

Superdiffusion in stochastic processes with long-range memory

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We investigate non-Markovian stochastic processes with long-range temporal memory described by a generalized Langevin (Mori-Zwanzig) equation with a power-law memory kernel. Special attention is paid to the critical regime where the system parameters lie on the boundary separating stable and unstable dynamics, leading to diffusionlike behavior with memory. We derive analytical expressions for the asymptotic behavior of the process and show that, under appropriate conditions on the memory kernel, the system exhibits superdiffusive scaling. In particular, we establish an explicit relation between the exponent of the power-law memory kernel and the growth rate of the variance. Analytical predictions are supported by numerical simulations, confirming the emergence of superdiffusive regimes in systems with long-range memory.

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I. INTRODUCTION

Stochastic integrodifferential equations have become a central tool for describing a broad class of diffusive processes in systems with nonlocal interactions, memory effects, or spatial heterogeneities. Such equations naturally arise in physics, biology, chemistry, ecology, and complex systems science, where the dynamics of a state variable at a given point depends not only on its immediate neighborhood but also on contributions from a distributed spatial region or from its past temporal states. This nonlocality leads to a wide spectrum of observable phenomena, including anomalous diffusion, long-range correlations, pattern formation, and multiscale relaxation dynamics [1–3].

For a long-time, stochastic phenomena were described by differential equations containing local memory effects, predicting the future of the dynamic system based only on its most recent state, i.e., using differential equations of the Markov type. Markov processes are the simplest and most widely used models for describing random systems. The two Gaussian Markov processes, the Ornstein-Uhlenbeck process, Ref. [4], and its particular type—Brownian motion have been extensively used in various applications, Refs. [5–7]. Their dynamics can be described by the classical Langevin equation,

$$\dot{X}(t) + \nu X(t) = F(t). \quad (1)$$

Here the derivative of the stochastic process $\dot{X}(t)$ is affected by the random force $F(t)$ with the known properties (for example, the white noise), and depends on the current value of the process $X(t)$ with the coefficient ν . So, the parameter ν characterizes the effect of the local (Markovian) memory in the process.

However, since the 1960s, such stochastic phenomena have been studied while taking into account the global effects of memory processes, which has led to the emergence of more complex integrodifferential equations containing integrals over the past dynamics of the system, for example, the generalized Langevin equation (GLE) for the dynamical variable $X(t)$,

$$\ddot{X}(t) + \int_0^t \mu(t - \tau) \dot{X}(\tau) d\tau + f(X) = F(t). \quad (2)$$

Here $F(t)$ is a time-dependent random force and $f(X)$ is a coordinate-dependent deterministic force. The integral kernel $\mu(t)$ is the so-called memory function. When the term $f(X) = 0$, Eq. (2) is transformed into the Mori-Zwanzig equation [8,9] with respect to the variable $\dot{X}(t)$. It has numerous applications, a fairly complete list of which can be found in Ref. [1].

The study of various models for describing stochastic phenomena has shown that the parameter domain of the memory function $\mu(t)$ is, as a rule, a set of regions with stable and unstable behavior, separated by some boundary lines (see Fig. 1). Some such cases have been considered in Refs. [10,11], where different forms of the memory functions, such as the retarded delta-shaped, stepwise, and exponential forms were considered. As one example, a solution to the equation

$$dX(t) = -\nu X(t)dt - \int_0^\infty \mu(t')X(t - t')dt'dt + \sigma dF(t) \quad (3)$$

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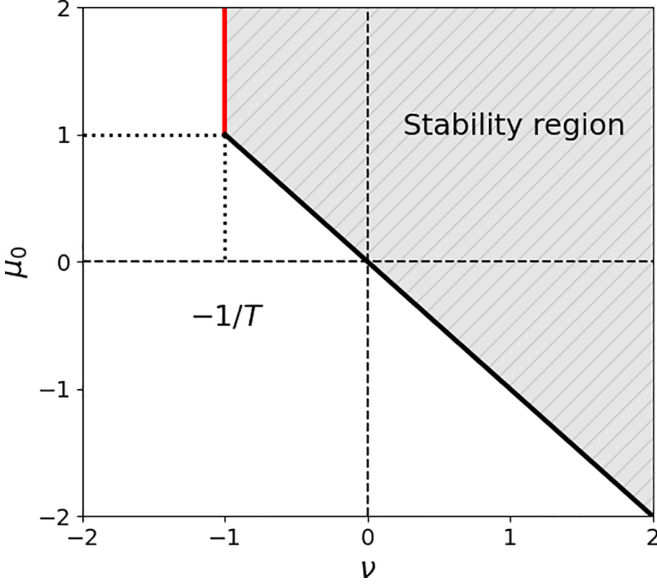


