

Viscoelastic toy model explaining the static friction paradox in stick-slip instability without friction laws

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The relaxation oscillation in sliding systems is termed stick-slip (SS) since the two surfaces in contact seem to repeat the stick and slip states. However, several precise measurements have revealed that extremely slow slip occurs in even apparently stick states before every stick-to-slip transition, which is here termed the static friction paradox. This study theoretically investigates instabilities induced by solid viscoelasticity using a minimal toy model without friction laws: a two-degrees-of-freedom sliding system consisting of a rigid probe and a Kelvin-Voigt viscoelastic foundation (VEF). Taking advantage of its simplicity, we discuss the mechanisms generating various system dynamics based on stability analysis and numerical simulation. As a result, surprisingly, even though the sliding system does not employ static friction, it exhibits intermittent oscillation resembling the typical SS instability. It comprises two slip states for the contact between the probe and the VEF, i.e., slow-slip and fast-slip states, providing a purely mechanical explanation for the static friction paradox.

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I. INTRODUCTION

The relaxation oscillation in sliding systems is one of the typical friction-induced instabilities, termed *stick-slip* (SS) [1–4], since the two surfaces in contact seem to repeat the *stick* and *slip* states. This fundamental physical phenomenon occurs across scales, from atomic scales (at a scanning probe in contact with a measured sample in atomic force microscopy [5,6]) to geological scales (at a continental plate in contact with an oceanic plate in earthquakes [7–9]). In our daily life scale, the SS instability of our fingertip in contact with an object surface provides our tactile sensations [10,11] and that of a string of bowed string instruments generates the Helmholtz wave, creating musical sounds [12,13]. It sometimes causes undesirable situations in mechanical engineering, such as uncomfortable vibrations and noise emissions from contact surfaces inside machines such as automobiles [14–16].

According to the classical interpretation, the stick and slip states are created by static and kinetic friction, respectively. For example, the famous object-on-slope experiment demonstrates the discontinuous stick-to-slip transition, where the object seems to start suddenly moving when the slope angle reaches a critical value [17], which certainly impresses upon us the existence of the two types of solid friction, as stated by the Coulomb friction law [18]. Contrary to this

macroscopic observation, several precise measurements have revealed that the transitions between the two states are continuous [19–21]. For example, extremely slow *slip* occurs in even apparently *stick* states before every stick-to-slip transition. This paradox, termed the *static friction paradox*, wherein two surfaces in contact nominally stick together but keep slipping from each other, is often discussed based on mathematically described friction laws, termed rate-and-state-dependent friction laws [22,23]. A mechanical explanation for the paradox has been theoretically provided based on a structurally asymmetric sliding model with a velocity-weakening friction law [24].

Although friction laws, such as those listed above, are convenient for dynamical analysis, it is evident that system dynamics depend strongly on the chosen friction law, and SS instability is no exception. Moreover, any friction law embeds the various (e.g., geometrical, physical, and sometimes chemical) properties of contact surfaces into the fitting parameters, such as friction coefficients, which are mandatory for describing it mathematically. Thus, using an explicit friction law in dynamical analysis makes many essential properties invisible, preventing deeper insight into friction mechanisms. That is why *a priori* methods, such as molecular dynamics simulation, have recently become popular for mechanistic studies in tribology [25–28]. However, the technique is usually not applicable to studying SS instabilities since the timescales of molecular dynamics and system dynamics are entirely different unless one is interested in the SS instability of nanoscale systems [5,6]. Thus, a different approach is required to understand SS instabilities across scales, especially in mesoscale to macroscale systems.

Viscoelasticity is one of the essential properties of materials, from soft to even hard materials, and viscoelastic friction is one of the most developed fields of friction modeling without using explicit friction laws. Specifically, experimental

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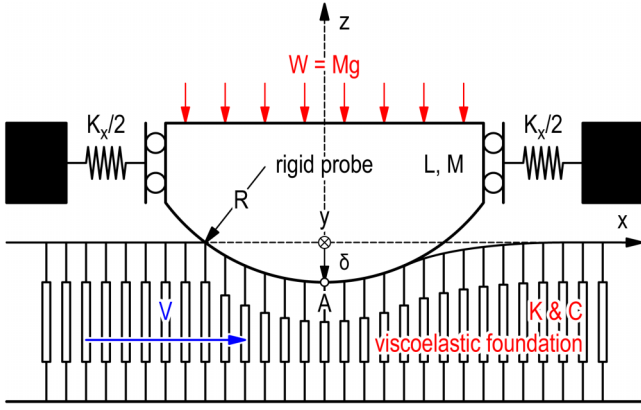


FIG. 1. Analytical model. A parabolic rigid probe (length L , tip curvature radius R , mass M) supported by horizontal springs (total stiffness K_x) is in contact with a planar Kelvin-Voigt viscoelastic foundation (stiffness per unit area K , damping constant per unit area C) moving at constant speed (drive speed V).

studies showed that friction in rubbery materials exhibits a bell-shaped velocity-dependent friction coefficient [29], and theoretical studies started at about the same time [30,31]. For steady sliding, viscoelastic friction has been theoretically described for various countersurface geometries, including cylindrical and spherical probes [32,33], fractal rough surfaces [34], and periodic surfaces [35]. As for unsteady dynamics of viscoelastic materials, several researchers have investigated probe dynamics by coupling with the boundary element method [36,37]. Although they applied the steady-state friction coefficients obtained from the technique to dynamical simulations to avoid massive numerical calculations, their results indicate the significance of the probe motion in unsteady states.

