

Directed extended-range percolation

Wenbo Liu ¹, Yiwen Zeng ¹, Xueming Liu ^{1,*} and Ginestra Bianconi ^{2,†}¹*School of Artificial Intelligence and Automation, Institute of Medical Equipment Science and Engineering, and State Key Laboratory of Digital Manufacturing Equipments and Technology, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China*²*School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom*

(Received 23 May 2026; accepted 8 October 2026; published 26 October 2026)

For standard percolation, directionality is known to increase the complexity of percolation, requiring the distinct statistical treatment of in messages and out messages for determining the percolation order parameters. Here we show that when connectivity is ensured by paths of length $R \geq 2$, network directionality, impeding backtracking, can significantly reduce the complexity of percolation. To illustrate this finding, we introduce directed extended-range percolation (DERP), defined on directed networks with nonreciprocal edges and motivated by applications in quantum communication. In this framework, message transmission is enabled between trusted nodes separated by a directed path of length at most R . Using a message-passing approach, we show that, on locally treelike structures, directionality enables an exact determination of the percolation threshold and the anomalous critical indices with a number of distinct statistical messages smaller than the naive expectation, which would require twice as many messages as in the undirected version. This reduction arises because directionality suppresses backtracking paths, thereby considerably simplifying the resulting equations. On random directed networks we find that the critical behavior of DERP depends sensitively on degree correlations. These analytical predictions are corroborated by extensive Monte Carlo simulations, highlighting the profound impact of directionality and correlations on long-range connectivity in complex networks.

DOI: [10.1103/hs2n-6rsn](https://doi.org/10.1103/hs2n-6rsn)

Introduction. Percolation theory [1–4] establishes the interplay between topology and dynamics by determining the conditions under which a network displays a giant component. Thus, percolation has wide application in robustness [5,6], contagion processes [7], brain research [8], and quantum communication [9]. While classical percolation typically assumes that connectivity is strictly ensured by links to active nearest neighbors, many physical and logical processes operate over paths of extended range [10–16]. For instance, in quantum communication networks, entanglement can be swapped between distant qubits via a sequence of quantum repeaters, provided the total path length does not exceed a coherence threshold distance R [17–20]. To capture these generalized connectivity rules, the concept of extended-range percolation (ERP) originally formulated for lattices [21–23] has been recently formulated for complex networks [13,14].

Existing studies on ERP have been largely confined to undirected graphs, leaving a significant gap in our understanding of intrinsically directed systems such as quantum communication networks. It is usually assumed that in directed networks, the percolation transition is more complex. Indeed, directionality of the links requires one to determine the size of the in giant component (IGC), out giant component

(OGC), and strongly connected giant component (SCGC), while undirected network percolation is described in terms of a single giant component and a single order parameter. Contrary to this intuition, in this work we will demonstrate that, when connectivity is defined on the basis of shortest paths, including directionality can reduce the complexity of the percolation process, as directed unreciprocated links will not allow backtracking of the paths.

In order to show this notable result, we propose directed extended-range percolation (DERP) defined on a directed network in which links are not reciprocated. We provide a theoretical framework based on a message-passing algorithm [3,24,25] to provide an analytic solution to DERP that is exact, provided that the network is locally tree like. While on undirected networks evaluating connectivity within a distance R requires considering backtracking of the paths connecting trusted nodes leading to a very complex message-passing algorithm [13,14], here we demonstrate the more straightforward formulation of DERP. Indeed, DERP explicitly tracks the propagation of connectivity through directed nonbacktracking paths passing through intermediate relay nodes, reducing the complexity of the underlying message-passing problem. This introduces a key tradeoff between DERP and ERP. Topologically, directionality requires accounting for the combined roles of the IGC, OGC, and SCGC, rather than a single giant component as in undirected ERP. Dynamically, however, enforcing directed nonbacktracking paths simplifies the message-passing algorithm, making the analytical treatment of DERP more tractable.