FIG. 1. The diagram (ν, μ_0) shows regions of stability (gray dashed area) and instability of the process (3) with the exponential memory function $\mu(t) = (\mu_0/T^2) \exp(-t/T)$ and the memory depth $T = 1$. At the borderline $\nu T + \mu_0 = 0$, $\mu_0 < 1$ (solid black line), the random process is the diffusion with memory. On the oscillatory borderline $\nu = -1/T$, $\mu_0 > 1$ (vertical red solid line), the process goes asymptotically into an oscillatory mode.

was considered with an exponential memory function, $\mu(t) = (\mu_0/T^2) \exp(-t/T)$, and the memory depth T . It was shown that there are two types of borderlines in the plane of parameters ν and μ_0 , diffusive and oscillatory ones. In the vicinity of the borderline,

$$\nu T + \mu_0 = 0, \quad \mu_0 < 1, \quad (4)$$

when approaching the instability region, the random process represents the diffusion with memory. In the vicinity of the oscillatory borderline,

$$\nu = -\frac{1}{T}, \quad \mu_0 > 1, \quad (5)$$

the process goes asymptotically into an oscillatory mode with some given frequency (see Fig. 1). Our attention is focused on the solution at the diffusion boundary, similar to that described by Eq. (4), but more precisely defined by Eqs. (25) and (12) with positive α .

In the present work we consider the Mori-Zwanzig equation (3) with the parameters lying precisely on the diffusive boundary line, leading to a Wiener-like process (neither stationary nor exponentially growing), but with long-range memory.

It is shown that the stochastic process on the diffusive boundary line can exhibit both normal and superdiffusive behavior. Our aim is to develop a simple, efficient model to describe the variety of the systems (mathematical, physical, economic, etc.) which can reveal anomalous diffusion. Analytical relations between the power-law memory function $\mu(t)$ of the process and its superdiffusion rate are obtained. The analytical results are confirmed by the numerical simulations.

The structure of the paper is as follows. In Sec. II, a master equation for the stochastic process with the long-range memory is considered. The solution of this equation in the long time limit is investigated. An analytical relation between the power-law memory function $\mu(t)$ and the superdiffusive variance $V(t)$ is obtained. In Sec. III, the numerical simulations to confirm the analytical results are performed. Several examples of the process with the different power exponents of the superdiffusion are presented. Sections IV and V contain conclusions and perspectives for further research.

II. SUPERDIFFUSION

The main goal of the paper is to investigate the statistical properties of the homogeneous stochastic process $X(t)$ described by the GLE on the diffusive stability boundary. The superdiffusive behavior of the process is assumed to be provided by the memory kernel $\mu(t)$ only, so it is not necessary to keep the external field term $f(X)$ in the driving equation (2). Consequently, the order of the equation can be reduced, by redefining $\dot{X}(t) \rightarrow X(t)$. This reduced form of the Langevin equation is named the Mori-Zwanzig equation. Another simplification consists in the fact that the noise term $F(t)$ is taken in its simplest form, namely, the δ -correlated white noise (its integral is the standard Wiener process).

Taking these points into account, we consider the random process $X(t)$, governed by the master equation of the form

$$dX(t) = - \int_0^\infty \mu(t') X(t-t') dt' + \sigma dW(t), \quad (6)$$

with the initial condition

$$X(t < 0) = 0. \quad (7)$$

Here $W(t)$ is the Wiener process and $\mu(t)$ is the memory function. The function $\mu(t)$, which describes the nonlocal memory, can also include the local memory δ -functional term, $\nu \delta(t)$.

We restrict our consideration to the diffusive solutions where the process $X(t)$ is neither stationary nor exponentially divergent and has an asymptotic martingalelike property. A martingale is a stochastic process in which the conditional expectation of the next value of X , independently of the past history, is equal to its last known value [12]. Just as martingales play an important role in the consideration of stochastic processes in the absence of memory, we see below that non-Markov processes with a given memory can also be described within the framework of a similar concept of a martingale that takes place in a system with memory.