Meanwhile, several researchers have developed fundamental models of viscoelastic friction, termed viscoelastic foundation (VEF) models [38–42], which mechanically represent viscoelastic friction and offer intuitive understandings of it. The VEF models represent viscoelastic friction by modeling viscoelastic solids as a one-dimensional (1D) array of independent 1D rheological elements in contact with a rigid probe. This model reveals steady sliding solutions for various probe geometries [38,42] and two origins for the bell-shaped velocity-dependent friction coefficient [41]. The main feature of VEF models is their simplicity, which provides a complete set of dimensionless numbers that govern the steady response of the system [40–42]. The obtained dimensionless numbers, independent of friction laws and without artificial parameters such as friction coefficients, help us understand phenomena deeply across scales in science and consider primitive

design guidelines from mechanical and material perspectives in engineering.

This article extends the previous VEF model for steady states to unsteady states by adding a horizontal finite stiffness, represented by K_x in Fig. 1, as stated in the following sections in detail. Taking advantage of its simplicity, we first theoretically derive the complete set of dimensionless numbers that govern system stability through the conventional linearization technique, revealing that the solid viscoelasticity acts not only to stabilize but also to destabilize the sliding system. Then, numerical simulations for the original nonlinear equations, surprisingly, find limit cycles that exhibit intermittent oscillation, such as the SS instability with a slow-slip event before a fast-slip event [19–21], even though the model does not consider any explicit friction laws. The above means that the present VEF model is the *minimal toy model* that provides a different explanation for the static friction paradox.

II. MODEL

This study analyzes the VEF model (Fig. 1) to investigate sliding system dynamics induced by solid viscoelasticity. A rigid probe of a parabolic column (y -directional length L , tip curvature radius R , and mass M) is in contact with a VEF under constant load $W = Mg$ and is supported by horizontal springs (total stiffness K_x), where g is the gravity constant. The VEF is composed of independent Kelvin-Voigt elements with the stiffness per unit area K and the damping constant per unit area C . Every element deforms only in the vertical direction, providing only a repulsive force to the probe [39], neglecting attractive and shear forces subject to the probe. When the VEF moves at constant drive speed V , the probe moves in the $x - z$ plane. Namely, it is a two-degrees-of-freedom problem.

Letting the probe tip position be $A(x_A, z_A)$, the shape function $h(x)$ is as follows:

$$h(x) = \frac{(x - x_A)^2}{2R}. \quad (1)$$

The vertical compression distribution $u(x)$ and vertical compression rate distribution Du/Dt of the VEF in contact with the probe are

$$u(x) = -h(x) - z_A = -\frac{(x - x_A)^2}{2R} - z_A, \quad (2)$$

$$\frac{Du}{Dt} = \left(\frac{\partial}{\partial t} + v_{\text{slip}} \frac{\partial}{\partial x} \right) u = -\frac{v_{\text{slip}}(x - x_A)}{R} - \dot{z}_A, \quad (3)$$

where $(\cdot)$ is the derivative with respect to time t , D/Dt is the material derivative along the VEF, and $v_{\text{slip}} = V - \dot{x}_A$ is the slip velocity. The vertical and horizontal components of the local stress distributions $f_z(x)$ and $f_x(x)$ are

$$f_z(x) = Ku + C \frac{Du}{Dt} = K \left(-\frac{(x - x_A)^2}{2R} - z_A \right) + C \left(-\frac{v_{\text{slip}}(x - x_A)}{R} - \dot{z}_A \right), \quad (4)$$

$$f_x(x) = -\frac{dh}{dx} \left(Ku + C \frac{Du}{Dt} \right) = -\frac{(x - x_A)}{R} \left[K \left(-\frac{(x - x_A)^2}{2R} - z_A \right) + C \left(-\frac{v_{\text{slip}}(x - x_A)}{R} - \dot{z}_A \right) \right]. \quad (5)$$

The macroscopic contact load F_z and friction force F_x are

$$F_z = L \int_{-a+x_A}^{b+x_A} f_z(x) dx = L \int_{-a+x_A}^{b+x_A} K \left(-\frac{(x-x_A)^2}{2R} - z_A \right) + C \left(-\frac{v_{\text{slip}}(x-x_A)}{R} - \dot{z}_A \right) dx, \quad (6)$$

$$F_x = L \int_{-a+x_A}^{b+x_A} f_x(x) dx = L \int_{-a+x_A}^{b+x_A} -\frac{(x-x_A)}{R} \left[K \left(-\frac{(x-x_A)^2}{2R} - z_A \right) + C \left(-\frac{v_{\text{slip}}(x-x_A)}{R} - \dot{z}_A \right) \right] dx, \quad (7)$$

where a and b are the contact semiwidth on the inlet and outlet sides, respectively. Then, the equations of motion are obtained as follows:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{pmatrix} \ddot{x}_A \\ \ddot{z}_A \end{pmatrix} = \begin{pmatrix} F_x - K_x x_A \\ F_z - W \end{pmatrix}. \quad (8)$$

Note that F_x and F_z show strong nonlinearity, which makes it difficult to solve Eq. (8) analytically.

III. STABILITY ANALYSIS

This section describes the stability analysis for the present model. First, Appendix A linearizes the equations of motion around the equilibrium point, assuming that the probe velocity is less than the drive speed (i.e., $\dot{x}_A < V$). Then, Appendix B derives the complete set of dimensionless numbers based on the linearized equations of motion. Finally, Appendix C conducts eigenvalue analysis to obtain the minimal modal damping ratio $\zeta_{\min}^*$, defined by Eq. (C6), which enables us to evaluate stability quantitatively.

