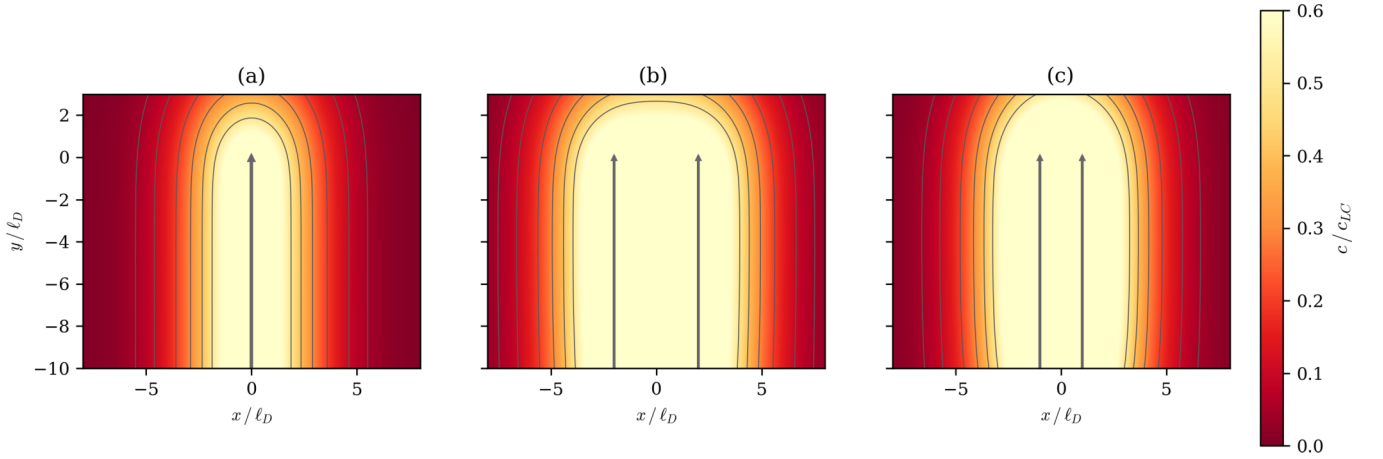


FIG. 11. Concentration field c/c_{LC} at $t = 4t_D$ from the finite-difference solution. (a) Isolated dendrite. (b) Two dendrites at $h = 4\ell_D$. (c) Two dendrites at $h = 2\ell_D$. Gray lines: Dirichlet half lines; triangles: tips; arrow: flow direction. As the spacing decreases, the two depletion wakes merge and the gradients feeding the inner flanks of the dendrites are progressively weakened. This finite distance shielding is the spatial mechanism measured by the process zone functional and responsible for competitive arrest.

the shape of the depletion wake and weakens the gradients that can deliver fresh solute into the process zone. Figure 11 therefore provides the geometric picture behind Figs. 9 and 10: the process zone functional Ψ_i is precisely the scalar quantity that measures this wake induced depletion in a form that can be inserted into the reduced growth law.

V. CONCLUSIONS

When two identical dendrites grow side by side in a uniform flow, the leading inverse square-root singularity at each tip is exactly the same as if the neighbor were absent, so the local tip amplitude carries no information about the competition that is nevertheless taking place a finite distance behind the tip. The exact Wiener-Hopf solution of the two dendrite advection-diffusion problem makes this statement quantitative: the singular amplitudes satisfy $A_1(t) = A_2(t) = A^{(1)}(t)$ for all lateral spacings h and all times. Any growth law tied solely to the local singular amplitude is therefore structurally blind to competition in the present exact two dendrite geometry. In particular, a local flux intensity factor extracted from this exact symmetric outer solution would carry no information capable of distinguishing the competitors.

The interaction is real, and the analysis identifies precisely where it lives. Behind the mathematical tip, the subleading terms of the single-layer density are sensitive to the depletion field of the neighbor, and it is this finite distance structure that the competing tips actually feel. The natural object that records this sensitivity is the finite zone process functional $\Psi_i(\ell, t)$ introduced in Sec. IV. Two scales appear in its construction and they should not be confused. The microscopic averaging scale $\delta_i = \Lambda\rho_i$ is the inner length over which the singular outer field is regularized against the capillary tip, and is forced on the theory by the inner solvability problem. The mesoscopic competition scale $\ell \sim h/\pi$ is the range over which the depletion wakes of the two dendrites overlap, and is set by the outer geometry. The first scale explains why a finite zone average is needed at all, while the second explains

why the dominant competitive correction is controlled by h/π rather than by h itself.

The exact transport results that support this picture are the technical core of the paper. The single-dendrite problem is solved exactly by Wiener-Hopf factorization of the parabolic dispersion relation and inverted to the time domain by a Cagniard-de Hoop deformation, yielding a closed form for the single-layer density in terms of the complex error function. The symmetric two-dendrite problem is reduced to a scalar Wiener-Hopf equation with the transcendental interaction kernel $(1 + e^{-\kappa h})^{-1}$, whose lower factor is constructed from a logarithmic Cauchy integral, and the resulting process zone functional is given as a regularized oscillatory integral on the real Fourier axis. The same integrand admits a Mittag-Leffler expansion in which the interaction resolves into a discrete hierarchy of exponentially screened modes with screening lengths

$$\ell_n = \frac{h}{(2n-1)\pi}, \quad n = 1, 2, \dots,$$

the slowest of which sets the dominant competitive scale. The exact transport input has been validated against independent finite-difference computations at three lateral spacings, and this validation is the genuine numerical test of the analytical results.

Three physical consequences follow from these exact statements without further modeling. First, no purely local arrest criterion can exist in this geometry, because the only object that distinguishes the two tips lives at finite distance, not at the tip itself. Second, competitive screening switches on with an intrinsic transient delay of order h^2/D , the time required for the depletion wakes to expand and overlap, any quasi steady reduction of the problem necessarily suppresses the intrinsic transient mechanism by which competition first switches on. Third, the interaction is not algebraic in the manner of a far-field multipole correction but is organized into a discrete tower of exponentially screened modes, of which only the longest is appreciable beyond the immediate neighborhood of the tips. None of these features is visible in steady or geometrically

reduced treatments and each emerges as a clean consequence of the time-dependent exact solution.

The growth dynamics presented in Sec. IV are of a different epistemic status and this work is careful to mark the boundary. They are not exact moving boundary solutions but reduced predictions obtained by evaluating the exact process zone functional on the instantaneous geometry and feeding it into a monotone kinetic law discussed in Eq. (126). Within this reduced dynamics, arrest occurs when the lagging dendrite's finite zone feeding falls below the threshold required to sustain growth, and a winner is selected only after a symmetry breaking perturbation generates a persistent feeding asymmetry. In the perfectly symmetric deterministic configuration no winner is selected, because symmetry enforces $\Psi_1 = \Psi_2$ for all time. The exact universality result concerns the local singular amplitudes, $A_1 = A_2$, and shows separately that no purely local tip law can break this tie. Competitive exclusion therefore emerges only after the symmetry has been broken, in agreement with experimental phenomenology. The sensitivity of the antisymmetric mode that drives this selection has been computed and is found numerically to be positive over the relevant parameter range, so that a tip slightly ahead of its neighbor receives more solute and pulls further ahead. A rigorous proof of the sign of this sensitivity coefficient Γ_ℓ directly from the exact representation remains open and is one of the natural next steps.

