

Bifurcation analysis of a birhythmic vibroimpact oscillator with two kinds of fractional derivative elements subjected to Gaussian white noise excitation

Boris A. Guimfack,^{1,*} Raoul Mbakob Yonkeu,^{2,†} Timoléon C. Kofané,^{3,4,‡} and Conrad B. Tabi^{3,§}

¹Laboratory of Biophysics, Department of Physics, Faculty of Science, [University of Yaoundé I](#), P.O. Box 812, Yaoundé, Cameroon

²Fundamental Physics Laboratory, Faculty of Science, [University of Bamenda](#), P.O. Box 39, Bambili, Northwest Region, Cameroon

³Complex Systems Research Group, Department of Physics and Astronomy, [Botswana International University of Science and Technology](#), Private Mail Bag 16, Palapye, Botswana

⁴Laboratory of Mechanics, Department of Physics, Faculty of Science, [University of Yaoundé I](#), P.O. Box 812, Yaoundé, Cameroon



(Received 3 January 2025; accepted 22 August 2025; published 17 September 2025)

This paper explores the stochastic bifurcations of a birhythmic vibroimpact oscillator incorporating two types of fractional derivative elements under Gaussian white noise excitation. The analysis begins with a nonsmooth coordinate transformation of the state variables, followed by the application of the stochastic averaging method to derive analytical solutions. To validate the proposed approach, numerical simulations based on the Monte Carlo method are conducted, demonstrating excellent agreement with the analytical predictions. A key finding of this study is that the fractional orders, fractional coefficients, and restitution coefficient play a significant role in triggering stochastic P bifurcations within the system.

DOI: [10.1103/jltq-f1mw](#)

I. INTRODUCTION

Mechanical or structural systems interacting with rigid obstacles, or with each other, often exhibit nonsmooth dynamics, typically modeled through impact phenomena. Vibroimpact (VI) systems, a distinct class of such nonsmooth systems, have garnered considerable attention in recent years. These systems exhibit rich dynamical behavior, including phenomena absent in smooth systems, such as corner bifurcations [1], grazing bifurcations [2–4], sliding bifurcations [5,6], and complex motions like chatter and sticking [7,8]. Given the pervasive presence of stochastic influences in real-world applications, it becomes essential to investigate VI systems under random excitations. Noise can significantly reshape the system's stability boundaries, modify the structure of the phase space, and alter the basins of attraction, effects that are especially pronounced in multistable systems [9,10]. This has fueled growing interest in the study of stochastic VI systems [11–17].

Parallel to this, fractional calculus has emerged as another vibrant area of research, particularly in the modeling of systems with memory and hereditary properties. Unlike integer-order models, fractional derivatives inherently account for memory through their convolutional structure, making

them especially suitable for describing complex dynamics in excitable media and engineering systems. Stochastic VI systems incorporating fractional derivatives have recently attracted growing attention, with applications spanning mechanics, control theory, engineering, biological systems, and applied mathematics [18,19]. For example, fractional vibroimpact oscillators are used in engineering to model the viscoelasticity of materials. Wang *et al.* [20] demonstrated this by analyzing the response of a vibroimpact system under random excitation using fractional derivatives. Such systems exhibit rich dynamical behaviors, including stochastic bifurcations. In this context, Yang *et al.* [21] highlighted the occurrence of complex transitions in a dynamical system by examining stochastic bifurcations in a fractional vibroimpact oscillator subjected to additive and multiplicative Gaussian white noise. The inclusion of fractional damping in vibroimpact oscillators can also induce resonance phenomena. For instance, Cocco *et al.* [22] showed that, for certain values of the fractional order, fractional damping could induce resonance in the Duffing oscillator. Fractional vibroimpact oscillators have also been employed in the design of vibration reduction systems. For example, Qu *et al.* [23] investigated the influence of the fractional order on superharmonic resonance and complex transitions in a nonlinear vibration isolation system with fractional derivatives.

Vibroimpact oscillators incorporating fractional derivatives have shown significant potential in biological applications, particularly in the modeling of complex physiological systems and the development of medical technologies. For instance, nanocantilevers employed as biosensors can detect specific biomolecular interactions through shifts in their resonance frequency. These frequency variations, induced by molecular binding events, can be effectively described using stochastically driven vibroimpact oscillator models, offering insight

*Contact author: boris.guimfack@yahoo.fr

†Contact author: raoulmbakob@yahoo.com

‡Contact author: tkofane@yahoo.com

§Contact author: tabic@biust.ac.bw

into the underlying biomolecular dynamics [24]. Moreover, modeling photoacoustic signals as damped oscillators within a mass-spring framework allows for the quantitative characterization of biomechanical properties of tissues, such as stiffness and relaxation times [25]. Additionally, experimental studies suggest that low-amplitude, high-frequency mechanical stimulation based on vibroimpact dynamics can promote bone regeneration in animal models of osteogenesis imperfecta, highlighting their therapeutic relevance in mechanobiology [26]. Biological processes such as anomalous transport in neural tissues, subdiffusive behavior in crowded environments, memory effects in calcium signaling, and nonlocal interactions can be effectively captured using vibroimpact oscillators incorporating fractional derivatives. These observations suggest that the dynamics of systems subject to both stochastic excitation and fractional-order effects are inherently rich and intricate. Consequently, a nonlinear stochastic system that integrates VI behavior and fractional derivatives is expected to exhibit even more complex and nuanced dynamical features.

The Van der Pol (VDP) oscillator and its generalizations represent classical paradigms of self-excited oscillators, widely employed to model biological and chemical rhythms. A particularly intriguing nonlinear feature is the generalized VDP damping, which fosters multistability, a phenomenon especially relevant in biochemical systems such as enzymatic reactions and glycolytic oscillations. This multistability gives rise to birhythmicity, characterized by the coexistence of two distinct oscillatory states with different amplitudes and frequencies. While birhythmicity is often advantageous and even essential in biological systems, it can be undesirable in certain engineering or physical contexts [27,28], making its analysis and control a subject of significant interest. Recent studies have explored the role of memory effects, particularly through fractional derivatives, in birhythmic models [29–31], revealing that fractional-order dynamics can effectively modulate the onset and persistence of birhythmicity.

A pivotal advancement in the theoretical study of VI systems is the introduction of nonsmooth coordinate transformations [32,33], which effectively recast a nonsmooth system into an equivalent smooth, impact-free form. These transformations enable the application of conventional analytical techniques to investigate the dynamics of VI systems. Among them, the stochastic averaging method, widely recognized for its efficacy in approximating complex systems, has been employed to analyze the response [14,34,35] and bifurcation behavior [21,36,37] of such systems. However, to the best of our knowledge, little attention has been given to the stochastic dynamics of birhythmic VI (BVI) systems incorporating fractional derivatives. This study aims to address that gap by examining the response of a nonlinear BVI oscillator with fractional-order dynamics subjected to Gaussian white noise. Our results demonstrate that changes in fractional coefficients, orders, and the restitution parameter trigger stochastic P bifurcations, marked by transitions between monorhythmic and birhythmic states, as evidenced by single- and triple-peak probability density functions (PDFs). The pseudopotential landscape highlights the dominant role of fractional orders in shaping these dynamics, confirming that fractional damping offers a tunable pathway to diverse oscillatory regimes.

The remainder of the paper is structured as follows. In Sec. II, the original VI oscillator is transformed into an equivalent impact-free system using a nonsmooth coordinate transformation, and the stochastic averaging method is applied to derive analytical solutions for the resulting system. Sections III and IV focus on the analysis of birhythmicity indicators, highlighting the influence of fractional orders and coefficients on the bifurcation structure. The evolution of the pseudopotential landscape and associated energy barriers under varying parameters is also examined. Section V investigates the occurrence of stochastic P bifurcations. Finally, Sec. VI summarizes the main findings and presents concluding remarks.

II. ANALYTICAL ESTIMATES FOR THE PROPOSED BVI MODEL

The differential equations governing a noisy BVI oscillator incorporating two distinct types of fractional derivative elements are expressed as follows:

$$\ddot{x} + a_1 D_t^{\lambda_1} x + a_2 D_t^{\lambda_2} x - \mu(c_1 - c_2 x^2 + \alpha x^4 - \beta x^6) \dot{x} + \omega_0^2 x = f(t), \quad x > 0, \quad (1a)$$

$$\dot{x}_p = -r \dot{x}_n, \quad x = 0, \quad (1b)$$

where x is the displacement state variable, and the overdots denote derivatives with respect to time t , corresponding to velocity and acceleration, respectively. The parameter ω_0 represents the undamped natural frequency of the linear oscillator. The coefficients α , β , and c_2 govern the nonlinear damping characteristics, while μ modulates the strength of this nonlinearity. The birhythmic behavior of the oscillator arises from the interplay between these parameters, particularly α , β , and the displacement x , making the system suitable for modeling certain biological processes. The term $f(t)$ represents Gaussian white noise with statistical properties $\langle f(t) \rangle = 0$ and $\langle f(t)f(t') \rangle = 2D\delta(t - t')$, where D is the noise intensity. The expressions $a_1 D_t^{\lambda_1} x(t)$ and $a_2 D_t^{\lambda_2} x(t)$ denote two fractional derivative elements, where a_1 and a_2 are the respective fractional coefficients, and λ_1 and λ_2 are the corresponding fractional orders. These two types of fractional derivative elements are defined in the Caputo sense as follows:

$$D_t^{\lambda_1} x(t) = \frac{1}{\Gamma(1 - \lambda_1)} \frac{d}{dt} \int_0^t \frac{x(t - \tau)}{\tau^{\lambda_1}} d\tau, \quad 0 < \lambda_1 \leq 1, \quad (2)$$

$$D_t^{\lambda_2} x(t) = \frac{1}{\Gamma(2 - \lambda_2)} \frac{d}{dt} \int_0^t \frac{x(t - \tau)}{\tau^{\lambda_2 - 1}} d\tau, \quad 1 < \lambda_2 \leq 2. \quad (3)$$

Equation (1b) describing the VI constraint consists of three elements: the subscripts n and p represent the velocities of the system before and after the instant t^* of impact, r , with $0 < r \leq 1$ reflecting the degree of energy loss when the impact occurs, and it is referred to as the restitution coefficient.

