

Inertial synchronization of networked oscillators in arbitrary dimensions

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The Kuramoto model provides a paradigmatic framework for studying the synchronization of interacting oscillators, and has been generalized to arbitrary dimensions to describe swarms, flocks, and multidimensional opinion dynamics. Yet, existing formulations neglect inertia, a key mechanism known to enhance information propagation and collective responsiveness. Here, we introduce and analyze an inertial Kuramoto model in arbitrary dimensions. We show that inertia fundamentally alters the nature of the synchronization transition, inducing a crossover from continuous to discontinuous behavior, with the onset of hysteresis depending explicitly on both inertia and dimensionality parity. Our analytical theory is supported by extensive numerical simulations. These results establish inertia as a crucial ingredient of high-dimensional collective dynamics and reveal a universal structure in synchronization phenomena.

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I. INTRODUCTION

The emergence of collective dynamics in large populations of interacting dynamical units is a subject of intense study in physics, sociology, biology, and engineering [1–12], with complex networks providing a suitable representation of the coupling structure of many nonequilibrium systems. In this framework, much interest has been devoted to the emergence of nontrivial collective dynamics in networks of elements whose evolution is intrinsically simple, such as phase oscillators, whose state is characterized by a point on a unit circle.

Particular attention has been given to the characterization of synchronized regimes [13–20], in which oscillators evolve coherently. Among the paradigmatic frameworks for investigating synchronization phenomena, the Kuramoto model and its generalizations play a central role. These models have been

successfully applied across a broad range of systems, from biological contexts, including cardiac pacemaker cells [21], flashing fireflies [22], circadian rhythms [23], and neuronal populations [24], to physical systems, both natural and engineered, such as power-grid networks [8], superconducting Josephson junction arrays [25], and quantum dipoles [26]. Other classic applications of the Kuramoto model have been the modeling of the members of a swarm or flock in two dimensions [27,28] and the characterization of the opinion of individuals in an interacting group [29].

However, all the aforementioned studies describe collective alignment through a single scalar phase variable, whereas in many realistic settings interactions occur in higher-dimensional spaces. For instance, velocity alignment in flocks of birds or schools of fish inherently involves three-dimensional vectors, and opinion dynamics may unfold in spaces of even higher dimensionality, depending on the range of social attributes considered. Motivated by these considerations, the Kuramoto framework has been generalized to arbitrary dimensions [30].

Moreover, a fundamental ingredient known to foster information propagation is inertia [31], which has been neglected up to now in modeling the collective behavior of insect swarms and bird flocks in high dimensions. Behavioral inertia

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in the rotations of individual velocities was first introduced to explain the propagation of collective turns in bird flocks [32,33], but experiments also found clear evidence of underdamped inertial relaxation in natural swarms of midges [34].

In this article, we bridge this gap by introducing an inertial extension of the Kuramoto model to arbitrary dimension D . Specifically, we demonstrate that the nature of the synchronization transition undergoes a crossover from continuous to discontinuous as inertia increases, with the minimal inertia required to induce hysteresis that depends explicitly on the dimensionality of the interaction space. In the vanishing-inertia limit, our framework naturally recovers the D -dimensional Kuramoto model, which emerges as a special case of the theory. Under reasonable approximations, we provide an analytical characterization of the transition and corroborate our findings through extensive numerical simulations.

II. RESULTS

We start by considering an ensemble of N rotators, each represented by a D -dimensional unit vector σ_i constrained to the surface of a unit D -sphere. Here, σ_i represents the orientation of an agent and the evolution equation for the i th rotator (agent) is given by

$$m(\ddot{\sigma}_i + \|\dot{\sigma}_i\|^2 \sigma_i) + \dot{\sigma}_i = \mathbf{W}_i \sigma_i + \frac{\lambda}{N} \sum_{j=1}^N [\sigma_j - (\sigma_j \cdot \sigma_i) \sigma_i], \quad (1)$$

