

## Second-order density-dependent geometric vector potentials: Modifying chiral soliton and ring-soliton dynamics in Bose-Einstein condensates with nonlocal interactions

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We consider a cigar-shaped Bose-Einstein condensate (BEC) of two-level atoms subjected to laser-induced Rabi coupling. By expanding the geometric vector potential to second order in the density-dependent fields and developing a quasi-one-dimensional (1D) reduction of the full three-dimensional problem via wave-function factorization, we derive an extended chiral Gross-Pitaevskii (GP) equation. We also derive traveling-wave solutions of the extended chiral GP equation using a projection (orthogonality) method, obtaining both bright and dark chiral solitons whose amplitudes and widths depend explicitly on the propagation direction. The chiral character of these solutions is confirmed through analytical analysis and numerical simulations. A BEC confined in a ring-shaped trap is also analyzed, leading to chiral soliton ring geometries. We show that a subtle interplay among density-dependent gauge potentials, confinement-induced interaction corrections, and higher-order current nonlinearities can give rise to alternative topological states, including chiral solitons and chiral ring solitons, in quasi-1D ultracold Bose gases. These results illustrate how higher-order corrections enrich BEC dynamics and provide means to control spatiotemporal structures in density-dependent condensates.

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### I. INTRODUCTION

As a system whose nonlinear interactions arise naturally from atomic collisions, a Bose-Einstein condensate (BEC) [1–3] provides a versatile platform for exploring nonlinear coherent matter waves, including dark solitons [4–6], bright solitons [7–9], vortices [10–12], vortex lattices [13,14], matter-wave amplification [15–21], and related phenomena. Understanding the physical mechanisms underlying solitary waves in BECs is particularly important, as their formation and dynamics play key roles in various applications, such as probing quantum phase effects [17,18], studying superfluid-Mott-insulator transitions [19,21], and advancing quantum information processing [22]. Furthermore, several experimental techniques, such as laser-assisted tunneling [23,24], Raman coupling schemes [25,26], and lattice shaking [27], now enable the realization of artificial gauge potentials in ultracold atomic gases. The interplay between these synthetic gauge fields and the atoms leads to a rich variety of physical effects, including the nucleation of quantized vortices in superfluids [10,11,28,29], the engineering of spin-orbit coupling

via non-abelian gauge fields [30], and access to the strong-magnetic-field regime in two-dimensional (2D) lattices [31].

Recently, advances in dynamical gauge fields [12–16] have shown that ultracold Bose gases can host density-dependent synthetic gauge potentials [17–24], opening the way to the study of chiral solitons [32–34]. A notable feature of these chiral solitons is the breaking of Galilean invariance [33], which implies that their amplitude, width, and chemical potential depend on the direction of propagation [35,36]. This has sparked significant interest in understanding the fundamental properties of chiral solitons, with particular emphasis on their unidirectional behavior [34,37–39]. The realization of direction-selective solitons is especially appealing because it offers a potential route toward implementing quantum time crystals in BEC. For instance, a rotating chiral bright soliton has been identified as the ground state of an atomic condensate confined in a ring and described by an interacting gauge theory [38]. Moreover, both chiral bright and dark solitons, solutions of a Korteweg–de Vries-type equation, display amplitudes and widths that depend on the direction of motion, becoming effectively unidirectional when the current-induced nonlinearity dominates [40,41].

Another interesting phenomena is the modulational instability (MI) which was investigated by Bhat *et al.* [42] in a related chiral extension of the Gross-Pitaevskii (GP) equation. They found that, although the current-nonlinear term did not affect the system's energy, it can partially suppress MI induced by attractive cubic interactions, thereby enabling the formation of soliton trains. Notably, within this zero-range framework, changing the sign of the current nonlinearity merely reverses the direction of soliton propagation. Similar

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behavior was observed by Tabi *et al.* [43] in a chiral model incorporating nonlocal interactions. Their study also reported the emergence of diverse dynamical structures, including soliton trains and breathing rogue waves. In addition, numerical scaling laws have been introduced that predict the shift of the central peak as a function of the propagation direction, determined by the sign of the current nonlinearity.

The purpose of this work is to investigate how second-order geometric vector potentials modify chiral nonlinear excitations in BEC of two-level atoms subjected to laser-induced Rabi coupling. While first-order synthetic gauge potentials are known to induce chirality and current nonlinearities, the impact of higher-order density-dependent corrections remains largely unexplored, particularly in reduced-dimensional settings. We therefore analyze a strongly confined cigar-shaped condensate and derive an effective quasi-one-dimensional (1D) model that incorporates these higher-order geometric contributions. The resulting extended chiral GP equation contains density-dependent synthetic gauge potentials, standard two-body interactions, zero-energy corrections, higher-order current nonlinearities, and additional interaction terms originating from confinement-induced shape effects. Using a projection (orthogonality) method, we derive traveling-wave solutions whose amplitudes and widths depend explicitly on the direction of propagation, thereby exhibiting intrinsic chirality. We then investigate the formation of both chiral soliton and chiral ring-soliton structures arising from the second-order geometric contributions.

The remainder of this paper is organized as follows. Section II introduces the physical model. By expanding the geometric vector potential to second order in the density-dependent fields and performing a quasi-1D reduction of the full three-dimensional (3D) system via wave-function factorization, we obtain an extended chiral GP equation. Section III examines density-dependent persistent currents within the framework of the derived extended chiral GP model, presenting the energy of stationary states in a ring geometry and identifying the critical density at which the ground-state transitions between different rotational states. Section IV addresses the free expansion dynamics that occur after the trap is switched off. We show, both analytically and numerically, that the current nonlinearities lead to a drift in the center-of-mass motion during expansion. In Sec. V, we derive traveling wave solutions of the model via the projection-orthogonality method, yielding both bright and dark chiral solitons whose amplitudes and widths explicitly depend on their propagation direction; this chirality is confirmed through analytical treatment and numerical simulations. Finally, Sec. VI provides concluding remarks.

## II. MODEL

### A. Second-order expansion of the geometric vector potential

We consider a system of two-level atoms interacting with light in which the interaction between atom  $q$  in state  $i$  and atom  $\ell$  occupying state  $j$  is given by [32]

$$\mathcal{V}_{q,\ell} = \begin{pmatrix} V_{11}(\mathbf{r}_q - \mathbf{r}_\ell) & 0 & 0 & 0 \\ 0 & V_{12}(\mathbf{r}_q - \mathbf{r}_\ell) & 0 & 0 \\ 0 & 0 & V_{12}(\mathbf{r}_q - \mathbf{r}_\ell) & 0 \\ 0 & 0 & 0 & V_{22}(\mathbf{r}_q - \mathbf{r}_\ell) \end{pmatrix}, \quad (1)$$

where  $i, j$  takes values 1,2. Atom  $q$  is at position  $\mathbf{r}_q$  from the origin, and likewise atom  $\ell$  is at position  $\mathbf{r}_\ell$  from the origin.

We begin from the microscopic description of a dilute gas of  $N$  interacting two-level bosonic atoms subject to coherent laser coupling. Each atom possesses two internal (pseudospin) states, denoted  $|1\rangle$  and  $|2\rangle$ , which are coupled via a Rabi process induced by an external light field. At ultracold temperatures, atomic interactions are governed by low-energy binary  $s$ -wave scattering processes, which may be characterized by the scattering length, and beyond the zero-range approximation, by effective-range corrections. In this regime, the many-body system can be described by single-particle kinetic and light-matter coupling terms together with pairwise interaction contributions between atoms occupying internal states  $i$  and  $j$ . Following the framework used in studies of density-dependent synthetic gauge fields in Bose gases [32], the corresponding Hamiltonian for the light-matter interaction takes the form

$$\hat{H} = \sum_{q=1}^N \left( \frac{\mathbf{p}_q^2}{2m} + \hat{U}_q + \hat{V}_q \right) \otimes \hat{I}_q + \sum_{q<\ell=1}^N \mathcal{V}_{q,\ell} \otimes \hat{I}_{q,\ell}. \quad (2)$$

Here,  $\mathbf{p}_q$  is the momentum operator on the atom  $q$ ,  $m$  is the mass of the individual atom,  $\hat{V}_q$  atom  $q$  external potential,

and  $\hat{U}_q$  is the Rabi potential describing light interacting with particle  $q$  given by [32]

$$\hat{U}_q = \frac{\hbar\Omega_r}{2} \begin{pmatrix} 0 & e^{-i\phi_q(\mathbf{r}_q)} \\ e^{i\phi_q(\mathbf{r}_q)} & 0 \end{pmatrix}, \quad (3)$$

where  $\Omega_r$  is the two-photon Rabi frequency characterizing the light-matter coupling, and  $\phi_q(\mathbf{r}_q)$  is the laser phase at atom  $q$ 's position.