In DERP, relay nodes play a fundamentally different role than in standard percolation: While untrusted nodes block connectivity when $R = 1$, they instead act as active relays,

*Contact author: xm_liu@hust.edu.cn†Contact author: ginestra.bianconi@gmail.com

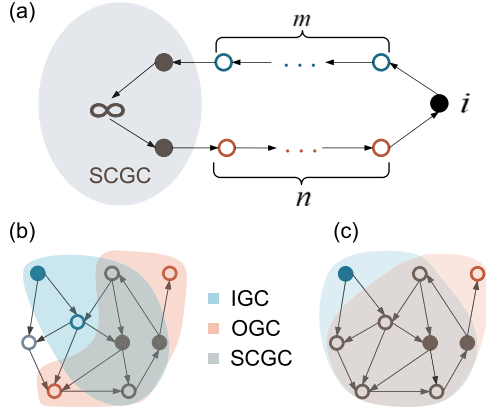


FIG. 1. DERP message-passing mechanism and network components. (a) Schematic of the DERP connectivity mechanism. Node i is trusted (closed circles), while the open circles are untrusted relay nodes. If both $m < R$ and $n < R$, node i connects to the SCGC through incoming and outgoing paths within range R and thus belongs to the SCGC together with the relay nodes. If only $m < R$ ($n < R$), node i and the corresponding relay nodes belong to the IGC (OGC). The spatial distribution of the IGC, OGC, and SCGC in a representative network is shown for (b) $R = 2$ and (c) $R = 3$, showing that increasing the interaction range expands all three giant components.

enabling long-range connections without altering the network structure. Our theory of DERP yields closed-form expressions for the percolation threshold and the DERP anomalous critical exponents. Notably, these results depend strongly on in-out degree correlations. For uncorrelated networks, DERP exhibits a finite threshold and mean-field exponents even with scale-free degrees. In contrast, for maximally correlated networks, the behavior depends on the degree distribution: The threshold vanishes for power-law exponents $\gamma \in (2, 3]$, and anomalous critical behavior emerges for $\gamma \leq 4$.

Directed extended-range percolation. We consider a directed graph $G = (V, E)$ composed of a set V of nodes and a set E of directed unreciprocated edges. Each node is independently assigned to be trusted with probability p or untrusted with probability $1 - p$. Directed extended-range percolation studies the emergence of macroscopic connectivity between trusted nodes when communication is allowed through directed paths of maximum length R , possibly traversing intermediate untrusted relay nodes. Figure 1 shows the mechanism underlying DERP. A trusted node i connects to the network through incoming and outgoing chains of untrusted relays of lengths m and n . If both $m < R$ and $n < R$, node i belongs to the SCGC, while satisfying only one of these conditions places it in the IGC or OGC.

Message-passing approach. Here we formulate the message-passing theory that predicts the size of the IGC, OGC, and SCGC of DERP on a generic locally treelike directed networks in which directed edges are not reciprocated. Due to the absence of reciprocal edges, directed paths cannot immediately backtrack, which considerably simplifies the message-passing formulation compared to the undirected extended-range percolation problem [13]. The message-passing algorithm implements the following

algorithm. If a node is trusted, it will connect a downstream node to the IGC if and only if the node is itself connected (by directed paths pointing to it) to trusted nodes at distance less than or equal to R . If a node is untrusted, the node will inform a downstream node that the minimum distance to an upstream trusted node (connected to the IGC via directed paths) is at distance $0 < r < R$. These simple arguments, as well as their generalization to treat the OGC, lead directly to the message-passing equations. These are recursive equations for the messages $\omega_{i \rightarrow j}^{r,(\pm)} \in [0, 1]$ indicating the probability that a generic node i connects its downstream (+) or upstream (−) neighbor node j to trusted nodes belonging to the IGC (+) or OGC (−) at minimum distance r (with $1 \leq r \leq R$) from node j . By generalizing the algorithm valid for undirected ERP to DERP, the message-passing equations for DERP can be derived to be

$$\begin{aligned} \omega_{i \rightarrow j}^{1,(\pm)} &= p \left[1 - \prod_{\ell \in N^{(\mp)}(i) \setminus j} \left(1 - \sum_{1 \leq q \leq R} \omega_{\ell \rightarrow i}^{q,(\pm)} \right) \right], \\ \omega_{i \rightarrow j}^{r+1,(\pm)} &= (1 - p) \left\{ \left[\prod_{\ell \in N^{(\mp)}(i) \setminus j} \left(1 - \sum_{1 \leq q \leq r-1} \omega_{\ell \rightarrow i}^{q,(\pm)} \right) \right] \right. \\ &\quad \left. - \prod_{\ell \in N^{(\mp)}(i) \setminus j} \left(1 - \sum_{1 \leq q \leq r} \omega_{\ell \rightarrow i}^{q,(\pm)} \right) \right\}, \end{aligned} \quad (1)$$

where we denote by $N^{(+)}(i)$ the set of outgoing neighbors of node i and by $N^{(-)}(i)$ the set of its incoming neighbors. This formulation explicitly delineates two distinct connectivity mechanisms: direct linkage via trusted nodes p and extended transmission through untrusted intermediates $1 - p$. Crucially, the $(1 - p)$ term in Eq. (1) represents a transmission channel rather than a vacancy. Unlike standard site percolation where unoccupied sites strictly terminate connectivity, here so-called untrusted nodes function as active relays, preserving the path continuity up to the range $r + 1$. This mechanism effectively enables the bridging of distant trusted clusters through an untrusted substrate, creating effective long-range correlations in the sparse network.