The scope of this work is therefore that the exact transport analysis applies to the canonical two-dimensional geometry of one or two semi-infinite internal Dirichlet lines in a uniform background flow, and the finite-difference validation tests precisely this exact transport input. The arrest trajectories and the winner selection dynamics are predictions of a reduced history free tip law built on the exact transport, not solutions of a fully resolved free boundary problem. The contribution of the present work is to establish the exact outer component of dendrite competition theory, to identify the universality of the local tip singularity as a structural obstruction to local tip laws, and to provide the explicit finite zone object through which competition does in fact enter. The case of transient dendrite-wall interaction is studied in Ref. [37].

Several extensions follow naturally. The most pressing is to complete the inner matching program so that the effective growth law is derived from the perturbed tip solvability problem rather than postulated. On the geometric side, it remains to determine which features of the present theory survive for finite length interfaces, for Robin or full Stefan boundary conditions, for unequal dendrites, and for fully three-dimensional needle crystals. On the computational side, the reduced arrest law should be tested against direct moving boundary or phase-field simulations to delineate the regime in which the instantaneous geometry approximation is quantitatively reliable. All of these extensions build on the same transport backbone constructed here: the competition field of two dendrites has been solved exactly, the local singularity has been shown to be universal, and the finite zone feeding through which competitive arrest actually proceeds has been made explicit, computable, and independently validated.

More broadly, the present work changes the analytical picture of dendritic competition. It shows that the decisive transport quantity is not a local tip singularity, but a finite zone

feeding functional that resolves how neighboring dendrites deplete one another over the mesoscopic region from which growth is actually sustained. In that sense, the paper does more than solve a canonical advection diffusion problem: it identifies the correct outer variable for competitive growth, makes that variable explicit in exact form, and thereby provides a transport framework within which arrest, spacing selection, and eventual multibody competition can be studied on a genuinely analytical basis. The hope is that this shifts dendrite competition from a largely heuristic and numerical subject toward one in which exact structure, reduced dynamics, and physical selection can be related in a single coherent theory.

DATA AVAILABILITY

The data that support the findings of this article are available from the authors upon reasonable request.

APPENDIX A: CAGNIARD-DE HOOP INVERSION OF THE TRANSFORMED DENSITY

1. Cagniard-de Hoop deformation of the Bromwich integral

The strategy is to deform the Bromwich contour to simplify the inversion integral and evaluate it in a single step via the Cagniard-de Hoop technique [33,41].

To that end, we define

$$v = \sqrt{p/D}, \quad p = Dv^2, \quad dp = 2Dv dv, \quad (A1)$$

with $\text{Re}(v) > 0$ when $\text{Re}(p) > 0$. Under this map, the Bromwich contour $\mathcal{B} : p = c + iy$ with $y \in \mathbb{R}$ is mapped to a contour $\mathcal{B}'$ in the right half of the v plane, which has the shape of a hyperbola in v . This is the so-called Cagniard contour. Indeed, writing $v = \alpha + i\tau$, one has

$$v^2 = \alpha^2 - \tau^2 + 2i\alpha\tau = p/D,$$

and therefore

$$\alpha^2 - \tau^2 = c/D, \quad 2\alpha\tau = y/D.$$

The first relation shows that $\mathcal{B}'$ is the right branch of a hyperbola, traversed from the fourth quadrant to the first as y increases from $-\infty$ to $+\infty$. This is the precise shape of the Cagniard contour; Fig. 12 is a schematic picture of the subsequent deformation together with the branch structure.

Substituting Eq. (40) into Eq. (39) gives

$$\begin{aligned} \mu(-s, t) &= \frac{1}{2\pi i} \int_{\mathcal{B}'} e^{Dv^2 t} \bar{\mu}(-s, Dv^2) 2Dv dv \\ &= \frac{1}{2\pi i} \int_{\mathcal{B}'} e^{Dv^2 t} \frac{2c_{LC}}{v} F(\beta + \lambda(v)) 2Dv dv, \end{aligned} \quad (A2)$$

where

$$F(a) = \text{erf}(\sqrt{sa}) + \frac{e^{-sa}}{\sqrt{\pi sa}}, \quad \lambda(v) = \sqrt{\beta^2 + v^2}.$$

The essential simplification is that the factor $1/v$ from $\sqrt{D/p} = 1/v$ cancels exactly against the factor v in the Jacobian $dp = 2Dv dv$. The Bromwich integral therefore reduces to

$$\mu(-s, t) = \frac{2Dc_{LC}}{\pi i} \int_{\mathcal{B}'} e^{Dv^2 t} F(\beta + \sqrt{\beta^2 + v^2}) dv. \quad (A3)$$

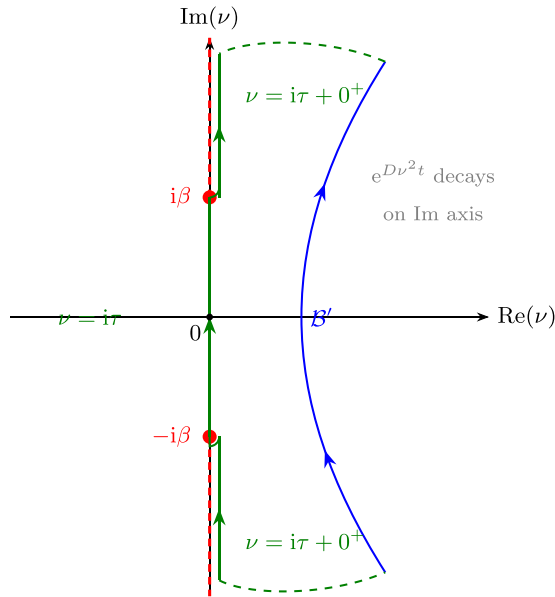


FIG. 12. Analytic structure in the ν plane ($\nu = \sqrt{p/D}$). The original parabolic contour $\mathcal{B}'$ (blue) lies in the right half plane. It is deformed leftward onto the imaginary axis (green), passing to the right of the branch cuts of $\lambda(\nu) = \sqrt{\beta^2 + \nu^2}$ (red dashed, running from $\pm i\beta$ to $\pm i\infty$). Between $-i\beta$ and $+i\beta$, λ is real and the integrand is smooth; beyond $|\tau| > \beta$, the contour runs along the right lip of the branch cut. The closing arcs at $|\nu| \rightarrow \infty$ (dashed) contribute nothing because $e^{D\nu^2 t} = e^{-D\tau^2 t} \rightarrow 0$ on the imaginary axis.