Next, we seek the effective potential  $V_{ij}(\mathbf{r}_q - \mathbf{r}_\ell)$ , from Eq. (1), which describes the interaction between state  $i$  of atom  $q$  and state  $j$  of atom  $\ell$ . It is well known that, for the range of energies always used in the literature [44,45], only the  $s$ -wave scattering is important. In this case, the scattering amplitude can be written in terms of the  $s$ -wave phase shift  $\delta_0(k)$  as [44,45]

$$f(k) = \frac{1}{k \cot \delta_0(k) - ik}, \quad (4)$$

where  $k$  is the wave number of the scattering particle and  $\delta_0(k)$  is the phase shift due to scattering. At low momentum, the effective-range expansion is given by

$$k \cot \delta_0(k) = -\frac{1}{a} + \frac{1}{2}r_e k^2 + \mathcal{O}(k^4), \quad (5)$$

where  $r_e$  is the effective range, this gives us

$$f(k) = \frac{-a}{1 - \frac{1}{2}ar_e k^2 + iak}. \quad (6)$$

We use  $(1+x)^{-1} \approx 1 - x + x^2$ , where  $x = -\frac{1}{2}ar_e k^2 + iak$ . Then, we expand it until  $x^2$  keeping terms up to  $\mathcal{O}(k^2)$ , we get [44]

$$f(k) \approx -a + ia^2k - \frac{1}{2}a^2r_e k^2 + a^3k^2 + \mathcal{O}(k^4), \quad (7)$$

of which we take the real part

$$\Re[f(k)] = -a - \frac{1}{2}a^2r_e k^2 + a^3k^2. \quad (8)$$

The two-body  $T$ -matrix is given by

$$T(k) = \frac{4\pi\hbar^2}{m}f(k). \quad (9)$$

Since the GP equation only keeps conservative interactions, one takes the effective (Hermitian) two-body  $T$  matrix

$$T_{\text{eff}}(k) = \frac{4\pi\hbar^2}{m}\Re[f(k)]. \quad (10)$$

For the interaction of state  $i$  of atom  $q$  with state  $j$  of atom  $\ell$ , the interaction potential  $V_{ij}(\mathbf{r}_q - \mathbf{r}_\ell)$  is derived from

$$T_{ij}(\mathbf{k}) = \int d^3r e^{-i\mathbf{k}\mathbf{r}} V_{ij}(\mathbf{r}), \quad (11)$$

$$\hat{\mathcal{V}} = \frac{1}{2} \begin{pmatrix} g_{11}|\Psi_1|^2 + g_{12}|\Psi_2|^2 + p_{11}\nabla^2(|\Psi_1|^2) + p_{12}\nabla^2(|\Psi_2|^2) \\ 0 \end{pmatrix}$$

Here  $p_{ij} = g_{ij}\sigma_{ij}$ . The coefficients  $p_{ij}$  are the higher-order coefficients that describe the shape-dependent confinement correction of the two-body collision potential [45].

After introducing the physical system under consideration (a dilute gas of two-level bosonic atoms with coherent Rabi coupling) and carefully presenting the microscopic origin of the interaction term, we employ the low-energy  $s$ -wave scattering expansion to include effective-range corrections. This allows us to proceed with the systematic derivation of the density-dependent geometric vector potential up to second order. For that purpose, we consider the regime of strong Rabi coupling, where the dressed-state basis diagonalizes the internal dynamics.

As in Ref. [32], we consider the regime of weak interactions where the Rabi coupling satisfies  $\hbar\Omega \gg g_{ij}, p_{ij}$ . In this limit, the internal coupling operator  $\hat{U}$  defines the dominant energy scale, while the interaction operator  $\hat{\mathcal{V}}$  can be treated perturbatively.

In contrast to the standard Bogoliubov treatment of a dilute Bose gas, the effective interaction here contains not only the usual mean-field contributions  $g_{ij}$ , but also additional

where  $\mathbf{r} = \mathbf{r}_q - \mathbf{r}_\ell$ . From Eq. (11), we infer that to find  $V_{ij}(\mathbf{r})$ , we take the inverse Fourier transform of Eq. (10). Hence,

$$V_{ij}(\mathbf{r}) = g_{ij}\delta(\mathbf{r}) - \sigma_{ij}g_{ij}\nabla^2\delta(\mathbf{r}), \quad (12)$$

where  $g_{ij} = \frac{4\pi\hbar^2}{m}a_{ij}$ , and  $\sigma_{ij} = (\frac{a_{ij}}{3} - \frac{1}{2}r_e)a_{ij}^2$ . The appearance of derivative (nonlocal) interaction terms follows from retaining the momentum dependence of the two-body  $T$  matrix beyond the zero-range approximation. In the usual GP theory, only the leading scattering-length contribution is kept, corresponding to a contact interaction proportional to  $\delta(\mathbf{r})$ . However, the inclusion of the effective-range correction in the low-energy expansion of the scattering amplitude introduces a quadratic momentum dependence in  $T_{ij}(k)$ . Upon inverse Fourier transformation, this momentum dependence generates derivative terms of the form  $\nabla^2\delta(\mathbf{r})$ , which represent leading-order finite-range corrections to the pseudopotential. Such effective-range-corrected interactions were derived and analyzed in Refs. [44,45], and provide a systematic extension of the zero-range GP model.

We assume all atoms occupy the same spinor orbital

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{q=1}^N \Psi(\mathbf{r}_q), \quad (13)$$

where the Hartree ansatz is given by

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \quad (14)$$

With this ansatz, Eq. (2) becomes

$$\hat{H} = \frac{\hat{p}^2}{2m} \otimes \hat{I} + \hat{\mathcal{V}} + \hat{U}, \quad (15)$$

with

$$\begin{pmatrix} 0 \\ g_{12}|\Psi_1|^2 + g_{22}|\Psi_2|^2 + p_{12}\nabla^2(|\Psi_1|^2) + p_{22}\nabla^2(|\Psi_2|^2) \end{pmatrix}. \quad (16)$$

higher-order residual terms  $p_{ij}$ . Since both interaction parameters satisfy

$$g_{ij}, p_{ij} \ll \hbar\Omega, \quad (17)$$

the dressed states may be expanded perturbatively as

$$|\chi_{\pm}\rangle = |\chi_{\pm}^{(0)}\rangle + |\chi_{\pm}^{(1)}\rangle. \quad (18)$$

By linearity of first-order perturbation theory, the correction separates into two additive contributions

$$|\chi_{\pm}^{(1)}\rangle = |\chi_{g\pm}^{(1)}\rangle + |\chi_{p\pm}^{(1)}\rangle, \quad (19)$$

arising respectively from the well-known mean-field interaction terms and the higher-order residual interaction terms that we introduce in the present work. In addition, the two additive contributions are proportional to the zeroth-order dressed spinors

$$|\chi_{g\pm}^{(1)}\rangle = \pm \frac{g_{11} - g_{22}}{8\hbar\Omega} |\Psi_{\pm}|^2 |\chi_{\pm}^{(0)}\rangle, \quad (20)$$

$$|\chi_{p\pm}^{(1)}\rangle = \pm \frac{p_{11} - p_{22}}{8\hbar\Omega} \nabla^2(|\Psi_{\pm}|^2) |\chi_{\pm}^{(0)}\rangle, \quad (21)$$

with the zeroth-order dressed spinors

$$|\chi_{\pm}^{(0)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm e^{i\phi}|2\rangle). \quad (22)$$

We observe that the mean-field contribution depends locally on the density  $|\Psi_{\pm}|^2$ , whereas the residual interaction term involves the Laplacian  $\nabla^2(|\Psi_{\pm}|^2)$  and therefore introduces second-order spatial derivative corrections. Although both arise from first-order perturbation theory with respect to the interaction parameters, the latter yields a genuinely second-order, density-dependent geometric contribution.