In Appendix B, we define the dimensionless time τ and the dimensionless x and z positions as follows:

$$\tau = \omega_{nx} t, \quad \tilde{x} = x_A K_x \left(\frac{4LR^2K}{W^4} \right)^{1/3}, \quad \tilde{z} = z_A \left(\frac{32L^2RK^2}{9W^2} \right)^{1/3}, \quad (9)$$

where $\omega_{nx} = (K_x/M)^{1/2}$ is the natural frequency in the x direction. The dimensionless form obtains the following complete set of dimensionless numbers that govern the stability of the system:

$$\begin{aligned} \Omega_1 &= \frac{CV}{K} \left(\frac{LK}{RW} \right)^{1/3}, \quad \Omega_2 = \left(\frac{RK_x}{W} \right)^{1/2}, \\ \Omega_3 &= \frac{C}{K} \left(\frac{W}{RM} \right)^{1/2}, \quad \Omega_4 = \left(\frac{LR^2K}{W} \right)^{1/3}. \end{aligned} \quad (10)$$

The four dimensionless numbers can be described by the following time constants: the retardation time of viscoelastic foundation, $t_1 = C/K$, the travel time within the contact, $t_2 = (RW/LK)^{1/3}/V$, the horizontal response time, $t_3 = (M/K_x)^{1/2}$, the vertical response time (at a representative position $\delta = R$), $t_4 = (RM/W)^{1/2}$, and the vertical response time (at the equilibrium point of stationary contact $\delta = \delta_0$), $t_5 = (M^3/L^2RWK^2)^{1/6}$. Using these time constants, the four dimensionless numbers are $\Omega_1 = t_1/t_2$, $\Omega_2 = t_4/t_3$, $\Omega_3 = t_1/t_4$, and $\Omega_4 = t_4/t_5$. Thus, we term Ω_1 the *dimensionless speed*, Ω_2 the *dimensionless horizontal stiffness*, Ω_3 the *material-to-mechanical time constant ratio*, and Ω_4 the *vertical mechanical time constant ratio*.

Figure 2 shows the stability analysis results, represented by the stability maps colored by the minimum modal damping ratio $\zeta_{\min}^*$. Their horizontal and vertical axes are the dimensionless speed Ω_1 and the dimensionless horizontal stiffness Ω_2 , respectively, under constant Ω_3 and Ω_4 . The blue and red areas show that they are stable ($\zeta_{\min}^* > 0$) and unstable ($\zeta_{\min}^* < 0$), respectively.

We find the distinguished boundaries of L_1 ($\Omega_1 = 1.6$) and L_2 ($\Omega_2 = 1.0$) in every stability map, suggesting that an unstable area (red area) appears in the high-speed region and that an extremely unstable area (deep-red area) is localized in the low-stiffness region. We also find that these two boundaries are independent of Ω_3 and Ω_4 . As clarified in the previous study [40], the friction force in steady sliding shows velocity-strengthening characteristics in the low-speed region ($\Omega_1 < 1.6$). In contrast, it shows velocity-weakening characteristics in the high-speed region ($\Omega_1 > 1.6$) due to the vertical lift of the probe [40]. Therefore, from boundary L_1 , we can conclude that the probe vertical lift is required for destabilizing the system. From boundary L_2 , we find that the lower the horizontal stiffness is, the more strongly the system is destabilized. Note that the stable area exists even in the high-speed region, which is generally thought to be unstable due to velocity-weakening friction without structural damping [1–3]. We should also note that the extremely unstable area is weakened in the high-speed region, as in previous studies [36]. We will discuss this elsewhere with an extended model.

IV. NUMERICAL SIMULATION

This section describes the numerical simulation used to explore the global system dynamics, which the stability analysis in the previous section could not estimate. For the vertical compression distribution $u_i(t)$ and vertical compression rate distribution $Du_i(t)/Dt$ of viscoelastic elements in contact with the probe, we calculate them according to Eqs. (2) and (3). For the elements not in contact with the probe, we use the following:

$$u_i(t) = u_i(t_0) \exp \left(-\frac{t-t_0}{t_c} \right), \quad (11)$$

$$\frac{Du_i(t)}{Dt} = -\frac{u_i(t_0)}{t_c} \exp \left(-\frac{t-t_0}{t_c} \right), \quad (12)$$

where t_0 is the time when the element detaches from the probe surface. The condition determining the transition from noncontact to contact is $u_i(t) < -h(x_i) - z_A(t)$, while the condition from contact to noncontact is $f_{zi}(t) < 0$. We solve the equations of motion (8) using the Euler method under sufficiently discretized time t and space x .