where dots denote temporal derivatives, m is the inertia, λ is the coupling strength, and $\mathbf{W}_i$ is a $D \times D$ antisymmetric matrix with elements $\omega_{j,k}^i \sim \mathcal{N}(0, 1)$, that represent the natural frequencies of the rotators. Specifically, the antisymmetric matrix $\mathbf{W}_i$ describes the intrinsic dynamics of oscillator i and constitutes the higher-dimensional generalization of the natural frequency of a phase oscillator. In the absence of coupling, a constant $\mathbf{W}_i$ generates a rigid rotation on the unit sphere through the evolution operator $e^{\mathbf{W}_i t}$. In two dimensions, an antisymmetric matrix is specified by a single scalar, which coincides with the conventional natural frequency. Although $\mathbf{W}_i$ could more generally depend on time or on σ_i , while remaining antisymmetric to preserve $\|\sigma_i\| = 1$, here we consider constant matrices in order to focus on the effects of collective interactions and inertia. In the following, while the analytical characterization of the synchronization transition is valid for all natural frequencies distributed according to rotational invariant distribution, numerical simulations are performed by choosing natural frequencies randomly distributed according to a unimodal Gaussian distribution with zero mean and unitary standard deviation. Moreover, the system is initialized with all the velocities tangent to the sphere, i.e., $\sigma_i(0) \cdot \dot{\sigma}_i(0) = 0$.

When the coupling structure among rotators is all to all, Eq. (1) can be conveniently reformulated in terms of the order parameter $R = |\rho| = |\frac{1}{N} \sum_{j=1}^N \sigma_j|$ as

$$m(\ddot{\sigma}_i + \sigma_i |\dot{\sigma}_i|^2) + \dot{\sigma}_i = \mathbf{W}_i \sigma_i + \lambda[\rho - (\rho \cdot \sigma_i) \sigma_i], \quad (2)$$

which admits a mean-field description for which we can derive the stationary solutions for the order parameter self-consistently. In this formulation it is easy to see that the coupling term acts as an external torque that tends to

align the orientation of each agent with the collective order parameter ρ .

For $D = 2$, this formulation reduces exactly to the standard Kuramoto model with inertia, confirming the model's consistency (see Supplemental Material [35] for the demonstration of the equivalence at $D = 2$). For $D = 2$, Eq. (2) corresponds to a damped driven pendulum equation which admits, for sufficiently small forcing frequency ω_i , two fixed points: a stable node and a saddle. At larger frequencies $|\omega_i| > \Omega_P \approx \frac{4}{\pi} \sqrt{\frac{\lambda R}{m}}$, a homoclinic bifurcation leads to the emergence of a limit cycle from the saddle. The stable limit cycle and the stable fixed point coexist until a saddle node bifurcation, taking place at $|\omega_i| = \Omega_D = \lambda R$, leads to the disappearance of the fixed points, and for $\omega_i > \Omega_D$ only the oscillating solution is present. This scenario is correct for sufficiently large masses, and at small m one has a direct transition from a stable node to a periodic oscillating orbit at $|\omega_i| = \Omega_D = \lambda R$ [36].

As pointed out in Tanaka *et al.* [37], the fully incoherent state ($R = 0$) and the fully synchronized one ($R \equiv 1$) correspond to qualitatively distinct dynamical regimes of the oscillators: in the asynchronous state all units drift with finite velocities $\langle \dot{\sigma}_i \rangle$; as the coupling strength increases, oscillators with sufficiently small natural frequencies $|\omega_i| < \Omega_P \approx \frac{4}{\pi} \sqrt{\frac{\lambda R}{m}}$ begin to lock ($\langle \dot{\sigma}_i \rangle = 0$), while the remaining ones continue to drift. This progressive locking mechanism ultimately leads to complete synchronization, i.e., $R = 1$. Conversely, when the system is initialized in the fully locked configuration, drifting motion sets in once the stable fixed-point solution ceases to exist and only a limit-cycle solution remains. This occurs, upon decreasing the coupling constant, whenever $|\omega_i| \geq \Omega_D = \lambda R$. In both the cases examined, there is a group of desynchronized oscillators and one of locked oscillators separated by a frequency, Ω_P in the first case and Ω_D in the second one. These groups contribute differently to the total level of synchronization of the system [37]: while drifting oscillators contribute to the total order parameter with a negative contribution, locked oscillators make a positive contribution, with the value of the natural frequency of each unit determining whether and when the unit becomes locked. In D dimensions, locked oscillators are those aligned with the average orientation of the population, whereas drifting ones rotate away from it.