Projecting Eq. (15) onto the upper dressed state  $|\chi_+\rangle$  yields the effective Hamiltonian

$$\hat{H}_+ = \frac{(\mathbf{p} - \mathbf{A}_+)^2}{2m} + V(\mathbf{r}) + W_+ + \frac{\hbar\Omega}{2} + g|\Psi_+|^2 + p\nabla^2|\Psi_+|^2. \quad (23)$$

In the Hamiltonian above Eq. (23), the first term describes the momentum operator with the contribution of the geometric vector potential  $\mathbf{A}_+$ , the second term describes the external potential, the third term is the geometric scalar potential, the fourth term is the Rabi-coupling energy, the fifth term describes two-body interaction, where  $g = \frac{1}{4}(g_{11} + g_{22} + 2g_{12})$ , and finally the last term is the higher-order residual nonlinearity, where  $p = \frac{1}{4}(p_{11} + p_{22} + 2p_{12})$ . In fact,  $p$  determines the nonlocal interaction in the dressed-state picture. We show that the higher-order residual nonlinearity arises from the energy-dependent effective interaction in a dilute many-body system. In the following, to obtain an equation that incorporates both the current nonlinearity and the higher-order current contribution, we derive the corresponding higher-order geometric scalar and vector potentials. The geometric vector potential is given as  $\mathbf{A}_+ = i\hbar\langle\chi_+|\nabla\chi_+\rangle$  and the geometric scalar potential is given as  $W_+ = \frac{\hbar^2}{2m}|\langle\chi_-|\nabla\chi_+\rangle|^2$ . Furthermore, we expand the geometric vector potential up to second order to get

$$\mathbf{A}_+ = \mathbf{A}_+^{(0)} + a_1|\Psi_+|^2 + a_2\nabla^2|\Psi_+|^2. \quad (24)$$

Here,

$$\mathbf{A}_+^{(0)} = -\frac{\hbar}{2}\nabla\phi, \quad a_1 = \frac{\nabla\phi}{8\Omega}(g_{11} - g_{22}), \quad \text{and} \quad a_2 = \frac{\nabla\phi}{8\Omega}(p_{11} - p_{22}). \quad (25)$$

That variation of the Dirac-Frenkel action leads to the extended GP equation [46–50]

$$i\hbar\partial_t\Psi_+ = \left[ \frac{(\mathbf{p} - \mathbf{A}_+^{(0)})^2}{2m} + V + W_+^{(0)} + \frac{\hbar\Omega}{2} + g|\Psi_+|^2 + p\nabla^2|\Psi_+|^2 + a_1\mathbf{J} + a_2\mathbf{K} \right] \Psi_+, \quad (26)$$

where  $\mathbf{J}$  is the current nonlinearity and  $\mathbf{K} = \mathbf{L} + \mathbf{N}$  the higher-order current term, given as

$$\mathbf{J} = \frac{\hbar}{2im} \left[ \Psi_+ \left( \nabla + \frac{i}{\hbar} \mathbf{A}_+^{(0)} \right) \Psi_+^* - \Psi_+^* \left( \nabla - \frac{i}{\hbar} \mathbf{A}_+^{(0)} \right) \Psi_+ \right], \quad (27)$$

and the associated higher-order current nonlinearity terms are given as

$$\mathbf{L} = \frac{\hbar}{2im} \left[ \nabla^2\Psi_+ \left( \nabla + \frac{i}{\hbar} \mathbf{A}_+^{(0)} \right) \Psi_+^* - \Psi_+^* \left( \nabla - \frac{i}{\hbar} \mathbf{A}_+^{(0)} \right) \nabla^2\Psi_+ \right], \quad (28)$$

$$\mathbf{N} = \frac{\hbar}{2im} \left[ \Psi_+ \left( \nabla + \frac{i}{\hbar} \mathbf{A}_+^{(0)} \right) \nabla^2\Psi_+^* - \nabla^2\Psi_+^* \left( \nabla - \frac{i}{\hbar} \mathbf{A}_+^{(0)} \right) \Psi_+ \right]. \quad (29)$$

The higher-order current nonlinearities Eqs. (28) and (29) are a consequence of the first order expansion of the geometric vector potential Eq. (24).

### B. Quasi-one-dimensional reduction

For a cigar-shaped trap with  $\omega_y = \omega_z = \omega_{\perp} \gg \omega_x$ , we use the factorized ansatz

$$\Psi_+(\mathbf{r}, t) = \psi(x, t)\psi_{\perp}(y, z) \exp\left[-\frac{i\mu t}{\hbar} + i\frac{\phi\ell}{2}\right], \quad (30)$$

where  $\psi_{\perp}$  is the transverse ground state. Integrating Eq. (26) over the transverse coordinates yields the effective 1D equation

$$i\hbar\partial_t\psi = \left[ \frac{(p_x - a_{1x}|\psi|^2 - a_{2x}\partial_x^2|\psi|^2)^2}{2m} + \frac{1}{2}m\omega_x^2x^2 + W + \frac{\hbar\Omega}{2} + g_x|\psi|^2 + p_x\partial_x^2|\psi|^2 + a_{1x}j_x + a_{2x}k_x \right] \psi, \quad (31)$$

with renormalized coefficients

$$a_{1x} = \frac{a_1}{2\pi\ell_{\perp}^2} + \frac{a_2}{4\pi\ell_{\perp}^3}, \quad a_{2x} = \frac{a_2}{2\pi\ell_{\perp}^2}, \quad g_x = \frac{g}{2\pi\ell_{\perp}^2} - \frac{p}{\pi\ell_{\perp}^3}, \quad \text{and} \quad p_x = \frac{p}{2\pi\ell_{\perp}^2}. \quad (32)$$

### C. Phase transformation

In the following, we derive the effective one-dimensional mean-field description incorporating both the density-dependent vector potential and the second-order density-dependent geometric contribution. To this end, we employ the standard procedure of dimensional reduction [32] and apply

the nonlinear phase transformation [33,42]

$$\psi(x, t) = \varphi(x, t) \exp \left[ \frac{i}{\hbar} a_{1x} \int_{-\infty}^x |\psi|^2 dx' + \frac{i}{\hbar} a_{2x} \int_{-\infty}^x \partial_{x'}^2 |\psi|^2 dx' - \frac{i}{\hbar} Wt - \frac{i}{\hbar} \frac{\hbar \Omega}{2} t \right]. \quad (33)$$

To proceed, we make use of two continuity relations

$$\partial_t \rho + \partial_x j = 0, \quad (34)$$

$$\partial_t (\partial_{xx} \rho) + \partial_x k = 0, \quad (35)$$

where  $\rho = |\psi|^2$  denotes the density. The current nonlinearity  $j$  and higher-order current nonlinearity  $k$  are defined in Appendix B. Equation (34) is the standard continuity equation associated with the usual current nonlinearity, while Eq. (35), obtained by taking the second spatial derivative of Eq. (34), governs the higher-order current contribution arising from the second-order density dependence. As a result, the equation of motion for  $\varphi$  takes the form

$$i\hbar \partial_t \varphi = -\frac{\hbar^2}{2m} \partial_x^2 \varphi + g_x |\varphi|^2 \varphi + p_x \varphi \partial_x^2 |\varphi|^2 - 2a_{1x} j \varphi - 2a_{2x} k \varphi. \quad (36)$$

#### D. Galilean transformation

Applying the transformation

$$\varphi(x, t) = \tilde{\varphi}(x - ut, t) \exp \left[ \frac{i}{\hbar} \left( mux - \frac{1}{2} mu^2 t \right) \right], \quad (37)$$

and dropping the tildes for clarity, one obtains

$$i\hbar \partial_t \varphi = -\frac{\hbar^2}{2m} \partial_x^2 \varphi + g_{\text{eff}} |\varphi|^2 \varphi + p_{\text{eff}} \varphi \partial_x^2 |\varphi|^2 - 2a_{1x} j \varphi - 2a_{2x} k \varphi, \quad (38)$$

where  $g_{\text{eff}} = g_x - 2ua_{1x}$  and  $p_{\text{eff}} = p_x - 2ua_{2x}$ .