The structural integrity of directed networks is characterized by the size of the SCGC. A node belongs to the SCGC if and only if it is part of both the IGC and the OGC. Let $s_i^{(\pm)}$ ($u_i^{(\pm)}$) denote the probability that a node i is trusted (untrusted) and belongs to the IGC (+) or OGC (−). This requires at least one neighbor to be connected to the respective component within range R :

$$s_i^{(\pm)} = p \left[1 - \prod_{\ell \in N^{(\mp)}(i)} \left(1 - \sum_{1 \leq r \leq R} \omega_{\ell \rightarrow i}^{r,(\pm)} \right) \right], \quad (2)$$

$$u_i^{(\pm)} = (1 - p) \left[1 - \prod_{\ell \in N^{(\mp)}(i)} \left(1 - \sum_{1 \leq r \leq R-1} \omega_{\ell \rightarrow i}^{r,(\pm)} \right) \right]. \quad (3)$$

Given that a node in the SCGC by definition belongs to both the IGC and the OGC, we obtain that the probability s_i that a node i is trusted and belongs to the SCGC and the probability u_i that a node is untrusted and belongs to the SCGC are

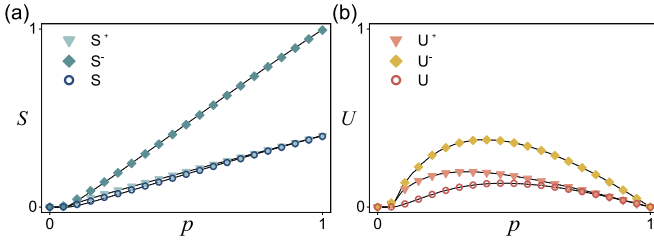


FIG. 2. DERP order parameters $S, S^\pm$ [panel (a)] and $U, U^\pm$ [panel (b)] in a real-world peer-to-peer network [26] ($N = 10\,876$ and $R = 2$). Symbols denote numerical simulations and solid lines represent theoretical predictions. Despite the inherent network asymmetry causing the sizes of IGC and OGC to be unequal, the theoretical fits remain excellent.

given by

$$s_i = \frac{1}{p} s_i^{(+)} s_i^{(-)}, \quad u_i = \frac{1}{(1-p)} u_i^{(+)} u_i^{(-)}. \quad (4)$$

The macroscopic order parameters S (fraction of trusted nodes in the SCGC) and U (fraction of untrusted nodes in the SCGC)

are finally given by

$$S = \frac{1}{N} \sum_i s_i, \quad U = \frac{1}{N} \sum_i u_i. \quad (5)$$

DERP leads to a continuous second-order phase transition well captured by the message-passing solution, as can be tested on locally treelike real-world networks, leading to excellent agreement between the simulation results and the theory (see Fig. 2). The critical properties of the DERP transition can be explored by linearizing the message-passing equations close to the trivial solution $S = U = 0$. The linearization of the message-passing equation (1) for $0 < \omega_{i \rightarrow j}^{r,(\pm)} \ll 1$, valid for $0 < p - p_c \ll 1$, leads to

$$\begin{aligned} \omega_{i \rightarrow j}^{1,(\pm)} &= p \sum_{\ell \in N^{(\mp)}(i) \setminus j} \sum_{1 \leq q \leq R} \omega_{\ell \rightarrow i}^{q,(\pm)}, \\ \omega_{i \rightarrow j}^{r+1,(\pm)} &= (1-p) \sum_{\ell \in N^{(\mp)}(i) \setminus j} \omega_{\ell \rightarrow i}^{r,(\pm)}. \end{aligned} \quad (6)$$