This property can be formulated in terms of homogeneity of the process. More precisely, for large times $t \rightarrow \infty$, the properties of the process must be invariant with respect to a constant displacement,

$$X(t) \rightarrow X(t) + X_0. \quad (8)$$

Substituting this relation into Eq. (6), we get for $t \rightarrow \infty$ the simple condition, imposed on the memory function,

$$\int_0^\infty \mu(t) dt = 0. \quad (9)$$

This relation can be considered as a diffusive condition, separating stable and unstable solutions to Eq. (6). Equality (9) imposes a restriction on the constant contained in the function $\mu(t)$, like that, e.g., in Eq. (4).

Note that for local memory, Eq. (9) leads to the condition $\nu = 0$ discussed above. Thus, here we consider a Wiener-type diffusion process, but containing nonlocal memory $\mu(t)$. In a sense, this process has an extended martingale property.

A. Solution to the master equation

Equation (6) with initial condition (7) can be solved using the Laplace transform, as it was done, for example, in Ref. [13]. As a result, we have an expression for $X(t)$ in the form of the Itô integral:

$$X(t) = \sigma \int_0^t h(t - \tau) dW(\tau), \quad (10)$$

where the function $h(t)$ satisfies the following equation and initial condition,

$$\frac{dh(t)}{dt} = - \int_0^t \mu(t') h(t - t') dt', \quad h(0) = 1. \quad (11)$$

It is easy to prove that expression (10) is indeed a solution to Eq. (6) by direct substitution, taking into account Eq. (11).

The function $h(t)$ is, in some sense, the “personal ID” of the process and defines all its statistical properties. It plays a role similar to that of Green’s function in the theory of differential equations: it is a solution of the homogeneous equation (6) in the absence of the driving force $W(t)$, and the integral (10) is a solution of the inhomogeneous equation.

Moreover, the function $h(t)$ has an important physical meaning. Indeed, to verify that a process is superdiffusion, it is not necessary to find a solution to Eq. (10) and then examine the moments of the random value $X(t)$. Instead, it is sufficient to study the behavior of the function $h(t)$.

So, using the relation (11), we can now study the properties of the process $X(t)$ with the definite memory function $\mu(t)$.

B. Power-law memory

It is natural to assume that the superdiffusive behavior of the process can be achieved by the heavy-tailed memory function $\mu(t)$ at $t \rightarrow \infty$,

$$\mu(t) \rightarrow \alpha t^{-\beta}, \quad \beta > 0. \quad (12)$$

The exact form of $\mu(t)$ at small t can be chosen arbitrary, but fulfilling the diffusive property (9).

Consider different cases for the parameter β . In the case $\beta < 1$, condition (9) cannot be fulfilled because the integral of the memory function $\mu(t)$ diverges. The function $h(t)$ of the process at $\beta < 1$ is exponentially growing with time [as it follows from Eq. (11)], and the process $X(t)$ is nonstationary. So, to achieve the superdiffusion, we restrict our analysis to the case of $\beta > 1$ only.

For the exponent $\beta > 1$, the integral in Eq. (9) converges, and this condition can be fulfilled by the appropriate choice of $\mu(t)$ at small t .

1. Laplace transform $\hat{\mu}(z)$ of the memory function at small z

Consider the Laplace transform $\hat{\mu}(z)$ of the memory function $\mu(t)$:

$$\hat{\mu}(z) = \mathcal{L}[\mu(t)] = \int_0^\infty \mu(t) \exp(-zt) dt. \quad (13)$$

To obtain $\hat{\mu}(z)$ at $z \rightarrow 0$ we use the asymptotic Abelian and Tauberian theorems, which state that the function $f(t)$ with the power asymptotics $f(t) \rightarrow t^q$ as $t \rightarrow \infty$ has the Laplace transform asymptotics as $z \rightarrow 0$,

$$\lim_{z \rightarrow 0} \mathcal{L}[f(t)] = \frac{\Gamma(q+1)}{z^{q+1}}, \quad q > -1. \quad (14)$$