In the ν plane there is now no pole at the origin, and the branch singularity associated with $\sqrt{\rho}$ has disappeared. The only singularities are the branch points of $\lambda(\nu) = \sqrt{\beta^2 + \nu^2}$ at $\nu = \pm i\beta$. We take the corresponding branch cuts to run along the imaginary axis from $i\beta$ to $i\infty$ and from $-i\beta$ to $-i\infty$, exactly as shown in Fig. 12. With this choice, the integrand in Eq. (A3) is analytic in the slit plane

$$\mathbb{C} \setminus ([i\beta, i\infty) \cup (-i\infty, -i\beta]).$$

2. Exact contour deformation

We now deform the contour $\mathcal{B}'$ onto the imaginary axis. To make the argument explicit, write

$$\Phi(v) = e^{Dv^2t} F(\beta + \sqrt{\beta^2 + v^2}),$$

so that Eq. (A3) becomes

$$\mu(-s, t) = \frac{2D_{\text{cLC}}}{\pi i} \int_{\mathcal{B}'} \Phi(v) dv.$$

Fix $R > \beta$, and let $\mathcal{B}'_R$ denote the portion of $\mathcal{B}'$ between its intersections with the lines $\text{Im}(v) = \pm R$. Let $\mathcal{I}_R$ denote the contour running from $-iR + 0^+$ to $iR + 0^+$ just to the right of the imaginary axis, exactly as in Fig. 12. Finally, let Γ_R^+ be a smooth arc in the upper half plane joining the upper endpoint of $\mathcal{B}'_R$ to $iR + 0^+$, and let Γ_R^- be a smooth arc in the lower half plane joining $-iR + 0^+$ to the lower endpoint of $\mathcal{B}'_R$. These arcs are chosen to remain entirely in the analytic region, that is, to the right of the branch cuts shown in Fig. 12.

With these definitions, the closed contour

$$\mathcal{C}_R = \mathcal{B}'_R \cup \Gamma_R^+ \cup (-\mathcal{I}_R) \cup \Gamma_R^-$$

lies in the slit ν plane and encloses no singularities of Φ . Therefore, Cauchy's integral theorem gives

$$\oint_{\mathcal{C}_R} \Phi(v) dv = 0,$$

that is,

$$\int_{\mathcal{B}_R} \Phi(v) dv + \int_{\Gamma_R^+} \Phi(v) dv - \int_{\mathcal{I}_R} \Phi(v) dv + \int_{\Gamma_R^-} \Phi(v) dv = 0. \quad (\text{A4})$$

Rearranging,

$$\int_{\mathcal{B}'_R} \Phi(\nu) d\nu = \int_{\mathcal{I}_R} \Phi(\nu) d\nu - \int_{\Gamma_R^+} \Phi(\nu) d\nu - \int_{\Gamma_R^-} \Phi(\nu) d\nu. \quad (\text{A5})$$

It remains to show that the two arc integrals vanish as $R \rightarrow \infty$. Along $\Gamma_R^\pm$, the contour may be chosen so that $\operatorname{Re}(\nu^2) \leq -\kappa R^2$ for some $\kappa > 0$ independent of R . Hence,

$$|e^{Dv^2t}| \leq e^{-\kappa DR^2t}.$$

However, as $|v| \rightarrow \infty$ one has $\sqrt{\beta^2 + v^2} \sim \pm iv$, so the factor $F(\beta + \sqrt{\beta^2 + v^2})$ grows at most algebraically in $|v|$. Therefore,

$$|\Phi(v)| \leq CR^m e^{-\kappa DR^2 t}$$

for some constants $C, m > 0$, uniformly on $\Gamma_R^\pm$. Since the lengths of $\Gamma_R^\pm$ are $O(R)$, it follows that

$$\int_{\Gamma_R^\pm} \Phi(v) dv \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Passing to the limit in Eq. (A5), we obtain

$$\int_{\mathcal{B}'} \Phi(v) dv = \int_{\mathcal{T}} \Phi(v) dv,$$

where $\mathcal{I}$ is the deformed contour

$$\nu = i\tau + 0^+, \quad -\infty < \tau < \infty. \quad (\text{A6})$$

Substituting this into Eq. (A3) gives

$$\mu(-s, t) = \frac{2D_{\text{CLC}}}{\pi} \int_{-\infty}^{\infty} e^{-D\tau^2 t} F(\beta + \lambda(i\tau)) d\tau, \quad (\text{A7})$$

where $\lambda(i\tau)$ is understood as the boundary value obtained by approaching the imaginary axis from the right half plane.

3. Boundary values on the imaginary axis

The path Eq. (A6) splits naturally into three pieces. For $|\tau| < \beta$, the point $i\tau$ lies between the branch points $\pm i\beta$, so no cut is crossed and the quantity $\beta^2 - \tau^2$ is positive. In this interval one simply has

$$\lambda(i\tau) = \sqrt{\beta^2 - \tau^2}, \quad |\tau| < \beta. \quad (\text{A8})$$

For $|\tau| > \beta$, the point $i\tau$ lies on the right lip of one of the cuts. To determine the corresponding boundary values, write $\nu = \varepsilon + i\tau$ with $\varepsilon > 0$ and let $\varepsilon \downarrow 0$. Then

$$\beta^2 + \nu^2 = \beta^2 + (\varepsilon + i\tau)^2 = (\beta^2 - \tau^2 + \varepsilon^2) + 2i\varepsilon\tau.$$

If $\tau > \beta$, then the real part tends to a negative number and the imaginary part tends to 0^+ , so the square root approaches the value on the upper lip of the cut:

$$\lambda(i\tau) = i\sqrt{\tau^2 - \beta^2}, \quad \tau > \beta.$$

If $\tau < -\beta$, then the real part again tends to a negative number but now the imaginary part tends to 0^- , so one obtains the boundary value on the lower lip:

$$\lambda(i\tau) = -i\sqrt{\tau^2 - \beta^2}, \quad \tau < -\beta.$$

Collecting the three cases,

$$\lambda(i\tau) = \begin{cases} \sqrt{\beta^2 - \tau^2}, & |\tau| < \beta, \\ i\sqrt{\tau^2 - \beta^2}, & \tau > \beta, \\ -i\sqrt{\tau^2 - \beta^2}, & \tau < -\beta. \end{cases} \quad (\text{A9})$$

This shows how the deformed contour samples the branch structure: on the central segment the square root remains real, whereas on the outer segments it takes conjugate purely imaginary values on the two lips of the cuts.

4. Reduction to the real part

We now explain carefully how the final $\text{Re}[\cdot]$ arises. The point is not that the integrand in Eq. (A7) is literally even in τ . Rather, the contributions from the upper and lower halves of the deformed contour are complex conjugates.