A. Nonsmooth transformation

Image transformation needs to be invoked to convert the VI constraint. Here the nonsmooth transformation [12,32] was

proposed as follows:

$$\begin{aligned}x &= x_1 = |V| = V \operatorname{sgn}(V), \\ \dot{x} &= x_2 = \dot{V} \operatorname{sgn}(V), \\ \ddot{x} &= x_3 = \ddot{V} \operatorname{sgn}(V),\end{aligned}\quad (4)$$

where $\operatorname{sgn}(V)$ is the signum function of V , such that

$$\operatorname{sgn}(V) = \begin{cases} -1 & \text{if } V < 0 \\ 0 & \text{if } V = 0 \\ +1 & \text{if } V > 0. \end{cases}$$

Obviously, the transformation in Eq. (4) maps the domain $x > 0$ of the original phase plane $(x, \dot{x})$ onto the whole phase plane $(V, \dot{V})$, and makes the transformed velocity $\dot{V}$ continuous when $r = 1$. Substituting Eqs. (4) into Eqs. (1), the equations of the new variable are written as follows:

$$\begin{aligned}\ddot{V} &+ a_1 \operatorname{sgn}(V) D_t^{\lambda_1}(|V|) + a_2 \operatorname{sgn}(V) D_t^{\lambda_2}(|V|) \\ &- \mu(c_1 - c_2 V^2 + \alpha V^4 - \beta V^6) \dot{V} \\ &+ \omega_0^2 V = \operatorname{sgn}(V) f(t), \quad t \neq t^*,\end{aligned}\quad (5a)$$

$$\dot{V}_p = r \dot{V}_n, \quad t = t^*. \quad (5b)$$

Here, Eq. (5b) is a constraint equation. Clearly, at $t = t^*$, the velocity has a discontinuity due to the impact when $r \neq 1$. The transformed velocity jump is reduced by an amount proportional to $(1 - r)$. It is possible to introduce this jump into the equation of motion using the Dirac δ function, and thus one avoids using the constraint equation. To this end, one may expand $V(t)$ around the time instant of impact, t^* , where $V(t^*) = 0$, $V(t) \approx V(t^*) + \dot{V}(t^*)(t - t^*) = \dot{V}(t^*)(t - t^*)$, for $V(t^*) = 0$. The above approximation implies that $\delta(t - t^*) \approx \delta(V(t)/\dot{V}(t^*))$. From the definition of the Dirac function, for a given function $g(V)$ that is continuous in the neighborhood of zero, one can write

$$\int_{-\infty}^{\infty} g(V) \delta(\zeta V) dV = \frac{1}{\zeta} \left[\int_{-\infty}^{\infty} g(u/\zeta) \delta(u) du \right] = \frac{g(0)}{\zeta}. \quad (6)$$

Thus, one can obtain

$$\delta(t - t^*) \approx |\dot{V}(t)| \delta(V). \quad (7)$$

The additional damping term due to impact may be written in the form

$$(\dot{V}_p - r \dot{V}_n) \delta(t - t^*) = (1 - r) \dot{V} \delta(t - t^*), \quad (8)$$

where V satisfies the double inequality $|\dot{V}_p| < |\dot{V}| < |\dot{V}_n|$ [38]. In this case, Eq. (8) can be written after using Eq. (7) as

$$(1 - r) \dot{V} \delta(t - t^*) \approx (1 - r) \dot{V} |\dot{V}| \delta(V). \quad (9)$$

For more information about differential operations with nonsmooth functions in terms of distributions, readers can consult Refs. [38–40]. Introducing the new impact term from Eq. (9), Eq. (5a) takes the form

$$\begin{aligned}\ddot{V} &+ a_1 \operatorname{sgn}(V) D_t^{\lambda_1}(|V|) + a_2 \operatorname{sgn}(V) D_t^{\lambda_2}(|V|) \\ &- \mu(c_1 - c_2 V^2 + \alpha V^4 - \beta V^6) \dot{V} \\ &+ (1 - r) \dot{V} |\dot{V}| \delta(V) + \omega_0^2 V = \operatorname{sgn}(V) f(t).\end{aligned}\quad (10)$$

B. Stochastic averaging approach

In terms of small VI losses, small random disturbances, and weak damping, according to the stochastic averaging method [41], Eq. (10) can be approximated as a quasiconservative system with a solution of the following form:

$$\begin{aligned}V_1 &= V(t) = A(t) \cos \phi(t), \\ V_2 &= \dot{V}(t) = -A(t) \omega_0 \sin \phi(t), \\ \phi(t) &= \omega_0 t + \theta(t),\end{aligned}\quad (11)$$

with A , ϕ , and θ being random processes. Therefore, the system equation is further simplified into

$$\begin{aligned}\ddot{V}_2 &+ a_1 \operatorname{sgn}(V_1) D_t^{\lambda_1}(|V_1|) + a_2 \operatorname{sgn}(V_1) D_t^{\lambda_2}(|V_1|) \\ &- \mu(c_1 - c_2 V_1^2 + \alpha V_1^4 - \beta V_1^6) V_2 \\ &+ (1 - r) V_2 |V_2| \delta(V_1) + \omega_0^2 V_1 = \operatorname{sgn}(V_1) f(t).\end{aligned}\quad (12)$$

Potential energy $E_p(V_1)$ and total energy E of the converted system are expressed as

$$\begin{aligned}E_p(V_1) &= \int_0^{V_1} \tilde{\omega}^2 x dx = \frac{1}{2} \omega_0^2 V_1^2, \\ E &= E_p(V_1) + \frac{1}{2} V_2^2.\end{aligned}\quad (13)$$

Based on this, one can obtain the equations for the amplitude A and the phase angle θ from Eqs. (13) as

$$\begin{aligned}\frac{dA}{dt} &= M_{11} + M_{12} + M_{13} + G_{11} f(t), \\ \frac{d\theta}{dt} &= M_{21} + M_{22} + M_{23} + G_{21} f(t),\end{aligned}\quad (14)$$

where

$$\begin{aligned}M_{11} &= + \frac{a_1 \sin \phi}{\omega_0} \operatorname{sgn}(A \cos \phi) D_t^{\lambda_1}(|A \cos \phi|), \\ M_{12} &= + \frac{a_2 \sin \phi}{\omega_0} \operatorname{sgn}(A \cos \phi) D_t^{\lambda_2}(|A \cos \phi|), \\ M_{13} &= -A \sin^2 \phi [-\mu(c_1 - c_2 A^2 \cos^2 \phi + \alpha A^4 \cos^4 \phi \\ &\quad - \beta A^6 \cos^6 \phi) + (1 - r)| - A \omega_0 \sin \phi | \delta(A \cos \phi)], \\ G_{11} &= - \frac{\sin \phi}{\omega_0} \operatorname{sgn}(A \cos \phi), \\ M_{21} &= + \frac{a_1 \cos \phi}{A \omega_0} \operatorname{sgn}(A \cos \phi) D_t^{\lambda_1}(|A \cos \phi|), \\ M_{22} &= + \frac{a_2 \cos \phi}{A \omega_0} \operatorname{sgn}(A \cos \phi) D_t^{\lambda_2}(|A \cos \phi|), \\ M_{23} &= -\sin \phi \cos \phi [-\mu(c_1 - c_2 A^2 \cos^2 \phi + \alpha A^4 \cos^4 \phi \\ &\quad - \beta A^6 \cos^6 \phi) + (1 - r)| - A \omega_0 \sin \phi | \delta(A \cos \phi)], \\ G_{21} &= - \frac{\cos \phi}{A \omega_0} \operatorname{sgn}(A \cos \phi).\end{aligned}\quad (15)$$

According to the stochastic averaging method [41], the variables A and θ can be approximated as a two-dimensional diffusion process. Consequently, the mean drift and diffusion coefficients of the corresponding Fokker-Planck-Kolmogorov (FPK) equation are obtained by performing stochastic averaging and integrating over one oscillation period $2\pi/\omega_0$. The

resulting Itô equation governing the amplitude $A(t)$ takes the form

$$dA = m(A)dt + \sigma(A)dW_1(t), \quad (16)$$

where $W_1(t)$ denotes normalized Wiener process. $m(A)$ and $\sigma(A)$ represent, respectively, the mean drift and mean diffusion coefficients expressed as

$$m(A) = \left\langle M_{11} + M_{12} + M_{13} + D \frac{\partial G_{11}}{\partial A} G_{11} + D \frac{\partial G_{11}}{\partial \theta} G_{21} \right\rangle_\phi, \quad (17a)$$

$$\sigma^2(A) = \langle 2DG_{11}^2 \rangle_\phi. \quad (17b)$$

To derive the explicit form of $m(A)$, the key step is the simplification of the first and second terms in Eq. (17a), and following the procedure detailed in the Appendix, the averaged drift coefficient $m(A)$ is explicitly given by

$$m(A) = \frac{D}{2A\omega_0^2} + \frac{\gamma}{2}A - \mu \left(\frac{c_2}{8}A^3 - \frac{\alpha}{16}A^5 + \frac{5\beta}{128}A^7 \right), \quad (18)$$

where

$$\begin{aligned} \gamma = & -\frac{64a_1}{\pi^2\omega_0} \sin \frac{\lambda_1\pi}{2} \sum_{n=1}^{15} \frac{n^2(2n\omega_0)^{\lambda_1-1}}{(1-4n^2)^2} \\ & - \frac{128a_2}{\pi^2\omega_0} \cos \frac{(\lambda_2-1)\pi}{2} \sum_{n=1}^{15} \frac{n^3(2n\omega_0)^{\lambda_2-2}}{(1-4n^2)^2} \\ & + \mu c_1 - \frac{2(1-r)\omega_0}{\pi}. \end{aligned} \quad (19)$$