### III. RING GEOMETRY AND DENSITY-DEPENDENT PERSISTENT CURRENT

In this section, we examine how second-order density-dependent geometric corrections influence persistent-current states in a ring geometry. Our analysis is intended to highlight the principal modifications introduced by these higher-order terms, rather than to provide an exhaustive treatment of all nonlinear stationary configurations.

A natural setting in which to highlight the dynamical consequences of density-dependent geometric vector potentials is a tightly confined ring trap of radius  $R$ . Writing the longitudinal coordinate as  $x = R\theta$  with periodic boundary condition  $\psi(x + 2\pi R, t) = \psi(x, t)$ , and using the azimuthal phase imprint  $\phi_{\text{azimuthal}} = \ell\theta$  from a Laguerre-Gaussian beam, the extended chiral GP Eq. (38) reduces to

$$i \frac{\partial \phi}{\partial \tau} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial \theta^2} + g_\theta |\phi|^2 \phi + p_\theta \phi \frac{\partial^2 (|\phi|^2)}{\partial \theta^2} - 2\alpha_1 J_\theta \phi - 2\alpha_2 K_\theta \phi, \quad (39)$$

where  $\tau = \hbar t / (mR^2)$ ,  $\phi(\theta, \tau) = \sqrt{R} \varphi(x, t)$ , and  $J_\theta$ ,  $K_\theta$ , are given in the Appendix. The coefficients  $g_\theta$ ,  $p_\theta$ ,  $\alpha_1$ , and  $\alpha_2$

incorporate the geometric vector-potential expansion to second order and depend on the microscopic interaction and confinement parameters (also given in the Appendix).

A stationary state on the ring has the form

$$\phi(\theta, \tau) = \sqrt{\rho} e^{iq\theta}, \quad (40)$$

where  $\rho = N/(2\pi)$  is the linear density and  $q \in \mathbb{Z}$  is the quantized winding number. Substituting Eq. (40) into the energy functional associated with Eq. (39) yields an energy ladder

$$E_q = \frac{\hbar^2 q^2}{2mR^2} - \frac{\hbar q}{mR} a_{1\rho} + \mathcal{O}(a_2, \partial_R^2 \rho), \quad (41)$$

where the  $a_{1\rho}$  term arises from the leading density-dependent vector potential and the  $a_2$ -dependent terms encode residual geometric corrections. For constant  $\rho$ , the  $a_2$ -dependent term evaluates to zero. The energy difference between neighboring circulation states becomes

$$E_{q+1} - E_q = \frac{1}{2m} \left[ \frac{2\hbar}{R} \left( \frac{\hbar q}{R} - a_{1\rho} \right) + \frac{\hbar^2}{R^2} \right], \quad (42)$$

showing that the ground-state angular momentum depends explicitly on the condensate density. The transition condition  $E_{q+1} = E_q$  yields

$$\rho_c(q) = \frac{8\hbar\Omega}{\ell(g_{11} - g_{22})} \left( q + \frac{1}{2} \right), \quad (43)$$

so that the density determines which quantized circulation state minimizes the energy.

This behavior differs fundamentally from the usual scenario of a ring BEC under rotation, where persistent currents are selected by an imposed angular velocity. Here, the density-dependent gauge potential itself selects the preferred circulation. Since the condensate density in this case is constant in space, the second-order geometric correction does not affect the results by Edmonds *et al.* [32].

In addition to the ring circumference  $L = 2\pi R$ , the system is characterized by a coherence (healing) length  $\xi$ , which determines the intrinsic spatial scale of nonlinear excitations. In the present model, this length is modified by the effective interaction parameters  $g_{\text{eff}}$  and  $p_{\text{eff}}$ , yielding a generalized coherence length that depends on both the current nonlinearity and higher-order geometric corrections.

The properties of solitary-wave solutions in a ring geometry are known to depend sensitively on the ratio  $R/\xi$ . In the regime  $R \gg \xi$ , localized soliton solutions persist and are well approximated by their infinite-line counterparts, justifying the use of the traveling-wave solutions that will be used in Sec. V. Conversely, when  $R \sim \xi$ , finite-size effects become important, leading to deformation or delocalization of the soliton profile. In the opposite limit  $R \ll \xi$ , the system supports only nearly uniform states corresponding to persistent currents.

In the present work, we focus on the regime of sufficiently large rings and weak finite-size effects ( $R \gg \xi$ ), where the density may be approximated as uniform in the stationary states. A detailed analysis of the crossover between other regimes ( $R \sim \xi$  and  $R \ll \xi$ ), particularly in the presence of second-order geometric corrections, is left for future work.

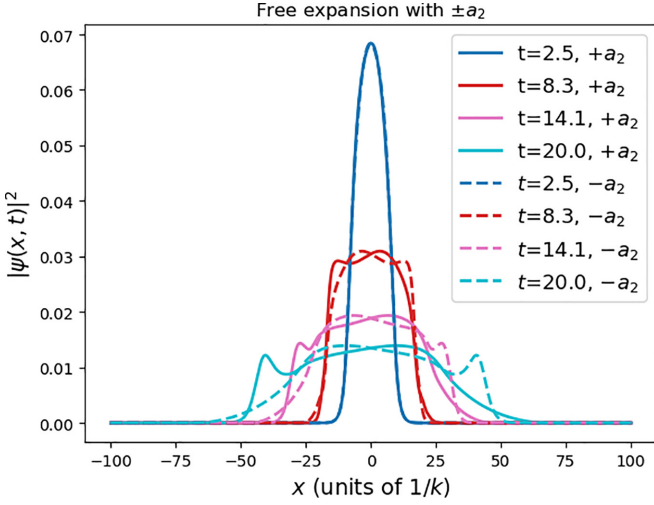


FIG. 1. Numerical simulation of the free expansion governed by the effective GP equation, Eq. (31). We plot the atomic density  $\rho(x, t)$  at several evolution times  $t$  (in units of  $\omega_0^{-1}$ ), following sudden release from initial harmonic confinement. The spatial coordinate  $x$  is measured in units of the oscillator length  $l_0$ . Although the initial wave packet is symmetric and has zero canonical momentum, the nonlinear current terms proportional to  $a_1$  and  $a_2$  generate a density-dependent vector potential, which induces a net drift of the expanding cloud. This asymmetric expansion is the hallmark of the chiral nonlinear dynamics. The center of mass acquires a finite velocity purely from the interaction-generated gauge field. The density is normalized to unity.

#### IV. FREE EXPANSION DRIFT

The extended chiral GP equation also predicts striking modifications to the free expansion of a condensate following trap release. In a conventional (non-gauge-coupled) condensate, the cloud expands symmetrically around its initial center of mass. By contrast, the density-dependent geometric vector potential induces an asymmetry in the expansion dynamics, causing the cloud to acquire a directed drift. Edmonds *et al.* [32] reported this effect for a system with only the leading current nonlinearity (that is,  $a_2 = 0$ ). When the second-order geometric correction is retained, the drift becomes substantially richer, that is, both the magnitude and even the sign of the drift now depend on the residual current nonlinearity  $a_2$ .