Indeed, from Eq. (A9) one sees immediately that

$$\lambda(-i\tau) = \overline{\lambda(i\tau)}, \quad \tau \in \mathbb{R}. \quad (\text{A10})$$

Since the function

$$F(a) = \text{erf}(\sqrt{sa}) + \frac{e^{-sa}}{\sqrt{\pi sa}}$$

has real coefficients and respects complex conjugation on the chosen branches, it follows that

$$F(\bar{a}) = \overline{F(a)}. \quad (\text{A11})$$

Combining Eqs. (A10) and (A11), and using that $e^{-D\tau^2 t}$ is real and even, gives

$$e^{-D(-\tau)^2 t} F(\beta + \lambda(-i\tau)) = \overline{e^{-D\tau^2 t} F(\beta + \lambda(i\tau))}.$$

Hence, the negative τ contribution in Eq. (A7) is the complex conjugate of the positive τ contribution. Therefore,

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-D\tau^2 t} F(\beta + \lambda(i\tau)) d\tau \\ &= 2\text{Re} \left[\int_0^{\infty} e^{-D\tau^2 t} F(\beta + \lambda(i\tau)) d\tau \right]. \end{aligned} \quad (\text{A12})$$

Substituting Eq. (A12) into Eq. (A7) yields the exact real integral representation

$$\mu(-s, t) = \frac{4Dc_{\text{LC}}}{\pi} \text{Re} \left[\int_0^{\infty} e^{-D\tau^2 t} F(\beta + \lambda(i\tau)) d\tau \right]. \quad (\text{A13})$$

APPENDIX B: EXPLICIT EVALUATION OF H_1

We carry out the Laplace inversion in Eq. (94) explicitly for $N = 1$. Substituting Eq. (100) into Eq. (94) gives

$$H_1(s, p; h) = \frac{1}{2} H_{\infty}(s, p) + \Delta H_1(s, p; h), \quad (\text{B1})$$

where the first term is one half of the isolated dendrite profile and the competitive correction is

$$\Delta H_1(s, p; h) = \mathcal{L}_{q \rightarrow s}^{-1} \left[\frac{2\kappa_q}{h(\gamma_1^2 - (q + \beta)^2)} \cdot \frac{1}{q} \sqrt{1 + \frac{q}{\beta + \lambda}} \right], \quad (\text{B2})$$

with $\kappa_q := \sqrt{\lambda^2 - (q + \beta)^2}$. The first term in Eq. (102) is already known in closed form from Eq. (98); it is the contribution from the $\frac{1}{2}$ in Θ_1 . The second term encodes the single competitive mode retained at this truncation order.

To evaluate ΔH_1 , observe that the denominator $\gamma_1^2 - (q + \beta)^2 = (\gamma_1 - q - \beta)(\gamma_1 + q + \beta)$ introduces two simple poles at $q = -\beta \pm \gamma_1$. Since $\gamma_1 > \lambda > \beta$, the pole $q_+ = -\beta + \gamma_1$ is a positive real number, and the pole $q_- = -\beta - \gamma_1$ is negative. Both lie to the left of the abscissa of convergence of the Laplace transform (which is at $\text{Re}(q) = 0$ for the full exact solution, but is shifted rightward by the truncation). The Bromwich contour for the Laplace inversion must therefore be placed to the right of both poles, at some $c > \gamma_1 - \beta$, and the inversion picks up residues at both q_+ and q_- together with the branch cut from $q = -(\beta + \lambda)$ to $-\infty$.

Define the shorthand

$$a := \beta + \lambda. \quad (\text{B3})$$

We combine the two square roots in the numerator. Since $\kappa_q = \sqrt{\lambda^2 - (q + \beta)^2} = \sqrt{(\lambda - q - \beta)(\lambda + q + \beta)}$ and $\sqrt{1 + q/a} = \sqrt{(q + a)/a} = \sqrt{(q + \beta + \lambda)/a}$, their product is

$$\kappa_q \sqrt{1 + \frac{q}{a}} = \frac{(q + \beta + \lambda)\sqrt{\lambda - \beta - q}}{\sqrt{a}}, \quad (\text{B4})$$

valid on the principal branch for $q < \lambda - \beta$. (For $q > \lambda - \beta$ the square root becomes imaginary, but the formula extends analytically.)

Substituting Eq. (B4) into Eq. (B2) gives the integrand for the Bromwich inversion:

$$\frac{2(q + \beta + \lambda)\sqrt{\lambda - \beta - q}}{h\sqrt{a}(\gamma_1^2 - (q + \beta)^2)q}. \quad (\text{B5})$$

a. Residue at $q_+ = -\beta + \gamma_1$. At this pole, $q + \beta = \gamma_1$, so $\gamma_1^2 - (q + \beta)^2$ has a simple zero with $\lim_{q \rightarrow q_+} (q - q_+)/(\gamma_1^2 - (q + \beta)^2) = -1/(2\gamma_1)$. The remaining factors evaluate to $q_+ = -\beta + \gamma_1$, $q_+ + \beta + \lambda = \gamma_1 + \lambda$, and $\lambda - \beta - q_+ = \lambda - \gamma_1 < 0$, so $\sqrt{\lambda - \beta - q_+} = i\sqrt{\gamma_1 - \lambda}$. Writing $\gamma_1^2 - \lambda^2 = \pi^2/h^2$, the residue is

$$R_+ e^{q_+ s} = \frac{-i(\gamma_1 + \lambda)\sqrt{\gamma_1 - \lambda}}{h\gamma_1\sqrt{a}(\gamma_1 - \beta)} e^{(-\beta + \gamma_1)s}. \quad (\text{B6})$$

b. Residue at $q_- = -\beta - \gamma_1$. At this pole, $q + \beta = -\gamma_1$, so $\lim_{q \rightarrow q_-} (q - q_-)/(\gamma_1^2 - (q + \beta)^2) = 1/(2\gamma_1)$. The remaining factors evaluate to $q_- = -\beta - \gamma_1$, $q_- + \beta + \lambda = \lambda - \gamma_1 < 0$, and $\lambda - \beta - q_- = \lambda + \gamma_1 > 0$. Hence,

$$R_- e^{q_- s} = \frac{(\lambda - \gamma_1)\sqrt{\lambda + \gamma_1}}{h\gamma_1\sqrt{a}(\beta + \gamma_1)} e^{-(\beta + \gamma_1)s}. \quad (\text{B7})$$

1. Branch cut contribution

The branch point at $q = -a = -(\beta + \lambda)$ generates a cut running to $-\infty$ along the real axis. Writing $q = -a - \sigma$ with

$\sigma > 0$, the discontinuity across the cut can be evaluated in the standard way, and the resulting integral has the same structure as the isolated dendrite inversion but with modified coefficients. We denote this contribution $\Delta H_1^{\text{cut}}(s, p; h)$. Its explicit form is

$$\Delta H_1^{\text{cut}}(s, p; h) = \frac{4}{h\sqrt{a\pi}} \int_0^\infty e^{-(\beta+\lambda+\sigma)s} \times \frac{(\lambda-\sigma)\sqrt{\sigma}}{(\gamma_1^2 - (\lambda+\sigma)^2)(\beta+\lambda+\sigma)} d\sigma, \quad (\text{B8})$$

where we have used the fact that the discontinuity of $\sqrt{\lambda-\beta-q}$ across the cut at $q = -a-\sigma$ is $2i\sqrt{\sigma}$.

Collecting all three contributions, the explicit $N = 1$ competitive correction is

$$\Delta H_1(s, p; h) = R_+ e^{(-\beta+\gamma_1)s} + R_- e^{-(\beta+\gamma_1)s} + \Delta H_1^{\text{cut}}(s, p; h), \quad (\text{B9})$$

with R_+ and R_- given by Eqs. (B6) and (B7), respectively.

