

One- and two-boundary problems for jump processes that slowly converge to Gaussian behavior

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We consider jump processes with finite variance which, from the point of view of experimental studies, apparently do not converge to the expected Gaussian behavior. A mathematically tractable example of such processes are those with a distribution of jump lengths in the form of a weighted sum of exponential distributions. For such processes on a half-line and on a finite interval, we calculate the probability densities of arrival at a given time at a given point on the line. The found probability densities allow us to calculate many quantities important for applications: coordinate probability density functions, first passage times, leapover-length distributions, splitting probabilities, conditional exit times, and others. The following results are obtained: for a given variance of the jump-length distribution, the ratio of the mean leapover length to the mean jump length can be arbitrarily large; for a given variance of the jump-length distribution and a given interval length, the transmission probability can be arbitrarily small; for a given variance of the jump-length distribution, a given interval length, and a given mean waiting time, the mean exit time can be arbitrarily large. These results are radically different from those previously obtained using the continuous limit for processes rapidly converging to Gaussian behavior.

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I. INTRODUCTION

Jump processes are relevant in various contexts [1]. They traditionally represent paradigmatic models of photon and neutron transport in scattering media [2–4]. Recently, they have regained interest in the context of self-propelled particles [5–9]. Jump processes are also suitable for describing discrete empirical time series [10].

Recently, a large number of theoretical works have appeared devoted to jump processes [11–17]. These works study various characteristics of such processes: first-passage times, leapover-length distributions, splitting probabilities, conditional exit times, and others. In all these works, with the exception of those that apply the big jump principle, the behavior of processes is considered only at large times, when the continuous limit is reached. Intermediate times, when the behavior of processes differs greatly from the behavior of limiting stable processes, have not been considered so far. But there are processes for which this transition period is very long. In some cases, it does not end throughout the entire experiment. The well-known truncated Lévy flights [18–23] belong to this type of processes. A characteristic feature of these processes is the “heavy-tailed” or “leptokurtic” nature of the jump-length distribution, the quantitative measure of which in applied research is considered to be the kurtosis of the distribution [24].

The kurtosis of the distribution of the sum of K independent identically distributed random variables is equal to κ/K where κ is the kurtosis of the distribution of an individual variable. Since the kurtosis of the Gaussian distribution is zero, then in order for the distribution of the sum of K variables to be close to Gaussian, it is necessary that the value of κ/K be close to zero, i.e., be less than some small value ϵ . For this, K must be greater than κ/ϵ . Therefore, for large kurtosis, the time to converge to Gaussian behavior will be large.

In this paper, the characteristics of jump processes are calculated for processes with a jump-length distribution in the form of a weighted sum of exponential distributions:

$$q(l) = \frac{1}{2} \sum_{i=1}^N \alpha_i v_i \exp(-v_i |l|), \quad (1)$$

where v_i are positive numbers and α_i are positive numbers satisfying condition $\sum_{i=1}^N \alpha_i = 1$. If we take a sufficiently large N , then any symmetric continuous distribution can be approximated by such a sum with any given accuracy. At large N , the expressions for the characteristics of the jump process become cumbersome, but to study the qualitative influence of the “heavy-tailed” nature of the jump-lengths distribution on these characteristics, N equal to two is sufficient. This can be seen from the following example; the distribution (1) with parameters $N = 2$:

$$v_1 = 1/(\bar{l}C), \quad (2)$$

$$v_2 = C/\bar{l}, \quad (3)$$

$$\alpha_1 = \frac{1 - 1/C^2}{C^2 - 1/C^2}, \quad (4)$$

$$\alpha_2 = \frac{C^2 - 1}{C^2 - 1/C^2}, \quad (5)$$

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where $\bar{l}$ and C are arbitrary positive numbers, has a variance equal to $2\bar{l}^2$ and a kurtosis equal to $6(C^2 + 1/C^2) - 9$. Since C is an arbitrary number, this kurtosis can take arbitrarily large values and therefore the time for the jump process with such a distribution to converge to Gaussian behavior can be arbitrarily large. Using this distribution, we have shown that the exact characteristics of the jump process can differ radically from those obtained in the continuous limit:

(1) In the continuous limit, there is no jump over the boundary (leapover) for distributions with finite variance. In reality, a leapover always exists, and the mean leapover length can be many times greater than the mean jump length. (2) In the continuous limit, all distributions of jump lengths with the same variance yield the same transmission probability value, say π . In reality, the transmission probability values for different distributions with the same variance fill the interval $(0, \pi)$. (3) In the continuous limit, all distributions of jump lengths with the same variance yield the same mean escape time value, say T . In reality, the mean escape time values for different distributions with the same variance fill the interval (T, ∞) .

The jump process used in this paper is continuous time random walks (CTRW) [25–30]. This model makes it possible to consider both Markov and non-Markov processes; the frequently used discrete-time random walks model is a special case of it [30].

The article is organized as follows. The first section considers the jump process on the half-line $x > 0$. The probability densities of arrival at a given time at a point on the half-line $x > 0$ and at a point on the half-line $x < 0$ are calculated. Then the first passage time distribution and the leapover-length distribution are calculated (The leapover length measures how far from the boundary the process ends.). The second section considers the process on the interval $[-L, L]$. The probability densities of arrival at a given time at a point on the half-line $x > L$, at a point on the half-line $x < -L$, and at a point on the interval $[-L, L]$ are calculated. Then the splitting probabilities and the mean conditional exit times are calculated. The third section is the conclusion.