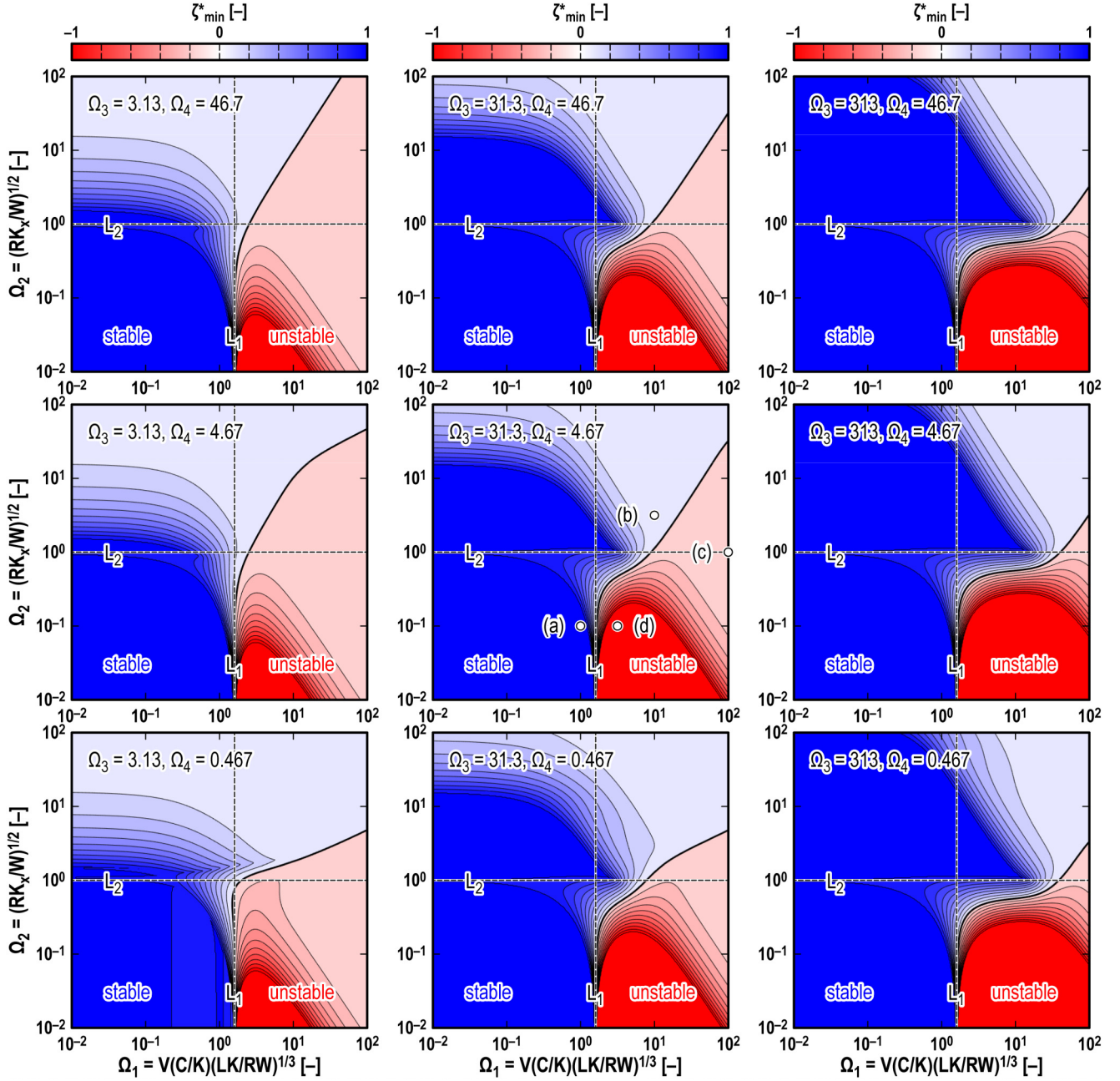


FIG. 2. Stability maps under various Ω_3 and Ω_4 . Points (a) to (d) on the central map represent the simulation parameter sets of (a) to (d) in Figs. 3 and 4.

Four conditions (a) to (d) represented in the middle graph of Fig. 2 were employed for numerical simulation based on the stability analysis results. The present model has six independent parameters (R , M/L , K_x/L , K , C , and V). Four parameters were constant in every calculation ($R = 10$ mm, $M/L = 10^3$ kg/m, $K = 10^{10}$ N/m³, and $C = 10^{10}$ Ns/m³). The remaining two were changed as follows: (a) $K_x/L = 10^4$ N/m² and $V = 2.14$ mm/s, (b) $K_x/L = 10^7$ N/m² and $V = 21.4$ mm/s, (c) $K_x/L = 10^6$ N/m² and $V = 214$ mm/s, and (d) $K_x/L = 10^4$ N/m² and $V = 6.77$ mm/s. These conditions keep Ω_3 and Ω_4 constant ($\Omega_3 = 31.3$ and $\Omega_4 = 4.67$) and change Ω_1 and Ω_2 as follows: (a) $\Omega_1 = 10^0$ and $\Omega_2 = 10^{-1}$, (b) $\Omega_1 = 10^1$ and $\Omega_2 = 10^{1/2}$, (c) $\Omega_1 = 10^2$ and $\Omega_2 = 10^0$, and (d) $\Omega_1 = 10^{1/2}$ and $\Omega_2 = 10^{-1}$. The $\zeta_{\min}^*$ values

obtained in the stability analysis were (a) $\zeta_{\min}^* = 1$, (b) $\zeta_{\min}^* = 0.03$, (c) $\zeta_{\min}^* = -0.01$, and (d) $\zeta_{\min}^* = -1$. The initial point of the probe was the stationary equilibrium point and its initial velocity was zero.

Figure 3 shows the numerical simulation results of the above four conditions. The upper graphs show the temporal changes of the x and z positions of the probe tip A, while the lower graphs show the trajectories in the $x-z$ plane. The open and solid circles represent the initial and equilibrium points, respectively. All axes are dimensionless using the dimensionless variables in Eq. (9). One can find movies that visualize the numerical results in the Supplemental Material [43]. Movies S1 to S4 correspond to Figs. 3(a)–3(d).