2. Structure of the result

The physical content of Eq. (B9) is as follows. The branch cut contribution ΔH_1^{cut} decays at least as fast as $e^{-as} = e^{-(\beta+\lambda)s}$ and therefore has the same exponential envelope as the isolated dendrite profile. The pole residue R_- decays as $e^{-(\beta+\gamma_1)s}$ with $\gamma_1 > \lambda$, so it decays faster than the isolated dendrite and constitutes a short-ranged correction localized near the tip. The residue R_+ grows as $e^{(\gamma_1-\beta)s}$; this is the spurious pole introduced by the Mittag-Leffler truncation (the exact Wiener-Hopf factor m_- is analytic in the right q -half plane, so no such pole exists in the exact solution). For small s (near the tip), R_+ is subdominant, and its contribution is partially canceled by ΔH_1^{cut} . In practice, the spurious growth becomes significant only for $s \gg h/\pi$, i.e., far behind the tip, where the truncation error is expected. For the process zone integrals with $\ell \lesssim h/\pi$ that are the primary objects of interest, the $N = 1$ approximation is quantitatively reliable.

The complete $N = 1$ Laplace domain density on each dendrite is therefore

$$\begin{aligned} \bar{\mu}_1(-s, p) &= \bar{\mu}_2(-s, p) \\ &= 2c_{\text{LC}} \sqrt{\frac{D}{p}} \left[\frac{1}{2} H_\infty(s, p) + \Delta H_1(s, p; h) \right], \end{aligned} \quad (\text{B10})$$

with $s > 0$, H_∞ given by Eq. (98), ΔH_1 by Eq. (103), and $\gamma_1 = \sqrt{\lambda^2 + \pi^2/h^2}$. The corresponding time-domain density follows from the Cagniard-de Hoop formula Eq. (108) with $N = 1$.

a. Truncation error of the Mittag-Leffler expansion. The error incurred by replacing the exact interaction factor $1/(1+e^{-z})$ by the N -term approximation $\Theta_N(z)$ is controlled by the tail of the Mittag-Leffler series, but with the essential caveat that the convergence is locally uniform on bounded intervals, not uniform on $[0, \infty)$. This is because every term $2z/(z^2 + a_n^2) \rightarrow 0$ as $z \rightarrow \infty$, so any finite truncation satisfies $\Theta_N(z) \rightarrow 1/2$ as $z \rightarrow \infty$, whereas the exact function tends to 1. In other words, for any fixed N , the approximation is good at moderate z but inevitably degrades at sufficiently large z .

For real $z > 0$, the absolute error is

$$\left| \frac{1}{1+e^{-z}} - \Theta_N(z) \right| = \sum_{n=N+1}^\infty \frac{2z}{z^2 + a_n^2} \leq \frac{2z}{\pi^2} \sum_{n=N+1}^\infty \frac{1}{(2n-1)^2}. \quad (\text{B11})$$

The tail sum is bounded by

$$\sum_{n=N+1}^\infty \frac{1}{(2n-1)^2} < \frac{1}{4N}, \quad (\text{B12})$$

using the integral comparison $\sum_{n=N+1}^\infty (2n-1)^{-2} < \frac{1}{2} \int_{2N}^\infty u^{-2} du$, giving

$$\left| \frac{1}{1+e^{-z}} - \Theta_N(z) \right| < \frac{z}{2\pi^2 N}. \quad (\text{B13})$$

This bound grows linearly in z , confirming that the approximation degrades at large z for any fixed N . The crossover occurs at $z \sim N\pi$: for $z \ll N\pi$, the bound is small and Θ_N is reliable; for $z \gg N\pi$, the bound exceeds $1/2$ and Θ_N has collapsed back toward $1/2$. In particular, even $N = 50$ is only accurate for $z \lesssim 150$; no finite N captures the saturation to unity at arbitrarily large z .

For the physical problem, however, this limitation is less severe than it appears. The argument $z = h\kappa$ enters the Cagniard-de Hoop quadrature through the integrand $e^{-D\tau^2 t} H_N(s, D(i\tau + 0^+)^2; h)$, and the Gaussian weight $e^{-D\tau^2 t}$ ensures that only values of τ (and hence κ) up to $O(1/\sqrt{Dt})$ contribute significantly. The effective range of z sampled by the quadrature is therefore bounded by $z_{\text{eff}} \sim h/\sqrt{Dt}$. The required truncation order is set by $N \gtrsim z_{\text{eff}}/\pi = h/(\pi\sqrt{Dt})$. For a typical solidification problem with $h \sim 10 \mu\text{m}$, $D \sim 10^{-9} \text{ m}^2/\text{s}$, and $t \sim 0.1 \text{ s}$, one finds $z_{\text{eff}} \sim 30$ and hence $N \gtrsim 10$ suffices. At early times ($t \ll h^2/D$), z_{eff} is large and more terms are needed; at late times the Gaussian truncation tightens and fewer terms suffice. For the $N = 1$ approximation specifically, the error is small only when $z_{\text{eff}} \ll \pi$, i.e., when $h \ll \pi\sqrt{Dt}$, which corresponds to the regime where the depletion zones have already overlapped thoroughly.

3. Liouville argument for the two dendrite case

The scalar Wiener-Hopf equation (61) of the two dendrite case is here solved exactly as in the one dendrite case. Define

$$F(k, p) = K_{s,+}(k, p)[G_-(k, p) + G_+(k, p)] - G_-(k, p). \quad (\text{B14})$$

Because $K_{s,+}(0, p) = 1$, the pole of G_- at $k = 0$ is canceled in $(K_{s,+} - 1)G_-$, so $F \in H^+$. Multiplying Eq. (61) by $K_{s,+}^{-1}$ gives

$$K_{s,-}(k, p)F(k, p) = \hat{\mu}(k, p) - K_{s,-}(k, p)G_-(k, p),$$

and the right-hand side belongs to H^- . Hence, F is a plus function and $K_{s,-}F$ is a minus function. The same Liouville argument as before therefore yields

$$F(k, p) \equiv 0.$$

Thus,

$$G_-(k, p) + G_+(k, p) = \frac{G_-(k, p)}{K_{s,+}(k, p)}, \quad (\text{B15})$$

and consequently,

$$\widehat{\mu}(k, p) = K_{s,-}(k, p)G_-(k, p). \quad (\text{B16})$$

4. Inversion of the Laplace transform solution via Cagniard-de Hoop for the two dendrite case

We now invert the Laplace transform in p by the same Cagniard-de Hoop change of variable as before. From Eq. (76),

$$\begin{aligned} \mu(-s, t) &= \frac{1}{2\pi i} \int_B e^{pt} \widehat{\mu}(-s, p) dp \\ &= \frac{1}{2\pi i} \int_B e^{p_t} 2c_{LC} \sqrt{\frac{D}{p}} H(s, p; h) dp. \end{aligned} \quad (\text{B17})$$