Because of the restriction on q in this formula we cannot use it directly for the power memory (12) with $\beta > 1$. Let β belong to the interval between two integers: $n < \beta < n+1$. Then consider the n th derivative of $\hat{\mu}(z)$ and use the property of the Laplace transform:

$$\frac{d^n \hat{\mu}(z)}{dz^n} = (-1)^n \mathcal{L}[t^n \mu(t)]. \quad (15)$$

Here the asymptotic power exponent $q = n - \beta$ of the function $t^n \mu(t)$ fulfills the condition $q > -1$. Then, from Eqs. (14) and (12), we obtain the $z \rightarrow 0$ approximation,

$$\lim_{z \rightarrow 0} \frac{d^n \hat{\mu}(z)}{dz^n} = (-1)^n \alpha \frac{\Gamma(n+1-\beta)}{z^{n+1-\beta}}, \quad (16)$$

and after n integrations, we get

$$\lim_{z \rightarrow 0} \hat{\mu}(z) = \alpha \Gamma(1-\beta) z^{\beta-1} + P_{n-1}(z), \quad (17)$$

where $P_{n-1}(z)$ is a polynomial of the degree $n-1$, determined by the exact form of $\mu(t)$ at small t . Since we are interested here in the $z \rightarrow 0$ region, the main contribution into sum (17) for $\hat{\mu}(z)$ gives the nonzero term with the smallest power.

To find the asymptotics of the function $h(t)$ as $t \rightarrow \infty$, rewrite Eq. (11) in the Laplace space,

$$\hat{h}(z)(\hat{\mu}(z) + z) = 1. \quad (18)$$

From this relation, one can conclude that $\hat{h}(z)$ at small z has a power form, $\hat{h}(z) \propto z^{-m}$, where m is the minimal exponent among the nonzero terms in $P_{n-1}(z)$, z , and $z^{\beta-1}$. According to relation (14), this corresponds to the power asymptotics of $h(t)$ at $t \rightarrow \infty$,

$$h(t) \propto t^{m-1}. \quad (19)$$

As it is shown below, in the case of superdiffusion $m = \beta - 1$, and we have

$$h(t) \propto t^{\beta-2}. \quad (20)$$

2. Variance

To characterize diffusive properties of the process, we consider the time-dependent *two-point increment variance*,

$$V_T(t) = \mathbb{E}[(X(T+t) - X(T))^2], \quad (21)$$

where symbol $\mathbb{E}$ denotes the expected value of a random variable, which can be calculated by an ensemble averaging over realizations of a random process.

Taking into account Eq. (10) for $X(t)$ and the properties of the Wiener process $W(t)$, one can express the variance via the function $h(t)$,

$$V_T(t) = \sigma^2 \int_0^T [h(\tau + t) - h(\tau)]^2 d\tau + \sigma^2 \int_0^t h^2(\tau) d\tau. \quad (22)$$

Consider this expression for the case of the power-law form (19) of $h(t)$ at $t \rightarrow \infty$. It can be seen that, if $m - 1 \geq 1/2$, then the first term depends essentially on T and grows infinitely at $T \rightarrow \infty$. This means that the two-point variance of the process is not independent of time T , passing from the starting point. In this case, the two-point variance depends explicitly on the reference time T , and the process loses asymptotic homogeneity with respect to the variance. Such regimes lie outside the scope of the present study, where we focus on processes with asymptotically T -independent variance.

This requirement leads to the condition $m < 3/2$, which, using $m = \beta - 1$, corresponds to the parameter range $\beta < 5/2$. In what follows, we restrict our analysis to the interval $2 < \beta < 5/2$, while the behavior near the boundary $\beta \rightarrow 5/2$ is discussed only qualitatively.

So, in order to obtain the process with the approximately time-invariant variance $V_T(t) \rightarrow V(t)$, we should consider the case $m < 3/2$, where m is the exponent of the main term in $\hat{\mu}(z) + z$, which means $\beta < 5/2$.

Note that in the case $\beta < 2$ we have the negative exponent in Eq. (19), so the function $h(t)$ is decreasing. It can be seen from Eq. (22) that, depending on the value β and parameters of the function $\mu(t)$ at small t , the process can exhibit either classical Brownian variance, $V_T(t) \propto t$, or subdiffusion.