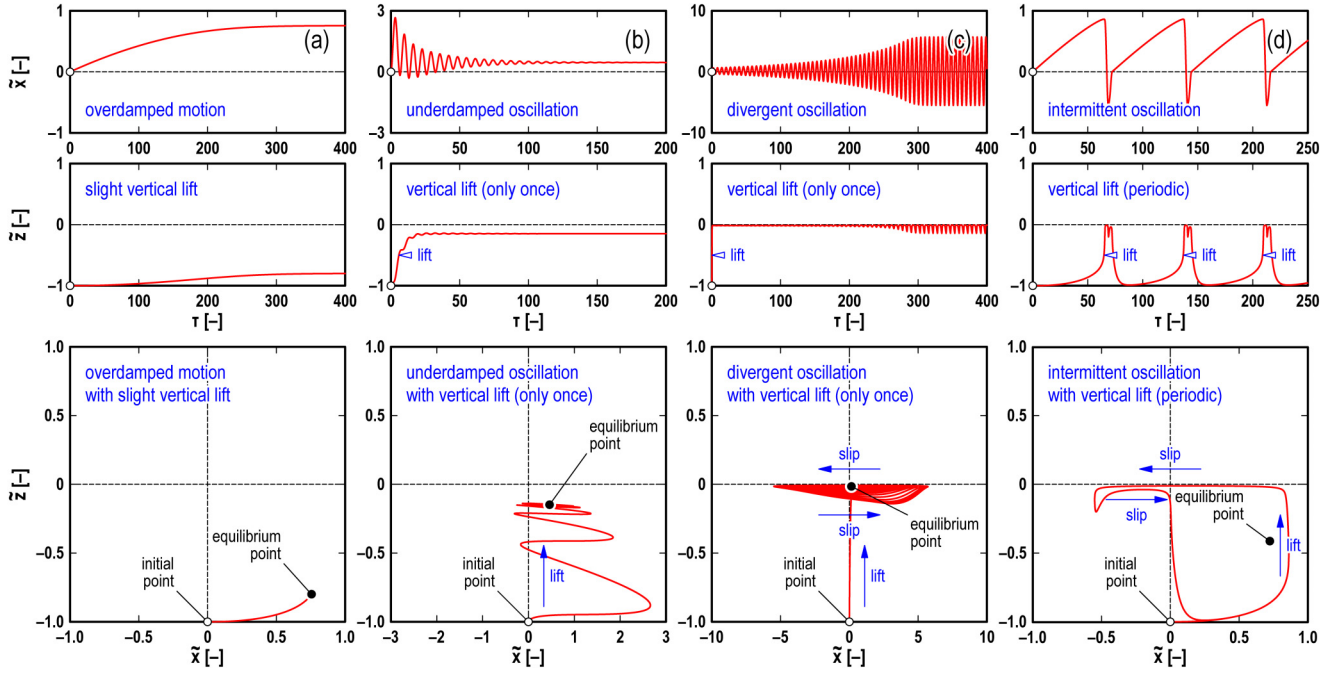


FIG. 3. Numerical results showing (a) overdamped motion, (b) underdamped oscillation, (c) divergent oscillation, and (d) intermittent oscillation. The upper graphs show the temporal changes of the x and z positions of the probe tip A in the dimensionless form. The lower graphs show the trajectories in the $x - z$ plane in the dimensionless form. The open and solid circles are the initial and equilibrium points. (See, also, Movies S1 to S4 in the Supplemental Material [43].)

Figure 3(a) shows extremely stable dynamics ($\zeta_{\min}^* = 1$), in which the probe exhibits overdamped motion in the x direction with slight motion in the z direction. (See, also, Movie S1 in the Supplemental Material [43].) Although the probe is lifted monotonically with increasing drive speed, the vertical lift is negligibly tiny at a low speed such as this condition ($\Omega_1 = 10^0 < 1.6$) [40].

Figure 3(b) shows the moderately stable dynamics ($\zeta_{\min}^* = 0.03$), in which the probe exhibits underdamped oscillation in the x direction with a gradual vertical lift in the z direction, and finally converges to the equilibrium point. (See, also, Movie S2 in the Supplemental Material [43].) Vertical lift occurs only once in the initial transient process. In this condition ($\Omega_1 = 10^1$ and $\Omega_2 = 10^{1/2}$), the system is stable despite the steady sliding friction being velocity weakening around the equilibrium point [40]. This situation is due to the high dimensionless horizontal stiffness ($\Omega_2 = 10^{1/2} > 1.0$), where the horizontal frequency is greater than the vertical frequency. During the vertical lift, intense horizontal oscillations occur (see the trajectory of Fig. 3(b) or Movie S2 in the Supplemental Material [43]), which forces VEF to deform, generating positive damping to stabilize the system.

Figure 3(c) shows the moderately unstable dynamics ($\zeta_{\min}^* = -0.01$), in which the probe exhibits divergent oscillation in the x direction with a quick vertical lift in the z direction, and finally makes a periodic trajectory. (See, also, Movie S3 in the Supplemental Material [43].) The horizontal waveform in the steady oscillation is sinusoidal, as in the solutions for high-speed conditions of conventional sliding models with friction laws [4].

Figure 3(d) shows the extremely unstable dynamics ($\zeta_{\min}^* = -1$), in which the probe motion immediately makes

a complicated trajectory with horizontal intermittent oscillation and periodic vertical lift. (See, also, Movie S4 in the Supplemental Material [43].) The horizontal waveform is sawtooth shaped, resembling the typical SS instability, even though the system does not employ static friction. Note that the periodic vertical lift is synchronized to the horizontal slip event.

The numerical results of the unstable two cases, given by Figs. 3(c) and 3(d), indicate that the system possesses a mechanism that inhibits unstable growth and produces a stable limit cycle even though it does not have lateral damping. Trajectories on phase planes can be clues to clarify the mechanism. Figure 4 shows the dimensionless phase planes of the four conditions, where the horizontal and vertical axes are the dimensionless probe position $\tilde{x}$ and its velocity $\tilde{x}' = d\tilde{x}/d\tau = \dot{x}_A K_x (4LR^2 K/W^4)^{1/3} / \omega_{nx}$, respectively. Although the trajectories of Figs. 4(a) and 4(b) start from the initial point (i.e., the open circle) and approach the stable equilibrium point (i.e., the solid circle), those of Figs. 4(c) and 4(d) form the stable limit cycle outside the vicinity of the unstable equilibrium point. We find that the drive speed, represented by the blue horizontal line, determines the size of the limit cycles in Figs. 4(c) and 4(d). When the probe's horizontal speed ($\dot{x}_A$) exceeds the VEF's drive speed (V), the macroscopic friction force (F_x) is flipped from the positive to negative direction, limiting the trajectory's unstable growth and generating the stable limit cycle. We note that in Figs. 4(c) and 4(d), the trajectory overshoots above the drive speed due to the probe's inertia. We also note that the trajectory of Fig. 4(d) resembles one of the SS instabilities in the sliding system that considers the tangential contact compliance and behaves under the Coulomb friction law with static and kinetic friction [44],