Along the same line, using Eq. (17b), the averaged diffusion coefficient $\sigma^2(A)$ is also obtained as

$$\sigma^2(A) = \langle 2DG_{11}^2 \rangle_\phi = \frac{D}{\omega_0^2}. \quad (20)$$

By closely inspecting the above equations, one sees immediately that both the drift and the diffusion coefficients do not explicitly depend on time t . Consequently, $A(t)$ is a time-homogeneous diffusion process. The corresponding FPK

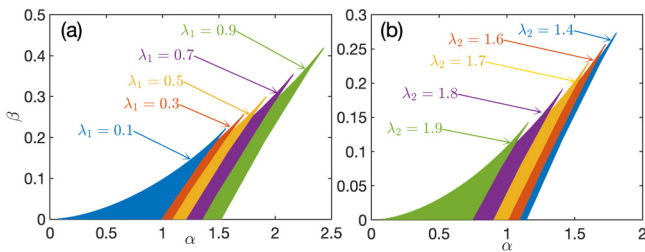


FIG. 1. Effect of the fractional orders λ_1 and λ_2 on the birhythmicity region in the parametric (α, β) plane, as derived from Eq. (24), with the natural frequency fixed at $\omega_0 = 1.0$. The remaining parameters are set to $\mu = 0.01$, $D = 0.1$, $a_1 = a_2 = 0.01$, $r = 0.95$, and $c_1 = c_2 = 1.0$. (a) $\lambda_2 = 1.4$ and (b) $\lambda_1 = 0.4$.

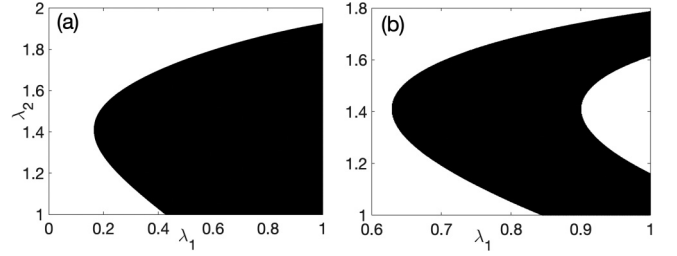


FIG. 2. Parameter domains for the existence of birhythmicity (black regions) in the (λ_1, λ_2) plane, based on Eq. (24), with fixed frequency $\omega_0 = 1.0$. The remaining parameters are $\mu = 0.01$, $D = 0.1$, $a_1 = a_2 = 0.01$, $r = 0.95$, and $c_1 = c_2 = 1.0$. (a) The case $(\alpha = 1.368, \beta = 0.144)$, and (b) the case $(\alpha = 1.89, \beta = 0.262)$. The white regions indicate monorhythmic behavior.

equation for the amplitude associated with Eq. (16) is

$$\begin{aligned} \frac{\partial P(A, t)}{\partial t} = & -\frac{\partial}{\partial A} [m(A)P(A, t)] \\ & + \frac{1}{2} \frac{\partial^2}{\partial A^2} [\sigma^2(A)P(A, t)]. \end{aligned} \quad (21)$$

Solving Eq. (21), one obtains the stationary PDF for the amplitude in the form

$$\begin{aligned} P(A) = & \frac{N_0}{\sigma^2(A)} \exp \left[\int_0^A \frac{2m(v)}{\sigma^2(v)} dv \right] \\ = & \frac{N_0 \omega_0^2 A}{D} \exp \left[\frac{\omega_0^2}{64D} \left(32\gamma A^2 \right. \right. \\ & \left. \left. - \mu \left(4c_2 A^4 - \frac{4}{3} \alpha A^6 + \frac{5}{8} \beta A^8 \right) \right) \right], \end{aligned} \quad (22)$$

where N_0 is the normalization constant which satisfies

$$\begin{aligned} N_0 = & \left[\int_0^\infty \frac{\omega_0^2 A}{D} \exp \left[\frac{\omega_0^2}{64D} \left(32\gamma A^2 \right. \right. \right. \\ & \left. \left. - \mu \left(4c_2 A^4 - \frac{4}{3} \alpha A^6 + \frac{5}{8} \beta A^8 \right) \right) \right] dA \right]^{-1}. \end{aligned} \quad (23)$$

By executing $\frac{\partial P(A)}{\partial A} = 0$, the extrema A_m of the distribution expression from Eq. (22) satisfies the following critical

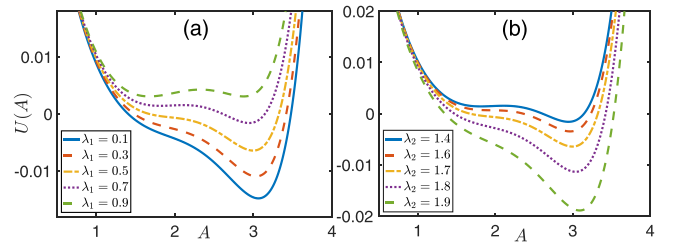


FIG. 3. Effects of the fractional orders λ_1 and λ_2 on the pseudopotential $U(A)$, as defined in Eq. (29), for the parameter set $(\alpha = 1.85, \beta = 0.242)$ at the natural frequency $\omega_0 = 1.0$. The remaining parameters are $\mu = 0.01$, $D = 0.1$, $a_1 = a_2 = 0.01$, $r = 0.95$, and $c_1 = c_2 = 1.0$. (a) The case $\lambda_2 = 1.4$; (b) the case $\lambda_1 = 0.7$.

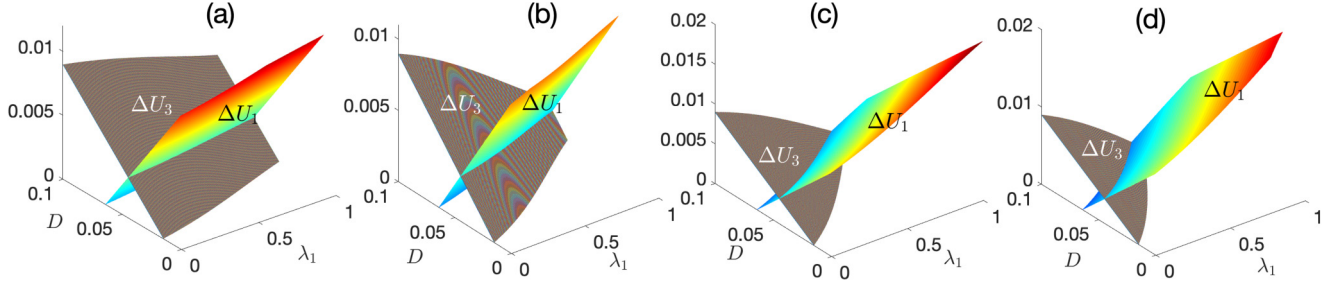


FIG. 4. Effect of the fractional coefficient a_1 on the energy barriers ΔU_1 and ΔU_3 for the parameter set ($\alpha = 1.85, \beta = 0.242$), with $\omega_0 = 1.0$ and $\lambda_2 = 1.7$: (a) $a_1 = 0.005$, (b) $a_1 = 0.01$, (c) $a_1 = 0.015$, and (d) $a_1 = 0.02$. The remaining parameters are fixed as $\mu = 0.01$, $a_2 = 0.01$, $r = 0.95$, and $c_1 = c_2 = 1.0$.

conditions:

$$\frac{D}{\omega_0^2} + \left[\gamma A_m^2 - \mu \left(\frac{c_2}{4} A_m^4 - \frac{1}{8} \alpha A_m^6 + \frac{5}{64} \beta A_m^8 \right) \right] = 0. \quad (24)$$

Then, the stationary PDF of the total energy $E = E_p(A) = \frac{1}{2} \omega_0^2 A^2$ can be retrieved so that

$$P(E) = P(A) \left| \frac{dA}{dE} \right| = \frac{P(A)}{\omega_0^2 A} \Big|_{A=E_p^{-1}(E)}. \quad (25)$$

According to Ref. [42], the corresponding first approximation of the joint stationary PDF of the displacement V_1 and velocity V_2 can be obtained as follows:

$$\begin{aligned} P_{V_1, V_2}(V_1, V_2) &= \frac{P(E)}{T(E)} \Big|_{E=\frac{1}{2}\omega_0^2 V_1^2 + \frac{1}{2}V_2^2} \\ &= \frac{N_0 \omega_0}{2\pi D} \exp \left[\frac{\omega_0^2}{64D} \left[32\gamma \left(V_1^2 + \frac{V_2^2}{\omega_0^2} \right) \right. \right. \\ &\quad \left. \left. - \mu \left\{ 4c_2 \left(V_1^2 + \frac{V_2^2}{\omega_0^2} \right)^2 - \frac{4}{3} \alpha \left(V_1^2 + \frac{V_2^2}{\omega_0^2} \right)^3 \right. \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{5}{8} \beta \left(V_1^2 + \frac{V_2^2}{\omega_0^2} \right)^4 \right\} \right] \right], \end{aligned} \quad (26)$$

with $T(E) = 2\pi/\omega_0$. The stationary PDF of the original variable x_1 and velocity x_2 can be derived as

$$\begin{aligned} P(x_1, x_2) &= 2P_{V_1, V_2}(x_1, x_2) \\ &= \frac{N_0 \omega_0}{\pi D} \exp \left[\frac{\omega_0^2}{64D} \left[32\gamma \left(x_1^2 + \frac{x_2^2}{\omega_0^2} \right) \right. \right. \end{aligned}$$

$$\begin{aligned} &\left. - \mu \left\{ 4c_2 \left(x_1^2 + \frac{x_2^2}{\omega_0^2} \right)^2 - \frac{4}{3} \alpha \left(x_1^2 + \frac{x_2^2}{\omega_0^2} \right)^3 \right. \right. \\ &\left. \left. + \frac{5}{8} \beta \left(x_1^2 + \frac{x_2^2}{\omega_0^2} \right)^4 \right\} \right]. \end{aligned} \quad (27)$$

