

Optimal finite-time thermodynamics of effective two-level systems

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The optimization of the conversion of thermal energy into work and the minimization of dissipation for nano- and mesoscopic systems is a complex challenge because of the important role fluctuations play on the dynamics of small systems. I generalize the work of Esposito *et al.* [*Europhys. Lett.* **89**, 20003 (2010)] to optimize at all driving speeds the control needed to extract the maximum amount of work from any effective two-level systems. These emerge when one coarse grains degrees of freedom, which is often unavoidable to obtain “real-world” two-level systems. In particular, I allow even for the system to have underlying quantum dynamics, as long as these allow for a coarse graining that leads to a Markovian master equation. I analyze the finite-time thermodynamics of these systems and find the thermodynamically optimal protocols, which depend on the size of the coarse graining needed to obtain a two-level system. Furthermore, I use these results to derive speed limits for any transformation performed on an effective two-level system.

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I. INTRODUCTION

The optimization of the work cost of physical tasks is a paradigmatic topic in thermodynamics, dating back to the very inception of the field [1]. Beyond its self-evident applicative aspects ranging from macroscopic to microscopic scales [1–4], this question also probes the fundamental thermodynamic cost of irreversibility [5–8]. The finite-time optimization of work has been extensively studied in systems close to equilibrium [8–19]. Yet, small systems—such as biological molecules or nanoscale devices—can easily be taken far from their equilibrium state, where they show large fluctuations in their behavior. In these regimes, fluctuation theorems [20,21] provide a very powerful machinery to study systems far from equilibrium in the stochastic setting. Still, many open questions remain, particularly when quantum dynamics cannot be neglected [7–9,22–28].

Two-level systems play a central role in the study of thermodynamic optimization in the mesoscopic regime as they reduce the problem to its most fundamental components [29–31]. However, most often the optimization problem remains still too complex to solve it in full generality. In this work, I adopt an optimal control framework to generalize the pioneering work of Ref. [32] to *effective* two-level systems, where each level may be arbitrarily degenerate. This

generalization allows for the implementation of this optimization procedure into a wide variety of nano- and mesoscopic two-level systems, for arbitrary driving speeds. In realistic settings, such “two-level” systems often arise by coarse graining multiple degenerate states into a single effective energy level. The resulting system has thermodynamic and dynamical properties that are significantly different from those of the microscopic case where there are no degeneracies. Therefore these differences must be taken into account when studying the out-of-equilibrium dynamics of these effective systems. A simple example of such a system is a quantum dot where the two logical states are defined via electronic occupation: the empty state is nondegenerate, while the occupied state exhibits twofold degeneracy due to the spin of the electron [33]. Furthermore, these different properties that effective two-level systems exhibit can also be desirable. For instance, they emerge naturally as the energy structure that maximizes the sensitivity of thermometers [34–36] and to approach reversibility in thermodynamic operations [17,37].

Here I consider scenarios where one can extract ($W < 0$) or inject ($W > 0$) some amount of work W from/into the system by changing the energy gap E between the two coarse-grained levels in some finite amount of time τ , while it interacts with a thermal reservoir. The amount of work can be split into a reversible and irreversible contribution $W = \Delta F_{\text{neq}} + W_{\text{diss}}$. ΔF_{neq} is the nonequilibrium free-energy difference, and depends exclusively on the initial and final conditions of the state and Hamiltonian. Whereas W_{diss} is the dissipated energy, which depends on the way the energy gap is changed and the state trajectory throughout the process. By the second law of thermodynamics, $W_{\text{diss}} \geq 0$, with equality in the reversible limit. For fixed boundary conditions, the task of minimizing the work cost is equivalent to minimizing the dissipated

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energy. In general, this is a hard objective because of the high degree of complexity in the dynamics of open quantum systems. Multiple approaches exist to obtain analytical results. For example, one can make assumptions on the driving speed of the Hamiltonian: either that the system does not have the time to evolve during the process [38–41], or that it remains close to its steady state [35,41–46]. Other approaches, such as optimal transport theory [18,47–51] and thermodynamic control [14,28,32,33,52] allow to also obtain analytical results by making different sets of assumptions. Here, similarly to Refs. [32,33], I restrict the system dynamics to a family of master equations, allowing to obtain exact results on the optimal driving of $E(t)$ without any assumption on the driving speed.