II. ONE-BOUNDARY PROBLEMS

A. Equations

In this section, we consider random walks on the half-line $x > 0$. A particle starts from a point x_0 lying on this half-line; if as a result of a jump it gets to a point lying on the half-line $x < 0$, then the random walks stop. We will deal with the forward and backward equations for the joint-probability density that at time t the particle arrives at point $x \in (-\infty, \infty)$, $P(t, x, x_0)$. Since this density has different analytical representations in the absorbing and nonabsorbing regions, these equations have to be solved separately for the part of this density corresponding to $x > 0$, $P_+(t, x, x_0)$, and the part corresponding to $x < 0$, $P_-(t, x, x_0)$.

The forward equation has the following form:

$$P(t, x, x_0) = \psi(t)q(x - x_0) + \int_0^t \int_0^\infty \psi(t - \tau)q(x - y) \times P(\tau, y, x_0)d\tau dy. \quad (6)$$

Here, $\psi(t)$ is the probability density to make a jump at time t . This equation has the following meaning. The probability of arrival at point x at time t consists of two parts: the probability of arrival at this point as a result of the first jump and the probability that at some time τ the particle arrives at some point y and then after time $t - \tau$ makes a jump from y to x . Since τ can be any number from the interval $(0, t)$, and y can be any number from the interval $(0, \infty)$, the integration is performed over these variables.

The backward equation looks like

$$P(t, x, x_0) = \psi(t)q(x - x_0) + \int_0^t \int_0^\infty \psi(\tau)q(y - x_0) \times P(t - \tau, x, y)d\tau dy. \quad (7)$$

This equation has the following meaning. The probability of arrival at point x at time t consists of two parts: the probability of arrival at this point as a result of the first jump and the probability that at some time τ the particle makes the first jump from x_0 to some point y and then after time $t - \tau$ arrives at x . Since τ can be any number from the interval $(0, t)$, and y can be any number from the interval $(0, \infty)$, the integration is performed over these variables.

Relation (6) is valid for any x , but it is an equation only for $P_+(t, x, x_0)$ (because the right side of this relation contains only $P_+(t, x, x_0)$). $P_-(t, x, x_0)$, can be found from this relation after $P_+(t, x, x_0)$ is known. If $P_-(t, x, x_0)$ is known (if it is found from the backward equation), then $P_+(t, x, x_0)$ can be found from Eq. (6) in the standard way using the Fourier-Laplace transform. In Eq. (7), the independent variable is x_0 , and x is a parameter, so both $P_+(t, x, x_0)$ and $P_-(t, x, x_0)$ can be found directly from this equation.

In this paper, only symmetric distributions $q(l)$ are considered, so the forward and backward equations for the function $P_+(t, x, x_0)$ coincide. It follows that this function is invariant with respect to the permutation of the variables x and x_0 and that this function can be found from either of the two equations with equal success.

We will find both $P_+(t, x, x_0)$ and $P_-(t, x, x_0)$ from the backward equation. First, we will find $P_-(t, x, x_0)$. This task is simpler because, in this case, condition $x < x_0$ is always satisfied and therefore $P_-(t, x, x_0)$ has a single analytical representation in the entire domain of its definition. At the same time, function $P_+(t, x, x_0)$ has different analytical representations in domains $x < x_0$ and $x > x_0$, which complicates the solution of the problem.

B. Function $P_-(s, x, x_0)$

In Laplace images, Eq. (7) has the following form (We will denote the Laplace images with the same symbols as the originals, distinguishing them by their arguments: $f(s) = \int_0^\infty \exp(-st)f(t)dt$):

$$P(s, x, x_0) = \psi(s)q(x - x_0) + \psi(s) \int_0^\infty q(y - x_0) \times P(s, x, y)dy. \quad (8)$$

We are looking for a solution to this equation in the region $x < 0$. Since x_0 is always greater than zero, then in this case the expression $|x - x_0|$ will be equal to $x_0 - x$. As is known, if the

distribution of jump lengths is a weighted sum of exponential distributions, then all characteristics of the process will have the form of a combination of exponential functions [31–33]. Therefore, we will look for a solution in the form

$$P_-(s, x, x_0) = \sum_{j=1}^N \beta_j(s, x) \exp(-\lambda_j x_0). \quad (9)$$

Substituting expressions (1) and (9) into Eq. (8) shows that this equation will be satisfied if the parameters λ_j are N different positive solutions of the equation

$$\sum_{i=1}^N \frac{\alpha_i v_i^2}{v_i^2 - \lambda_j^2} = \frac{1}{\psi(s)}, \quad (10)$$

and the parameters β_j are the solution to the linear system of equations

$$\sum_{j=1}^N \frac{\beta_j}{v_i - \lambda_j} = \exp(v_i x), \quad i = 1, \dots, N. \quad (11)$$

For $N = 2$, positive solutions to Eq. (10) are

$$\lambda_1 = \sqrt{d - c} \quad (12)$$

and

$$\lambda_2 = \sqrt{d + c}, \quad (13)$$

where

$$d = \frac{v_1^2 + v_2^2 - \psi(s)(\alpha_1 v_1^2 + \alpha_2 v_2^2)}{2}, \quad (14)$$

$$c = \sqrt{d^2 - (1 - \psi(s))v_1^2 v_2^2}. \quad (15)$$

The solutions to Eqs. (11) are

$$\beta_1 = \frac{(\lambda_1 - v_1)(\lambda_1 - v_2)}{(v_1 - v_2)(\lambda_1 - \lambda_2)} \{(\lambda_2 - v_1) \exp(v_1 x) - (\lambda_2 - v_2) \exp(v_2 x)\}, \quad (16)$$

$$\beta_2 = \frac{(\lambda_2 - v_1)(\lambda_2 - v_2)}{(v_1 - v_2)(\lambda_2 - \lambda_1)} \{(\lambda_1 - v_1) \exp(v_1 x) - (\lambda_1 - v_2) \exp(v_2 x)\}. \quad (17)$$

In the Specific Characteristics section, this expression will be used to analyze the dependence of the leapoverlength distribution on the parameter C (i.e., on the kurtosis of the jump-length distribution).