III. ANALYTICAL TREATMENT AND BIRHYTHMICITY INDICATORS

The Langevin equation [Eq. (16)] of the Brownian motion of the particle in a double-well potential can, therefore, be approximated by

$$\dot{A} = -\frac{dU(A)}{dA} + \sigma(A)\dot{W}_1, \quad (28)$$

and the effective pseudopotential $U(A)$ is given by

$$\begin{aligned} U(A) &= \frac{1}{128} \left[-32\gamma A^2 + \mu \left(4c_2 A^4 - \frac{4}{3} \alpha A^6 + \frac{5}{8} \beta A^8 \right) \right] \\ &\quad - \frac{D \ln(A)}{2\omega_0^2}. \end{aligned} \quad (29)$$

It is found that the restitution coefficient r , the noise intensity D , and the fractional orders λ_1 and λ_2 influence the pseudopotential $U(A)$. Additionally, $U(A)$ exhibits a weak, logarithmic dependence on the oscillation amplitude.

In Fig. 1, we report the influence of the fractional-order parameters λ_1 and λ_2 on the stability map. Specifically, Fig. 1(a) displays the effect of the fractional-order parameter λ_1 , whose increase causes the domain for birhythmicity to expand. The impact of the fractional-order parameter λ_2 is shown by Fig. 1(b), where it is found that increasing the values of λ_2

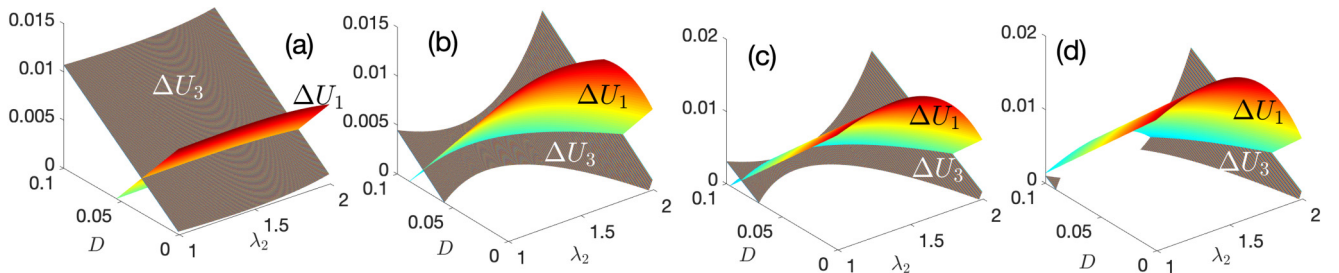


FIG. 5. Effect of the fractional coefficient a_2 on the energy barriers ΔU_1 and ΔU_3 , for the parameter set ($\alpha = 1.85, \beta = 0.242$) at the natural frequency $\omega_0 = 1.0$, with $\lambda_1 = 0.7$: (a) $a_2 = 0.001$, (b) $a_2 = 0.01$, (c) $a_2 = 0.012$, and (d) $a_2 = 0.015$. The remaining parameters are fixed as $\mu = 0.01$, $a_1 = 0.01$, $r = 0.95$, and $c_1 = c_2 = 1.0$.

causes the domain for birhythmicity to reduce. Obviously, with increasing λ_1 , multimodal structures of the probability distribution are expected, while with increasing λ_2 , unimodal structures of the probability distribution can be found, when fractional coefficients are equal. Otherwise, a couple of parameters (α, β) out of such regions will not induce the stochastic bifurcation of the PDF. Figure 2 represents the bifurcation diagram against the system parameter plane (λ_1, λ_2) , where Figs. 2(a) and 2(b) correspond to the respective values $(\alpha = 1.368, \beta = 0.144)$ and $(\alpha = 1.89, \beta = 0.262)$ of the system parameters. We should also stress that white areas are those where monorhythmic events are expected, while black areas are supposed to support the appearance of birhythmic dynamics. By comparing these two figures, we see that only one bifurcation is possible in Fig. 2(a), while two bifurcations are expected in Fig. 2(b), when λ_2 increases, for a fixed value of λ_1 .

In Fig. 3, a two-dimensional view of the pseudopotential is plotted as a function of amplitude A for the set of parameters $(\alpha = 1.85, \beta = 0.242)$, when noise intensity $D = 0.1$. We analyze the effects of fractional orders on the existence and the number of wells in the pseudopotential. We underline that the depth of the well embodies the stability of the system. The effective energy barrier becomes more important when the well is deeper. It is found that fractional orders λ_1 and λ_2 change the dynamics of the system. Precisely, increasing λ_1 leads to the appearance of two wells [see Fig. 3(a)], while increasing λ_2 leads to the disappearance of the birhythmic behavior [see Fig. 3(b)].

IV. EVALUATION OF THE EFFECTIVE ENERGY BARRIERS VERSUS λ_1, λ_2 AND THE NOISE INTENSITY D

The transition between unimodal and bimodal distributions, which can be monitored through the appropriate competition between the fractional orders λ_1 and λ_2 , as well as the noise intensity D , can significantly alter the behavior of potential barriers from the oscillations within potential wells. From Eq. (29), the behaviors of the potential barriers ΔU_1 (associated with the inner orbit) and ΔU_3 (related to the outer orbit) can be calculated. These are illustrated in Figs. 4 and 5. In Fig. 4, it is observed that as the fractional coefficient a_1 increases, ΔU_3 , in relation to λ_1 and D , tends to disappear, while the barrier ΔU_1 remains unchanged. This also signifies the transition between bimodal and unimodal distributions. A similar behavior is seen in Fig. 5 as the fractional coefficient a_2 increases. Note that $\Delta U_j (j = 1, 3)$ represents the difference between the maximum and minimum values of an effective potential $U(A)$.

V. STOCHASTIC P BIFURCATION

In this section, we will explore the possible transition scenarios of the oscillator, in conjunction with numerical techniques to analyze the effects of fractional orders, fractional coefficients, and the restitution coefficient on the transformations of the stationary PDF from one form to another.

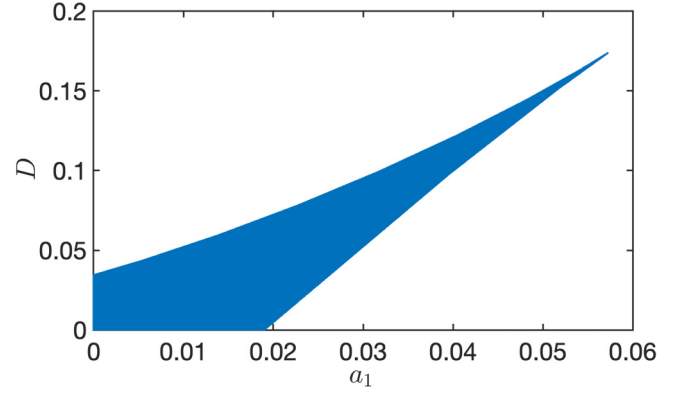


FIG. 6. Bifurcation diagram in the (a_1, D) -plane parameter space based on Eq. (24), for the parameter set $(\alpha = 1.85, \beta = 0.242)$, with natural frequency $\omega_0 = 1.0$. The remaining parameters are fixed as $\mu = 0.01$, $\lambda_1 = 0.7$, $\lambda_2 = 1.7$, $a_2 = 0.01$, $r = 0.97$, and $c_1 = c_2 = 1.0$.

A. Effect of fractional coefficients a_1 and a_2

The influence of the first fractional coefficients a_1 on the stochastic response of the fractional-order BVI system is examined in Fig. 6. In this diagram, the blue region corresponds to a bimodal stationary amplitude distribution, indicating birhythmic behavior, while the white region signifies a unimodal distribution associated with monorhythmic dynamics.

As illustrated in Fig. 7, the stationary probability density function $P(A)$ is used to examine bifurcation control under varying fractional coefficients a_1 . For $a_1 = 0.001$, $P(A)$ displays a single right-hand peak associated with the outer limit cycle. As a_1 increases, this peak diminishes, while a left-hand peak, corresponding to the inner cycle, gradually emerges. At $a_1 = 0.02$, both peaks coexist, forming a bimodal distribution. Further increasing a_1 to 0.04 suppresses the right peak entirely, yielding a unimodal distribution centered on the inner cycle. These transitions between unimodal and bimodal

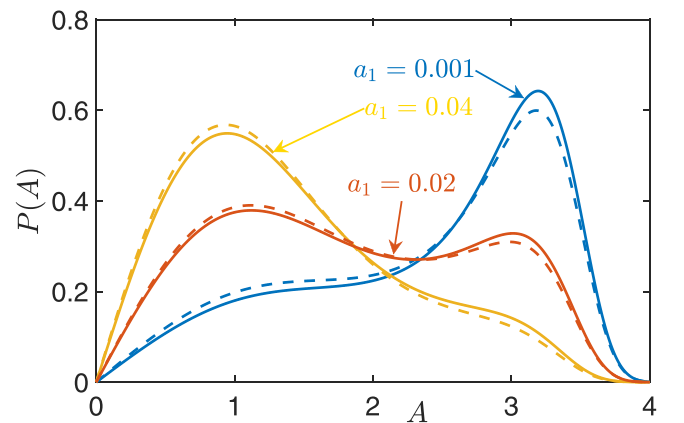


FIG. 7. Analytical (solid lines) and Monte Carlo simulation (dashed lines) results for the stationary PDF as a function of the amplitude A , for various values of the fractional coefficient a_1 . The parameter set is $(\alpha = 1.85, \beta = 0.242)$ with $\mu = 0.01$. Other parameters are fixed as $\omega_0 = 1.0$, $\lambda_1 = 0.7$, $\lambda_2 = 1.7$, $r = 0.97$, $a_2 = 0.01$, $c_1 = c_2 = 1.0$, and $D = 0.04$.