II. FRAMEWORK

I consider an open (finite-dimensional) quantum system in contact with a thermal bath, where its energetic structure is equivalent to that of a two-level system $\hat{H}(t) = E_e(t)\hat{\Pi}_e + E_g(t)\hat{\Pi}_g$, where $\hat{\Pi}_e$ is a rank n projector and $\hat{\Pi}_g$ is a rank m projector, with $\hat{\Pi}_g\hat{\Pi}_e = \hat{\Pi}_e\hat{\Pi}_g = 0$. These projectors define what I refer to as the *excited* sector $\mathcal{H}_e$ and the *ground* sector $\mathcal{H}_g$, so that I can decompose the Hilbert space of the system as a direct sum $\mathcal{H} = \mathcal{H}_g \oplus \mathcal{H}_e$. Defining the energy gap as $E(t) := E_e(t) - E_g(t)$ and denoting the state of the system at time t as $\hat{\rho}(t)$, I can express the expected work cost of a transformation as

$$W = \int_0^\tau dt \dot{E}(t) \text{Tr}[\hat{\Pi}_e \hat{\rho}(t)]. \quad (1)$$

I want to restrict attention to cases where each sector can be coarse grained into a single effective state, while preserving the Markovianity of the dynamics in the coarse-grained description. However, it is not guaranteed that a coarse-grained description of a system remains Markovian, even if I were to start from a fully stochastic and classical description (cf. Appendix). One needs that the dynamics across effective energy levels (i.e., the energy sectors) remain unaffected from the internal dynamics of the level. A natural scenario where this property can emerge would be when one encodes a quantum dot with the presence (or absence) of an electron. Then, the spin degeneracy provides an additional internal degree of freedom in the excited (or ground) sector [33,53,54]. Another relevant example would be encoding a two-level system in terms of photon number in a time-bin encoding; $\hat{\Pi}_e$ and $\hat{\Pi}_g$ become projectors over different Fock space sectors, which have internal degrees of freedom in the form of the photon polarization [55,56]. Essentially, I am requiring that the jumps across sectors do not distinguish the states within a sector. The detailed derivation and mathematical requirements for the system to satisfy this property are presented in Appendix, while here I keep a simplified description.

In the interest to keep the derivation as general as possible, I start from a generic open quantum-system description where the system is weakly coupled to the thermal environment. The state's evolution is then given by a Lindblad master

equation with one bath at inverse temperature β :¹

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)] + \sum_{\omega \in \Omega} f(-\omega) \mathcal{D}_\omega[\hat{\rho}(t)], \quad (2)$$

where $\Omega = \{-E(t), 0, E(t)\}$ are the system's energy gaps, $f(\omega) = (1 + e^{\beta\omega})^{-1}$ is the Fermi distribution, and the dissipator is given by

$$\mathcal{D}_\omega[\hat{\rho}] = \sum_{i,j} \hat{L}_i(\omega) \hat{\rho} \hat{L}_j(\omega)^\dagger - \frac{1}{2} \{ \hat{L}_j(\omega)^\dagger \hat{L}_i(\omega), \hat{\rho} \}. \quad (3)$$

For any $|1\rangle \in \mathcal{H}_e$ and $|0\rangle \in \mathcal{H}_g$, the stochastic transition rates between the states are given by

$$\Gamma_{1 \rightarrow 0} = \gamma_{1,0}(1 - f(E(t))), \quad \Gamma_{0 \rightarrow 1} = \gamma_{0,1}f(E(t)), \quad (4)$$

for $\gamma_{1,0} = \gamma_{0,1} = \sum_{i,j} \langle 1 | \hat{L}_i | 0 \rangle \langle 0 | \hat{L}_j^\dagger | 1 \rangle$.

At this point I can coarse grain the system by defining the probability of excitation to be $p(t) := \text{Tr}[\hat{\Pi}_e \hat{\rho}(t)]$, so that the ground-state probability is simply $1 - p(t)$. From Eq. (2) I can obtain the dynamical equation for $p(t)$. However, as I noted above, it will not always be Markovian. The minimal assumptions such that the dynamics remain Markovian are (cf. Appendix)

(i) No coherences between the two sectors

$$\hat{\rho}(t) = \hat{\rho}_g(t) \oplus \hat{\rho}_e(t).$$

(ii) Jump operators that act across sectors are such that the intersector rates become uniform:

$$\gamma := \gamma_{1,0} = \gamma_{0,1}, \quad \Gamma_{e \rightarrow g} := \Gamma_{1 \rightarrow 0}, \quad \Gamma_{g \rightarrow e} := \Gamma_{0 \rightarrow 1},$$

$\forall |1\rangle \in \mathcal{H}_e$ and $\forall |0\rangle \in \mathcal{H}_g$.

It is worth noting that these assumptions do not affect jump operator that can be written as a direct sum over the two sectors.

With these assumptions, I can write the Pauli master equation $\dot{p}(t) = \Gamma_{g \rightarrow e}(1 - p(t)) - \Gamma_{e \rightarrow g}p(t)$ [57]. By defining $n = \text{Tr}[\hat{\Pi}_e]$ and $m = \text{Tr}[\hat{\Pi}_g]$, I obtain

$$\dot{p}(t) = \gamma(nf(E(t)) - [m + (n - m)f(E(t))]p(t)). \quad (5)$$

The presence of the degeneracies modifies the thermal steady-state probability from $f(E)$ to

$$p_{\text{eq}} = \frac{1}{1 + r e^{\beta E}}, \quad (6)$$

where I defined the *degeneracy ratio* $r = \frac{m}{n}$. It is worth noting that this immediately shows how the presence of degeneracies changes nontrivially the thermodynamic aspects of the system when compared to a true two-level system ($n = m = 1$). In particular, Eq. (6) shows that the equilibrium properties are modified whenever $r \neq 1$. Despite that, I can still emulate the