C. Function $P_+(s, x, x_0)$

To simplify the calculations, we will isolate the nonanalytical component from the desired function. We will represent it in the form

$$P_+(s, x, x_0) = P^\infty(s, x, x_0) - R(s, x, x_0), \quad (18)$$

where $R(s, x, x_0)$ is the auxiliary function to be determined and $P^\infty(s, x, x_0)$ is a function similar to $P(s, x, x_0)$ but for an

infinite line. It satisfies the equation

$$P^\infty(s, x, x_0) = \psi(s)q(x - x_0) + \psi(s) \int_{-\infty}^{\infty} q(y - x_0) \times P^\infty(s, x, y) dy \quad (19)$$

and, in the case of the distribution $q(l)$ given by formula (1), has the following form:

$$P^\infty(s, x, x_0) = \sum_{k=1}^N d_k \exp(-\lambda_k |x - x_0|), \quad (20)$$

where

$$d_k = \frac{\prod_{m=1}^N (v_m^2 - \lambda_k^2)}{2\lambda_k \prod_{m \neq k} (\lambda_m^2 - \lambda_k^2)}. \quad (21)$$

Substituting expression (18) into Eq. (8) and using Eq. (19), we obtain the following equation for the auxiliary function $R(s, x, x_0)$:

$$R(s, x, x_0) = \psi(s) \int_{-\infty}^0 q(y - x_0) P^\infty(s, x, y) dy + \psi(s) \int_0^{\infty} q(y - x_0) R(s, x, y) dy. \quad (22)$$

The first term on the right side of this equation after calculating the integral takes the form

$$\sum_{i=1}^N \sum_{k=1}^N \frac{\psi(s) \alpha_i v_i d_k}{2(v_i + \lambda_k)} \exp(-v_i x_0 - \lambda_k x). \quad (23)$$

Now we see that Eq. (22) differs from the equation for the function P_- only in the coefficients in front of the functions $\exp(-v_i x_0)$ in the first term on the right-hand side. It follows that the solution to this equation will have the form (9) with parameters λ_j determined from Eq. (10) and parameters β_j , determined from the system of equations with the same matrix as in Eq. (11) but with different right-hand sides:

$$\sum_{j=1}^N \frac{\beta_j}{v_i - \lambda_j} = \sum_{k=1}^N \frac{d_k}{v_i + \lambda_k} \exp(-\lambda_k x), \quad i = 1, \dots, N. \quad (24)$$

For $N = 2$, we obtain the following expression for P_+ :

$$\begin{aligned} P_+(s, x, x_0) = & \frac{(v_1^2 - \lambda_1^2)(v_2^2 - \lambda_1^2)}{2\lambda_1(\lambda_2^2 - \lambda_1^2)} \exp(-\lambda_1 |x - x_0|) \\ & + \frac{(v_1^2 - \lambda_2^2)(v_2^2 - \lambda_2^2)}{2\lambda_2(\lambda_1^2 - \lambda_2^2)} \exp(-\lambda_2 |x - x_0|) \\ & + \frac{(v_1 - \lambda_1)(v_2 - \lambda_1)(v_1 - \lambda_2)(v_2 - \lambda_2)}{(\lambda_2^2 - \lambda_1^2)(\lambda_2 - \lambda_1)} \\ & \times \{\exp(-\lambda_1 x - \lambda_2 x_0) + \exp(-\lambda_2 x - \lambda_1 x_0)\} \\ & - \sum_{i=1}^2 \frac{(\lambda_i - v_1)^2 (\lambda_i - v_2)^2}{2\lambda_i (\lambda_2 - \lambda_1)} \exp(-\lambda_i x - \lambda_i x_0). \end{aligned} \quad (25)$$

In the next section, the consistency of this expression with expression (17) will be demonstrated, thereby confirming the correctness of both of these expressions.

D. Specific characteristics

Integrating the function $P_-(t, x, x_0)$ over time from zero to infinity yields the density of the leapover-length distribution $Le(y, x_0)$. In Laplace domain, this distribution is equal to the value of the function $P_-(s, x, x_0)$ at $s = 0$. Considering that $\lambda_1 = 0$ and $\lambda_2 = \nu_1 \nu_2 \bar{l}$ for $s = 0$, we find from formulas (9), (16), and (17)

$$\begin{aligned} Le(y, x_0) &= \exp(-\nu_1 y) \frac{\nu_1(\nu_2 \bar{l} - 1)}{\bar{l}(\nu_1 - \nu_2)} \{(1 - \nu_1 \bar{l}) \exp(-\lambda_2 x_0) - 1\} \\ &+ \exp(-\nu_2 y) \frac{\nu_2(\nu_1 \bar{l} - 1)}{\bar{l}(\nu_1 - \nu_2)} \{(\nu_2 \bar{l} - 1) \exp(-\lambda_2 x_0) + 1\}. \end{aligned} \quad (26)$$

Here, $y = -x$ is the leapover length; $\bar{l} = \sqrt{\alpha_1/\nu_1^2 + \alpha_2/\nu_2^2}$. The average jump length, $\bar{y}(x_0) = \int_0^\infty y Le(y, x_0) dy$, is

$$\begin{aligned} \bar{y}(x_0) &= \frac{(\nu_2 \bar{l} - 1)}{\nu_1 \bar{l}(\nu_1 - \nu_2)} + \frac{(1 - \nu_1 \bar{l})}{\nu_2 \bar{l}(\nu_1 - \nu_2)} \\ &- \frac{(\nu_2 \bar{l} - 1)(1 - \nu_1 \bar{l})}{\nu_1 \nu_2 \bar{l}} \exp(-\lambda_2 x_0). \end{aligned} \quad (27)$$