Finally, we conclude that for the superdiffusion the exponent of the memory function (12) must belong to the interval

$$2 < \beta < \frac{5}{2}. \quad (23)$$

3. The asymptotics of function $h(t)$

Let us consider the case when the parameter n in Eq. (17) is 2, and β belongs to the interval (23). Here we have

$$\hat{\mu}(z)|_{z \rightarrow 0} + z = C_0 + (C_1 + 1)z + \alpha\Gamma(1 - \beta)z^{\beta-1}. \quad (24)$$

Since we need the main term in the right-hand side of Eq. (24) to be related to the memory function, the first two terms must be zero: $C_0 = 0$ and $C_1 = -1$. In terms of the memory function, this corresponds to two integral conditions:

$$\int_0^\infty \mu(t) dt = 0, \quad (25a)$$

$$\int_0^\infty t\mu(t) dt = 1. \quad (25b)$$

The first condition here coincides with the condition (9) for the diffusive regime. The second one has a more complicated meaning. If Eq. (25a) is fulfilled, while Eq. (25b) is not, then the term z in the left-hand side of Eq. (24) is not compensated, so $\hat{h}(z) \propto 1/z$ and the function $h(t)$ tends to some constant. As it can be seen from Eq. (22), this means that the variance $V_T(t)$ is linear in t . Therefore, the violation of Eq. (25b) results in normal diffusion.

So, if Eqs. (25) are fulfilled due to definite behavior of the memory function $\mu(t)$ at small t , then, from Eq. (18), we obtain $\hat{h}(z)$ at $z \rightarrow 0$,

$$\hat{h}(z) \approx \frac{z^{1-\beta}}{\alpha\Gamma(1 - \beta)}. \quad (26)$$

Finally, taking into account Eq. (14), we have

$$h(t) \approx \frac{(\beta - 1) \sin \pi \beta}{\pi \alpha} t^{\beta-2}, \quad t \rightarrow \infty. \quad (27)$$

It should be noted that the considered case of a power-law memory function on the diffusive boundary, Eq. (9), is qualitatively different from the previously considered, in Ref. [11], case of an exponential memory function. It can be shown that the fulfillment of conditions (25) for an exponential memory (see the caption to Fig. 1) leads to the expression $\mu(z) = -z/(1 + Tz)$, which corresponds to the point of tangency of the two boundary lines in Fig. 1. In this case, unlike the case of a power-law memory, in the expression $\mu(z) + z$ the term at $z \rightarrow 0$ is quadratic, $\mu(z) + z \rightarrow Tz^2$, so $h(t) \rightarrow t/T$ and the variance $V_0(t)$ depends significantly on t_0 , according to Eq. (22). Therefore, the exponential memory does not result in superdiffusion.

C. Position-independent variance $V(t)$

As we have shown above, despite the memory of the process being heavy-tailed, the diffusive process is asymptotically homogeneous at $2 < \beta < 5/2$; i.e., the two point variance $V_T(t)$ is independent of T at $t \rightarrow \infty$. So, we consider below simple one-point variance,

$$V(t) = \mathbb{E}[X^2(t)]. \quad (28)$$

Taking into account the property of the Wiener process, $\mathbb{E}[dW(t_1)dW(t_2)] = \delta(t_1 - t_2)dt_1dt_2$, we can obtain from Eq. (10) the variance in terms of the function $h(t)$,

$$V(t) = \sigma^2 \int_0^t h^2(\tau) d\tau. \quad (29)$$

For the power-law memory (12) with $2 < \beta < 5/2$ under conditions (25), using Eq. (27), we obtain finally the variance,

$$V(t) = ct^{2\beta-3}, \quad c = \sigma^2 \frac{(\beta - 1)^2 \sin^2 \pi \beta}{\pi^2 \alpha^2 (2\beta - 3)}. \quad (30)$$

It follows that the memory function $\mu(t)$ with the exponent $\beta \rightarrow 2$ results in the ordinary linear diffusion, and higher β values give the superdiffusive variance with the parameters (30).

D. Discussion of the result

The main result (30) connects the asymptotics of variance $V(t)$ of the superdiffusive process at $t \rightarrow \infty$ with the parameters α and β of the memory function (12) under the condition that its local part is chosen to fulfill the integral restrictions (25).

So, changing the memory exponent β from 2 up to $5/2$ results in changing through all possible superdiffusion rates, from 1 (the Brownian motion) up to 2 (the ballistic regime).