¹Here I am not including chemical potentials in the description, because I am refraining from assigning a meaning of particle number to the energy levels. However, if I were to assign $\mathcal{H}_e$ and $\mathcal{H}_g$ to different Fock space sectors, then I can introduce a chemical potential μ to the thermal bath. By construction the Hamiltonian commutes with the particle-number operator, and therefore I can reabsorb μ in the energy gap $E(t)$ since there is only one bath present.

equilibrium properties of a true two-level system with energy gap ε with an effective two-level system by introducing an energy bias $E = \varepsilon - k_B T \ln r$. However, while the equilibrium properties can be emulated, the dynamics are fundamentally changed. Using that $\dot{p}_{\text{eq}} = 0$ I can rewrite Eq. (5) as

$$\dot{p}(t) = n\gamma[r + (1-r)f(E(t))](p_{\text{eq}} - p(t)). \quad (7)$$

In the case of a true two-level system I would have $\dot{p}(t) = \gamma(p_{\text{eq}} - p(t))$. When the degeneracy ratio is equal to 1 the only difference lies in a rescaling of the thermalization time scale γ . For $r \neq 1$, however, the result is a thermalization rate that is dependent on the energy gap. For systems where the energy gap is not driven, this is also equivalent to a simple time-scale rescaling. However, for driven systems this rescaling breaks down as it becomes time dependent and it changes the dynamics in a fundamental way—in particular for optimizations and, therefore, the derivation of fundamental bounds. For example, in the limit of $r \rightarrow 0$ I observe a critical slowdown of thermalization for large energy gaps.

In what follows, I will use time units such that $n\gamma = 1$ to lighten the notation. Then Eq. (5) becomes

$$\dot{p}(t) = f(E(t)) - [r + (1-r)f(E(t))]p(t). \quad (8)$$

The thermodynamic optimization of this type of systems has already been studied in the literature for specific degeneracy ratios. In particular, Ref. [32] covers the case $r = 1$ and Ref. [33] treats $r = \frac{1}{2}$.

III. THERMODYNAMIC OPTIMIZATION

The thermodynamic optimization of a protocol is the functional minimization of the work functional $W[p(t), E(t)] = \int_0^\tau dt \dot{E}(t)p(t)$ for some given (arbitrary) boundary conditions on both the probability and the energy [i.e., fixing the values of $p(0)$, $p(\tau)$, $E(0)$, and $E(\tau)$]. Via Eq. (8) I can express the energy $E(t)$ as a function of $p(t)$ and $\dot{p}(t)$, therefore I can express the optimization problem as a Lagrangian extremization. Following the same steps as in Refs. [32,33], I integrate the work by parts to avoid having to express $\dot{E}(t)$ in terms of $p(t)$ and $\dot{p}(t)$:

$$W = \Delta E - \int_0^\tau dt E(t)\dot{p}(t), \quad (9)$$

where $\Delta E = E(\tau)p(\tau) - E(0)p(0)$ and the second term (including the minus sign) corresponds to the heat. Since ΔE is set by the boundary conditions, minimizing the work cost is equivalent to minimizing the dissipated heat. Inverting Eq. (8) I find that I can express the energy gap as

$$E(t) = k_B T \ln \left[\frac{1 - p(t) - \dot{p}(t)}{\dot{p}(t) + rp(t)} \right]. \quad (10)$$

Therefore the Lagrangian to extremize is

$$\mathcal{L}[p(t), \dot{p}(t)] = \dot{p}(t) \ln \left[\frac{1 - p(t) - \dot{p}(t)}{\dot{p}(t) + rp(t)} \right]. \quad (11)$$

Since this Lagrangian is not explicitly time dependent, by Noether's theorem, its extremizer will conserve the quantity $K = \mathcal{L}[p, \dot{p}] - \dot{p} \frac{\partial \mathcal{L}[p, \dot{p}]}{\partial \dot{p}} = \dot{p}^2 \frac{1 - (1-r)p}{(1-p-\dot{p})(\dot{p}+rp)}$ along the whole

trajectory of $p(t)$ and $\dot{p}(t)$ [58]. By solving for $\dot{p}$ I obtain

$$\dot{p} = \frac{1}{2} \frac{K[1 - (1+r)p] \pm \sqrt{\Delta}}{1 - (1-r)p + K}, \quad (12)$$

with $\Delta = K^2[1 - (1-r)p]^2 + 4Krp(1-p)[1 - (1-r)p]$. The $\pm$ sign matches the sign of $\dot{p}$, which is determined by the boundary conditions: if $p(\tau) - p(0) > 0 (< 0)$ then I take the $+$ ($-$) sign. Interestingly, in Eq. (12) the time and probability are separable, therefore I can integrate to obtain

$$t = 2 \int_{p(0)}^{p(\tau)} dp \frac{1 - (1-r)p + K}{K[1 - (1+r)p] \pm \sqrt{\Delta}}. \quad (13)$$

By defining $F_K(p)$ as the integrand² of Eq. (13) I obtain a solution for the optimal trajectory in terms of an inverse function $p(t) = F_K^{-1}(t)$. With the boundary conditions, Eq. (13) sets the integration constant to $K = \kappa_\tau$, which is defined by the following equation

$$F_{\kappa_\tau}(p(\tau)) = \tau. \quad (14)$$

It is therefore a function of $p(0)$, $p(\tau)$, and τ only.