In the particular case of the distribution $q(l)$ given by formulas (2)–(5), it takes the form more convenient for analysis:

$$\bar{y}(x_0) = \bar{l} \left\{ \frac{C^3 + 1}{C^2 + C} - \frac{(C - 1)^2}{C} \exp\left(-\frac{x_0}{\bar{l}}\right) \right\}. \quad (28)$$

From here, we find that for $x_0 = 0$ the average leapover length does not depend on C and is equal to $\bar{l}$, and for $x_0 \rightarrow \infty$ it is approximately equal to $\bar{l}C$ for large C . At the same time, the average jump length (equal to $\alpha_1/\nu_1 + \alpha_2/\nu_2$) is approximately $\bar{l}/C$ for large C . Therefore, for a fixed variance of the jump-lengths distribution (equal to $2\bar{l}^2$), the ratio of the average leapover length to the average jump length can take on arbitrarily large values both for $x_0 \rightarrow \infty$ and for $x_0 = 0$. Previously, a similar result was obtained in [34] for distributions $q(l)$ decreasing at infinity according to a power law. In the case considered here, this seemingly paradoxical phenomenon is easily explained.

Let us assume that the particle makes all jumps to region $x < 0$ from point x^* . Then, the leapover-length distribution will have the form

$$Le(y) = \frac{\sum_{i=1}^2 \alpha_i \nu_i \exp(-\nu_i(x^* + y))}{\sum_{i=1}^2 \alpha_i \nu_i \int_0^\infty \exp(-\nu_i(x^* + y)) dy}. \quad (29)$$

This can be written as $Le(y) = \sum_{i=1}^2 \alpha'_i \nu_i \exp(-\nu_i y)$, where $\alpha'_i = \frac{\alpha_i \exp(-\nu_i x^*)}{\sum_{i=1}^2 \alpha_i \exp(-\nu_i x^*)}$. The ratio $\frac{\alpha'_1}{\alpha'_2}$ is equal to $\frac{\alpha_1}{\alpha_2} \exp((\nu_2 - \nu_1)x^*)$. If ν_2 is greater than ν_1 , then this ratio is greater than $\frac{\alpha_1}{\alpha_2}$, and if ν_2 is less than ν_1 , then it is less than $\frac{\alpha_1}{\alpha_2}$. But in both cases, the weight of the wider term in the sum $\sum_{i=1}^2 \alpha'_i \nu_i \exp(-\nu_i y)$ is greater than in the sum

$\frac{1}{2} \sum_{i=1}^2 \alpha_i \nu_i \exp(-\nu_i |l|)$. Consequently, the width of the leapover-length distribution is always greater than the width of the jump-length distribution, and it is greater the greater x^* . Since, in a real process, jumps to region $x < 0$ are made from different points, the real leapover-length distribution will be a weighted sum (integral) of distributions (29) with different x^* . For some values of parameters α_i and ν_i , the ratio of the average leapover length to the average jump length becomes arbitrarily large.

Integrating the function $P_-(t, x, x_0)$ over x from $-\infty$ to 0 yields the density of the first passage time distribution: $F(t, x_0) = \int_0^\infty P_-(t, x, x_0) dx$. In Laplace domain, we have from formulas (9), (16), and (17)

$$\begin{aligned} F(s, x_0) &= \frac{\lambda_2(\lambda_1 - \nu_1)(\lambda_1 - \nu_2)}{\nu_1 \nu_2 (\lambda_2 - \lambda_1)} \exp(-\lambda_1 x_0) \\ &- \frac{\lambda_1(\lambda_2 - \nu_1)(\lambda_2 - \nu_2)}{\nu_1 \nu_2 (\lambda_2 - \lambda_1)} \exp(-\lambda_2 x_0). \end{aligned} \quad (30)$$

The function $P_+(s, x, x_0)$ is closely related to the coordinate probability density function $\rho(s, x, x_0)$:

$$\rho(s, x, x_0) = \frac{1 - \psi(s)}{s} \{\delta(x - x_0) + P_+(s, x, x_0)\}. \quad (31)$$

Integrating the coordinate probability density function over x gives the survival probability: $Q(s, x_0) = \int_0^\infty \rho(s, x, x_0) dx$. Taking into account expression (25), we obtain

$$\begin{aligned} Q(s, x_0) &= \frac{1}{s} - \frac{\lambda_2(\lambda_1 - \nu_1)(\lambda_1 - \nu_2)}{s \nu_1 \nu_2 (\lambda_2 - \lambda_1)} \exp(-\lambda_1 x_0) \\ &+ \frac{\lambda_1(\lambda_2 - \nu_1)(\lambda_2 - \nu_2)}{s \nu_1 \nu_2 (\lambda_2 - \lambda_1)} \exp(-\lambda_2 x_0). \end{aligned} \quad (32)$$

Expressions (30) and (32) satisfy the necessary relationship between the first passage time distribution and survival probability: $F = 1 - sQ$. Expression (32) was previously obtained in a different way in [35].

III. TWO-BOUNDARY PROBLEMS

A. Function $P_-(s, x, x_0)$

In this section, we consider random walks on the interval $[-L, L]$. The forward and backward equations for function $P(s, x, x_0)$ in this case differ from the equations considered in the previous section only in the limits of integration over the spatial variable. In particular, the backward equation in the Laplace space has the form

$$P(s, x, x_0) = \psi(s) q(x - x_0) + \psi(s) \int_{-L}^L q(y - x_0) P(s, x, y) dy. \quad (33)$$

Function $P(s, x, x_0)$ will have different analytical representations in regions $x < -L$, $x > L$, and $x \in [-L, L]$. We denote by $P_+(s, x, x_0)$ the part of the function $P(s, x, x_0)$ corresponding to the region $[-L, L]$, and by $P_-(s, x, x_0)$ the part of the function $P(s, x, x_0)$ corresponding to the region $x > L$. The part of the function $P(s, x, x_0)$ corresponding to the region $x < -L$ is simply expressed through $P_-(s, x, x_0)$: $P(s, x, x_0) = P_-(s, -x, -x_0)$ for $x < -L$.