Thus this linear system can be expressed in matrix form for the vector $\boldsymbol{\omega}_{i \rightarrow j}^{(\pm)} = (\omega_{i \rightarrow j}^{1,(\pm)}, \omega_{i \rightarrow j}^{2,(\pm)}, \dots, \omega_{i \rightarrow j}^{R,(\pm)})$ as

$$\boldsymbol{\omega}_{i \rightarrow j}^{(\pm)} = \mathcal{A}^{(\pm)} \boldsymbol{\omega}_{i \rightarrow j}^{(\pm)}, \quad (7)$$

with the matrix $\mathcal{A}^{(\pm)}$ having an $R \times R$ block form of the type

$$\mathcal{A}^{(\pm)} = \begin{pmatrix} p\mathcal{B}^{(\pm)} & p\mathcal{B}^{(\pm)} & p\mathcal{B}^{(\pm)} & p\mathcal{B}^{(\pm)} & \dots & p\mathcal{B}^{(\pm)} & p\mathcal{B}^{(\pm)} \\ 0 & q\mathcal{B}^{(\pm)} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & q\mathcal{B}^{(\pm)} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & q\mathcal{B}^{(\pm)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & q\mathcal{B}^{(\pm)} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & q\mathcal{B}^{(\pm)} \end{pmatrix}, \quad (8)$$

where $q = 1 - p$ and where $\mathcal{B}^{(\pm)}$ indicate the nonbacktracking matrices of elements given by

$$[\mathcal{B}^{(\pm)}]_{(i \rightarrow j); (r \rightarrow s)} = p \delta_{j,r} (1 - \delta_{i,s}), \quad (9)$$

where $i \xrightarrow{(\pm)} j$ indicates links going in the same (+) or in the opposite (−) direction of the link (i, j) and here and in the following $\delta_{x,y}$ indicates the Kronecker delta. The percolation threshold $p = p_c$ satisfies the condition $\Lambda^{(\pm)}(p_c) = 1$, where $\Lambda^{(\pm)}$ is the maximum eigenvalue of $\mathcal{A}^{(\pm)}$ studied as a function of p . Note that, given the structure of the matrices $\mathcal{A}^{(\pm)}$, we have $\Lambda^{(+)}(p) = \Lambda^{(-)}(p) = \Lambda(p)$. This maximum eigenvalue can be expressed in terms of the maximum eigenvalue λ of the nonbacktracking matrix $\mathcal{B}^{(+)}$ which coincides with the maximum eigenvalue λ of the nonbacktracking matrix $\mathcal{B}^{(-)}$. Exploiting the block structure of the matrices $\mathcal{A}^{(\pm)}$, we thus obtain

$$\Lambda(p_c) = p_c \lambda \frac{[(1 - p_c)\lambda]^R - 1}{(1 - p_c)\lambda - 1} = 1. \quad (10)$$

DERP on random networks with given degree distribution. For directed networks in the random network ensemble with given degree distribution $P(\mathbf{k}) = P(k_{\text{in}}, k_{\text{out}})$, the local

message-passing equations provide results that are self-averaging in the infinite network limit. Let us define $W_r^{(\pm)}$ as the average of the message $\omega_{i \rightarrow j}^{r,(\pm)}$ over the probability distribution of the random network, i.e., $W_r^{(\pm)} = \langle \omega_{i \rightarrow j}^{r,(\pm)} \rangle$. The ensemble-average equations of the message-passing equation (1) valid for DERP yield the self-consistent system of equations for $W_r^{(+)}$,

$$\begin{aligned} W_1^{(+)} &= p \sum_{\mathbf{k}} \frac{k_{\text{out}}}{\langle k_{\text{out}} \rangle} P(\mathbf{k}) \left[1 - \left(1 - \sum_{q=1}^R W_q^{(+)} \right)^{k_{\text{in}}} \right], \\ W_{r+1}^{(+)} &= (1-p) \sum_{\mathbf{k}} \frac{k_{\text{out}}}{\langle k_{\text{out}} \rangle} P(\mathbf{k}) \left[\left(1 - \sum_{q=1}^{r-1} W_q^{(+)} \right)^{k_{\text{in}}} \right. \\ &\quad \left. - \left(1 - \sum_{q=1}^r W_q^{(+)} \right)^{k_{\text{in}}} \right], \end{aligned} \quad (11)$$

while for $W_r^{(-)}$ the analogous equations hold [see Supplemental Material (SM) [27] for details].