At this point, with the integration constant set by Eq. (14), the optimization problem is solved. It is worth noting that applications can be done efficiently as it requires solving three scalar equations numerically³ instead of a boundary-value problem. By combining Eqs. (10) and (12) I can obtain the optimal control of the energy gap as a function of the probability.

$$E_{\kappa_\tau}(p) = k_B T \ln \left[\frac{[1 - (1-r)p](\kappa_\tau + 2(1-p)) \mp \sqrt{\Delta}}{[1 - (1-r)p](\kappa_\tau + 2rp) \pm \sqrt{\Delta}} \right]. \quad (15)$$

Therefore the optimal driving as a function of time is given by $E_{\kappa_\tau}(F_{\kappa_\tau}^{-1}(t))$ and the minimal work cost can be expressed as function of the boundary conditions

$$W_{\min} = \Delta E - \int_{p(0)}^{p(\tau)} dp E_{\kappa_\tau}(p). \quad (16)$$

It is interesting to note that, in the above analysis, the only place where the boundary condition on the energies enters is in the work cost, while the bulk of the optimal protocol and heat exchanged are completely independent of it. Therefore the optimal protocol is continuous in its bulk ($0 < t < \tau$), but features quenches at its edges. In particular, at $t = 0$ the energy control jumps from $E(0)$ to $E_{\kappa_\tau}(p(0))$, and at $t = \tau$ it jumps from $E_{\kappa_\tau}(p(\tau))$ to $E(\tau)$. This remains true even when I assume the system to start and end at equilibrium: from Eq. (6) I have that $E = k_B T \ln \frac{1-p}{rp}$ at equilibrium, which does not match Eq. (15). While the system is being driven it is “lagging behind” its instantaneous equilibrium state, as it can be seen from its trajectory, which I obtain by inserting Eq. (8) into the

²Since it is defined up to a constant I set $F_K(p(0)) = 0$.

³Except for a few special values of r , Eq. (13) cannot be integrated analytically. Furthermore, even in those exceptional cases, Eq. (14) cannot be solved analytically.

definition of K :

$$p(t) = p_{\text{eq}}(t) \left[1 - A(t) \mp \sqrt{A(t)^2 + r\kappa_\tau \frac{1 + e^{\beta E(t)}}{r + e^{-\beta E(t)}}} \right], \quad (17)$$

where I made the time dependence of the driven equilibrium explicit and defined $A(t) = \frac{\kappa_\tau}{2} \frac{1-r}{r + e^{-\beta E(t)}}$.⁴ Therefore, when making an equilibrium-to-equilibrium transformation, the optimal protocol will overshoot the final value of the energy to then quench back to it, regardless of the $\pm$ sign.

Before moving on to the study of limiting cases and a few numerical examples, note that not all scenarios impose boundary conditions on the probabilities. It is often the case that one wishes to drive only the system's energy to some specific value $E(\tau)$ and let it relax to equilibrium after the driving. This allows for one more layer of optimization and removes the constraint imposed by Eq. (14). Here, the constant κ_τ is determined by minimizing directly Eq. (16):

$$\kappa_\tau = \arg \min_K F_K^{-1}(\tau) E(\tau) - \int_{p(0)}^{F_K^{-1}(\tau)} dp E_K(p), \quad (18)$$

where I neglected terms that do not depend on K . In this case I observe a different behavior of the optimal driving of the energy to the one described in the paragraph above: since the system does not have to reach its boundary condition on the probability at $t = \tau$, it can lag behind while the energy reaches its final value. Therefore, the optimal protocol will not be overshooting the final energy, but rather quench to it from an intermediary value.