Let us find a solution for region $x > L$. In this region, the first term on the right side of Eq. (33) has the form $\frac{\psi(s)}{2} \sum_{i=1}^N \alpha_i v_i \exp(v_i(x_0 - x))$. We seek a solution in the form

$$P_-(s, x, x_0) = \sum_{j=1}^N \{\beta_j(x) \exp(-\lambda_j x_0) + \gamma_j(x) \exp(\lambda_j x_0)\}. \quad (34)$$

Substituting expressions (1) and (34) into Eq. (33) shows that this equation will be satisfied if the parameters λ_j are N different solutions of Eq. (10) and the parameters β_j and γ_j are the solution to the linear system of equations

$$\begin{aligned} \sum_{j=1}^N \beta_j \frac{\exp(\lambda_j L)}{v_i - \lambda_j} + \gamma_j \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} &= 0, \\ \sum_{j=1}^N \beta_j \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} + \gamma_j \frac{\exp(\lambda_j L)}{v_i - \lambda_j} &= \exp(v_i(L - x)), \\ i &= 1, 2, \dots, N. \end{aligned} \quad (35)$$

This system of $2N$ equations with $2N$ unknowns β_j and γ_j is transformed into two systems of N equations with N unknowns $\sigma_j = \beta_j + \gamma_j$ and $\delta_j = \beta_j - \gamma_j$:

$$\begin{aligned} \sum_{j=1}^N \sigma_j \left\{ \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} + \frac{\exp(\lambda_j L)}{v_i - \lambda_j} \right\} &= \exp(v_i(L - x)), \\ i &= 1, 2, \dots, N \end{aligned} \quad (36)$$

and

$$\begin{aligned} \sum_{j=1}^N \delta_j \left\{ \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} - \frac{\exp(\lambda_j L)}{v_i - \lambda_j} \right\} &= \exp(v_i(L - x)), \\ i &= 1, 2, \dots, N. \end{aligned} \quad (37)$$

We present the solution of these equations only for $N = 1$, since in the case of $N = 2$ it is rather cumbersome. Some particular characteristics of the process for $N = 2$ will be found below from the equations that follow from Eq. (33). In the case of $N = 1$ we obtain

$$\begin{aligned} P_-(s, x, x_0) &= (v^2 - \lambda^2) \exp(v(L - x)) \\ &\times \frac{(v + \lambda) \exp(\lambda(L + x_0)) - (v - \lambda) \exp(-\lambda(L + x_0))}{(v + \lambda)^2 \exp(2\lambda L) - (v - \lambda)^2 \exp(-2\lambda L)}, \end{aligned} \quad (38)$$

where $\lambda = v\sqrt{1 - \psi(s)}$.

Using the known function $P_-(s, x, x_0)$, the following quantities can be found:

(1) Laplace images of the distributions of exit times through the right and left boundaries: $F_r(s, x_0) = \int_L^\infty P_-(s, x, x_0) dx$ and $F_l(s, x_0) = F_r(s, -x_0)$.

(2) Splitting probabilities, which are equal to the probabilities that the exit occurs through the right or left boundary: $\pi_r(x_0) = F_r(0, x_0)$ and $\pi_l(x_0) = F_l(0, x_0)$.

(3) Mean conditional exit times, which are equal to the mean exit times calculated under the conditions that the exit occurs through the right or left boundary: $T_r(x_0) =$

$$-\frac{1}{\pi_r(x_0)} \frac{d}{ds} F_r(s, x_0)|_{s=0} \quad \text{and} \quad T_l(x_0) = -\frac{1}{\pi_l(x_0)} \frac{d}{ds} F_l(s, x_0)|_{s=0} \quad [36].$$

(4) First hit distributions that are equal to the probabilities that the process ends at a point belonging to region $x > L$ or at a point belonging to region $x < -L$: $H_r(x, x_0) = P_-(0, x, x_0)$ and $H_l(x, x_0) = P_-(0, -x, -x_0)$ [37].

(5) Leapover-length distribution: $L(y, x_0) = H_r(L + y, x_0) + H_l(-L - y, x_0)$ [38].

B. Function $P_+(s, x, x_0)$

As in the previous section, we will look for the function $P_+(s, x, x_0)$ in the form (18). For function $R(s, x, x_0)$ we have the equation

$$\begin{aligned} R(s, x, x_0) &= \psi(s) \int_{-\infty}^{-L} q(y - x_0) P^\infty(s, x, y) dy \\ &+ \psi(s) \int_L^\infty q(y - x_0) P^\infty(s, x, y) dy \\ &+ \psi(s) \int_{-L}^L q(y - x_0) R(s, x, y) dy. \end{aligned} \quad (39)$$

The first and second terms on the right side of this equation after calculating the integrals take the form

$$\sum_{i=1}^N \sum_{k=1}^N \frac{\psi(s) \alpha_i v_i d_k}{2(v_i + \lambda_k)} \exp(-v_i(L + x_0) - \lambda_k(L + x)) \quad (40)$$

and

$$\sum_{i=1}^N \sum_{k=1}^N \frac{\psi(s) \alpha_i v_i d_k}{2(v_i + \lambda_k)} \exp(-v_i(L - x_0) - \lambda_k(L - x)). \quad (41)$$