Introduce

$$v = \sqrt{\frac{p}{D}}, \quad p = Dv^2, \quad dp = 2Dv dv, \quad (\text{B18})$$

with $\text{Re}(v) > 0$ when $\text{Re}(p) > 0$. The factor $\sqrt{D/p} = 1/v$ cancels against the v in the Jacobian, yielding

$$\mu(-s, t) = \frac{2Dc_{LC}}{\pi i} \int_{B'} e^{Dv^2 t} H(s, Dv^2; h) dv. \quad (\text{B19})$$

APPENDIX C: NUMERICAL EVALUATION OF THE EXACT TWO-DENDRITE PROCESS ZONE FUNCTIONAL

The Cagniard-de Hoop representation Eq. (108) is the natural analytical route to the time domain: It replaces the Bromwich inversion in p by an explicit real-axis integral in τ . For the single-dendrite problem, the CdH integrand is entirely explicit and the quadrature converges rapidly. For the two-dendrite problem, however, the CdH contour lies on the imaginary p axis ($p_\tau = -D\tau^2$), which passes through the Mittag-Leffler interaction poles at $\tau_n = (2n+1)\pi/h$. These poles are features of the interaction factor $(1 + e^{-\kappa h})^{-1}$ at $k=0$; they do not appear in the exact lower Wiener-Hopf factor $M_-(\omega, p)$ for $\text{Re}(p) > 0$, but they lie directly on the CdH integration path and cause severe numerical cancellation. The CdH representation is therefore excellent analytically, as it yields the exact inversion formulas of Sec. IIF and the Mittag-Leffler mode decomposition, but it is the worst choice for the direct numerical evaluation of the two dendrite $\Psi(\ell, t; h)$.

The resolution is to abandon the CdH contour for numerics and work directly in the Laplace p plane with $\text{Re}(p) > 0$. On any contour in the open right half p plane, the interaction factor $m_-(0, p; h) = 1/(1 + e^{-h\sqrt{p/D}})$ is bounded and smooth, because $\text{Re}(\sqrt{p/D}) > 0$ forces $|e^{-h\sqrt{p/D}}| < 1$. The Mittag-Leffler poles, which lie on the imaginary p axis, are never encountered.

The numerical procedure is as follows:

Step 1: Talbot contour. Choose a set of quadrature nodes $\{p_n\}_{n=1}^{2N}$ on the Talbot contour [52,53], parametrized by

$$p_n(\theta) = \frac{N}{t} (0.5017\theta \cot(0.6407\theta) - 0.6122 + 0.2645i\theta), \quad (\text{C1})$$

with $\theta_n = -\pi + (n - \frac{1}{2})\pi/N$. All nodes satisfy $\text{Re}(p_n) > 0$.

Step 2: Wiener-Hopf factor. At each p_n , evaluate $M_-(\omega^-, p_n)$ on the real ω axis via the Plemelj boundary value Eq. (124). The principal value integral is computed by Gauss-Legendre quadrature on the regularized integrand $[f(\xi) - f(\omega)]/(\xi - \omega)$ over $[-\Xi, \Xi]$, with $\Xi = \max(12, 3|\sqrt{p_n/D}|)$.

Step 3: Process zone integral. Compute $J(\ell, p_n; h) = J^{(1)}(\ell, p_n) + \Delta J(\ell, p_n; h)$, where $J^{(1)}$ is the exact single-dendrite integral (known in closed form via the error function) and

$$\begin{aligned} \Delta J(\ell, p; h) &= \int_0^\ell \Delta H(s, p; h) ds, \quad \Delta H = (F_0 - 1) \\ &+ \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\omega s} \frac{[M_-(\omega^-) - 1]\sqrt{1 + \frac{i\omega}{a}} - (F_0 - 1)}{i\omega} d\omega. \end{aligned} \quad (\text{C2})$$

The key advantage of this decomposition is that $[M_-(\omega) - 1] \rightarrow 0$ as $|\omega| \rightarrow \infty$ for $\text{Re}(p) > 0$, so the ω -integrand decays much faster than the raw integrand (which decays only as $|\omega|^{-1/2}$ due to the $\sqrt{1 + i\omega/a}$ factor). In practice, $W = 20$ and $n_\omega = 100$ Gauss-Legendre nodes on $[-W, W]$ suffice. The s integral is evaluated by Gauss-Legendre on $[0, \ell]$.

Step 4: Laplace domain functional. Form $\bar{\Psi}(\ell, p_n; h) = c_{LC}\sqrt{D/(p_n\ell)}J(\ell, p_n; h)$.

Step 5: Time domain inversion. Recover $\Psi(\ell, t; h)$ by the Talbot midpoint rule:

$$\Psi(\ell, t; h) = \text{Re} \left[\frac{1}{2Ni} \sum_{n=1}^{2N} e^{p_n t} \bar{\Psi}(\ell, p_n; h) \frac{dp_n}{d\theta} \frac{\pi}{N} \right]. \quad (\text{C3})$$

With $N = 24$ (i.e., 48 function evaluations), the Talbot inversion reproduces the exact single-dendrite CdH benchmark to machine precision.

This pipeline is validated in three stages. First, setting $h \rightarrow \infty$ (so $M_- \rightarrow 1$ and $\Delta J \rightarrow 0$) recovers the exact single dendrite $\Psi^{(1)}$ to machine precision. Second, the Talbot inversion of $\bar{\Psi}_{\text{single}}^{(1)}$ matches the CdH quadrature at all times. Third, the two dendrite $\Psi(\ell, t; h)$ at $h/\ell_D = 2, 4, 8$ matches the independent (albeit far costlier) finite-difference solution (Fig. 10).

APPENDIX D: NESTED MICROSCOPIC AND MESOSCOPIC AVERAGING SCALES

The reduced competition law of the present paper involves two distinct lengths which should not be identified. The first is a microscopic averaging length inherited from the sharp interface tip problem. The second is a mesoscopic process zone length associated with competitive shielding in the outer line problem. The first answers the question of how the inner tip problem samples the outer feeding field. The second answers the question of over what distance behind the tip the neighboring dendrite materially alters that feeding field. The purpose of this Appendix is to make this distinction explicit and to show that, in the slender competition regime, the microscopic scale is naturally nested inside the mesoscopic one.

1. Microscopic averaging scale from sharp interface matching

Any sharp interface selection law which samples the incoming flux over a tip neighborhood of size $O(\rho_i)$ reduces, in the outer line limit, to a weighted average of the exact outer density μ_i . Writing $s = L_i(t) - y > 0$ for the distance behind the tip, one introduces the weighted flux functional

$$J_i^\phi(\rho_i, t) := \int_0^\infty \phi\left(\frac{s}{\rho_i}\right) \mu_i(L_i(t) - s, t) ds, \quad (\text{D1})$$

with associated normalized amplitude

$$\Psi_i^\phi(\rho_i, t) := \frac{J_i^\phi(\rho_i, t)}{M_{-1/2}[\phi]\rho_i^{1/2}}, \quad M_a[\phi] := \int_0^\infty r^a \phi(r) dr. \quad (\text{D2})$$

Using the exact near-tip expansion Eq. (111), one obtains

$$\Psi_i^\phi(\rho_i, t) = A_i(t) + \alpha_\phi B_i(t) \sqrt{\rho_i} + \beta_\phi C_i(t) \rho_i + O(\rho_i^{3/2}), \quad (\text{D3})$$

where $\alpha_\phi = M_0[\phi]/M_{-1/2}[\phi]$ and $\beta_\phi = M_{1/2}[\phi]/M_{-1/2}[\phi]$. Thus, the inner problem does not sample the outer field pointwise. It samples a regularized flux over a microscopic neighborhood of size $O(\rho_i)$.