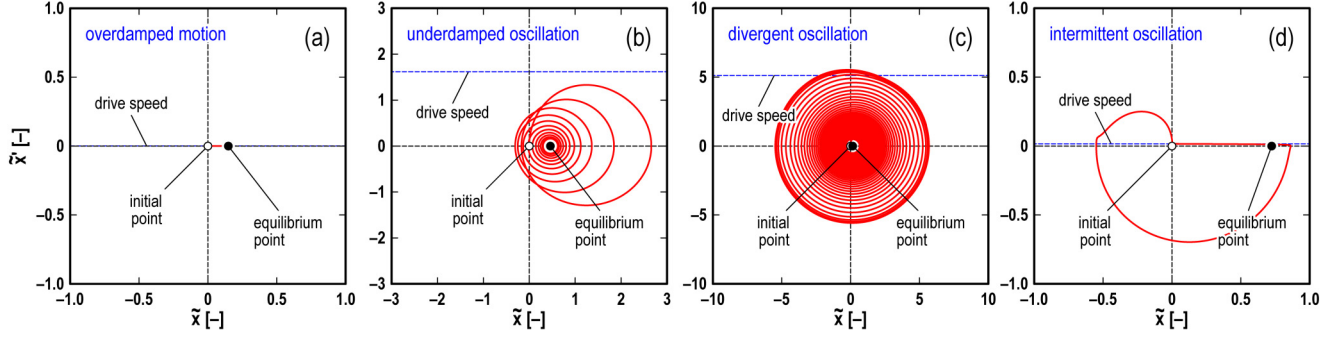


FIG. 4. Phase planes for Figs. 3(a)–3(d). The open and solid circles are the initial and equilibrium points, respectively. The blue horizontal line denotes the drive speed, inhibiting unstable growth in Figs. 3(c) and 3(d).

where the horizontal portion overlapping the blue line corresponds to the stick state.

V. DISCUSSION

Previous sections investigated local stability theoretically based on the linearized equations (Sec. III) and global system dynamics numerically based on the full equations (Sec. IV). Although the stability analysis results are limited around the equilibrium point, we here discuss the relationships between the minimum modal damping ratio $\zeta_{\min}^*$ and the types of dynamics. Figure 5 shows the stability map and various numerical results under $\Omega_3 = 31.3$ and $\Omega_4 = 4.67$. Temporal changes of the x position in the dimensionless form for 12 conditions are shown with their $\zeta_{\min}^*$ values. From the forms of numerical results and the $\zeta_{\min}^*$ values of stability analysis,

we can say that $\zeta_{\min}^*$ corresponds to the system dynamics: $\zeta_{\min}^* = 1$ is overdamped motion, $0 < \zeta_{\min}^* < 1$ is underdamped oscillation, $-1 < \zeta_{\min}^* < 0$ is divergent oscillation, and $\zeta_{\min}^* = -1$ is intermittent oscillation.

In the rest of this section, we focus on the intermittent oscillation observed in the numerical result of Fig. 3(d), which exhibits a sawtooth-shaped waveform resembling the typical SS instability. The condition is $\Omega_1 = 10^{1/2}$, $\Omega_2 = 10^{-1}$, $\Omega_3 = 31.3$, and $\Omega_4 = 4.67$, showing $\zeta_{\min}^* = -1$ in the stability analysis. (See, also, Movies S4 and S5 in the Supplemental Material [43].)

Figure 6(a) shows the temporal changes of the dimensionless slip velocity, $\tilde{v}_{\text{slip}} = (V - \dot{x})(4LK_x^3 R^2 K / W^4)^{1/3} (M / K_x)^{1/2}$, obtained by the derivative of the dimensionless position $\tilde{x}$ with respect to dimensionless time τ . In the upper graph, one may find periodic, discontinuous transitions from stick ($\tilde{v}_{\text{slip}} = 0$)