The solution of Eq. (39) has the form (34) with the same parameters λ_j and with parameters β_j and γ_j determined from the system of equations

$$\begin{aligned} \sum_{j=1}^N \beta_j \frac{\exp(\lambda_j L)}{v_i - \lambda_j} + \gamma_j \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} &= \sum_{k=1}^N \frac{d_k}{v_i + \lambda_k} \exp(-\lambda_k(L + x)), \\ \sum_{j=1}^N \beta_j \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} + \gamma_j \frac{\exp(\lambda_j L)}{v_i - \lambda_j} &= \sum_{k=1}^N \frac{d_k}{v_i + \lambda_k} \exp(-\lambda_k(L - x)), \quad i = 1, 2, \dots, N. \end{aligned} \quad (42)$$

This system of $2N$ equations is reduced, as in the case of function P_- , to two systems of N equations with N unknowns $\sigma_j = \beta_j + \gamma_j$ and $\delta_j = \beta_j - \gamma_j$. In the case of $N = 1$, the following expression is obtained for function $P_+(s, x, x_0)$

when $x > x_0$:

$$P_+(s, x, x_0) = \frac{v^2 - \lambda^2}{2\lambda} \{ (v + \lambda) \exp(\lambda(L - x)) - (v - \lambda) \exp(-\lambda(L - x)) \} \frac{(v + \lambda) \exp(\lambda(L + x_0)) - (v - \lambda) \exp(-\lambda(L + x_0))}{(v + \lambda)^2 \exp(2\lambda L) - (v - \lambda)^2 \exp(-2\lambda L)} \quad (43)$$

The expression for $P_+(s, x, x_0)$ in the case $x < x_0$ is obtained from Eq. (43) by replacing x and x_0 with $-x$ and $-x_0$. A special case of this expression when x_0 lies on the boundary of the interval for a process with discrete time was previously obtained in [15].

Using the known function $P_+(s, x, x_0)$, the coordinate probability density function, survival probability, and first passage time distribution can be calculated. In addition, it can be used in the study of extreme value statistics [15,39].

C. Splitting probabilities

The splitting probability and the mean conditional exit time can be found directly, without calculating the function $P_-(s, x, x_0)$. The splitting probability $\pi_r(x_0)$ is found from equation

$$\pi_r(x_0) = \int_{L-x_0}^{\infty} q(y) dy + \int_{-L}^L q(y - x_0) \pi_r(y) dy, \quad (44)$$

which is obtained from Eq. (33) as a result of integrating over x from L to infinity and substituting $s = 0$ (i.e., $\psi(s) = 1$). Since the splitting probability must be of the form $1/2 + f(x_0)$, where $f(x_0)$ is an odd function, it is sought in the form

$$\pi_r(x_0) = \frac{1}{2} + bx_0 + 2 \sum_{j=1}^{N-1} \beta_j \sinh(\lambda_j x_0), \quad (45)$$

where λ_j , $j = 1, 2, \dots, N-1$ are the nonzero positive roots of Eq. (10) with $\psi(s)$ equal to one. Substitution of this expression into Eq. (44) shows that the parameters b and β_j should be found from the system of equations

$$\begin{aligned} & \sum_{j=1}^{N-1} \beta_j \left\{ \frac{\exp(\lambda_j L)}{v_i - \lambda_j} - \frac{\exp(-\lambda_j L)}{v_i + \lambda_j} \right\} \\ &= b \left\{ \frac{L}{v_i} + \frac{1}{v_i^2} \right\} - \frac{1}{2v_i}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (46)$$

For $N = 2$, we find

$$b = \frac{1}{2L + \frac{v_1 d_1 - v_2 d_2}{v_1 v_2 (d_1 - d_2)}}, \quad (47)$$

$$\beta_1 = b \frac{v_2 - v_1}{v_1 v_2 (d_1 - d_2)}, \quad (48)$$

where

$$\lambda_1 = \sqrt{\alpha_1 v_2^2 + \alpha_2 v_1^2}, \quad (49)$$

$$d_i = \frac{v_i \exp(\lambda_i L)}{v_i - \lambda_i} - \frac{v_i \exp(-\lambda_i L)}{v_i + \lambda_i}, \quad i = 1, 2. \quad (50)$$

If the condition $\lambda_1 L \gg 1$ is met, then the expressions for the parameters b and β_1 are simplified: $b = 1/[2(A\bar{l} + L)]$,

$\beta_1 = b(A - 1)\bar{l} \exp(-\lambda_1 L)$ and the expression for $\pi_r(x_0)$ takes the form

$$\pi_r(x_0) = \frac{A\bar{l} + L + x_0}{2(A\bar{l} + L)} + \frac{2(A - 1)\bar{l} \exp(-\lambda_1 L) \sinh(\lambda_1 x_0)}{2(A\bar{l} + L)}. \quad (51)$$

Here, we introduce the notation $l_1 = 1/v_1$, $l_2 = 1/v_2$,

$$\bar{l} = \sqrt{\alpha_1 l_1^2 + \alpha_2 l_2^2}, \quad (52)$$

$$A = l_1/\bar{l} + l_2/\bar{l} - l_1 l_2 / \bar{l}^2. \quad (53)$$

The parameters of distribution $q(l) = \frac{1}{2} \sum_{i=1}^2 (\alpha_i / l_i) \exp(-l/l_i)$ are expressed through the parameters λ_1 , $\bar{l}$, A according to the formulas

$$l_{1,2} = \frac{A\bar{l}}{2} + \frac{1}{2\lambda_1} \pm \sqrt{\left\{ \frac{A\bar{l}}{2} \right\}^2 + \frac{A\bar{l}}{2\lambda_1} + \frac{1}{4\lambda_1^2} - \frac{\bar{l}}{\lambda_1}}, \quad (54)$$

$\alpha_1 = (\bar{l}^2 - l_2^2)/(l_1^2 - l_2^2)$, $\alpha_2 = 1 - \alpha_1$. It can be seen that for any positive λ_1 and $\bar{l}$, and A greater than one, these formulas yield real positive l_1 and l_2 and coefficients α_1 and α_2 lying in the interval $(0, 1)$. Therefore, there exist distributions of the form $q(l) = \frac{1}{2} \sum_{i=1}^2 (\alpha_i / l_i) \exp(-l/l_i)$, giving any values of the parameters λ_1 , $\bar{l}$, A , satisfying the conditions $\lambda_1 > 0$, $\bar{l} > 0$, $A > 1$. In particular, for any fixed λ_1 and $\bar{l}$, there exist distributions that give arbitrarily large A .