For the flat window $\phi_\Lambda(r) = \mathbf{1}_{[0, \Lambda]}(r)$, the microscopic averaging length is

$$\delta_i := \Lambda \rho_i. \quad (\text{D4})$$

The corresponding regularized amplitude may be written as

$$\Psi_i(\delta_i, t) := \frac{1}{2\sqrt{\delta_i}} \int_0^{\delta_i} \mu_i(L_i(t) - s, t) ds, \quad (\text{D5})$$

with near-tip expansion

$$\Psi_i(\delta_i, t) = A_i(t) + \frac{1}{2} B_i(t) \sqrt{\delta_i} + \frac{1}{3} C_i(t) \delta_i + O(\delta_i^{3/2}). \quad (\text{D6})$$

We therefore interpret δ_i as the microscopic tip averaging scale. It is not a free process zone parameter. It is the outer representative of the inner solvability sampling length.

2. Why a larger scale is needed for competition

In the symmetric two dendrite geometry, the exact outer solution shows that the leading singular amplitude is universal: $A_1(t) = A_2(t) = A^{(1)}(t)$. Competitive shielding first enters only through the subleading coefficients $B_i, C_i, \dots$. Define the antisymmetric feeding signal

$$\Sigma(\ell, t) := \Psi_1(\ell, t) - \Psi_2(\ell, t). \quad (\text{D7})$$

Using the near-tip expansion,

$$\Sigma(\ell, t) = \frac{1}{2} (B_1(t) - B_2(t)) \sqrt{\ell} + \frac{1}{3} (C_1(t) - C_2(t)) \ell + O(\ell^{3/2}), \quad (\text{D8})$$

so $\Sigma(\ell, t) \rightarrow 0$ as $\ell \downarrow 0$. This is the precise universality law underlying the failure of any purely local tip law.

Evaluating at the microscopic scale δ_i gives $\Sigma(\delta_i, t) = \frac{1}{2} (B_1 - B_2) \sqrt{\delta_i} + O(\delta_i)$, which is parametrically small in the slender regime. The immediate tip neighborhood is dominated by the universal local singularity, whereas neighbor-induced depletion is an outer effect which accumulates over a longer distance behind the tip.

3. Mesoscopic process zone from the outer interaction kernel

The sharp interface reduction justifies finite averaging, but it does not by itself determine how large the averaging interval must be when one asks specifically about competitive shielding. That second question is controlled by the outer interaction kernel. As discussed in Sec. IV, the dominant competitive mode is associated with the screening length h/π . The physically natural outer process zone is therefore

$$\ell_{\text{comp}} \sim \frac{h}{\pi}. \quad (\text{D9})$$

We stress that ℓ_{comp} is not an inner matching length. It is the mesoscopic distance over which the outer wake of one dendrite materially modifies the feeding of the other.

It is therefore conceptually cleaner to reserve distinct symbols for the two lengths:

$$\delta_i = \Lambda \rho_i, \quad \ell = \ell_{\text{comp}} \sim \frac{h}{\pi}. \quad (\text{D10})$$

The first is inherited from inner tip sampling. The second is chosen to resolve the dominant competitive mode in the outer problem.

4. Nesting in the slender competition regime

In the regime considered throughout the paper, $\varepsilon_i := \rho_i/\ell_{\text{out},i} \ll 1$. Writing $\delta_i = \Lambda \rho_i$ and $\ell_{\text{comp}} = h/\pi$, one has

$$\frac{\delta_i}{\ell_{\text{comp}}} = \pi \Lambda \frac{\rho_i}{h}. \quad (\text{D11})$$

Hence, whenever $\rho_i \ll h$,

$$\delta_i \ll \ell_{\text{comp}}. \quad (\text{D12})$$

The microscopic averaging window is contained inside the mesoscopic process zone. The former regularizes the endpoint singularity so that a sharp interface selection law is well defined. The latter is the larger interval over which competitive shielding becomes measurable in the reduced outer model. The model does not contain two competing definitions of the same length, but a nested microscopic regularization and a mesoscopic competition filter.

APPENDIX E: POSITIVITY OF THE FULL ANTISYMMETRIC SENSITIVITY

In this Appendix we prove that the full process zone sensitivity coefficient is strictly positive. The proof uses only the exact prescribed-geometry transport problem together with translation invariance, parabolic comparison, the Hopf boundary lemma, and the exact first variation of the antisymmetric forcing. No truncation of the Mittag-Leffler expansion is used.

a. Setup. For fixed-tip positions L_1, L_2 , let $c^{L_1, L_2}(x, y, t)$ denote the exact solution of the prescribed-geometry advection-diffusion problem with internal Dirichlet data $c = c_{\text{LC}}$ on the two dendrite lines and $c(x, y, 0) = 0$, $c \rightarrow 0$ at infinity. By the parabolic maximum principle,

$$0 \leq c^{L_1, L_2}(x, y, t) \leq c_{\text{LC}} \quad (t > 0)$$

throughout the liquid domain.

Let $\mu_i^{L_1, L_2}(y, t)$ denote the associated feeding density on dendrite i , and define the process zone functional

$$\Psi_i(\ell, t; L_1, L_2, h) = \frac{1}{2\sqrt{\ell}} \int_0^\ell \mu_i^{L_1, L_2}(L_i - s, t) ds. \quad (\text{E1})$$

Lemma E1 (Translation invariance). For every $a \in \mathbb{R}$,

$$\Psi_i(\ell, t; L_1 + a, L_2 + a, h) = \Psi_i(\ell, t; L_1, L_2, h).$$

Hence, there exists a scalar function $F(\ell, t; g, h)$, with $g = L_1 - L_2$, such that

$$\Psi_1(\ell, t; L_1, L_2, h) = F(\ell, t; L_1 - L_2, h),$$

$$\Psi_2(\ell, t; L_1, L_2, h) = F(\ell, t; L_2 - L_1, h).$$

Proof. The advection-diffusion equation has constant coefficients and is invariant under common translation in the y direction. If $c^{L_1, L_2}(x, y, t)$ solves the problem for tip positions (L_1, L_2) , then

$$\tilde{c}(x, y, t) := c^{L_1, L_2}(x, y - a, t)$$

solves the same problem for $(L_1 + a, L_2 + a)$. Since the process zone functional is expressed in the tip-relative coordinate $s = L_i - y$, the common shift cancels exactly in Eq. (E1). Exchanging the two dendrites gives the second identity. ■

By Lemma E1, it is enough to fix the first tip at $L_1 = 0$ and write

$$F(\ell, t; g, h) := \Psi_1(\ell, t; 0, -g, h), \quad g \in \mathbb{R}.$$

Lemma E2 (Strict monotonicity under competitor retreat). Fix $t > 0$, $\ell > 0$, and $h > 0$. If $0 \leq g_1 < g_2$, then

$$F(\ell, t; g_2, h) > F(\ell, t; g_1, h).$$

Proof. Consider the two configurations

$$(L_1, L_2) = (0, -g_1), \quad (L_1, L_2) = (0, -g_2), \quad 0 \leq g_1 < g_2.$$

In the second configuration the first dendrite is unchanged and the second dendrite is shorter. Write

$$c^{(1)} := c^{0, -g_1}, \quad c^{(2)} := c^{0, -g_2}.$$

Let Ω_1 denote the liquid domain for configuration g_1 . Since the competitor is shorter in configuration g_2 , one has $\Omega_1 \subset \Omega_2$.