As an example, let us consider a specific form of the memory function $\mu(t)$ where the power-law part is taken in combination with local memory δ term and zero gap on the time interval $t < T$,

$$\mu(t) = \nu\delta(t) + \begin{cases} 0, & t < T, \\ \alpha t^{-\beta}, & t \geq T. \end{cases} \quad (31)$$

Here, the zero gap of the function $\mu(t)$ at $t < T$ is introduced to prevent the divergence of the power function at $t \rightarrow 0$.

Consider the near-Brownian case, $\beta \rightarrow 2$, more precisely. From Eq. (30), one can see that in this case the exponent $2\beta - 3$ of the variance tends to the Brownian value 1, but the coefficient c depends on both parameters β and α . If the memory function $\mu(t)$ has the power form at $t > T$, then to fulfill the second condition in Eq. (25) these parameters have to be connected as

$$\alpha = (\beta - 2)T^{\beta-2}. \quad (32)$$

Then the coefficient c in Eq. (30) tends to $c = \sigma^2 T^{2(2-\beta)}$, and the variance is

$$V(t) = \frac{\sigma^2}{T^{2(\beta-2)}} t; \quad (33)$$

so, at the point $\beta = 2$ we have the transition to the ordinary Brownian motion.

It is also interesting that the memory coefficient α is squared in Eq. (30). This means that the negative nonlocal memory in $\mu(t)$ must also give the superdiffusive process, as well as the positive one. But, as it is known from the previous investigations, such a type of memory corresponds to an oscillatory border and may cause the stochastic resonance, which now will have a superdiffusive amplitude. This case is out of the present consideration.

Here we were interested in the limit $t \rightarrow \infty$ mainly, but one also can obtain a simple asymptotics at $t \rightarrow 0$. From Eq. (11), we have approximately $h(t) \approx 1$, and then

$$V(t) \approx \sigma^2 t, \quad (34)$$

so at small times the process reveals the normal diffusion properties.

III. NUMERICAL SIMULATIONS

To confirm the obtained analytical results, the numerical simulations were performed. For this, Eq. (6) was solved with the initial condition (7). Thus, samples of the random process $X(t)$ were constructed with the time discreteness $\Delta t = 0.01$ and the process duration $t_{\text{proc}} = 100$.

The memory function $\mu(t)$ was taken in the form of Eq. (31). For the fixed parameter β and chosen $T = 1$, the coefficients ν and α are defined by the conditions (25). The noise amplitude in Eq. (6) is chosen as $\sigma = 1$.

In Fig. 2(a), a sample realization of the random process $X(t)$ for the exponent $\beta = 2.1$ is presented. One can see that for $\beta \approx 2$ the process shape is visually similar to the Brownian one.

Figure 2(b) represents in double logarithmic scale the function $h(t)$ of the process (green dots), constructed by the

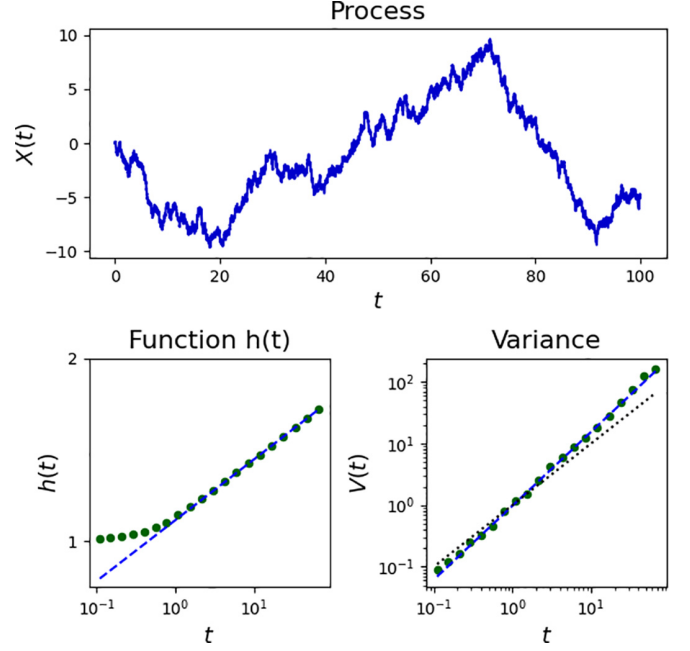


FIG. 2. (a) A sample of the process $X(t)$, generated using Eq. (6) with $\beta = 2.1$. (b) The corresponding $h(t)$ function (dots) with its asymptotics (blue dashed line). (c) The superdiffusive variance $V(t)$ (solid circles) with asymptotics (straight blue dashed line) in comparison with the normal diffusion (black dotted line).

numerical solving of Eq. (11). The blue dashed line indicates the analytical asymptotics of $h(t)$ at $t \rightarrow \infty$, Eq. (27).