Since the distribution of $\mathbf{W}_i$ is rotationally invariant, being multivariate Gaussian, the macroscopic behavior of the system depends only on the magnitude of ρ , and not on its direction. Accordingly, we assume that R takes a constant value $\tilde{R}$, associated with a vector $\tilde{\rho} = \tilde{R} \hat{\rho}$, where $\hat{\rho}$ is a unit vector. Then, we derive equations for $\tilde{R}$ for the forward and backward transitions that can be solved self-consistently, determining the transition curves between the asynchronous state and the partially synchronized one.

For $D = 3$ the transition curves can be derived explicitly. We briefly outline the derivation here, and refer the reader to the Supplemental Material [35] for full details. To start, we make a few assumptions, namely that:

- (1) The distribution of the oscillators is invariant under rotations around $\tilde{\rho}$.
- (2) The oscillators always switch collectively between a rotating periodic solution and a single stationary point, corre-

sponding to the low and high values of the order parameter, respectively.

(3) When the system is in the asynchronous state, the global effect of all interactions is negligible.

(4) When the system is in the asynchronous state, any perturbation dissipates on a timescale that is of the same order of magnitude of the order parameter.

(5) For the case of $D = 2$, when the system is in the asynchronous state, the average of the rotation speeds of the oscillators is negligible.

(6) In three dimensions, each oscillator is either in a fixed point or is rotating in a fixed plane with almost constant speed.

Then, since $\mathbf{W}_i$ is antisymmetric, for each rotator there exists a basis in which $\mathbf{W}_i$ assumes the block-diagonal form

$$\begin{pmatrix} 0 & \omega_i & 0 \\ -\omega_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

Projecting Eq. (2) onto the third basis vector yields

$$m(\ddot{\sigma}_{i,3} + \sigma_{i,3}|\dot{\sigma}_i|^2) + \dot{\sigma}_i, 3 = \lambda \tilde{R}[\hat{\rho}_3 - (\hat{\rho} \cdot \sigma_i)\sigma_{i,3}], \quad (4)$$

where $(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i,3})$ and $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ denote the coordinates of σ_i and $\hat{\rho}$ in this basis. For a fixed $\hat{\rho}$, Eq. (2) admits a solution with constant $\sigma_{i,3}$ satisfying

$$\hat{\rho}_3 = \left[\hat{\rho} \cdot \sigma_i + \frac{m}{\lambda \tilde{R}} \omega_i^2 (1 - \sigma_{i,3}^2) \right] \sigma_{i,3}, \quad (5)$$

which characterizes the rotating solution in the transformed reference frame [37].

To determine the transition points, Eq. (5) is solved to identify the value of $\sigma_{i,3}$ corresponding to the quasistationary third component of the periodic rotating solution. One then assesses whether the rotating motion in the first two coordinates constitutes a unique stable periodic solution of the rescaled system. For the backward transition, if this condition holds, the average projection of σ_i onto $\hat{\rho}$ reads

$$\hat{\rho} \cdot \sigma_i = -\frac{1}{2} \lambda m \tilde{R} \frac{r^2}{(m|\omega_i|)^3} + \hat{\rho}_3 \sigma_{i,3}, \quad (6)$$

where r denotes the norm of the subvector formed by the first two components of $\hat{\rho}$. If instead a stable fixed point exists, the contribution becomes

$$\hat{\rho} \cdot \sigma_i = r \sqrt{1 - \sigma_{i,3}^2} \sqrt{1 - \left(\frac{\omega_i^2}{\lambda \tilde{R}} \right)^2}, \quad (7)$$

which corresponds to the stationary fixed-point solution. The expected value of the order parameter along the backward threshold is then obtained by integrating the projection over the distribution of natural frequencies, accounting for all oscillators, and solving the resulting equation self-consistently.

Similarly, for the forward transition, if the stability criterion holds, the average projection of σ onto $\hat{\rho}$ is

$$\hat{\rho} \cdot \sigma_i = -\frac{1}{2} \frac{\lambda m \tilde{R} r^2}{1 + (m\omega_i)^2} + \hat{\rho}_3 \sigma_{i,3}; \quad (8)$$

otherwise, Eq. (7) holds. Again, the order parameter is obtained by integrating the projection and solving the resulting equation in a self-consistent way.