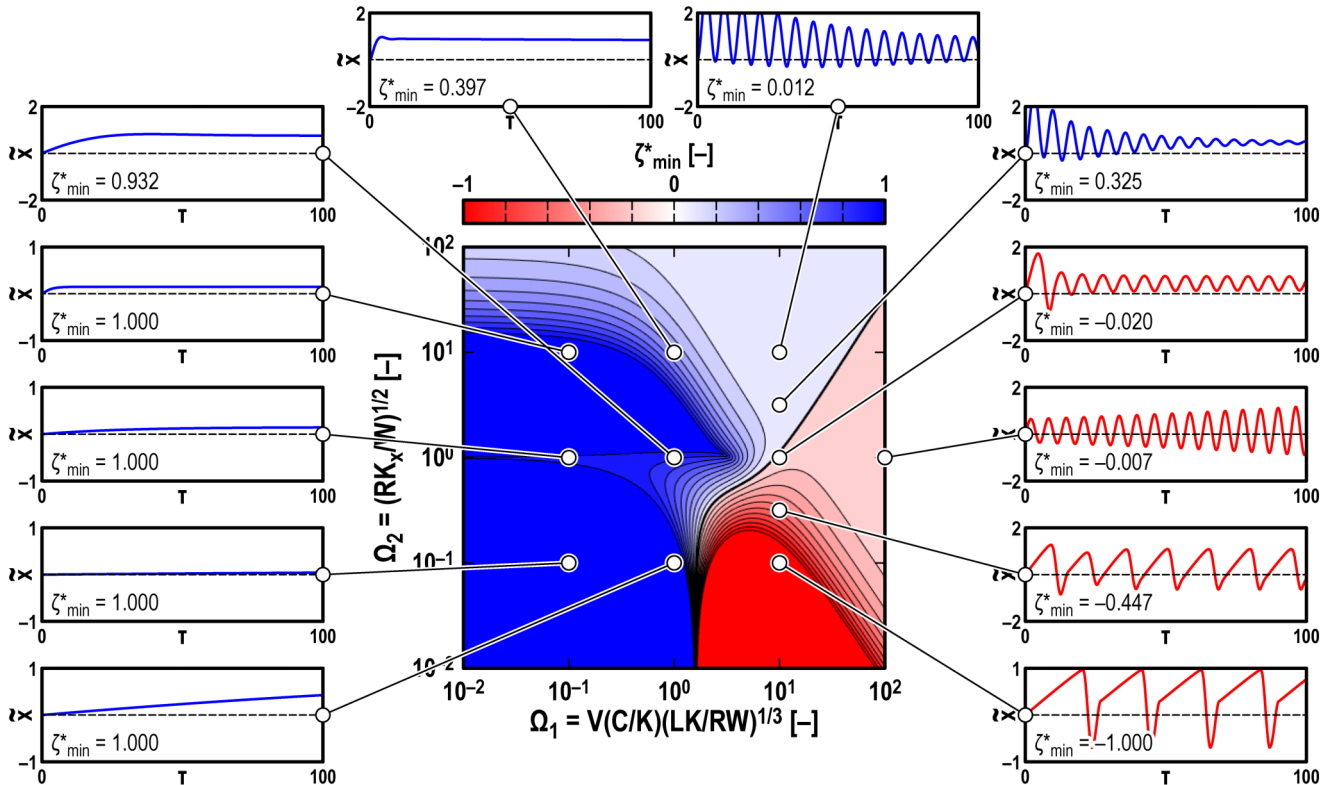


FIG. 5. Stability map and various numerical results under $\Omega_3 = 31.3$ and $\Omega_4 = 4.67$. Temporal changes of the x position in the dimensionless form are shown with their $\zeta_{\min}^*$ values. The blue is stable ($\zeta_{\min}^* > 0$) and the red is unstable ($\zeta_{\min}^* < 0$).

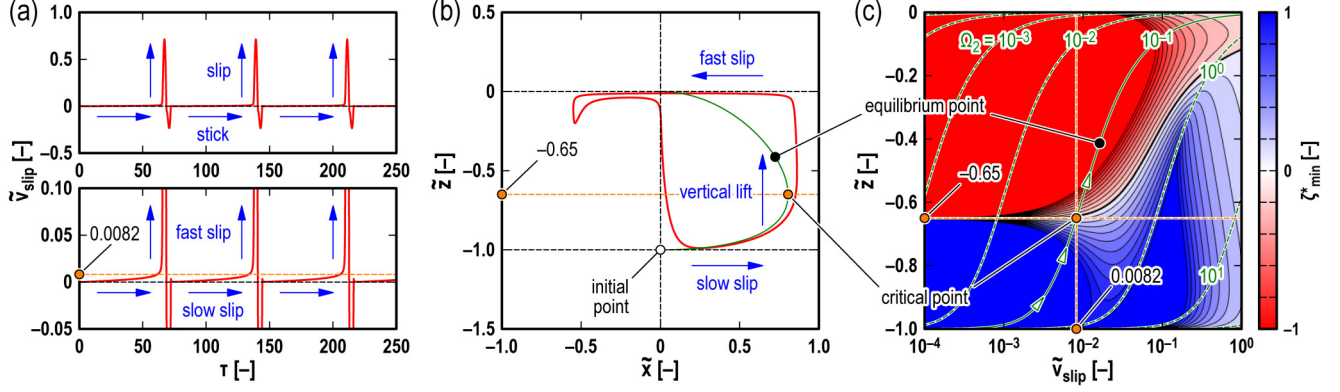


FIG. 6. Mechanism of slow-to-fast slip transition. (a) Temporal changes of the dimensionless slip velocity $\tilde{v}_{\text{slip}}$. The upper graph shows the macroscale dynamics. The lower graph shows the microscale dynamics with the horizontal axis extended by a factor of 10. (b) Probe tip trajectory in the dimensionless $x - z$ plane. The green line is the equilibrium orbit. (c) Relationship among $\tilde{v}_{\text{slip}}$, $\tilde{z}$, and $\zeta_{\min}^*$ during the slow-slip period under $\Omega_3 = 31.3$ and $\Omega_4 = 4.67$. The S-shaped green curves are equilibrium orbits for different Ω_2 values. (See, also, Movies S4 and S5 in the Supplemental Material [43].)

to slip ($\tilde{v}_{\text{slip}} > 0$) states. However, in the lower graph with the vertical axis extended by a factor of 10, we clearly find that during the stick state, the probe keeps slipping on the VEF at low speed, which is here termed the *slow-slip state*. (The observer on the VEF can more easily recognize the slow slip, as shown in Movie S5 in the Supplemental Material [43].) After a gradual acceleration, the slow-slip state transits smoothly to the *fast-slip state* at around $\tilde{v}_{\text{slip}} = 0.0082$. This smooth state transition is due to the continuous governing equations and does not happen when using the discontinuous Coulomb friction law with static and kinetic friction [4]. Note that similar slow-slip events have been experimentally observed [19–21] and attract significant interest in several fields, such as seismology [7–9].

Figure 6(b) shows the probe tip trajectory (in red) in the dimensionless $x - z$ plane. The green line represents the orbit of the equilibrium point when $\tilde{v}_{\text{slip}}$ changes from zero to infinite. In the slow-slip state, the probe moves along the equilibrium orbit and then gradually deviates from it. Then, when the z position reaches the critical value ($\tilde{z} = -0.65$), the sliding system is destabilized, leading to an abrupt vertical lift followed by a fast slip. (See, also, Movie S4 in the Supplemental Material [43].)