The transmission probability is defined as the probability of exiting through the right boundary given that the starting point is the left boundary [12]. Formula (51) gives the following value for this quantity: $\pi_r(-L) = \bar{l}/[2(A\bar{l} + L)]$. From this it is clear that, since the parameter A can take any value in the interval $(1, \infty)$, the transmission probability can take any value in the interval $0, \bar{l}/[2(\bar{l} + L)]$. This contrasts with the result obtained using the continuous limit [12]. Taking the continuous limit results in all distributions with the same value of the parameter $\bar{l}$ yielding the same transmission probability equal to $\bar{l}/(2L)$.

D. Mean conditional exit times

To find the mean conditional exit time it is necessary to have, in addition to the splitting probability, the derivative $\frac{d}{ds} F_r(s, x_0)|_{s=0}$. The equation for this derivative is obtained by integrating Eq. (33) with respect to x from L to infinity, differentiating it with respect to s , and equating s to zero.

Denoting the value of $-\frac{d}{ds} F_r(s, x_0)|_{s=0}$ through $D_r(x_0)$, we write the resulting equation as

$$D_r(x_0) = \xi \pi_r(x_0) + \int_{-L}^L q(y - x_0) D_r(y) dy, \quad (55)$$

where ξ is the mean waiting time ($\xi = \int_0^\infty t\psi(t)dt = \lim_{s \rightarrow 0} \frac{1-\psi(s)}{s}$). Since this equation is linear, its solution for an inhomogeneous term equal to the sum of several terms can be written as the sum of the solutions corresponding to individual terms. In the expression for the splitting probability (45), there are three types of terms: c_1 , c_2x_0 , and $c_3 \sinh(\lambda_k x_0)$. Let us consider the solution of Eq. (55) separately for an inhomogeneous term of each type.

Inhomogeneous term of type c_1 . In this case, the solution must be an even function, so it is sought in the form

$$D_r(x_0) = a + bx^2 + 2 \sum_{j=1}^{N-1} \beta_j \cosh(\lambda_j x_0). \quad (56)$$

Substituting this expression into Eq. (55) shows that the parameter b must be found from the relation

$$b \sum_{i=1}^N \frac{2\alpha_i}{v_i^2} + c_1 = 0, \quad (57)$$

and the parameters a and β_j must be found from the system of equations

$$\begin{aligned} \sum_{j=1}^{N-1} \beta_j \left\{ \frac{\exp(\lambda_j L)}{v_i - \lambda_j} + \frac{\exp(-\lambda_j L)}{\lambda_j + v_i} \right\} + \frac{a}{v_i} \\ = -b \left\{ \frac{L^2}{v_i} + \frac{2L}{v_i^2} + \frac{2}{v_i^3} \right\}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (58)$$

Inhomogeneous term of type c_2x_0 . In this case, the solution must be an odd function, so it is sought in the form

$$D_r(x_0) = ax_0 + bx_0^3 + 2 \sum_{j=1}^{N-1} \beta_j \sinh(\lambda_j x_0). \quad (59)$$

Substituting this expression into Eq. (55) shows that the parameter b must be found from the relation

$$b \sum_{i=1}^N \frac{6\alpha_i}{v_i^2} + c_2 = 0, \quad (60)$$

and the parameters a and β_j must be found from the system of equations

$$\begin{aligned} \sum_{j=1}^{N-1} \beta_j \left\{ \frac{\exp(\lambda_j L)}{v_i - \lambda_j} - \frac{\exp(-\lambda_j L)}{\lambda_j + v_i} \right\} - a \left\{ \frac{L}{v_i} + \frac{1}{v_i^2} \right\} \\ = b \left\{ \frac{L^3}{v_i} + \frac{3L^2}{v_i^2} + \frac{6L}{v_i^3} + \frac{6}{v_i^4} \right\}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (61)$$

Inhomogeneous term of type $c_3 2 \sinh(\lambda_k x_0)$. In this case, the solution must be an odd function, so it is sought in the form

$$D_r(x_0) = ax_0 + 2bx_0 \cosh(\lambda_k x_0) + 2 \sum_{j=1}^{N-1} \beta_j \sinh(\lambda_j x_0). \quad (62)$$

Substituting this expression into Eq. (55) shows that the parameter b must be found from the relation

$$b \sum_{i=1}^N \frac{2\alpha_i v_i^2 \lambda_k}{(v_i - \lambda_k)^2 (v_i + \lambda_k)^2} + c_3 = 0, \quad (63)$$

and the parameters a and β_j must be found from the system of equations

$$\begin{aligned} \sum_{j=1}^{N-1} \beta_j \left\{ \frac{\exp(\lambda_j L)}{v_i - \lambda_j} - \frac{\exp(-\lambda_j L)}{\lambda_j + v_i} \right\} - a \left\{ \frac{L}{v_i} + \frac{1}{v_i^2} \right\} \\ = b \left\{ \frac{L \exp(-\lambda_k L)}{v_i + \lambda_k} + \frac{\exp(-\lambda_k L)}{(v_i + \lambda_k)^2} \right. \\ \left. + \frac{L \exp(\lambda_k L)}{v_i - \lambda_k} + \frac{\exp(\lambda_k L)}{(v_i - \lambda_k)^2} \right\}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (64)$$