IV. SPEED LIMIT

In the limit of τ going to 0, it is clear from Eq. (13) that it is not possible to realize the transformation of a fixed $p(\tau)$. Therefore there must exist a minimum time $\tau_{\min} > 0$ for the system to reach $p(\tau)$ from a given $p(0) \neq p(\tau)$. It is determined by $\tau_{\min} = \min_K F_K(p(\tau))$. It is straightforward to verify that Eq. (13) is monotonically decreasing as a function of K . Therefore I can obtain the minimum amount of time required to realize the transformation by taking the limit $K \rightarrow \infty$ and solving the integral

$$\tau_{\min} = 2 \int_{p(0)}^{p(\tau)} dp \frac{1}{1 - (1+r)p \pm [1 - (1-r)p]}, \quad (19)$$

which leads to

$$\tau_{\min} = \begin{cases} \frac{1}{n\gamma} \ln \frac{1-p(0)}{1-p(\tau)}, & \dot{p} > 0, \\ \frac{1}{m\gamma} \ln \frac{p(0)}{p(\tau)}, & \dot{p} < 0, \end{cases} \quad (20)$$

where I explicitly reintroduced the time units $n\gamma$. It is worth noting that these speed limits depend only on the degeneracy of the sector being filled and not on the degeneracy of the sector being emptied: when I am cooling⁵ the system ($\dot{p} < 0$)

the dependence on n disappears and, conversely, the dependence on m drops when I am heating the system ($\dot{p} > 0$). Interestingly, in the specific case of cooling, I can see that as I approach the ground state $\tau_{\min}$ scales as the corresponding energy gap $\Delta E/m\gamma$ (such that the target probability is thermal). Therefore, it seems that by changing the degeneracy I can reduce at will the necessary time to perfectly cool the system to its ground state—seemingly breaking the third law. However, this speed limit is valid within the Lindblad regime, which stops being valid on short time scales. To go beyond it one should use techniques that yield universally valid speed limits, such as Ref. [59]. In the limit where I achieve the fastest possible protocol ($K \rightarrow \infty$) I can also compute the optimal probability trajectory.⁶ I obtain that the fastest I can cool/heat the system is

$$p(t) = \begin{cases} p(0)e^{-n\gamma t} + 1 - e^{-n\gamma t}, & \dot{p} > 0, \\ p(0)e^{-m\gamma t}, & \dot{p} < 0. \end{cases} \quad (21)$$

However, it is worth noting that this limit is only realized for a diverging control of the energy $E_\infty(p) = \mp\infty$ and therefore also has a diverging thermodynamic cost.

By letting go of the boundary condition on the state I can use the fact that at short times the state barely evolves $p(t) = p(0) + \dot{p}(0)t + \mathcal{O}(t^2)$. Since the curve of the optimal energy control is as a function of the probability that I obtain for fast protocol, the optimal curve is almost a constant (regardless of the boundary conditions). Therefore, at first order, the optimal protocol will consist of a large quench at the start, then keeping the energy at an intermediary value from $t = 0^+$ to $t = \tau^-$, and finally a quench to its final value. I thus reobtain, in this specific context, the “bang-bang” protocols of Refs. [40,41].

V. VERY DEGENERATE SYSTEMS

In the limit where the degeneracy of one sector is much larger than the other (e.g., Refs. [17,34–36]), I can obtain some simplifications of the results presented above, which lead to interesting dynamics. The evolution of the system towards equilibrium is given by Eq. (7):

$$\dot{p}(t) = \begin{cases} n\gamma f(E(t))(p_{\text{eq}} - p(t)), & r \ll 1, \\ m\gamma [1 - f(E(t))](p_{\text{eq}} - p(t)), & r \gg 1. \end{cases} \quad (22)$$

I can notice that these two regimes behave in the same way under the transformation $n \leftrightarrow m$, $E \rightarrow -E$, and $p \rightarrow 1 - p$. Therefore I will focus on the case $r \ll 1$.

It is worth noting that in this limit the time scale of relaxation is given by $n\gamma f(E)$. Since $n \gg 1$, it is generally large. However it can be exponentially suppressed by taking a gap of the order $-k_B T \ln(r)$, leading to a critical slowdown of the dynamics in this regime.

By expanding Eq. (13) to the leading order in r I find

$$F_K(p(\tau)) = \tau_{\min} \pm \frac{1}{K}(p(\tau) - p(0)) + \mathcal{O}(r), \quad (23)$$

⁴In the limit of $r \rightarrow 1$ I recover the result of Ref. [32].

⁵Since both reducing the temperature and increasing the energy gap have the same effect on a system, I refer to any operation that decreases $p(t)$ as cooling (and conversely for heating).

⁶Note that the fastest protocol is also optimal in energy consumption for fixed τ , as it is the only protocol that realizes the transformation in that given amount of time τ .

where I returned to units such that $n\gamma = 1$. Note that $\tau_{\min}$ contains a $\mathcal{O}(r^{-1})$ term for $\dot{p} < 0$. Therefore, for fixed boundary conditions on the probability I can solve Eq. (14):

$$\kappa_\tau = \frac{|p(\tau) - p(0)|}{\tau - \tau_{\min}}. \quad (24)$$

Consider the scaling of κ_τ with respect to r . Despite the fact that, in principle, τ is a free parameter, as I will argue in the next section, I should pick it to be a multiple of $\tau_{\min}$. Therefore, I find that for heating protocols ($\dot{p} > 0$) $\kappa_\tau = \mathcal{O}(1)$, while for cooling protocols ($\dot{p} < 0$) I have $\kappa_\tau = \mathcal{O}(r)$. Now, I can invert Eq. (23) to obtain the optimal protocol as an explicit function of time,

$$p(t) = \begin{cases} 1 - \kappa_\tau w\left(\frac{1-p(0)}{\kappa_\tau} e^{\frac{1-p(0)}{\kappa_\tau} - t}\right), & \dot{p} > 0, \\ \frac{\kappa_\tau}{r} w\left(\frac{rp(0)}{\kappa_\tau} e^{\frac{rp(0)}{\kappa_\tau} - rt}\right), & \dot{p} < 0, \end{cases} \quad (25)$$

with $w(z)$ denoting the principal solution of $z = xe^x$ units such that $n\gamma = 1$.