The fractions of nodes $S^{(+)}$ and U^{+} that are trusted and untrusted, respectively, and belong to the IGC are given by

$$S^{(+)} = p \sum_{\mathbf{k}} P(\mathbf{k}) \left[1 - \left(1 - \sum_{q=1}^R W_q^{(+)} \right)^{k_{\text{in}}} \right],$$

$$U^{(+)} = (1-p) \sum_{\mathbf{k}} P(\mathbf{k}) \left[1 - \left(1 - \sum_{q=1}^{R-1} W_q^{(+)} \right)^{k_{\text{in}}} \right], \quad (12)$$

while the analogous expressions hold for the fractions of nodes $S^{(-)}$ and U^{-} that are trusted and untrusted, respectively, and belong to the OGC (see the SM [27] for details). Finally, the order parameters S and U indicating the fraction of nodes that are trusted and untrusted, respectively, and belong to the SCGC are obtained from the joint probability that nodes are in the IGC and OGC, thus obtaining

$$S = \frac{1}{p} S^{(+)} S^{(-)}, \quad U = \frac{1}{1-p} U^{(+)} U^{(-)}. \quad (13)$$

For random networks with a given degree distribution $P(\mathbf{k})$, the analytical condition for criticality, determination of the percolation threshold p_c is obtained by linearizing the self-consistent equations (11) and their analogous equations for $W_r^{(-)}$ by taking into account that for the directed network we have $\langle k_{\text{in}} \rangle = \langle k_{\text{out}} \rangle$. In this way we obtain that the percolation threshold p_c satisfies

$$p_c \kappa \sum_{h=0}^{R-1} [(1-p_c)\kappa]^h = p_c \kappa \frac{[(1-p_c)\kappa]^R - 1}{(1-p_c)\kappa - 1} = 1, \quad (14)$$

where κ is the branching factor $\kappa = \langle k_{\text{in}} k_{\text{out}} \rangle / \langle k_{\text{in}} \rangle$. Interestingly, the expression for the percolation threshold on random directed networks given by Eq. (14) has the same structure as the expression derived in Eq. (10) for individual locally treelike directed networks via the message-passing approach, where λ is identified with the branching factor κ . Overall, this theoretical treatment of DERP elucidates the physical role of the interaction range R : It acts as an effective amplifier of the effective connectivity. Even for sparse networks where standard percolation ($R=1$) would fail ($p_c > 1$), a finite range $R > 1$ can drive p_c to much lower values, significantly increasing the network robustness.

The critical behavior of DERP is characterized by the set of critical indices indicating the scaling of the order parameters for $0 < p - p_c \ll 1$:

$$U^{(\pm)} \propto (p - p_c)^{\beta_U^{(\pm)}}, \quad S^{(\pm)} \propto (p - p_c)^{\beta_S^{(\pm)}},$$

$$U \propto (p - p_c)^{\beta_U}, \quad S \propto (p - p_c)^{\beta_S}. \quad (15)$$

DERP on uncorrelated and maximally correlated directed networks. Our theory predicts a distinct difference in the critical properties of DERP defined on uncorrelated (UC) and maximally correlated (MC) directed networks (see Fig. 3). While for UC random directed networks the in-degree and out-degree of each node are uncorrelated, the MC random networks impose that the in-degree of any node i matches its out-degree, i.e., $k_i^{\text{in}} = k_i^{\text{out}}$. This maximal correlation changes drastically the critical properties of DERP.

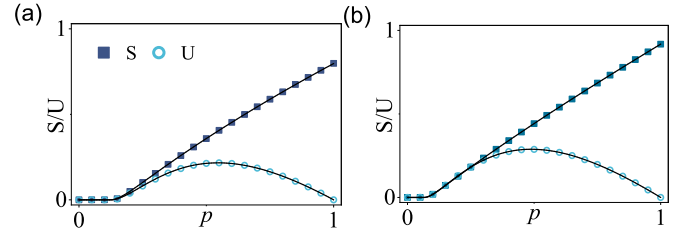


FIG. 3. DERP order parameters are shown for directed networks with varying degree of correlation. The figure compares the order parameters obtained with Monte Carlo simulations (symbols) with the theoretical predictions (solid lines) for (a) UC (Poisson) and (b) MC (Poisson) networks. The order parameters S and U are shown as functions of p for $R = 2$. The network parameters are $N = 20\,000$ and $c = 2.5$. Data are averaged over 100 realizations.