For $N = 2$, Eqs. (58), (61), and (64) are easily solved. We will not write out the resulting cumbersome expressions; we will only give an expression for the mean conditional exit time corresponding to the initial condition $x_0 = 0$, which coincides with the mean unconditional exit time found in [35]. For the distribution $q(l) = \frac{1}{2} \sum_{i=1}^2 (\alpha_i/l_i) \exp(-l/l_i)$ with parameters from Eqs. (2)–(5) the mean exit time is expressed (when the conditions $C \gg 1$ and $L \gg \bar{l}$ are satisfied) by the formula

$$T_r(0) = \frac{\xi(L^2 + 2L\bar{l}C + 2\bar{l}^2 C^2)}{2\bar{l}^2}. \quad (65)$$

Since, for a fixed $\bar{l}$, the parameter C can take any value in the interval $(1, \infty)$, the mean exit time can take any value in the interval $(\xi(L^2 + 2L\bar{l} + 2\bar{l}^2)/(2\bar{l}^2), \infty]$. At the same time, if the transition to the continuous limit is used, then for a fixed $\bar{l}$ the mean exit time is determined uniquely. As can be seen from Eq. (65), this unique value is equal to $\xi L^2/(2\bar{l}^2)$ and is achieved in cases where the interval width satisfies the condition $L \gg 2\bar{l}C$. In the case of distributions with large C this condition is not satisfied and the continuous approximation will give incorrect results.

IV. DISCUSSION

The authors of [18] showed that the physical processes that are often interpreted as Levy flights are, in fact, processes that slowly converge to Gaussian behavior. To describe such processes, they proposed a model of truncated Levy flights. However, this model is difficult to study analytically and, therefore, has not received wide acceptance. Subsequently, proposed modifications of this model are also difficult to use. In addition, all these models are intended to describe only processes of a certain type. They assume that the central part of the jump-length distribution has the form of a stable distribution. However, physical systems are very diverse and there is no reason to assume in advance that all of them will have jump-length distributions of this type.

Until now, the characteristics of jump processes have been calculated approximately using the transition to the continuous limit. In this approximation, the exact form of the jump-length distribution is ignored. The process is assumed to be determined by only two parameters, μ and $\bar{l}$, which appear in the small k -expansion of the Fourier transform of

the jump-length distribution

$$q(k) = 1 - (\bar{l}|k|)^\mu + o(|k|^\mu), \quad (66)$$

where $q(k) = \int_{-\infty}^{\infty} \exp(ikl)q(l)dl$. In the case of a jump-length distribution with finite variance, the parameter μ is equal to 2 and the jump process is reduced to ordinary diffusion. But ordinary diffusion is a process with continuous trajectories that is qualitatively different from the jump process. Some characteristics, such as transmission probability and leapover, are completely absent in ordinary diffusion. Other characteristics of jump processes can be found using the diffusion approximation only in the limit of large times, which is not achieved in the case of processes slowly converging to Gaussian behavior [18]. The authors of the works [12,13] improved the diffusion approximation so that it became suitable to calculate the splitting probabilities and conditional exit times in cases where the starting point is near or on the boundary. But this improved approximation is also inapplicable to processes slowly converging to Gaussian behavior. This is evident from a comparison of the results obtained in this approximation with the exact results obtained in this paper.

The main result of this paper is the solutions of integral equations that allow one to calculate many characteristics of jump processes, including processes slowly converging to Gaussian behavior. Three results are also obtained on specific characteristics.

The first result concerns the leapover-length distribution. This characteristic was previously calculated only for jump-length distributions with infinite variance [34]. For distributions with finite variance, the method used in [34] predicts the absence of leapover. This paper shows that for processes slowly converging to Gaussian behavior (i.e., processes with large C), the leapover-length distribution is always wider than the jump-length distribution, and the larger C , the wider it is. As C increases infinitely, the ratio of the mean leapover length to the mean jump length becomes infinitely large.

The second result concerns the transmission probability. In [12], it is shown that for a sufficiently large interval

width ($2L \gg \bar{l}$) for distributions with finite variance, the transmission probability is equal to $\frac{\bar{l}}{2L}$, that is, the transmission probability is uniquely determined by the value of the parameter $\bar{l}$. In this paper, in contrast, it is shown that for the same value of the parameter $\bar{l}$, the transmission probability can take any value in the interval $(0, \frac{\bar{l}}{2L})$ depending on the value of parameter C .

The third result concerns the mean escape times. In the diffusion approximation, the escape times are uniquely determined by the diffusion coefficient. This means that in all processes with jump-length distributions having the same variance, the escape times are the same. The improved approximation proposed in [13] does not differ from the usual approximation in this respect. In this paper, in contrast, it is shown that different distributions with the same variance can produce mean escape times that fill an interval of infinite length.

Thus, this paper shows that the characteristics of jump processes slowly converging to Gaussian behavior depend significantly on the form of the jump-length distribution. Therefore, approximate methods that ignore the distribution form are not applicable to such processes.

In conclusion, we note that the processes considered in this article can be interpreted as processes with intermittent dynamics. In particular, in the Markov case, a process with jump-length distribution [Eq. (1)] and jump frequency κ can be interpreted as a sum of N Markov processes with jump-length distributions $q_i(x) = \frac{\nu_i}{2} \exp(-\nu_i|x|)$ and frequencies $\kappa_i = \alpha_i \kappa$. Composite processes are often encountered in practice [40–44]. Various combinations of different models have been proposed to describe them [45–49]. The model considered here is one of the possible combinations. This means that the results obtained in this article can be applied to systems with intermittent dynamics.

DATA AVAILABILITY

No data were created or analyzed in this study.

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