III. NUMERICAL VALIDATION

To validate the analytical predictions, we performed extensive numerical simulations by varying the coupling strength λ adiabatically. For the forward transition, simulations were initialized from random configurations, and λ was increased from 0 up to a maximum value λ_M in increments of $\Delta\lambda$. Subsequently, the protocol was reversed, decreasing λ stepwise by $\Delta\lambda$ until returning to 0. Each simulation, except for the first one, was initialized from the final state of the preceding run. For every value of λ , the system was evolved over a transient time T_t , followed by an additional interval T_s during which the time-averaged value of R was computed.

The results, reported in Fig. 1, demonstrate that the synchronization transition is continuous for small inertia and becomes hysteretic beyond a critical inertia threshold. In particular, in Fig. 1 are reported the cases $D = 3$ [Figs. 1(a)–1(d)] and $D = 5$ [Figs. 1(e)–1(h)]. However, this phenomenology is robust across dimensions from 2 to 7, and consistent with previous findings [38,39]. The theoretical curves (dashed lines) reported in Figs. 1(a)–1(d) exhibit excellent agreement with numerical simulations in the discontinuous jumps and shape of the coherent branch, confirming the validity of the self-consistent analysis and accurately capturing the dependence of the transition on m . Also, for fixed m , the width of the hysteresis loop increases with D . It is important to notice that the theoretical curves for the forward and backward transitions are predicted through an extension of the Kuramoto self-consistent theory for a second-order system with dimension $D = 3$. The theory is able to predict the discontinuity of the transition with respect to the coupling strength regardless of the fact that it approximates a large but still finite-size system (1) by the continuum (infinite-size) limit. However, the self-consistent equations for the order parameter are not necessary a good approximations in the vanishing coupling strength limit, as already shown in Ref. [37] for $D = 2$, and the end point of the lower backward branch is not captured properly since the self-consistent theory does not provide the instability properties of the incoherent state. For larger m ($m > 1$), although the decreasing backward branch shows good agreement to the theoretical prediction, the forward branch generally shows a growing deviation from the theoretical prediction due to the break of the independence between the whirling oscillators, and the emergence of a secondary synchronization of groups of the whirling oscillators with lower natural frequencies (drifting clusters).

Additionally, the synchronization transition exhibits finite-size effects, as illustrated in Fig. 2(a). In particular, the forward critical coupling λ_c^F increases with the system size and saturates to an asymptotic value $\lambda_c^F \approx 5.5$, in agreement with established results for $D = 2$ [40]. As mentioned above, and similarly to what is known for $D = 2$ [37,40,41], sufficiently large inertia leads to the formation of drifting clusters with finite average velocities within the partially synchronized regime. Furthermore, the transition between the asynchronous state, shown in Fig. 2(c), and the synchronized one, shown in Fig. 2(e), proceeds through an intermediate cluster state characterized by pronounced oscillations of the order parameter, shown in Fig. 2(d). These oscillations originate from large secondary clusters undergoing coherent rotation with