Figure 1 highlights this effect for  $(p_{11} - p_{22})kN/(S_t\hbar\Omega) = \pm 1.5$ . Reversing the sign of  $a_2$  reverses the direction of the drift, demonstrating the sensitivity of the expansion dynamics to higher-order geometric corrections. This behavior reflects the asymmetric momentum redistribution generated by the density-dependent gauge coupling.

To gain analytical insight, we adopt a time-dependent Gaussian variational ansatz

$$\phi(x, t) = \left( \frac{N}{\sqrt{\pi}\sigma(t)} \right)^{\frac{1}{2}} \exp \left( -\frac{[x - x_0(t)]^2}{2\sigma(t)^2} + i\frac{m}{\hbar}\dot{x}_0(x - x_0) \right). \quad (44)$$

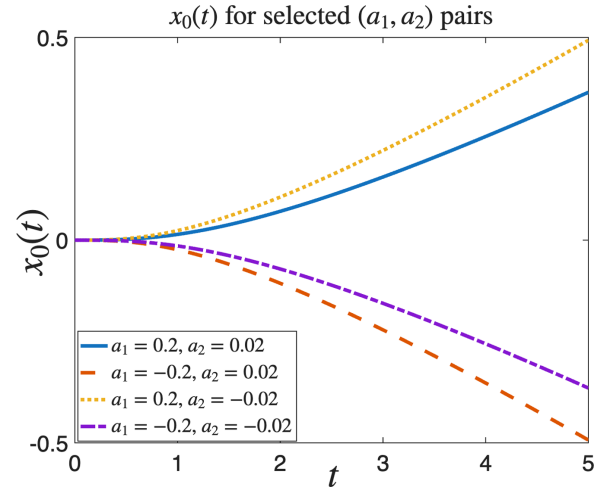


FIG. 2. Center-of-mass drift  $x_0(t)$  during free expansion. Solid curves correspond to the analytical prediction Eq. (46), while symbols denote results from full numerical simulations of the extended chiral GP equation, Eq. (31). Different curves correspond to different values of the nonlinear coefficients  $a_1$  and  $a_2$ . The excellent agreement confirms the validity of the variational description.

Substituting the ansatz into the effective Lagrangian yields the equation of motion for the center-of-mass coordinate

$$\ddot{x}_0 = \frac{\sqrt{2}N}{\sqrt{\pi}m} \left( a_1 - 3\frac{a_2}{\sigma} \right) \frac{\dot{\sigma}(t)}{\sigma(t)^2}. \quad (45)$$

Two features immediately follow.

(i) The term proportional to  $a_1$  represents an effective gauge force whose sign is fixed by the polarity of the current nonlinearity.

(ii) The  $a_2$  term scales as  $\sigma^{-2}(t)$ , making it dominant at early times when the cloud is still narrow. Depending on the sign and magnitude of  $a_2$ , this residual coupling enhances, weakens, or even reverses the drift predicted from the leading-order gauge term.

To assess the validity of the assumed width dynamics, we perform a full time-dependent variational calculation using the Gaussian ansatz Eq. (44), treating both  $x_0(t)$  and  $\sigma(t)$  as dynamical variables. The resulting Euler-Lagrange equations show that the width obeys the standard free-expansion equation, while the density-dependent gauge terms contribute only to the center-of-mass dynamics. (See the Appendix.)

Integrating Eq. (45) gives the full trajectory

$$x_0 = \frac{\sqrt{2}a_1N\tau}{m\sqrt{\pi}\sigma_0} \left[ \frac{t}{\tau} - \operatorname{arcsinh} \left( \frac{t}{\tau} \right) \right] - \frac{\sqrt{2}a_2N\tau}{m\sqrt{\pi}\sigma_0^3} \left[ \frac{t}{\tau} - \frac{t}{\tau} \frac{1}{\sqrt{1 + \frac{t^2}{\tau^2}}} \right]. \quad (46)$$

The resulting center-of-mass trajectory is shown in Fig. 2, where the analytical prediction Eq. (46) is compared with full numerical simulations of the extended GP Eq. (31). Excellent agreement is observed over the entire evolution time. Figure 3 summarizes the dependence of the drift position on the nonlinear coefficients  $a_1$  and  $a_2$  at a fixed

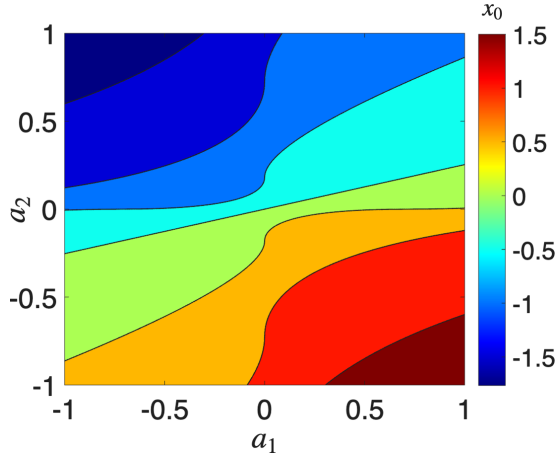


FIG. 3. Drift position of the center of mass at fixed time  $t = 2.0$  as a function of the nonlinear coefficients  $a_1$  and  $a_2$ , obtained from the analytical expression (46). The sign and magnitude of the drift depend sensitively on the product  $a_1 a_2$ , illustrating the role of higher-order current nonlinearities.

time, highlighting the crucial role of the second-order current nonlinearity. Equation (46) agrees quantitatively with our numerical simulations: the condensate drifts in a direction set by  $a_1$ , with a contribution from  $a_2$  that modifies the drift most strongly at early times. Physically, the drift originates from an imbalance in how different momentum components couple to the density-dependent gauge field. The inclusion of the second-order geometric term amplifies this imbalance by modifying the current and residual-current responses of the expanding wave packet. Chiral soliton dynamics in the presence of density-dependent gauge fields has also been demonstrated experimentally [51,52]. The present analysis shows how higher-order geometric corrections modify the existence and propagation properties of such solitary waves.

## V. TRAVELING-WAVE SOLUTIONS

We now investigate solitary-wave solutions of the extended chiral GP equation. Our objective is to demonstrate how second-order geometric vector-potential corrections affect the structure and propagation properties of chiral solitons. We therefore focus on representative traveling-wave solutions that capture the leading dynamical consequences of these higher-order terms, without attempting a complete nonlinear classification. We look for a constant-velocity soliton solution of Eq. (36). We assume the traveling-wave ansatz [32]

$$\phi(x, t) = \Phi(x - ut) \exp \left[ \frac{i}{\hbar} \left[ mu x - \left( \frac{1}{2} mu^2 + \mu \right) t \right] \right], \quad (47)$$

where  $\Phi$  is real and  $\mu$  is the chemical potential. Then  $j = u \Phi^2$  and  $k = u (\Phi^2)''$ . We obtain the differential equation

$$-\frac{\hbar^2}{2m} \Phi'' + (g - 2a_1 u) \Phi^3 + (p - 2a_2 u) (\Phi^2)'' \Phi = \mu \Phi. \quad (48)$$

Equation (48) differs from the integrable GP equation due to the presence of the derivative nonlinearity  $(\Phi^2)'' \Phi$  and the velocity-dependent effective coefficients  $g_{\text{eff}}$  and  $p_{\text{eff}}$ . The

combined effect of current-induced terms and finite-range corrections modifies the Galilean symmetry of the standard model and removes its complete integrability. As a consequence, solitary-wave solutions cannot be constructed via inverse-scattering methods and must instead arise from a balance between dispersion, the effective cubic interaction, and the higher-order residual nonlinearity.

It is instructive to examine limiting regimes. When  $p_{\text{eff}} \rightarrow 0$ , Eq. (48) reduces to the previously studied chiral GP equation [32], for which sech and tanh soliton families are recovered with velocity-dependent effective interaction  $g_{\text{eff}} = g - 2a_1 u$ . Conversely, when  $g_{\text{eff}} \rightarrow 0$ , localization requires  $p_{\text{eff}}$  to possess the appropriate sign so that the residual nonlinear term compensates dispersion. This indicates that second-order geometric corrections may either enhance or suppress soliton formation depending on the propagation velocity and interaction parameters.