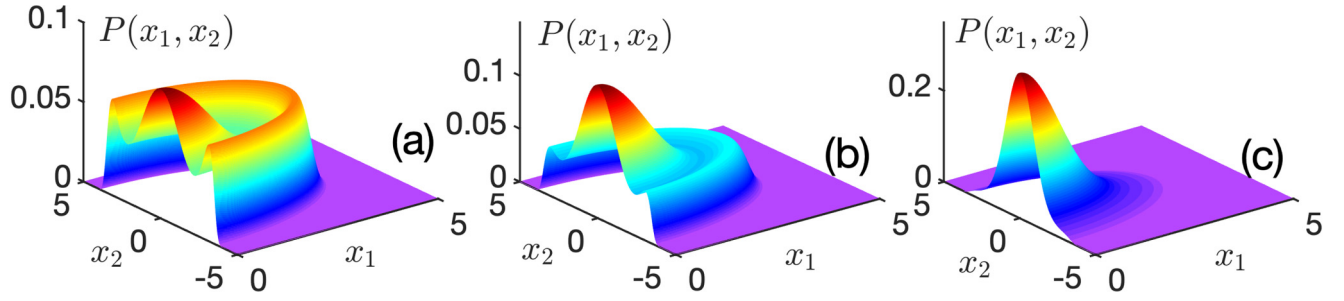


FIG. 8. Effect of the fractional coefficient a_1 on the joint PDFs, for the parameter set ($\alpha = 1.85$, $\beta = 0.242$), with $D = 0.04$ and $\mu = 0.01$. The values of a_1 are (a) $a_1 = 0.001$, (b) $a_1 = 0.02$, and (c) $a_1 = 0.04$. The remaining parameters are fixed as $\omega_0 = 1.0$, $\lambda_1 = 0.7$, $\lambda_2 = 1.7$, $a_2 = 0.01$, $r = 0.97$, and $c_1 = c_2 = 1.0$.

profiles signify a stochastic P bifurcation, indicating that a_1 effectively governs the bifurcation dynamics. This behavior is corroborated in Fig. 8, where the joint PDF $P(x_1, x_2)$ evolves from a craterlike shape with a single peak at $a_1 = 0.001$ to a topologically distinct unimodal structure at $a_1 = 0.04$, confirming the occurrence of bifurcation.

Considering the fractional coefficient a_2 as a bifurcation parameter, Fig. 9 presents the bifurcation diagram in the (a_2, D) plane, where the blue region denotes birhythmic behavior characterized by the coexistence of two stable limit cycles, while the white region corresponds to monorhythmic dynamics with a single stable limit cycle. It is evident that for $D \leq 0.0263$ a single bifurcation occurs as a_2 increases from 0 to 0.045. In contrast, when $0.0263 < D \leq 0.1734$, two bifurcations emerge within the same interval of a_2 , indicating a more intricate transition between rhythmic regimes under stronger noise intensity.

The P bifurcation induced by the second type of fractional coefficient a_2 is illustrated in Fig. 10, and dashed lines correspond to numerical results obtained from simulations of system (1). As shown, the PDF initially exhibits a unimodal structure with a pronounced peak at a large amplitude when $a_2 = 0.001$. Increasing a_2 to 0.015 leads to the emergence of a secondary peak, marking a transition from unimodality to

bimodality. When a_2 exceeds 0.05, the outer peak disappears, and the PDF reverts to a unimodal profile centered around the inner stable limit cycle, indicating the occurrence of a second P bifurcation.

The corresponding joint PDFs of displacement x_1 and velocity x_2 are shown in Fig. 11. Figure 11(a) reveals a craterlike distribution with a shallow central hump. In contrast, Fig. 11(c) for $a_2 = 0.05$ displays a sharp peak at the equilibrium, confirming the return to monorhythmic behavior. These topological transitions in the joint PDF substantiate the presence of a stochastic P bifurcation as a_2 increases from 0.001 to 0.05.

B. Effect of fractional orders λ_1 and λ_2

We now examine the stochastic bifurcation induced by the first type of fractional-order parameters λ_1 and λ_2 . The bifurcation diagram in the (λ_1, D) parameter space is presented in Fig. 12, where the blue region denotes birhythmic behavior and the white region corresponds to monorhythmic dynamics. When the noise intensity $D < 0.02$, the system consistently exhibits monorhythmic behavior as λ_1 increases. For

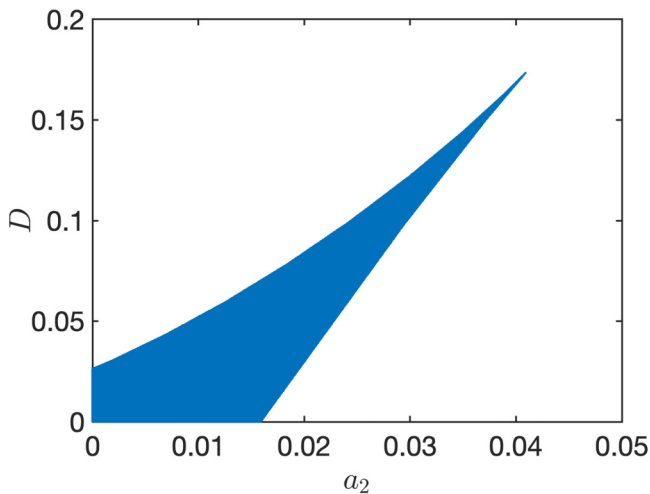


FIG. 9. Bifurcation diagram in the (a_2, D) -plane parameter space, derived from Eq. (24), for the fixed set ($\alpha = 1.85$, $\beta = 0.242$) at the natural frequency $\omega_0 = 1.0$. The remaining parameters are $\mu = 0.01$, $\lambda_1 = 0.7$, $\lambda_2 = 1.7$, $a_1 = 0.01$, $r = 0.97$, and $c_1 = c_2 = 1.0$.

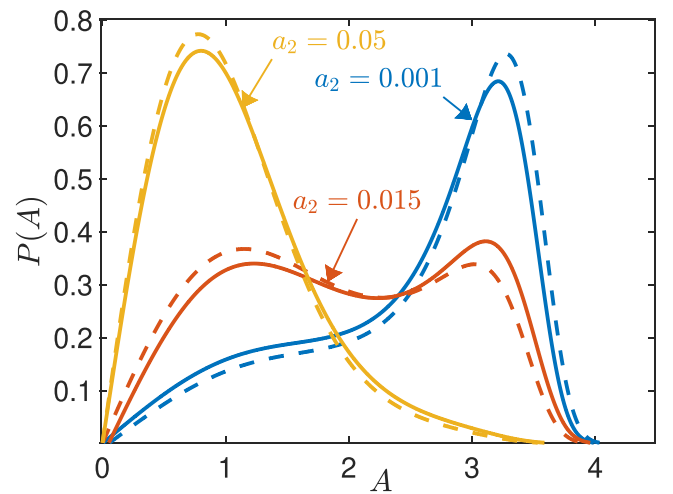


FIG. 10. Analytical (solid lines) and Monte Carlo simulation (dashed lines) results for the stationary PDF as a function of the amplitude A , for various values of the fractional coefficient a_2 . The parameters are fixed at ($\alpha = 1.85$, $\beta = 0.242$) and $\mu = 0.01$, with $\omega_0 = 1.0$, $\lambda_1 = 0.7$, $\lambda_2 = 1.7$, $r = 0.97$, $a_1 = 0.01$, $c_1 = c_2 = 1.0$, and $D = 0.04$.

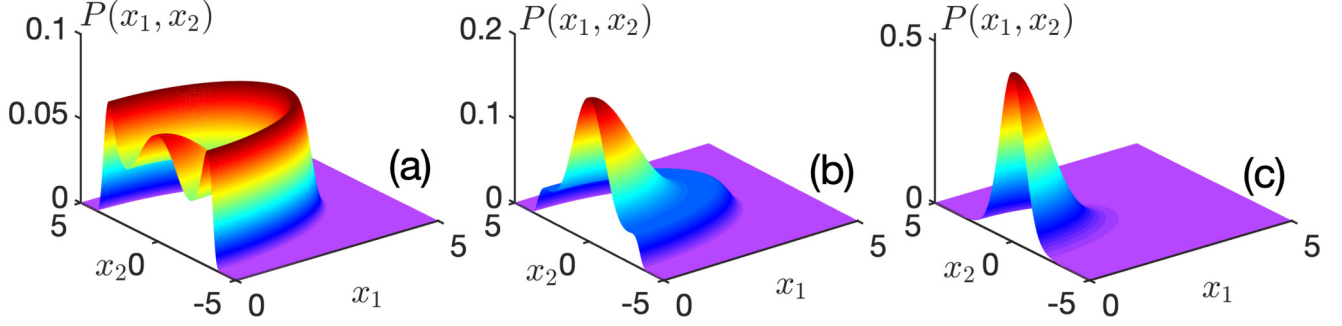


FIG. 11. Effect of the fractional coefficient a_2 on the joint PDFs for the parameter set ($\alpha = 1.85$, $\beta = 0.242$), with $D = 0.04$ and $\mu = 0.01$: (a) $a_2 = 0.001$, (b) $a_2 = 0.015$, and (c) $a_2 = 0.05$. The remaining parameters are fixed as $\omega_0 = 1.0$, $\lambda_1 = 0.7$, $\lambda_2 = 1.7$, $a_1 = 0.01$, $r = 0.97$, and $c_1 = c_2 = 1.0$.

intermediate noise levels $0.02 < D < 0.078$, a single transition is observed, from birhythmicity to monorhythmicity, as λ_1 increases from 0 to 1. However, when $D > 0.078$, two distinct transitions emerge, reflecting a more complex bifurcation structure driven by the interplay between noise and fractional-order effects.