Finally, I note that at leading order the optimal energy curve is given by

$$E_{\kappa_\tau}(p) = \begin{cases} k_B T \ln\left[\frac{1-p}{\kappa_\tau}\right] + \mathcal{O}(r), & \dot{p} > 0, \\ k_B T \ln\left[\frac{rp(0)}{\kappa_\tau}\right] + \mathcal{O}(r), & \dot{p} < 0. \end{cases} \quad (26)$$

This equation can be easily integrated analytically to obtain the corresponding minimal amount of work with Eq. (16). I observe that for cooling protocols the work cost diverges as $\sim \ln(r)$.

VI. BIT ERASURE

I will now showcase the difference in the optimal protocols for different values of r with the paradigmatic task of erasing a bit of information in finite time. This allows to benchmark the quantitative and qualitative differences between the effective two-level systems with different degeneracies. Bit erasure is defined by the boundary conditions $p(0) = \frac{1}{2}$ and $p(\tau) \approx 0$ ⁷ [61]. For the system to be at equilibrium at the start and the end of the protocol I have $E(0) = -k_B T \ln r$ and $E(\tau) = k_B T \ln \frac{1-p(\tau)}{rp(\tau)}$. In order to have a fair comparison between systems with different degeneracy ratios, I would want all the relaxation time scales to be identical. However, Eq. (7) shows that the relaxation time scale depends on the energy gap (except for $r = 1$). Therefore, I cannot ensure that the relaxation time scale remains the same between systems with different degeneracies while the gap is being driven. It is worth noting that, in general, relaxation time scales depend on the energetic structure of the system, and therefore one has to take extra care when considering a driven system. In this case the quantity that can be compared between systems is the speed limit to realize the task, given by Eq. (20): $\tau_{\min} = r^{-1} \ln \frac{p(0)}{p(\tau)}$. Indeed, I cannot compare the finite-time erasure of systems with different degeneracy ratios if, for some values of r , the task cannot even be realized in that duration of time.

⁷We require only approximate erasure, as perfect erasure in finite time is impossible [60]. In this case, I can see that Eq. (20) diverges for $p(\tau) = 0$.

Therefore, here I compare the finite-time erasure protocols and their energetic cost while varying r and keeping $\tau/\tau_{\min}$ fixed.

I showcase the obtained optimal protocols $E(t)$ for finite-time erasure $p(\tau) = 10^{-5}$ for a relatively slowly driven protocol ($\tau = 5\tau_{\min}$, which is slow, but not slow enough to fully be in the quasistatic regime) in Fig. 1(a) and their corresponding state trajectories $p(t)$ in Fig. 1(b). These correspond to the cooling branch of Eqs. (15) and (17). Similarly to the findings of Refs. [32,33] I observe quenches at the start and end of the protocol for the optimal energy control. First note that the shift in the extremal points of the energy protocols (and thus in the bulk) is due to the fact that I am imposing boundary conditions on the probabilities. From Eq. (6), I can see that, to keep the same equilibrium probability, a change in r needs to be compensated by a shift in the energy gap. Beyond this shift, I can notice that the curves' shape remains relatively unchanged as r changes, in particular for small values of r . This can be seen also from Eq. (26), where the dependence on r results only in a constant shift at leading order. However, even after correcting for the energy shift, the differences in instantaneous thermalization rates lead to different probability trajectories. Ultimately, the differences in the probabilities result in a dependence of the minimal energetic cost of erasure on the degeneracy ratio, as I show in Fig. 1(c). Each line corresponds to a different ratio of protocol time to the speed limit. I can note the cost of erasure decreases for smaller degeneracy ratios, i.e., increasing the degeneracy of the excited sector brings the minimal cost of erasure at a minimum for a fixed protocol time. And, as expected, the cost is lower for slower protocols and is ultimately bounded by the Landauer bound $W \geq k_B T \ln 2$ [61] which I show with the dashed black line as it is achieved for $\tau \rightarrow \infty$.