Accordingly, the existence and spatial width of solitary waves are governed by the competition between the effective cubic nonlinearity and the higher-order geometric contribution. This interplay provides a mechanism for direction-dependent localization beyond the leading-order chiral model.

We enforce the orthogonality of the residual higher-order nonlinearity term to the two weights  $S$  and  $S^3$ :

$$I_1(A, \kappa) := \int_{-\infty}^{\infty} S(\xi) \mathcal{R}(\xi; A, \kappa) d\xi = 0, \quad (49)$$

$$I_3(A, \kappa) := \int_{-\infty}^{\infty} S^3(\xi) \mathcal{R}(\xi; A, \kappa) d\xi = 0, \quad (50)$$

where

$$\mathcal{R} := -\frac{\hbar^2}{2m} \Phi'' + g_{\text{eff}} |\Phi|^2 \Phi + p_{\text{eff}} (|\Phi|^2)'' \Phi - \mu \Phi. \quad (51)$$

Evaluating the integrals where  $S(x - ut) = A \text{sech}[\kappa(x - ut)]$  gives

$$\frac{\hbar^2 \kappa^2}{3m} + \frac{4}{3} g_{\text{eff}} A^2 - \frac{16}{15} p_{\text{eff}} A^2 \kappa^2 - 2\mu = 0, \quad (52)$$

$$\frac{2}{5} \frac{\hbar^2}{m} \kappa^2 + \frac{16}{15} g_{\text{eff}} A^2 - \frac{128}{105} p_{\text{eff}} A^2 \kappa^2 - \frac{4}{3} \mu = 0, \quad (53)$$

for these two equations we get

$$\mu = \frac{1}{2} \left( \frac{\hbar^2}{3m} + \frac{4}{3} g_{\text{eff}} A^2 - \frac{16}{15} p_{\text{eff}} A^2 \kappa^2 \right), \quad (54)$$

$$k = \sqrt{\frac{28mA^2 g_{\text{eff}} - 35\hbar^2}{80mA^2 p_{\text{eff}} - 63\hbar^2}}. \quad (55)$$

The requirement  $\kappa^2 > 0$  therefore defines a nontrivial existence domain in the  $(g_{\text{eff}}, p_{\text{eff}}, A)$  parameter space, showing explicitly how second-order geometric corrections reshape the admissible soliton region.

This projection-orthogonality method is standard in soliton-perturbation theory [53,54], and follows from the fact that translations of the stationary profile form a neutral mode of the unperturbed equation. Projecting the dynamics onto this mode imposes a solvability condition and yields closed algebraic equations for the collective coordinates  $A$  and  $\kappa$ .

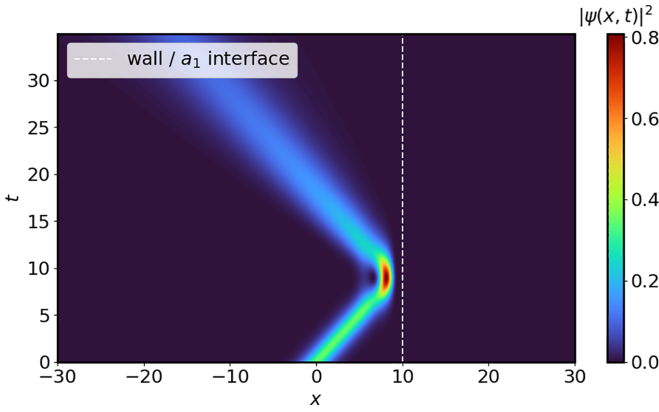


FIG. 4. Reflection of a chiral soliton from a hard-wall boundary. The soliton propagates unidirectionally due to the current nonlinearity and retains its shape before the collision. After reflection, its width increases and peak amplitude decays (demonstrated in Fig. 5). Here we use  $\hbar = m = 1$ ,  $g = -1.0$ ,  $a_1 = 0.2$ , and  $u = 1.0$ .

This approach ensures that the center-of-mass motion extracted from Eq. (46) is consistent with the symmetry of the underlying system.

(i) For  $p_{\text{eff}} \ll 1$ . Solving Eqs. (52) and (53), we get

$$\mu = \frac{96A^4 g_{\text{eff}} m p_{\text{eff}} - 35A^2 g_{\text{eff}} \hbar^2}{200A^2 m p_{\text{eff}} + 7\hbar^2}, \quad (56)$$

$$\kappa = \sqrt{\frac{7mA^2 g_{\text{eff}}}{20mA^2 p_{\text{eff}} + 7\hbar^2}}. \quad (57)$$

A bright soliton solution is

$$\begin{aligned} \phi(x, t) = & \sqrt{\frac{2\mu}{g - 2a_1 u}} \text{sech}[\kappa(x - ut)] \\ & \times e^{\frac{i}{\hbar}[mu x - (\frac{1}{2} mu^2 + \mu)t]}, \end{aligned} \quad (58)$$

with  $\mu > 0$ , and  $g_{\text{eff}} = g - 2a_1 u < 0$ . A dark soliton solution is

$$\begin{aligned} \phi(x, t) = & \sqrt{\frac{2\mu}{g - 2a_1 u}} \tanh[\kappa(x - ut)] \\ & \times e^{\frac{i}{\hbar}[mu x - (\frac{1}{2} mu^2 + \mu)t]}, \end{aligned} \quad (59)$$

with  $g - 2a_1 u > 0$ . The solutions Eqs. (58) and (59) are chiral. That is, the solitons can only propagate in a specific direction for a chosen velocity. As shown by Fig. 4. The soliton propagates at constant width and amplitude until the soliton collides with the wall. After the collision, the width increases and the amplitude decays exponentially (as illustrated in Fig. 5).

(ii) For  $g_{\text{eff}} \ll 1$ . Here we get the leading asymptotic

$$\kappa^2 \approx \frac{28mA^2 g_{\text{eff}} - 35\hbar^2}{80mA^2 p_{\text{eff}}}. \quad (60)$$

From Eq. (60), one might notice that the inverse of the width approaches zero as  $p_{\text{eff}}$  approaches infinity, that is, the soliton broadens. Huge residual coupling tends to delocalize an  $A \text{sech}(\kappa\xi)$ -type ansatz unless a finer balance with other terms

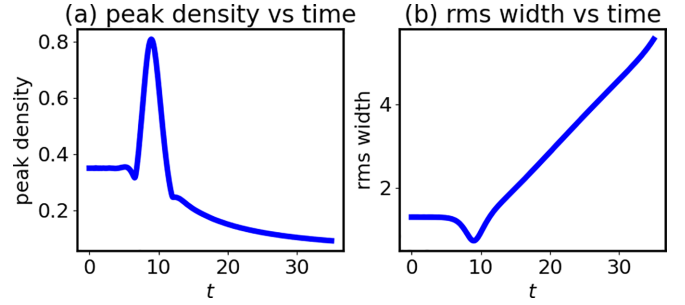


FIG. 5. Amplitude and width relations obtained numerically from the simulation of the model Eq. (36). The solid curves show the solutions for the peak amplitude and inverse width as functions of the effective interaction parameters  $g_{\text{eff}}$  and  $p_{\text{eff}}$ . The change in the amplitude and the inverse width after  $t \approx 10$  demonstrates the chirality of the solution since at this time the direction of the traveling wave changes.

occurs. For this case (bright soliton), we need  $\kappa^2 > 0$ . In the limit  $g_{\text{eff}} \rightarrow 0$ , the numerator becomes  $-35\hbar^2$ , so to get positive  $\kappa^2$ , the denominator must also be negative. That is,

$$80mA^2 p_{\text{eff}} - 63\hbar^2 < 0, \quad (61)$$

that is,

$$p_{\text{eff}} < \frac{63\hbar^2}{80mA^2}. \quad (62)$$

Figure 6 shows the evolution of the bright soliton. If  $p_{\text{eff}}$  is large and positive, the denominator is positive while the numerator is negative, hence  $\kappa^2 < 0$ , which implies no localized bright soliton. Large positive  $p_{\text{eff}}$  therefore tends to prevent the bright soliton solution. However, if  $p_{\text{eff}}$  is large and negative, the denominator  $\sim 80mA^2 p_{\text{eff}}$  is negative and the numerator is negative in the limit  $g_{\text{eff}} \rightarrow 0$  allowing  $\kappa^2 > 0$ . This case allows for localized structure formation.