Figure 13 presents the PDF of the amplitude for various values of the fractional-order parameter λ_1 , with the noise intensity fixed at $D = 0.1$. As shown, the amplitude distribution is unimodal for $\lambda_1 = 0.2$ and $\lambda_1 = 0.9$, whereas a bimodal structure emerges at $\lambda_1 = 0.6$. These qualitative changes in the topology of the PDF, driven by variations in λ_1 , reflect the occurrence of a phenomenological bifurcation in the system's stochastic dynamics.

Figure 14 reveals that the parameter space (λ_2, D) is partitioned into two distinct regions: a white region associated with monorhythmic behavior and a blue region corresponding to birhythmic dynamics. As the fractional-order parameter λ_2 increases from 1 to 2, multiple transitions between these rhythmic states emerge. For instance, when the noise intensity $D < 0.06$, a single transition is observed. As D increases to the interval $0.06 < D < 0.12$, two transitions occur. In the range $0.12 \leq D < 0.133$, three transitions are evident, and a

fourth emerges when D lies between 0.133 and 0.154. These results underscore the intricate interplay between stochastic fluctuations and fractional dynamics in shaping the system's rhythmic behavior.

The PDF of the amplitude is depicted in Fig. 15 for various values of the second fractional-order parameter λ_2 , with the noise intensity fixed at $D = 0.1$. As shown, the PDFs corresponding to $\lambda_2 = 1.5$ and $\lambda_2 = 1.95$ exhibit unimodal profiles, whereas the case $\lambda_2 = 1.83$ reveals a bimodal distribution. These observations suggest that variations in λ_2 can induce a stochastic phenomenological bifurcation, highlighting its crucial role in shaping the system's dynamic behavior.

C. Effect of restitution coefficient r

Equation (24) suggests that the number of real roots varies between one and three as the parameters change, since the amplitude A_m cannot be negative. Consequently, the probability distribution for the amplitude will either be unimodal or bimodal, indicating the occurrence of a type of stochastic bifurcation. By determining the roots of Eq. (24), we present the corresponding bifurcation diagram of system (1) in the (r, D) space in Fig. 16. In Fig. 16(a), we observe that the

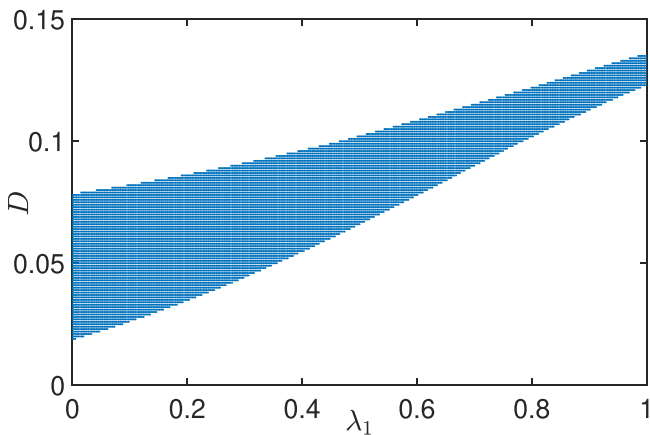


FIG. 12. Bifurcation diagram in the (λ_1, D) parameter space, based on Eq. (24), for the fixed set ($\alpha = 1.85$, $\beta = 0.242$) and natural frequency $\omega_0 = 1.0$. The remaining parameters are $\mu = 0.01$, $\lambda_2 = 1.4$, $a_1 = 0.015$, $a_2 = 0.01$, $r = 0.95$, and $c_1 = c_2 = 1.0$.

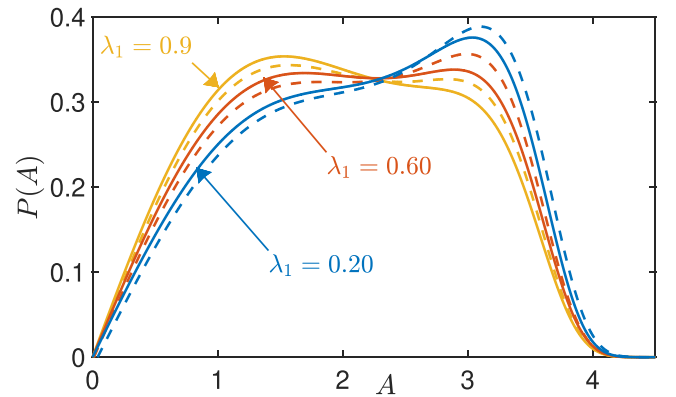


FIG. 13. Analytical (solid lines) and Monte Carlo simulation (dashed lines) results for the stationary PDF as a function of amplitude A , for varying values of the fractional order λ_1 . The parameters are fixed at ($\alpha = 1.85$, $\beta = 0.242$) and $\mu = 0.01$, with $\omega_0 = 1.0$, $\lambda_2 = 1.4$, $a_1 = 0.015$, $a_2 = 0.01$, $r = 0.95$, $c_1 = c_2 = 1.0$, and $D = 0.1$.

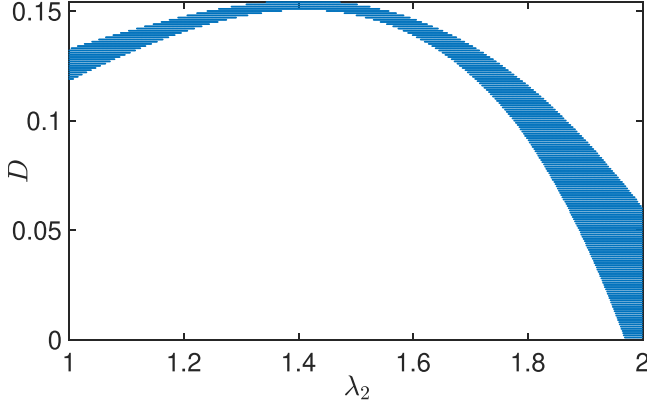


FIG. 14. Bifurcation diagram in the (λ_2, D) parameter plane based on Eq. (24), for the set $(\alpha = 1.85, \beta = 0.242)$, with the natural frequency $\omega_0 = 1.0$. The remaining parameters are $\mu = 0.01$, $\lambda_1 = 0.7$, $a_1 = 0.01$, $a_2 = 0.02$, $r = 0.95$, and $c_1 = c_2 = 1.0$.

stationary amplitude distribution is bimodal in the blue region. This diagram further implies that when the noise intensity is below a certain threshold, an increase in the parameter r results in two types of transitions: from monorhythmic to birhythmic behavior and from birhythmic to monorhythmic structures. Figure 16(b) illustrates the influence of the modulated nonlinear parameter μ on the birhythmicity region, revealing that an increase in μ significantly expands the area of birhythmicity.

The bifurcation analysis highlights the pivotal role of the restitution coefficient r in shaping the dynamics of stochastic system (1). In this section, r is treated as a bifurcation parameter to examine its influence on the behavior of the stochastic VI system. Figure 17 presents both the approximate stationary solution and the Monte Carlo simulation results for the stationary PDF, with the noise intensity fixed at $D = 0.05$. When $r = 0.92$, the stationary distribution remains unimodal in both analytical and numerical cases. However, as r increases to 0.96 and 0.98, the distribution undergoes a qualitative change, transitioning from unimodal to bimodal, and then reverting

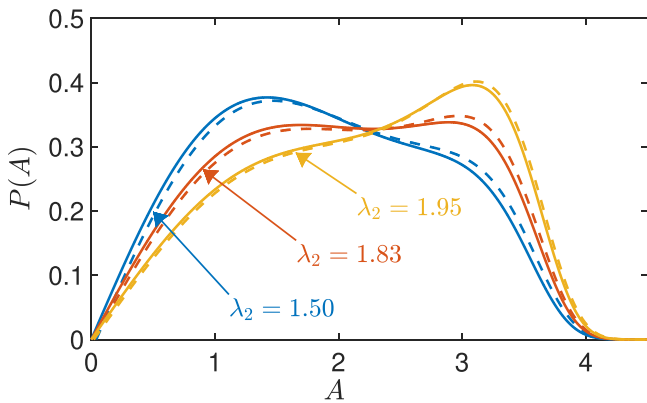


FIG. 15. Analytical (solid lines) and Monte Carlo simulation (dashed lines) results for the stationary PDF as a function of amplitude A , for different values of the fractional order λ_2 . The parameters are set as $(\alpha = 1.85, \beta = 0.242)$ with $\mu = 0.01$, $\omega_0 = 1.0$, $\lambda_1 = 0.7$, $a_1 = 0.01$, $a_2 = 0.02$, $r = 0.95$, $c_1 = c_2 = 1.0$, and $D = 0.1$.

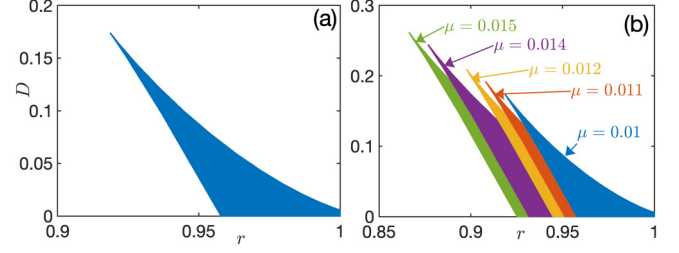


FIG. 16. (a) Bifurcation diagram in the (r, D) parameter plane derived from Eq. (24) for the fixed set $(\alpha = 1.85, \beta = 0.242)$ with $\mu = 0.01$. (b) Influence of the nonlinear parameter μ on the birhythmicity region (shaded area) in the (r, D) plane. The remaining parameters are $\omega_0 = 1.0$, $\lambda_1 = 0.4$, $\lambda_2 = 1.6$, $a_1 = a_2 = 0.01$, and $c_1 = c_2 = 1.0$.

to a monorhythmic form. This sequence of changes clearly signals the occurrence of a stochastic P bifurcation, induced by the variation of the restitution coefficient.