VII. CONCLUSION

In this work, I investigated how internal degeneracies in the energy levels of a two-level system influence minimally dissipating thermodynamic operations. Extending the framework of Ref. [32], I considered two-level systems with internal degeneracies and analyzed the optimization of these *effective* two-level systems, which play a central role in multiple frameworks ranging from theoretical metrology [34–36,46] to experimental quantum technologies [3,8]. I derived closed-form analytical expressions for the minimally dissipating state trajectories and control protocols in terms of the degeneracy ratio between the two levels and provided explicit formulas for the corresponding work cost. The performance of these optimal protocols was further illustrated and benchmarked in the context of bit erasure. Starting from a microscopic description, I demonstrated that this optimization applies to any system that can be coarse grained into a (Markovian) effective two-level description, while still allowing for internal quantum dynamics within each effective level. I further showed that degeneracies alter the dynamical and equilibrium properties of the system in nontrivial ways. In particular, the degeneracy of the level being populated during a process imposes a fundamental bound on the achievable system evolution.

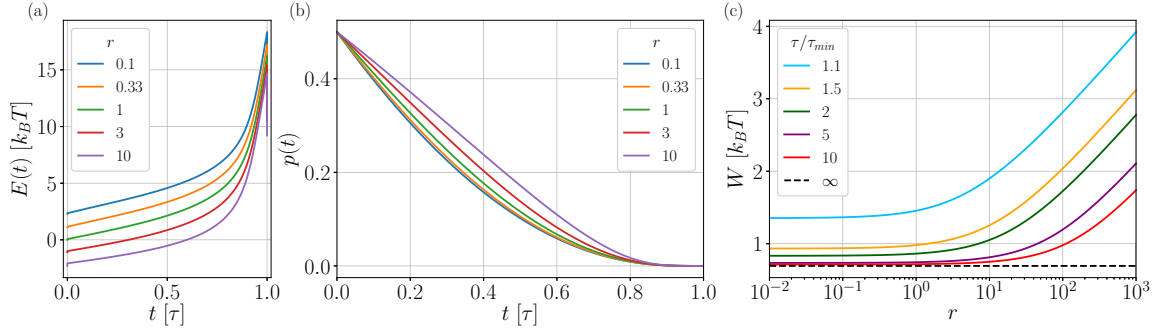


FIG. 1. Optimal finite-time bit-erasure protocols $p(\tau) = 10^{-5}$ for different values of $r = m/n$ (small r corresponding to a larger degeneracy in the excited sector and large r to a larger degeneracy in the ground sector). (a), (b) Optimal energy and probability trajectory for finite time erasure protocols $\tau = 5\tau_{\min}$ for different values of r . (c) Minimal finite-time erasure cost as a function of r for different protocol durations.

Looking ahead, these results provide a blueprint to implement minimally dissipating protocols in both classical and quantum two-level systems, enabling a wide range of applications. For example, theoretically, they can be used to design resource-optimal, ultraprecise thermometers. Experimentally, they enable thermodynamically optimal operations in mesoscopic systems such as quantum dots. Moreover, this analysis can be exploited to characterize the presence of degeneracies in the system and benchmark the degree of control over experimental platforms.

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DATA AVAILABILITY

The implementation of the optimization presented in the previous sections and the code used to obtain the results in this section are publicly available [62].

APPENDIX: COARSE GRAINING TO A PAULI MASTER EQUATION

I start from a Lindblad master equation with one bath

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)] + \sum_{\omega \in \Omega} f(-\omega)\mathcal{D}_{\omega}[\hat{\rho}(t)], \quad (\text{A1})$$

where $\Omega = \{-E(t), 0, E(t)\}$ are the system's energy gaps, $f(\omega) = (1 + e^{\beta\omega})^{-1}$ is the Fermi distribution, and the dissipator is given by

$$\mathcal{D}_{\omega}[\hat{\rho}] = \sum_{i,j} \hat{L}_i(\omega)\hat{\rho}\hat{L}_j(\omega)^{\dagger} - \frac{1}{2}\{\hat{L}_j(\omega)^{\dagger}\hat{L}_i(\omega), \hat{\rho}\}. \quad (\text{A2})$$

The operators $\hat{L}_i = \sum_{\omega \in \Omega} \hat{L}_i(\omega) = \sum_{\omega \in \Omega} \hat{L}_i(\omega)^{\dagger}$ are the jump operators, which originate from the description of the microscopic interaction between system and bath [57,63]. In particular, $\hat{L}_i$ corresponds precisely to the interaction term

of the Hamiltonian in the original unitary description that includes the bath. I can obtain from it the energy-dependent jump operators that are featured in the Lindblad master equation

$$\hat{L}_i(E(t)) = \hat{\Pi}_g \hat{L}_i \hat{\Pi}_e, \quad (\text{A3})$$

$$\hat{L}_i(-E(t)) = \hat{\Pi}_e \hat{L}_i \hat{\Pi}_g, \quad (\text{A4})$$

$$\hat{L}_i(0) = \hat{\Pi}_g \hat{L}_i \hat{\Pi}_g + \hat{\Pi}_e \hat{L}_i \hat{\Pi}_e. \quad (\text{A5})$$