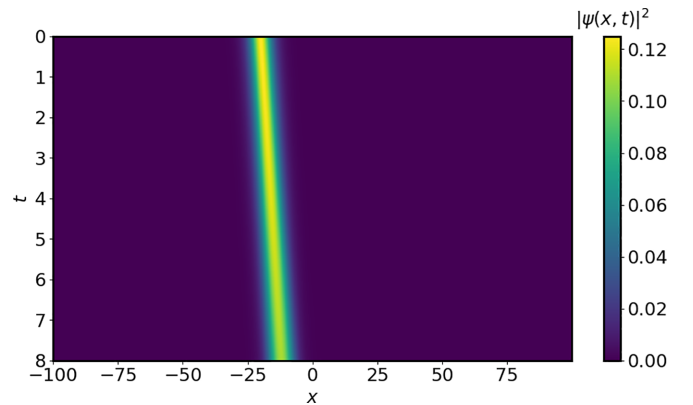


FIG. 6. Time evolution of the traveling-wave solution predicted by Eqs. (55) and (54) for  $|g_{\text{eff}}| \ll |p_{\text{eff}}|$ . Shown is  $|\psi(x, t)|^2$  at successive times under the full dynamical equation, Eq. (36). As the wave packet propagates, it maintains its shape over long timescales, confirming the dynamical stability of the analytic solution. Here we used  $\hbar = m = 1$ ,  $g = -0.02$ ,  $p = -8.0$ ,  $a_1 = 0.0$ , and  $a_2 = 0.02$ .

## VI. CONCLUSION

We analyzed the impact of second-order density-dependent geometric vector potentials on the nonlinear dynamics of a quasi-1D BEC with laser-induced Rabi coupling. By deriving an extended chiral GP equation that incorporates higher-order gauge corrections and finite-range interaction effects, we showed that these additional geometric contributions fundamentally modify the structure and propagation of nonlinear excitations.

The resulting model exhibits intrinsically chiral solitary waves whose amplitudes and widths depend explicitly on the direction of motion, reflecting the breakdown of Galilean invariance induced by density-dependent gauge fields. Using a projection (orthogonality) approach, we constructed bright and dark traveling-wave solutions and demonstrated their robustness through numerical simulations. In addition, the ring-geometry analysis revealed density-dependent persistent-current states and chiral ring-soliton configurations arising from the interplay between curvature and nonlinear gauge coupling. Our results highlight how higher-order geometric terms reshape localization, transport, and symmetry properties in gauge-coupled condensates. Beyond their relevance to ultracold atomic gases, these findings contribute more broadly to the understanding of nonlinear wave propagation in systems with dynamical gauge structures and current-induced interactions.

The reader should, however, be aware that in BECs a quadratic phase (chirp) term naturally arises from spatial variations of the condensate phase  $\phi(x, t)$ , which determines the local velocity field  $v = (\hbar/m)\nabla\phi$  [55,56]. Such a phase curvature is typically generated by nonlinear interactions or by hydrodynamic expansion in which the wavepacket acquires a position-dependent velocity profile. In the weakly interacting or short-time regime considered here, the nonlinear interaction energy  $g|\psi|^2$  remains small compared with the kinetic energy scale  $\hbar^2/(2mL^2)$  [56,57]. Consequently, the condensate phase remains approximately uniform,  $\phi(x, t) \approx -\mu t/\hbar$ , suppressing the development of any significant chirp [55]. The internal phase curvature, and thus breathing dynamics, is therefore strongly reduced. In contrast, when the quadratic phase term is fully included, it is known that the center-of-mass motion no longer oscillates at the trap frequency but instead acquires a nonlinear forcing term that depends explicitly on the width of the 1D cloud [58]. Likewise, the equation of motion for the width contains an additional contribution proportional to the center-of-mass velocity. Small-amplitude oscillations around the equilibrium points then provide access to the condensate's low-lying excitation spectrum. A variational treatment including the chirp term will be addressed in the presence of second-order density-dependent geometric vector potentials in future contributions.

## DATA AVAILABILITY

No new data was generated for this research.

## ACKNOWLEDGMENTS

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## APPENDIX A: EXPLICIT EVALUATION OF THE $\mathcal{O}(a_2, \partial_R^2 \rho)$ TERMS

This appendix shows explicitly why the second-order geometric corrections denoted  $\mathcal{O}(a_2, \partial_R^2 \rho)$  vanish for the uniform-ring stationary state  $\phi(\theta) = \sqrt{\rho} e^{iq\theta}$ . We consider Eq. (31) with the external potential  $V(x) = 0$ . On a ring of radius  $R$ , the steady-state solution

$$\psi(\theta) = \sqrt{\frac{N}{2\pi R}} e^{iq\theta} \quad (\text{A1})$$

the density is uniform. The terms  $g|\psi|^2$ ,  $p(|\psi|^2)_{\theta\theta}$ ,  $a_1 j$ , and  $a_2 k$  are independent of  $q$ , hence when taking the difference

$$E_{q+1} - E_q, \quad (\text{A2})$$

they vanish. Thus, the relevant energy term for the  $q$  dependence is

$$\begin{aligned} E_q &= \frac{1}{2m} [\hbar q/R - a_1 |\psi|^2 - a_2 (|\psi|^2)_{\theta\theta}]^2 \\ &= \frac{1}{2m} (\hbar q/R - a_1 |\psi|^2)^2. \end{aligned} \quad (\text{A3})$$

This concludes with the energy difference and the critical density derived in Ref. [32].

## APPENDIX B: DEFINITIONS OF THE CURRENT NONLINEARITIES

### 1. 1D current nonlinearities with density-dependent gauge potentials

After quasi-1D reduction, the current nonlinearity in Eq. (31) is given as

$$\begin{aligned} j_x &= \frac{\hbar}{2im} \psi \left( \partial_x + \frac{i}{\hbar} a_{1x} |\psi|^2 + \frac{i}{\hbar} a_{2x} \partial_{xx} (|\psi|^2) \right) \psi^* \\ &\quad - \frac{\hbar}{2im} \psi^* \left( \partial_x - \frac{i}{\hbar} a_{1x} |\psi|^2 - \frac{i}{\hbar} a_{2x} \partial_{xx} (|\psi|^2) \right) \psi, \end{aligned} \quad (\text{B1})$$

and the higher-order current nonlinearity is given as  $k_x = l_x + n_x$  where

$$\begin{aligned} l_x &= \frac{\hbar}{2im} \partial_{xx} \psi \left( \partial_x + \frac{i}{\hbar} a_{1x} |\psi|^2 + \frac{i}{\hbar} a_{2x} \partial_{xx} (|\psi|^2) \right) \psi^* \\ &\quad - \frac{\hbar}{2im} \psi^* \left( \partial_x - \frac{i}{\hbar} a_{1x} |\psi|^2 - \frac{i}{\hbar} a_{2x} \partial_{xx} (|\psi|^2) \right) \partial_{xx} \psi, \\ n_x &= \frac{\hbar}{2im} \psi \left( \partial_x + \frac{i}{\hbar} a_{1x} |\psi|^2 + \frac{i}{\hbar} a_{2x} \partial_{xx} (|\psi|^2) \right) \partial_{xx} \psi^* \\ &\quad - \frac{\hbar}{2im} \partial_{xx} \psi^* \left( \partial_x - \frac{i}{\hbar} a_{1x} |\psi|^2 - \frac{i}{\hbar} a_{2x} \partial_{xx} (|\psi|^2) \right) \psi. \end{aligned} \quad (\text{B2})$$