To further validate these findings, the influence of the restitution coefficient r on the shape of the joint PDF is illustrated in Fig. 18, along with the corresponding density plots. As r increases, the joint PDF undergoes a clear transformation, changing from a single peak at $r = 0.92$ to a triple-peaked structure at $r = 0.98$, reflecting the emergence of complex stochastic dynamics driven by variations in impact restitution.

VI. CONCLUSION

In this paper, we explored the stochastic bifurcations of a nonlinear BVI oscillator incorporating two types of fractional derivative elements and subjected to Gaussian white noise excitation. By employing a nonsmooth transformation along with the stochastic averaging method, we derived approximate analytical expressions for both the stationary PDF of the amplitude and the joint PDF of displacement and velocity. To validate these analytical results, we conducted numerical simulations based on the Monte Carlo method, yielding excellent agreement between theory and computation. Our analysis shows that the probability distributions are highly

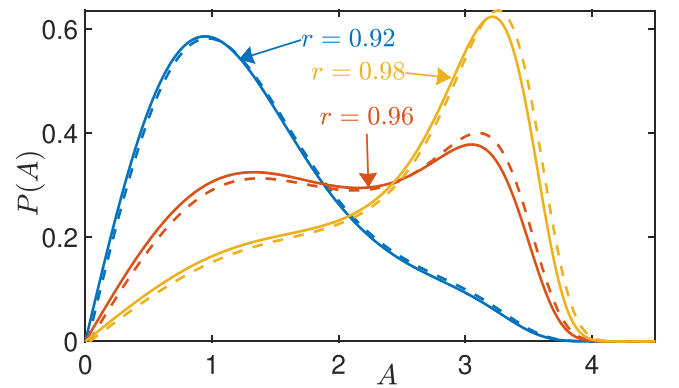


FIG. 17. Analytical (solid lines) and Monte Carlo simulation (dashed lines) results for the PDF as a function of amplitude A , for various values of the restitution coefficient r . The parameters are set to $(\alpha = 1.85, \beta = 0.242)$ with $\mu = 0.01$, $\omega_0 = 1.0$, $\lambda_1 = 0.4$, $\lambda_2 = 1.6$, $a_1 = a_2 = 0.01$, $c_1 = c_2 = 1.0$, and $D = 0.05$.

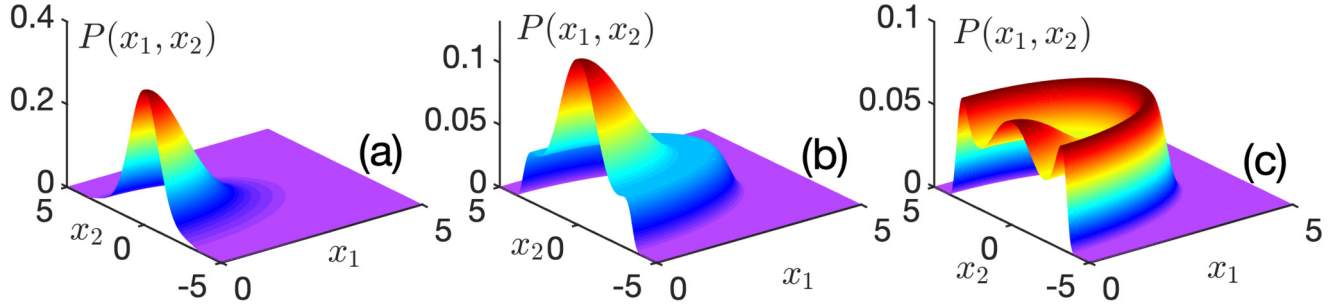


FIG. 18. Influence of the restitution coefficient r on the joint PDFs for the parameter set $(\alpha = 1.85, \beta = 0.242)$ with $\mu = 0.01$: (a) $r = 0.92$, (b) $r = 0.96$, and (c) $r = 0.98$. The remaining parameters are fixed as $\omega_0 = 1.0$, $\lambda_1 = 0.4$, $\lambda_2 = 1.6$, $a_1 = a_2 = 0.01$, $c_1 = c_2 = 1.0$, and $D = 0.05$.

sensitive to variations in the fractional coefficients, fractional-order parameters, and the restitution parameter. Specifically, increasing the coefficients a_1 and a_2 causes a transition from monorhythmic to birhythmic behavior, and then back to monorhythmicity in the amplitude PDFs, as evidenced by a shift from single- to triple-peak and back to single-peak structures in the three-dimensional joint PDF, characterizing a stochastic P bifurcation. Similar trends are noted for the fractional orders λ_1 and λ_2 , which also govern the transition between rhythmic regimes in the amplitude distributions. Furthermore, the evolution of the pseudopotential and energy barriers concerning these bifurcation parameters supports the conclusion that fractional orders play a dominant role in controlling the onset and nature of bifurcations. These findings suggest that with appropriate tuning of system parameters, a broad spectrum of oscillatory dynamics can be engineered in systems with fractional damping and nonlinear effects. These models highlight the relevance of such systems as powerful frameworks for analyzing the dynamics of complex biological phenomena and for designing innovative biomedical devices.

ACKNOWLEDGMENTS

C.B.T. thanks the Kavli Institute for Theoretical Physics (KITP), University of California, Santa Barbara, where this work was supported in part by the National Science Foundation, Grant No. NSF PHY-2309135.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: KEY STEPS TO CALCULATE EXPRESSIONS (18)–(20) USING EQS. (15) AND (17)

The full calculation of Eqs. (18)–(20) is based on the determination of the following expressions:

$$\langle M_{11} \rangle_\phi = \left\langle \frac{a_1 A \sin \phi}{\omega_0 \Gamma(1-\lambda_1)} \operatorname{sgn}(A \cos \phi) \frac{d}{dt} \int_0^t \frac{|\cos(\phi - \omega_0 \tau)|}{\tau^{\lambda_1}} d\tau \right\rangle_\phi. \quad (\text{A1})$$

Then, the cosine function in Eq. (A1) can be replaced by the following Fourier series:

$$|\cos \phi| = \frac{2}{\pi} + \sum_{n=1}^{\infty} B_n \cos(2n\phi), \quad (\text{A2})$$

where

$$B_n = \frac{4}{\pi} \frac{(-1)^n}{1-4n^2}. \quad (\text{A3})$$

According to Eqs. (A2) and (A3), and Refs. [42,43],

$$\begin{aligned} \langle M_{11} \rangle_\phi &= -\frac{32a_1 A}{\pi^2 \omega_0} \sin \frac{\lambda_1 \pi}{2} \sum_{n=1}^{\infty} \frac{n^2 (2n\omega_0)^{\lambda_1-1}}{(1-4n^2)^2} \\ &\approx -\frac{32a_1 A}{\pi^2 \omega_0} \sin \frac{\lambda_1 \pi}{2} \sum_{n=1}^{15} \frac{n^2 (2n\omega_0)^{\lambda_1-1}}{(1-4n^2)^2}. \end{aligned} \quad (\text{A4})$$

Following the same procedure, $\langle M_{12} \rangle_\phi$ can be obtained as follows:

$$\begin{aligned} \langle M_{12} \rangle_\phi &= -\frac{64a_2 A}{\pi^2 \omega_0} \cos \frac{(\lambda_2 - 1)\pi}{2} \sum_{n=1}^{\infty} \frac{n^3 (2n\omega_0)^{\lambda_2-2}}{(1-4n^2)^2} \\ &\approx -\frac{64a_2 A}{\pi^2 \omega_0} \cos \frac{(\lambda_2 - 1)\pi}{2} \sum_{n=1}^{15} \frac{n^3 (2n\omega_0)^{\lambda_2-2}}{(1-4n^2)^2}. \end{aligned} \quad (\text{A5})$$

Equations (A6) and (A7) can be obtained through mathematical calculation:

$$\begin{aligned} \langle M_{13} \rangle_\phi &= \langle -A \sin^2 \phi [-\mu(c_1 - c_2 A^2 \cos^2 \phi + \alpha A^4 \cos^4 \phi - \beta A^6 \cos^6 \phi) + (1-r)] - A \omega_0 \sin \phi |\delta(A \cos \phi)| \rangle_\phi \\ &= -\mu \left(-\frac{c_1}{2} A + \frac{c_2}{8} A^3 - \frac{\alpha}{16} A^5 + \frac{5\beta}{128} A^7 \right) - \frac{\omega_0 (1-r) A^2}{2\pi} \int_0^{2\pi} \sin^2 \phi |\sin \phi| \delta(A \cos \phi) d\phi \end{aligned}$$

$$\begin{aligned}
&= -\mu \left(-\frac{c_1}{2}A + \frac{c_2}{8}A^3 - \frac{\alpha}{16}A^5 + \frac{5\beta}{128}A^7 \right) - \frac{\omega_0(1-r)A^2}{2\pi} \int_0^{2\pi} \frac{\sin^2 \phi |\sin \phi|}{A} \delta(\cos \phi) d\phi \\
&= -\mu \left(-\frac{c_1}{2}A + \frac{c_2}{8}A^3 - \frac{\alpha}{16}A^5 + \frac{5\beta}{128}A^7 \right) - \frac{(1-r)\omega_0}{\pi} A,
\end{aligned} \tag{A6}$$

$$\begin{aligned}
\left\langle D \frac{\partial G_{11}}{\partial A} G_{11} + D \frac{\partial G_{11}}{\partial \theta} G_{21} \right\rangle_\phi &= \left\langle D \left[-\frac{\cos \phi}{\omega_0} \text{sgn}(A \cos \phi) + \frac{A \sin^2 \phi}{\omega_0} \delta(\cos \phi) \right] \left[-\frac{\cos \phi}{A \omega_0} \text{sgn}(A \cos \phi) \right] \right\rangle_\phi \\
&= \frac{D}{2\pi} \int_0^{2\pi} \frac{\cos^2 \phi}{A \omega_0^2} \text{sgn}^2(A \cos \phi) d\phi = \frac{D}{2A \omega_0^2}.
\end{aligned} \tag{A7}$$