In Fig. 2(c), the process variance $V(t)$ in the log-log scale is presented. To perform an ensemble averaging, a set of $N = 100$ samples of the process was generated, and the one-point variance (28) was calculated (green dots). The analytical asymptotics (30) is plotted by the blue dashed line. The black dotted line corresponds to the Brownian variance $V(t) \propto t$. One can see that, despite the Brownian-like visible shape of the process $X(t)$, it reveals the clearly predicted superdiffusion property of $V(t)$.

In Fig. 3 the analogous plots are presented for the memory exponent, close to the ballistic regime of the process, $\beta = 2.4$. Other parameters are the same, as in Fig. 2. Here the change of the process type is visually distinguishable. At this β , the superdiffusive variance $V(t)$ are in good agreement with the analytical asymptotics.

The implementation of the numerical algorithms together with example datasets has been deposited in Zenodo [14].

IV. CONCLUSION

In this paper, the stochastic process with the nonlocal memory is investigated. The generalized Langevin equation is used to model such a process and to obtain its main statistical characteristics, including its two-point increment variance $V_T(t)$ and one-point variance $V(t)$.

It is shown that these functions are defined by the memory function $\mu(t)$ of the process. Some forms of the memory function can cause the diffusive (semimartingale) behavior of the system. Although the process $X(t)$ itself is nonstationary in these conditions, some of its characteristics, for example,

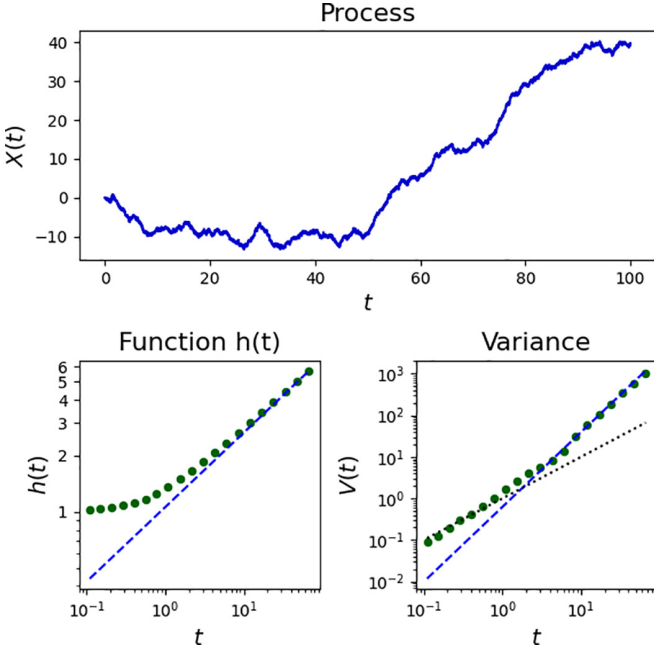


FIG. 3. (a) A sample of the process $X(t)$, generated using Eq. (6) with $\beta = 2.4$. (b) The corresponding $h(t)$ function (dots) with its asymptotics (blue dashed line). (c) The superdiffusive variance $V(t)$ (solid circles) with asymptotics (straight blue dashed line) in comparison with the normal diffusion (black dotted line).

$V_T(t)$, are asymptotically stationary and independent of the elapsed time T .

It is shown that, while reaching the conditions for the diffusive regime, the process can reveal superdiffusive properties; namely, its variance takes the power form $V(t) \rightarrow t^\gamma$ with $\gamma > 1$, which means that the process variance grows faster with time than in the classical Brownian motion.