### 2. 1D Current nonlinearities after phase transformation and their associated continuity equations

The current nonlinearities we get after performing the phase transformation on Eq. (31) are given as

$$j = \frac{i\hbar}{2m} (\psi \partial_x \psi^* - \psi^* \partial_x \psi), \quad (\text{B4})$$

where the higher-order current nonlinearity is given as  $k = l + n$  for

$$l = \frac{i\hbar}{2m}(\partial_{xx}\psi\partial_x\psi^* - \psi^*\partial_{xxx}\psi), \quad (\text{B5})$$

$$n = \frac{i\hbar}{2m}(\psi\partial_{xxx}\psi^* - \partial_{xx}\psi^*\partial_x\psi). \quad (\text{B6})$$

We use the continuity equations

$$\dot{\rho} + \partial_x j = 0, \quad (\text{B7})$$

where  $\rho = |\psi|^2$ . In addition, when higher-order density-gradient terms are retained, it is useful to introduce the associated higher-order current  $k$ , which satisfies

$$\partial_{xx}\dot{\rho} + \partial_x k = 0. \quad (\text{B8})$$

This second relation Eq. (B8) does not represent an independent conservation law, but rather provides a convenient and systematic definition of the higher-order current entering the extended GP equation. One can show that  $k = \partial_{xx}j$ . The Madelung transform shows that the speed of a plane wave  $\phi = \sqrt{\rho}e^{-i(\omega t + \kappa x)}$  is given by

$$v \equiv \frac{j}{\rho} = \frac{\hbar\kappa}{m} \quad (\text{B9})$$

for the former continuity equation. For the latter continuity equation, we get

$$v = \frac{\hbar\kappa}{m} + C_k \int^x \frac{dy}{\rho(y)}, \quad (\text{B10})$$

given that  $\rho$  is a function of  $x$ .

### 3. Current nonlinearity in the ring geometry

If we perform the ring geometry on our 1D model, the associated current nonlinearities are

$$J_\theta = \frac{i}{2}(\phi\partial_\theta\phi^* - \phi^*\partial_\theta\phi), \quad (\text{B11})$$

$$K_\theta = L_\theta + N_\theta, \quad (\text{B12})$$

$$L_\theta = \frac{i}{2}(\partial_{\theta\theta}\phi\partial_\theta\phi^* - \phi^*\partial_{\theta\theta\theta}\phi), \quad (\text{B13})$$

$$N_\theta = \frac{i}{2}(\phi\partial_{\theta\theta\theta}\phi^* - \partial_{\theta\theta}\phi^*\partial_\theta\phi), \quad (\text{B14})$$

and the associate coefficients are given as

$$g_\theta = \frac{mR^2}{\hbar^2}(g_x - 2a_{1x}u), \quad \alpha_1 = \frac{R}{\hbar}a_{1x}, \quad (\text{B15})$$

$$p_\theta = \frac{m}{\hbar^2}(p_x - 2a_{2x}u), \quad \alpha_2 = \frac{a_{2x}}{R\hbar}. \quad (\text{B16})$$

## APPENDIX C: VARIATIONAL DERIVATION OF THE CENTER-OF-MASS AND WIDTH DYNAMICS

In this appendix, we present a time-dependent variational derivation of the expansion dynamics associated with the effective one-dimensional chiral GP equation, Eq. (36). This analysis clarifies the validity of the assumed width dynamics used in Sec. IV and demonstrates that the interaction-induced gauge terms primarily affect the center-of-mass motion.

### 1. Variational ansatz and Lagrangian

We consider the Gaussian trial wave function

$$\phi(x, t) = \left(\frac{N}{\sqrt{\pi}\sigma(t)}\right)^{\frac{1}{2}} \times \exp\left[-\frac{[x - x_0(t)]^2}{2\sigma(t)^2} + i\frac{m}{\hbar}\dot{x}_0(t)[x - x_0(t)]\right], \quad (\text{C1})$$

where  $x_0(t)$  and  $\sigma(t)$  are treated as independent variational parameters representing the center-of-mass position and the condensate width, respectively.

The dynamics follows from the effective Lagrangian

$$L = \int dx \left[ \frac{i\hbar}{2}(\phi^*\partial_t\phi - \phi\partial_t\phi^*) - \mathcal{H} \right], \quad (\text{C2})$$

where  $\mathcal{H}$  is the Hamiltonian density associated with Eq. (36) given as

$$\mathcal{H} = \frac{\hbar^2}{2m}|\partial_x\phi|^2 + \frac{g_x}{2}|\phi|^4 + p_x|\phi|^2\partial_x^2(|\phi|^2) - a_{1x}j_x|\phi|^2 - a_{2x}k_x|\phi|^2, \quad (\text{C3})$$

where  $j$  and  $k = l + n$  are the current and higher-order current nonlinearities defined in Eqs. (B1) to (B3).

### 2. Effective Lagrangian

Substituting the ansatz Eq. (C1) into Eq. (C2) and performing the spatial integration yields the reduced Lagrangian

$$L_{\text{eff}} = \frac{1}{2}mN\dot{x}_0^2 + \frac{mN}{4}\dot{\sigma}^2 - \frac{\hbar^2N}{4m\sigma^2} - \frac{g_xN^2}{2\sqrt{2\pi}\sigma} + L_{\text{gauge}}, \quad (\text{C4})$$

where the first four terms correspond to the kinetic, dispersive, and interaction contributions.

The gauge-induced contribution takes the form

$$L_{\text{gauge}} = -N\sqrt{\frac{2}{\pi}}\left(a_1 - \frac{3a_2}{\sigma^2}\right)\dot{x}_0, \quad (\text{C5})$$

arising from the current nonlinearities proportional to  $a_1$  and  $a_2$ .

### 3. Euler-Lagrange equations

Applying the Euler-Lagrange equations to the variational parameters yields two coupled equations of motion.

For the center-of-mass coordinate  $x_0(t)$ ,

$$\frac{d}{dt}\left(\frac{\partial L_{\text{eff}}}{\partial \dot{x}_0}\right) = \frac{\partial L_{\text{eff}}}{\partial x_0}, \quad (\text{C6})$$

which leads to

$$\ddot{x}_0 = \sqrt{\frac{2}{\pi}}\frac{N}{m}\left(a_1 - \frac{3a_2}{\sigma^2}\right)\frac{\dot{\sigma}}{\sigma^2}. \quad (\text{C7})$$

This expression coincides with Eq. (45) in the main text.

For the width  $\sigma(t)$

$$\frac{d}{dt}\left(\frac{\partial L_{\text{eff}}}{\partial \dot{\sigma}}\right) = \frac{\partial L_{\text{eff}}}{\partial \sigma}, \quad (\text{C8})$$

which yields

$$\ddot{\sigma} = \frac{\hbar^2}{m^2 \sigma^3} + \frac{g_x N}{\sqrt{2\pi} m \sigma^2} + \mathcal{O}(a_2 \dot{x}_0). \quad (\text{C9})$$

#### 4. Interpretation and consistency

Equation (C9) shows that, to leading order, the width dynamics is governed by the familiar free-expansion equation, with interaction-induced corrections identical to those obtained in conventional GP theory. The gauge-induced terms do not introduce a direct force on  $\sigma(t)$ , and their contribution enters only indirectly through higher-order couplings proportional to  $\dot{x}_0$ .

As a result, under the conditions considered in Sec. IV, the width evolution is well approximated by the standard free-expansion solution

$$\sigma(t) = \sigma_0 \sqrt{1 + \frac{t^2}{\tau^2}}, \quad (\text{C10})$$

where  $\tau = m\sigma_0^2/\hbar$ , while the interaction-generated gauge field produces a nontrivial drift of the center-of-mass motion governed by Eq. (C7)

This variational analysis confirms that the asymmetric expansion discussed in Sec. IV originates from the density-dependent geometric vector potential, rather than from modifications of the width dynamics.

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