-
- [1] C. J. Budd and P. T. Piiroinen, Corner bifurcations in non-smoothly forced impact oscillators, *Physica D* **220**, 127 (2006).
- [2] M. di Bernardo, A. Nordmark, and G. Olivar, Discontinuity-induced bifurcations of equilibria in piecewise-smooth and impacting dynamical systems, *Physica D* **237**, 119 (2008).
- [3] J. de Weger, D. Binks, J. Molenaar, and W. van de Water, Generic behavior of grazing impact oscillators, *Phys. Rev. Lett.* **76**, 3951 (1996).
- [4] W. Chin, E. Ott, H. E. Nusse, and C. Grebogi, Grazing bifurcations in impact oscillators, *Phys. Rev. E* **50**, 4427 (1994).
- [5] M. di Bernardo, P. Kowalczyk, and A. Nordmark, Bifurcations of dynamical systems with sliding: Derivation of normal-form mappings, *Physica D* **170**, 175 (2002).
- [6] M. di Bernardo, P. Kowalczyk, and A. Nordmark, Sliding bifurcations: A novel mechanism for the sudden onset of chaos in dry friction oscillators, *Int. J. Bifurcation Chaos* **13**, 2935 (2003).
- [7] D. Wagg and S. Bishop, Chatter, sticking and chaotic impacting motion in a two-degree-of-freedom impact oscillator, *Int. J. Bifurcation Chaos* **11**, 57 (2001).
- [8] J. Q. Feng, W. Xu, and Y. J. Niu, Chattering bifurcations in a Duffing unilateral vibro-impact system, *Acta Phys. Sin.* **59**, 157 (2010).
- [9] F. T. Arecchi, R. Badii, and A. Politi, Generalized multistability and noise-induced jumps in a nonlinear dynamical system, *Phys. Rev. A* **32**, 402 (1985).
- [10] S. Kim, S. H. Park, and C. S. Ryn, Noise-enhanced multistability in coupled oscillator systems, *Phys. Rev. Lett.* **78**, 1616 (1997).
- [11] H. G. Davies, Random vibrations of a beam impacting stops, *J. Sound Vib.* **68**, 479 (1980).
- [12] M. F. Dimentberg and D. V. Lourtchenko, Random vibrations with impacts: A review, *Nonlinear Dyn.* **36**, 229 (2004).
- [13] J. Feng and W. Xu, Analysis of bifurcation for nonlinear stochastic non-smooth vibro impact systems via top Lyapunov exponent, *Appl. Math. Comput.* **213**, 577 (2009).
- [14] J. Feng, W. Xu, and R. Wang, Stochastic response of vibro impact Duffing oscillator excited by additive Gaussian noise, *J. Sound Vib.* **309**, 730 (2008).
- [15] J. Q. Feng, W. Xu, H. W. Rong, and R. Wang, Stochastic response of Duffing-Van der Pol vibro impact system under additive and multiplicative random excitation, *Int. J. Nonlinear Mech.* **44**, 51 (2009).
- [16] H. Rong, X. Wang, W. Xu, and T. Feng, Resonant response of a nonlinear vibro impact system to combined deterministic harmonic and random excitations, *Int. J. Nonlinear Mech.* **45**, 474 (2010).
- [17] H. T. Zhu, Stochastic response of vibro-impact Duffing oscillator under eternal and parametric Gaussian white noises, *J. Sound Vib.* **333**, 954 (2014).
- [18] Y. Yang, W. Xu, Y. Sun, and Y. Xiao, Stochastic bifurcations in the nonlinear vibroimpact system with fractional derivative under random excitation, *Commun. Nonlinear Sci. Numer. Simul.* **42**, 62 (2017).
- [19] Y. Yang, Y. Sun, and W. Xu, Stochastic bifurcation analysis of a friction-damped system with impact and fractional derivative damping, *Nonlinear Dyn.* **105**, 3131 (2021).
- [20] J. Wang, Z. Yang, W. Sun, and J. Zhang, Response analysis of the vibro-impact system under fractional-order joint random excitation, *J. Low Freq. Noise Vib. Act. Control* **43**, 1065 (2024).
- [21] Y.-G. Yang, Y.-H. Sun, and W. Xu, Stochastic bifurcations of a fractional-order vibro-impact system driven by additive and multiplicative Gaussian white noises, *Complexity* **2019**, 6737139 (2019).
- [22] M. Coccolo, J. M. Seoane, and M. A. F. Sanjuan, Fractional damping induces resonant behavior in the Duffing oscillator, *Commun. Nonlinear Sci. Numer. Simul.* **133**, 107965 (2024).
- [23] M. Qu, Q. Yang, S. Wu, W. Ding, J. Li, and G. Li, Analysis of super-harmonic resonance and periodic motion transition of fractional nonlinear vibration isolation system, *J. Low Freq. Noise Vib. Act. Control* **42**, 771 (2023).
- [24] P. Snyder, A. Joshi, and J. D. Serna, Modeling a nanocantilever based biosensor using a stochastically perturbed harmonic oscillator, *Int. J. Nanosci.* **13**, 1450011 (2014).
- [25] F. Gao, X. Feng, and Y. Zheng, Photoacoustic elastic oscillation and characterization, *Opt. Express* **23**, 20617 (2015).
- [26] M. Vanleene, and S. J. Shefelbine, Therapeutic impact of low amplitude high frequency whole body vibrations on the osteogenesis imperfecta mouse bone, *Bone* **53**, 507 (2013).
- [27] C. A. K. Kwuimy and C. Nataraj, Recurrence and joint recurrence analysis of multiple attractors energy harvesting system, *Structural Nonlinear Dynamics and Diagnosis* (Springer, Cham, 2015), pp. 97–123.
- [28] R. Yamapi and G. Filatrella, Noise effects on a birhythmic Josephson junction coupled to a resonator, *Phys. Rev. E* **89**, 052905 (2014).
- [29] B. A. Guimfack, R. Mbakob Yonkeu, C. B. Tabi, and T. C. Kofané, On stochastic response of fractional-order generalized birhythmic Van der Pol oscillator subjected to delayed feedback displacement and Gaussian white noise excitation, *Chaos Solitons Fract.* **157**, 111936 (2022).

- [30] Q. Guo, Z. Sun, and W. Xu, Bifurcations in a fractional birhythmic biological system with time delay, *Commun. Nonlinear Sci. Numer. Simul.* **72**, 318 (2019).
- [31] R. M. Yonkeu, R. Yamapi, and G. Filatrella, Time, delay feedbacks enhanced bifurcation in the birhythmic fractional self-sustained system subjected to correlated noise, *Int. J. Nonlinear Mech.* **154**, 104429 (2023).
- [32] V. Zhuravlev, A method for analyzing vibration-impact systems by means of special functions, *Mech. Solids* **11**, 23 (1976).
- [33] A. Ivanov, Impact oscillations: Linear theory of stability and bifurcations, *J. Sound Vib.* **178**, 361 (1994).
- [34] J. Qian and L. Chen, Random vibration of SDOF vibroimpact oscillators with restitution factor related to velocity under wide-band noise excitations, *Mech. Syst. Signal Process.* **147**, 107082 (2021).
- [35] D. Liu, M. Li, and J. Li, Probabilistic response and analysis for a vibro-impact system driven by real noise, *Nonlinear Dyn.* **91**, 1261 (2018).
- [36] P. Kumar, S. Narayanan, and S. Gupta, Bifurcation analysis of a stochastically excited vibro-impact Duffing-Van der Pol oscillator with bilateral rigid barriers, *Int. J. Mech. Sci.* **127**, 103 (2017).
- [37] P. Kumar, S. Narayanan, and S. Gupta, Stochastic bifurcations in a vibro-impact Duffing-Van der Pol oscillator, *Nonlinear Dyn.* **85**, 439 (2016).
- [38] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides* (Kluwer Academic Publishers, Berlin, 1988).
- [39] R. Ibrahim, Recent advances in vibro-impact dynamics and collision of ocean vessels, *J. Sound Vib.* **333**, 5900 (2014).
- [40] V. Pilipchuk, Non-smooth spatio-temporal coordinates in non-linear dynamics, [arXiv:1101.4597](https://arxiv.org/abs/1101.4597).
- [41] Z. L. Huang and X. L. Jin, Response and stability of a SDOF strongly nonlinear stochastic system with light damping modeled by a fractional derivative, *J. Sound Vib.* **319**, 1121 (2009).
- [42] Y. Yang, W. Xu, and G. Yang, Bifurcation analysis of a noisy vibro-impact oscillator with two kinds of fractional derivative elements, *Chaos* **28**, 043106 (2018).
- [43] D. Yurchenko, A. Burlon, M. Di Paola, G. Failla, and A. Pirrotta, Approximate analytical mean-square response of an impacting stochastic system oscillator with fractional damping, *ASCE-ASME J. Risk Uncert. Eng. Syst. Pt. B Mech. Eng.* **3**, 30903 (2017).