The relation between the superdiffusion rate and the exponent of the power-law memory function is obtained. The results are illustrated by numerical simulations. By setting the power form of the memory function, the process samples $X(t)$ were generated with a fixed time step. The variance $V(t)$ of the process sample was calculated using the ensemble averaging. The results indicate clear superdiffusive properties of the processes generated.

It is important to distinguish the present model from Lévy processes.

Although the variance exhibits superdiffusive scaling, the stochastic process remains Gaussian-distributed, since it is constructed as a linear functional of the Wiener process. Therefore, the anomalous behavior originates from long-range temporal correlations rather than heavy-tailed jump distributions. In this sense, the process is closer in spirit to fractional Brownian motion.

V. PERSPECTIVES

This work is the first step in studying diffusive solutions of stochastic Mori-Zwanzig integrodifferential equations. In this regard, we outline the range of tasks that require a priority study.

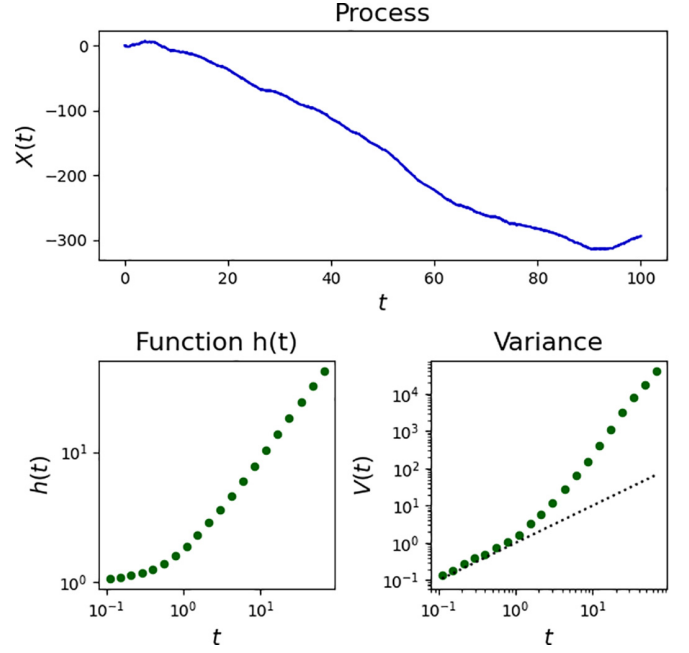


FIG. 4. (a) A sample of the process $X(t)$, generated using Eq. (6) with $\beta = 3.5$. (b) The corresponding $h(t)$ function. (c) The superballistic variance $V(t) \propto t^{2.5}$ (solid circles) in comparison with the normal diffusion (black dotted line).

In Ref. [11], it was discovered that there exist two types of fundamentally different borderlines, separating stable and unstable regions of system dynamics. On boundaries of the first type, corresponding to the positive nonlocal memory, a *diffusion with memory* takes place, and on the boundaries of the second type, corresponding to the negative memory, the phenomenon of *noise-induced resonance* occurs. In the present work we considered the diffusive boundary only, although the study of solutions on the second type of boundary should also be an important task.

It would be interesting to investigate which type of a process one can obtain in the case $\beta > 5/2$. Preliminary consideration suggests that such a memory may also result in anomalous diffusion, with the rate of variance faster than 2. As already noted in Sec. II B 2, for such a memory the process is not homogeneous relative to the two-point variance $V_T(t)$. However, the one-point variance $V(t)$ still can be an indicator of the superdiffusion in this case. In Fig. 4, an example of the stochastic process with the exponent $\beta = 3.5$ of the memory function is presented. Indeed, it can be seen that the variance $V(t)$ reveals the power-law asymptotics, evaluated as $V(t) \propto t^{2.5}$, and the process is superballistic.

We hope that the theory presented here can be applied to the description of different complex phenomena, in particular, financial [15,16] and meteorological ones, such as seasonal variability in prediction and/or 24-hour-ahead temperature forecasts [17,18].

The considered model describes stochastic dynamics with long-range memory kernels, which naturally arise in several classes of systems: particle motion in fluid media [19], transport in percolating clusters [20], anomalous relaxation, heat transport, and heat diffusion [2,3].

It is important to note that here the power-law memory $\mu(t)$ was analyzed only. The possibility of achieving superdiffusion using other forms of memory remains an open question.

DATA AVAILABILITY

The datasets and source code that support this article are available at Zenodo [14].

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