Specifically, for UC random directed networks the joint degree distribution $P(\mathbf{k})$ factorizes into the in-degree distribution $P_{\text{in}}(k_{\text{in}})$ and the out-degree distribution $P_{\text{out}}(k_{\text{out}})$, i.e., $P(\mathbf{k}) = P_{\text{in}}(k_{\text{in}})P_{\text{out}}(k_{\text{out}})$; thus the branching ratio is simply given by $\kappa = \langle k_{\text{in}} \rangle = \langle k_{\text{out}} \rangle$. This implies that the critical threshold p_c given by Eq. (14) is finite for any finite value of R , regardless of the considered in-degree and out-degree distributions $P_{\text{in}}(k_{\text{in}})$ and $P_{\text{out}}(k_{\text{out}})$. Moreover, in the case of UC random directed networks, the critical $\beta^{(\pm)}$ and β determining the critical behavior of the order parameters $S^{(\pm)}$ ($U^{(\pm)}$) and S (U) remain always mean-field critical exponents (see Table I and the SM [27] for details of the derivation).

In contrast, for MC random directed networks with joint degree distribution $P(\mathbf{k}) = P(k)\delta_{k,k_{\text{out}}}\delta_{k,k_{\text{in}}}$ having branching ratio $\kappa = \langle k^2 \rangle / \langle k \rangle$, we observe that the critical behavior is strongly dependent on the degree distribution $P(k)$. In particular, for well-behaved distributions $P(k)$ having finite moments of order up to 3, we recover a finite percolation threshold p_c for any finite value of R and mean-field critical exponents. However, for power-law $P(k)$ distributions with $\gamma \leq 3$ defining scale-free networks, the branching ratio diverges in the

TABLE I. Critical exponents for UC (with identical in- and out-degree distributions) and MC directed networks for well-behaved (WB) and power-law (PL) in- and out-degree distributions. WB directed networks have in- and out-degree distributions with finite moments of order up to 3 and include Poisson directed networks. PL directed networks have power-law in- and out-degree distributions with power-law exponent $\gamma \in (2, 4)$. Power-law networks with exponent $\gamma \in (2, 3]$ are scale-free. The critical indices are expressed in terms η_{R-1} and η_R , given by $\eta_{R-1} = (\gamma - 2)^{R-1}$ and $\eta_R = (\gamma - 2)^R$, respectively.

Model	Condition	$\beta_S^{\pm}$	$\beta_U^{\pm}$	β_S	β_U
UC (WB)		1	1	2	2
MC (WB)		1	1	2	2
UC (PL)	$\gamma > 3$	1	1	2	2
	$\gamma \in (2, 3)$	$\frac{1}{\gamma-2}$	$\frac{1}{\gamma-2}$	$\frac{2}{\gamma-2}$	$\frac{2}{\gamma-2}$
MC (PL)	$\gamma > 4$	1	1	2	2
	$\gamma \in (3, 4)$	$\frac{1}{\gamma-3}$	$\frac{1}{\gamma-3}$	$\frac{2}{\gamma-3}$	$\frac{2}{\gamma-3}$
	$\gamma \in (2, 3)$	$1 + \frac{\eta_{R-1}}{1-\eta_R}$	$\frac{\eta_R-2}{1-\eta_R}$	$1 + 2\frac{\eta_{R-1}}{1-\eta_R}$	$\frac{2\eta_R-2}{1-\eta_R}$

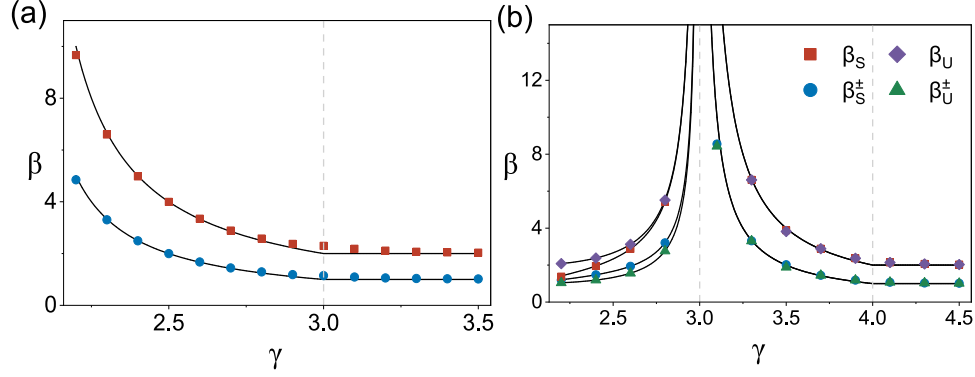


FIG. 4. Dependence of the critical exponent β on the parameter γ for $R=2$ in the DERP model. (a) In the UC (PL) model, a single transition occurs at $\gamma=3$ (vertical dashed line). For $\gamma > 3$, the system exhibits mean-field behavior. (b) In the MC (PL) model, four exponents β_S , β_U , β_S^+ , and β_U^+ show boundaries at $\gamma=3$ and $\gamma=4$, becoming nonanalytic at $\gamma=3$. In both panels, symbols represent numerical simulations ($N=50,000$) and solid lines are theoretical curves.