Define

$$w := c^{(1)} - c^{(2)}$$

on Ω_1 . Then w satisfies

$$w_t + v w_y = D \Delta w$$

in Ω_1 . On the common Dirichlet boundary, both solutions equal c_{LC} , hence

$$w = 0.$$

On the extra competitor segment

$$S := \{(x_2, y) : -g_2 < y < -g_1\},$$

which is part of the dendrite in configuration g_1 but lies in the liquid of configuration g_2 , one has

$$c^{(1)} = c_{LC}, \quad 0 \leq c^{(2)} < c_{LC} \quad (t > 0),$$

so

$$w > 0 \quad \text{on } S.$$

Initially and at infinity one has

$$w = 0.$$

By the parabolic maximum principle, $w \geq 0$ in Ω_1 for $t > 0$. Since $w \not\equiv 0$, the strong maximum principle gives

$$w > 0 \quad \text{in the interior of } \Omega_1 \quad (t > 0).$$

Now consider the unchanged first dendrite. On that boundary segment $w = 0$, while $w > 0$ in the adjacent liquid. Let n_1 denote the inward liquid normal to dendrite 1. By the Hopf boundary lemma, $\partial_{n_1} w > 0$ on the common process zone segment of dendrite 1. Since the feeding density is

$$\mu_1 = -2D \partial_{n_1} c$$

on dendrite 1, it follows that

$$\mu_1^{(2)}(y, t) > \mu_1^{(1)}(y, t)$$

for every y in the common process zone segment. Integrating over $0 < s < \ell$ yields

$$F(\ell, t; g_2, h) > F(\ell, t; g_1, h). \quad \blacksquare$$

Lemma E3 (Exact first variation of the antisymmetric forcing). Let

$$G_{a,-}(k, p; \delta) = \frac{c_{LC}}{2ip(k - i0)} (e^{ik(\bar{L} + \delta)} - e^{ik(\bar{L} - \delta)}). \quad (\text{E2})$$

Then, in the sense of tempered distributions,

$$\partial_\delta G_{a,-}(k, p; \delta)|_{\delta=0} = \frac{c_{LC}}{p} e^{ik\bar{L}}. \quad (\text{E3})$$

Proof. Differentiating Eq. (E2) at $\delta = 0$ gives

$$\partial_\delta G_{a,-}|_{\delta=0} = \frac{c_{LC}}{p} \frac{k}{k - i0} e^{ik\bar{L}}.$$

Using

$$\frac{1}{k - i0} = \text{PV} \frac{1}{k} + i\pi \delta(k),$$

one finds

$$\frac{k}{k - i0} = k \text{PV} \frac{1}{k} + i\pi k \delta(k) = 1,$$

since $k\delta(k) = 0$. This proves Eq. (E3). ■

Lemma E4 (Strict positivity of the derivative at the symmetric branch). For fixed $t > 0$, $\ell > 0$, and $h > 0$, the map

$$g \mapsto F(\ell, t; g, h)$$

is differentiable at $g = 0$, and

$$\partial_g F(\ell, t; 0, h) > 0.$$

Proof. The differentiability follows from standard shape differentiability of linear parabolic Dirichlet problems with respect to endpoint perturbations. In the present problem it may be seen directly from the exact boundary-integral formulation: the antisymmetric forcing is differentiable by

Lemma E3, and the antisymmetric Wiener-Hopf problem is linear with uniquely solvable factorized kernel, so the first variation exists and is unique.

Let

$$v := \partial_g c^{0,-g}|_{g=0}.$$

By Lemma E2,

$$\frac{c^{0,-g} - c^{0,0}}{g} \geq 0 \quad (g > 0)$$

throughout the common symmetric-domain liquid region. Passing to the limit $g \downarrow 0$ gives

$$v \geq 0.$$

Moreover, $v \neq 0$, because the first variation of the boundary forcing is nonzero by Lemma E3.

The function v satisfies the homogeneous advection-diffusion equation in the symmetric liquid domain, vanishes on the unchanged first dendrite, and carries a nontrivial boundary-source contribution at the competitor tip generated by the antisymmetric perturbation. By positivity of the parabolic Green function, $v > 0$ in the interior for $t > 0$. Since $v = 0$ on dendrite 1 and $v > 0$ in the adjacent liquid, the Hopf boundary lemma yields

$$\partial_{n_1} v > 0$$

on the first dendrite, where n_1 is the inward liquid normal.

Using $\mu_1 = -2D \partial_{n_1} c$, the first variation of the feeding density on dendrite 1 is

$$\partial_g \mu_1^{0,-g}|_{g=0} = -2D \partial_{n_1} v > 0.$$

Integrating over the process zone gives

$$\partial_g F(\ell, t; 0, h) > 0.$$

Theorem E1 (Positivity of the full antisymmetric sensitivity).

Fix $t > 0$, $\ell > 0$, and $h > 0$. Let

$$\begin{aligned} \Psi_1(\ell, t; \bar{L} + \delta, \bar{L} - \delta, h) &= \Psi_*(\ell, t; \bar{L}, h) \\ &\quad + \Gamma_\ell(\bar{L}, h, t) \delta + O(\delta^2) \end{aligned}$$

denote the linearization of the process zone feeding on the symmetric branch. Then

$$\Gamma_\ell(\bar{L}, h, t) > 0.$$

Proof. By Lemma E1,

$$\Psi_1(\ell, t; \bar{L} + \delta, \bar{L} - \delta, h) = F(\ell, t; 2\delta, h).$$

Therefore,

$$\begin{aligned} \Gamma_\ell(\bar{L}, h, t) &= \partial_\delta \Psi_1(\ell, t; \bar{L} + \delta, \bar{L} - \delta, h)|_{\delta=0} \\ &= 2 \partial_g F(\ell, t; 0, h). \end{aligned}$$

Lemma E4 gives $\partial_g F(\ell, t; 0, h) > 0$, hence

$$\Gamma_\ell(\bar{L}, h, t) > 0.$$

Remark E1. The proof is exact at the level of the prescribed geometry transport problem. It does not rely on a truncation of the Mittag-Leffler expansion. The analytical inputs beyond the explicit transport solution are standard: translation invariance, comparison on nested Dirichlet sets, the strong maximum principle, the Hopf boundary lemma, and differentiability of the process zone functional with respect to endpoint perturbations.

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