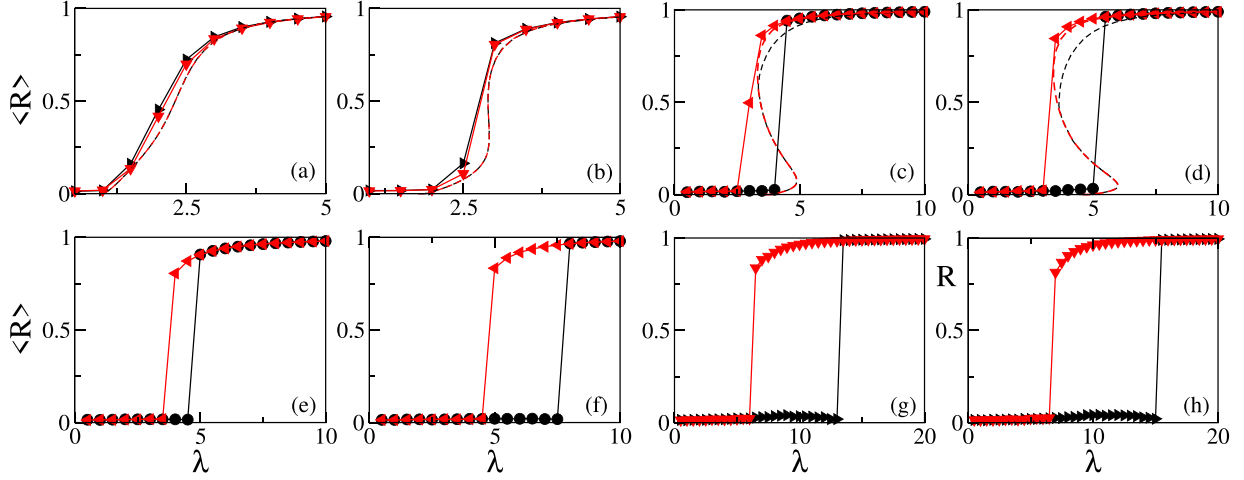


FIG. 1. Change in order of the transition. The time average of the order parameter $\langle R \rangle$ vs the coupling strength λ shows that the transition is continuous for small enough values of m . As m or the number of dimensions D increase, a hysteresis loop appears. In all panels, the black dots indicate the results of forward simulations and the red ones those of backward simulations, the dashed lines with corresponding colors are the theoretical predictions, and the simulations were carried out for $N = 1000$ oscillators, for $T_s = 100$ time units after a transient time $T_t = 400$, varying $\Delta\lambda = 0.5$. Simulations were performed applying a Runge-Kutta fourth-order integration scheme with integration step 0.001. The values of inertia are (a), (e) $m = 0.3$, (b), (f) $m = 0.5$, (c), (g) $m = 0.85$, and (d), (h) $m = 1$. (a)–(d) refer to $D = 3$; (e)–(h) to $D = 5$.

finite velocities. Remarkably, the emergence of such macroscopic oscillations is anomalous: while typically associated with critical (continuous) transitions, here they arise in a discontinuous, first-order synchronization transition.

To investigate the dependence of hysteresis on inertia and dimensionality, we performed systematic simulation cycles in which the coupling strength λ was first increased until the

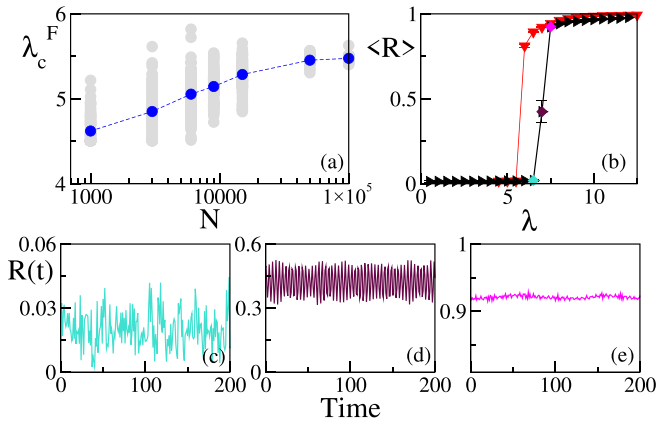


FIG. 2. The high inertia regime. (a) The critical coupling λ_c^F increases with system size, eventually reaching approximately 5.5. Blue dots denote the mean values over different realizations, which range from 100 for $N = 1000$ to 20 for $N = 100\,000$. (b) Time average of the order parameter $\langle R \rangle$ vs the coupling strength λ . The black and red triangles refer to simulations performed by following protocols (I) and (II). The color code of the solid diamonds corresponds to the color of the time traces shown in (c)–(e), that identify the asynchronous state (turquoise), cluster state (maroon), and partial synchronization (magenta). (c)–(e) Time behavior of the order parameter $R(t)$ for three different values of λ (6.5, 7, and 7.5, respectively), exhibiting wide oscillations due to the emergence of clusters close to the critical coupling in panel (c). In all panels, the dimensionality is $D = 3$ and the inertia is $m = 5$.

order parameter reached 0.8, and subsequently decreased back to 0. For each value of λ , we evaluated the difference between the forward and backward branches of the order parameter, R^F and R^B , identifying hysteresis whenever $|R^F - R^B| > 0$ along the transition. Repeating this protocol for different masses m , we determined, for each dimension D , the minimum mass m_m required to induce a first-order transition. Specifically, we define m_m as the smallest value of m such that $|R^F - R^B| > 0.2$. As shown in Fig. 3, the minimum inertia necessary for hysteresis decreases systematically with increasing dimension, with the notable exception of $D = 3$. The nonmonotonic behavior observed in Fig. 3(b) can be explained on the basis of the emergence of drifting clusters during the forward transition to synchronization, which is present only for $D = 2, 3$. The presence (absence) of intermediate steps in the forward branch can result in an increasing (decreasing) dependence on the inertia for increasing dimensionality.