infinite network limit, i.e., $\kappa \rightarrow \infty$ as $N \rightarrow \infty$. This implies that the percolation threshold p_c vanishes in the infinite network limit as

$$p_c \simeq \kappa^{-R} = \left(\frac{\langle k \rangle}{\langle k^2 \rangle} \right)^R \rightarrow 0. \quad (16)$$

Therefore, in DERP the convergence of the percolation threshold to zero is remarkably faster for large values of R than in standard percolation on UC directed networks, recovered for $R=1$ (see the SM [27] for the perfect agreement between these predictions and our numerical simulations). Additionally, also the critical exponents of DERP are anomalous (see Table I and the SM [27] for details of the derivation) when DERP is defined on MC directed networks with power-law degree distribution $P(k)$ with $\gamma \leq 4$ as captured in full by our theoretical predictions (see the SM [27] for details). To compare these results, we have used advanced numerical techniques (see the SM [27]) for the validation of the critical indices, finding excellent agreement between numerical results and analytics predictions (see Fig. 4 and the SM [27] for details).

Conclusion. Generalized path-based percolation models have attracted growing interest, yet most work focuses on undirected networks. While directionality typically increases complexity, we showed that for extended-range percolation,

its directed counterpart instead simplifies both the analytical treatment and the message-passing formulation, opening alternative directions for studying such systems. We investigated DERP on directed networks, where connectivity between trusted nodes within distance R is mediated by paths that may include untrusted intermediates. The absence of reciprocal edges suppresses backtracking, enabling a simple analytical characterization of the strongly connected giant component via message passing. As a result, DERP admits a more tractable treatment than undirected ERP, yielding exact expressions for the percolation threshold and critical exponents. Directionality also has a strong impact on the percolation transition. The threshold decays exponentially with R at large branching ratio κ , showing that even short-range extensions significantly enhance connectivity. Moreover, critical behavior depends sensitively on degree correlations. Uncorrelated networks retain mean-field behavior and a finite threshold, whereas maximally correlated scale-free networks can exhibit a vanishing threshold for $\gamma \leq 3$ and anomalous critical exponents for $\gamma \leq 4$.

Acknowledgments. This work was supported by the National Natural Science Foundation of China under Grants No. T2422010 and No. 62172170 and by the Fundamental Research Funds for the Central Universities.

Data availability. Codes will be available at a public repository [28] upon publication of the work.

- [1] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Critical phenomena in complex networks, *Rev. Mod. Phys.* **80**, 1275 (2008).
- [2] N. Araújo, P. Grassberger, B. Kahng, K. Schrenk, and R. M. Ziff, Recent advances and open challenges in percolation, *Eur. Phys. J.: Spec. Top.* **223**, 2307 (2014).
- [3] G. Bianconi, *Multilayer Networks: Structure and Function* (Oxford University Press, Oxford, 2018).
- [4] A. P. Millán, H. Sun, L. Giambagli, R. Muolo, T. Carletti, J. J. Torres, F. Radicchi, J. Kurths, and G. Bianconi, Topology shapes dynamics of higher-order networks, *Nat. Phys.* **21**, 353 (2025).
- [5] O. Artime and M. D. Domenico, Percolation on feature-enriched interconnected systems, *Nat. Commun.* **12**, 2478 (2021).
- [6] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Catastrophic cascade of failures in interdependent networks, *Nature (London)* **464**, 1025 (2010).
- [7] H. Sun, D. Saad, and A. Y. Lokhov, Competition, collaboration, and optimization in multiple interacting spreading processes, *Phys. Rev. X* **11**, 011048 (2021).
- [8] H. Sun, F. Radicchi, J. Kurths, and G. Bianconi, The dynamic nature of percolation on networks with triadic interactions, *Nat. Commun.* **14**, 1308 (2023).