For any $|1\rangle \in \mathcal{H}_e$ and $|0\rangle \in \mathcal{H}_g$, the stochastic transition rates between the states are given by

$$\Gamma_{1 \rightarrow 0} = \gamma_{1,0}(1 - f(E(t))), \quad (\text{A6})$$

$$\Gamma_{0 \rightarrow 1} = \gamma_{0,1}f(E(t)), \quad (\text{A7})$$

where

$$\gamma_{1,0} = \gamma_{0,1} = \sum_{i,j} \langle 1|\hat{L}_i|0\rangle \langle 0|\hat{L}_j^{\dagger}|1\rangle. \quad (\text{A8})$$

From this starting point, in order to reach a stochastic two-level effective description of the system that remains Markovian I need to make two main assumptions. First, I will assume that the two sectors are completely dephased, i.e., that the state contains no coherences between the ground and excited sectors

$$\hat{\rho}(t) = \hat{\rho}_g(t) \oplus \hat{\rho}_e(t). \quad (\text{A9})$$

Second, I assume that the resulting transition rates across a sector are uniform. That is, the probability of a jump happening from $\mathcal{H}_e$ to $\mathcal{H}_g$ is independent of which state within $\mathcal{H}_e$ the system is jumping from and of which state within $\mathcal{H}_g$ the system is jumping to:

$$\gamma = \gamma_{1,0} = \gamma_{0,1}, \quad \Gamma_{e \rightarrow g} = \Gamma_{1 \rightarrow 0}, \quad \Gamma_{g \rightarrow e} = \Gamma_{0 \rightarrow 1} \quad (\text{A10})$$

for all $|1\rangle \in \mathcal{H}_e$ and $|0\rangle \in \mathcal{H}_g$. This implies that the dynamics between the energy sectors do not distinguish the internal structure of the system within an energy sector. There are multiple ways in which this requirement can be satisfied by the jump operators. We list here a few examples: suppose I have a four-level system, with $\{|0a\rangle, |0b\rangle\}$ forming a basis of $\mathcal{H}_g$ and $\{|1a\rangle, |1b\rangle\}$ forming a basis of $\mathcal{H}_e$. Then the following collections of jump operators satisfy Eq. (A10):

(i) $\hat{L}$ acts identically on all of $\mathcal{H}_e$ and maps to a single state in $\mathcal{H}_g$

$$\hat{L} = \gamma |0a\rangle (\langle 1a| + \langle 1b|).$$

(ii) $\hat{L}$ acts identically on all of $\mathcal{H}_e$ and maps to multiple states in $\mathcal{H}_g$

$$\hat{L} = \gamma (|0a\rangle\langle 1a| + |0b\rangle\langle 1b|).$$

(iii) $\hat{L}_1$ and $\hat{L}_2$ map all of $\mathcal{H}_e$ onto a single state in $\mathcal{H}_g$

$$\hat{L}_1 = \gamma |0a\rangle\langle 1a|, \quad \hat{L}_2 = \gamma |0a\rangle\langle 1b|.$$

(iv) $\hat{L}_1$ and $\hat{L}_2$ map all of $\mathcal{H}_e$ onto multiple states in $\mathcal{H}_g$

$$\hat{L}_1 = \gamma |0b\rangle\langle 1a|, \quad \hat{L}_2 = \gamma |0a\rangle\langle 1b|.$$

Both these assumptions are needed to obtain a valid effective two-level description of the system. In principle, one could relax Eq. (A9) by claiming that the dynamics are such that any coherence is quickly suppressed by the environment. However, Eq. (A10) is necessary to preserve the Markovianity of the system. I can illustrate this point with a simple and fully

classical example: suppose that I have a Markovian stochastic system consisting of the states $|0\rangle$, $|1a\rangle$, and $|1b\rangle$. I want to coarse grain the levels $|1a\rangle$ and $|1b\rangle$ into a macrostate $|1\rangle$. If the rate $\Gamma_{0 \rightarrow 1a}$ is different from $\Gamma_{0 \rightarrow 1b}$ (or $\Gamma_{1a \rightarrow 0} \neq \Gamma_{1b \rightarrow 0}$) then the resulting effective description of the system cannot be Markovian. This is because the effective rate $\Gamma_{0 \rightarrow 1}$ (or $\Gamma_{1 \rightarrow 0}$) will depend on the internal population of the coarse-grained states, and therefore becomes time dependent. In light of this, I can fully justify the assumption brought by Eq. (A10) since I am interested only in cases where an effective Markovian description exists.

By defining $p(t) := \text{Tr}[\hat{\Pi}_e \hat{\rho}(t)]$, I can turn the Lindblad master equation into a Pauli master equation

$$\dot{p}(t) = \gamma(nf(E(t)) - [m + (n - m)f(E(t))]p(t)), \quad (\text{A11})$$

for $n = \text{Tr}[\hat{\Pi}_e]$ and $m = \text{Tr}[\hat{\Pi}_g]$. The steady state is given by

$$p_{\text{eq}} = \frac{1}{1 + re^{\beta E}}. \quad (\text{A12})$$

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