Finally, we analyzed the backward transition in the limit of vanishing inertia. While in even dimensions, the system

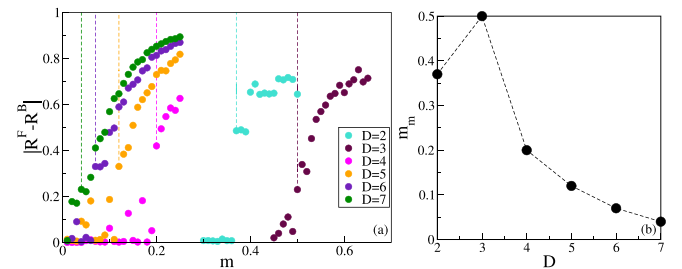


FIG. 3. The inertia needed to generate a first-order transition decreases with the dimensions. (a) Difference between the order parameter values in the two transition directions (when positive for the first time) during the backward transition. (b) Minimum value of the mass m_m needed to observe a discontinuous transition as a function of D .

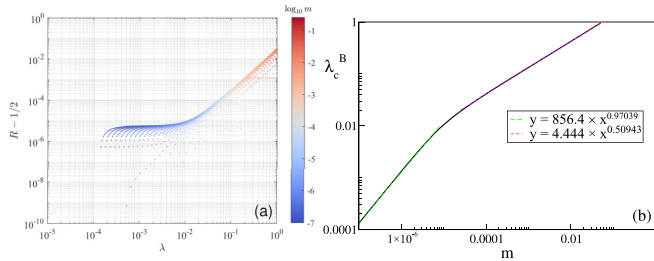


FIG. 4. The low inertia regime: recovering the classic D -dimensional Kuramoto model. (a) For vanishing λ , the order parameter approaches $R = \frac{1}{2}$ for small-enough masses. (b) The scaling exponent of the smallest coupling λ_c^B for which partial synchronization is observed with the mass crosses over from approximately 0.5 to approximately 1 as the mass decreases.

reaches the asynchronous state with vanishing R in a continuous manner independently on the inertia value, for odd dimensions the transition is more complex, and for vanishing inertia the system must recover the behavior of the standard high-dimensional Kuramoto model, whose transition is discontinuous at zero coupling where R reaches $1/2$ [30,42]. More specifically, the system approaches the asynchronous state through a continuous transition in even dimensions. In contrast, odd dimensions exhibit a qualitatively different behavior: the transition is discontinuous in the limit of vanishing inertia value, as shown in Fig. 4. As illustrated in Fig. 4(a), the order parameter converges to $R = \frac{1}{2}$ as both λ and m approach zero. Moreover, in the vanishing-inertia regime, the minimum coupling λ_c^B required to sustain partial synchronization scales approximately linearly with m [see Fig. 4(b)]. Note that for nonvanishing, finite masses, the system is not able to reach $R = 1/2$ at $\lambda = 0$ and the order parameter remains finite for $\lambda_c^B > 0$, scaling approximately with $\sqrt{m}$.

IV. CONCLUSIONS

In conclusion, we generalized the Kuramoto model with inertia to arbitrary dimensions, describing a system of interacting, orientable rotators, whose state is completely described by D -dimensional unit vectors. We showed that inertia fundamentally reshapes the macroscopic dynamics of the system: the synchronization transition is continuous at low inertia and becomes discontinuous beyond a dimension-dependent threshold. In general, the inertia required to induce a first-order transition decreases with increasing dimensionality D . In the limit of vanishing inertia, our framework recovers the D -dimensional Kuramoto model, including its characteristic dependence on dimensional parity. Furthermore, we derived analytical predictions for both forward and backward synchronization transitions for the case $D = 3$ and validated them through large-scale numerical simulations. Altogether, our results establish inertia as a key control parameter of high-dimensional collective dynamics and provide a unified theoretical framework for synchronization phenomena beyond the phase-only description.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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