- [9] X. Meng, J. Gao, and S. Havlin, Concurrence percolation in quantum networks, *Phys. Rev. Lett.* **126**, 170501 (2021).
- [10] X. Meng, B. Hao, B. Ráth, and I. A. Kovács, Path percolation in quantum communication networks, *Phys. Rev. Lett.* **134**, 030803 (2025).
- [11] M. Kim and F. Radicchi, Shortest-path percolation on random networks, *Phys. Rev. Lett.* **133**, 047402 (2024).
- [12] M. Kim, L. Cirigliano, C. Castellano, H. Sun, R. Jankowski, A. Poggialini, and F. Radicchi, Shortest-path percolation on scale-free networks, *Phys. Rev. E* **113**, 014314 (2026).
- [13] L. Cirigliano, C. Castellano, and G. Bianconi, General theory for extended-range percolation on simple and multiplex networks, *Phys. Rev. E* **110**, 034302 (2024).
- [14] L. Cirigliano, C. Castellano, and G. Timár, Extended-range percolation in complex networks, *Phys. Rev. E* **108**, 044304 (2023).
- [15] L. Cirigliano, V. Brosco, C. Castellano, C. Conti, and L. Piloizzi, Optimal quantum key distribution networks: capacitance versus security, *npj Quantum Inf.* **10**, 44 (2024).
- [16] X. Hu, G. Dong, K. Christensen, H. Sun, J. Fan, Z. Tian, J. Gao, S. Havlin, R. Lambiotte, and X. Meng, Unveiling the importance of nonshortest paths in quantum networks, *Sci. Adv.* **11**, eadt2404 (2025).
- [17] Y. Huang, Y. Yang, H. Li, J. Wang, J. Qiu, Z. Qi, Y. Zhang, Y. Li, Y. Zheng, and X. Chen, Quantum fusion of independent networks based on multi-user entanglement swapping, *Nat. Photon.* **20**, 87 (2026).
- [18] S. Zhang, J. Shi, Y. Liang, Y. Sun, Y. Wu, L. Duan, and Y. Pu, Fast delivery of heralded atom-photon quantum correlation over 12 km fiber through multiplexing enhancement, *Nat. Commun.* **15**, 10306 (2024).
- [19] L. Feng, Y.-Y. Huang, Y.-K. Wu, W.-X. Guo, J.-Y. Ma, H.-X. Yang, L. Zhang, Y. Wang, C.-X. Huang, C. Zhang, L. Yao, B.-X. Qi, Y.-F. Pu, Z.-C. Zhou, and L.-M. Duan, Realization of a crosstalk-avoided quantum network node using dual-type qubits of the same ion species, *Nat. Commun.* **15**, 204 (2024).
- [20] C. M. Knaut, A. Suleymanzade, Y.-C. Wei, D. R. Assumpcao, P.-J. Stas, Y. Q. Huan, B. Machielse, E. N. Knall, M. Sutula, G. Baranes, N. Sinclair, C. De-Eknamkul, D. S. Levonian, M. K. Bhaskar, H. Park, M. Lončar, and M. D. Lukin, Entanglement of nanophotonic quantum memory nodes in a telecom network, *Nature (London)* **629**, 573 (2024).
- [21] Z. Xun, D. Hao, and R. M. Ziff, Site percolation on square and simple cubic lattices with extended neighborhoods and their continuum limit, *Phys. Rev. E* **103**, 022126 (2021).
- [22] Z. Xun and R. M. Ziff, Bond percolation on simple cubic lattices with extended neighborhoods, *Phys. Rev. E* **102**, 012102 (2020).
- [23] Z. Xun, D. Hao, and R. M. Ziff, Extended-range site and bond percolation in five dimensions, *J. Stat. Mech.* (2025) 123301.
- [24] M. Newman, Message passing methods on complex networks, *Proc. R. Soc. A* **479**, 20220774 (2023).
- [25] B. Karrer, M. E. Newman, and L. Zdeborová, Percolation on sparse networks, *Phys. Rev. Lett.* **113**, 208702 (2014).
- [26] J. Leskovec and A. Krevl, SNAP datasets: Stanford large network dataset collection, 2014, <https://snap.stanford.edu/data>.
- [27] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/hs2n-6rsn> for details of the derivation and further numerical results.
- [28] See the repository “Directed-extended-range-percolation” for numerical codes for DERP implementation (GitHub, San Francisco, 2026), <https://github.com/ginestrab/Directed-Extended-Range-Percolation>.