The mechanism of slow-to-fast slip transition is summarized in Fig. 6(c), depicted from the stability map and the master curve of indentation depth for the steady state of the one-degree-of-freedom sliding system [40], assuming that the slow-slip state at a slip velocity v_{slip} can be regarded as the steady sliding state at $V = v_{\text{slip}}$. During the slow-slip state, the probe moves toward the equilibrium point along the S-shaped equilibrium orbit of $\Omega_2 = 10^{-1}$. The system is first in the deep-blue region ($\zeta_{\min}^* = 1$). When $\tilde{v}_{\text{slip}}$ gradually increases and reaches the critical point ($\tilde{v}_{\text{slip}} = 0.0082$ and $\tilde{z} = -0.65$), across the dense contour lines, $\zeta_{\min}^*$ abruptly decreases into the deep-red region ($\zeta_{\min}^* = -1$). Note that the more abruptly $\zeta_{\min}^*$ changes with increasing $\tilde{v}_{\text{slip}}$ (e.g., for small Ω_2 or low horizontal stiffness K_x), the more intensively the slow-to-fast slip transition occurs. In summary, the probe vertical lift controlled by gradual slow-slip growth destabilizes the sliding system, causing the continuous

transition from slow-slip to fast-slip states. We should stress that this scenario is not only for specific soft materials since, as the simple toy model (Fig. 1) shows, it can happen if any sliding system has two degrees of freedom and solid viscoelasticity.

VI. CONCLUSIONS

This study theoretically investigated instabilities induced by solid viscoelasticity using a minimal toy model without friction laws: a two-degrees-of-freedom sliding system consisting of a rigid probe and a Kelvin-Voigt VEF. Taking advantage of its simplicity, we discussed the mechanisms generating various system dynamics based on stability analysis and numerical simulation. The main conclusions drawn from this study are as follows:

(1) Four dimensionless numbers (dimensionless speed Ω_1 , dimensionless horizontal stiffness Ω_2 , material-to-mechanical time constant ratio Ω_3 , and vertical mechanical time constant ratio Ω_4) determine the stability of the sliding system. The vertical lift of the probe leads to velocity-weakening friction, destabilizing the system. The lower Ω_2 is, the more intensively the system is destabilized.

(2) Minimum modal damping ratio $\zeta_{\min}^*$ classifies the sliding system dynamics into four types: overdamped motion ($\zeta_{\min}^* = 1$), underdamped oscillation ($0 < \zeta_{\min}^* < 1$), divergent oscillation ($-1 < \zeta_{\min}^* < 0$), and intermittent oscillation ($\zeta_{\min}^* = -1$). In principle, solid viscoelasticity works as a negative damping due to the vertical lift. It works as a positive damping to a limited extent when Ω_2 is large.

(3) Even though the sliding system does not employ static friction, it exhibits intermittent oscillation resembling the typical SS instability. It comprises two slip states, i.e., slow-slip and fast-slip states, providing a purely mechanical explanation for the *static friction paradox*. The probe vertical lift, controlled by gradual slow-slip growth, regulates the slow-to-fast slip transition in the sliding system. During slip velocity growth, the more abruptly $\zeta_{\min}^*$ changes with increasing slip velocity, the more intensively the slow-to-fast slip transition occurs.

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K.N. conceived this study; T.W. and K.N. conducted the theoretical analysis; T.W. conducted the numerical analysis; T.W. and K.N. wrote the manuscript and composed the figures.

APPENDIX A: LINEARIZED EQUATIONS OF MOTION

We here linearize the equations of motion (8) around the equilibrium point for stability analysis. Integrating Eqs. (6) and (7), the contact load F_z and the friction force F_x are

$$F_z = L \left[-\frac{K}{6R}(b^3 + a^3) - \frac{v_{\text{slip}}C}{2R}(b^2 - a^2) + \delta K(b + a) + \dot{\delta}C(b + a) \right], \quad (\text{A1})$$

$$F_x = L \left[\frac{K}{8R^2}(b^4 - a^4) + \frac{v_{\text{slip}}C}{3R^2}(b^3 + a^3) - \frac{\delta K}{2R}(b^2 - a^2) - \frac{\dot{\delta}C}{2R}(b^2 - a^2) \right]. \quad (\text{A2})$$

Assuming $\dot{x}_A < V$, we obtain the following contact semiwidths [39]:

$$a = \sqrt{2R\delta}, \quad (\text{A3})$$

$$b = \sqrt{2R\delta + 2Rt_c\dot{\delta} + v_{\text{slip}}^2t_c^2 - v_{\text{slip}}t_c}, \quad (\text{A4})$$

where $\delta = -z_A$ is the indentation depth of the probe (see Fig. 1), and $t_c = C/K$ is the retardation time of the viscoelastic foundation. From Eqs. (A1) to (A4), F_z and F_x are the functions of slip velocity v_{slip} , indentation rate $\dot{\delta}$, and indentation depth δ . Introducing the disturbances of $\Delta x_A = x_A - x_{\text{eq}} \sim 0$, $\Delta z_A = z_A - z_{\text{eq}} \sim 0$, $\Delta \dot{x}_A \sim 0$, and $\Delta \dot{z}_A \sim 0$:

$$F_z(v_{\text{slip}}, \dot{\delta}, \delta) \sim F_z(V, 0, \delta_{\text{eq}}) + \frac{\partial F_z}{\partial v_{\text{slip}}}(V, 0, \delta_{\text{eq}})(v_{\text{slip}} - V) + \frac{\partial F_z}{\partial \dot{\delta}}(V, 0, \delta_{\text{eq}})(\dot{\delta} - 0) + \frac{\partial F_z}{\partial \delta}(V, 0, \delta_{\text{eq}})(\delta - \delta_{\text{eq}}), \quad (\text{A5})$$

$$F_x(v_{\text{slip}}, \dot{\delta}, \delta) \sim F_x(V, 0, \delta_{\text{eq}}) + \frac{\partial F_x}{\partial v_{\text{slip}}}(V, 0, \delta_{\text{eq}})(v_{\text{slip}} - V) + \frac{\partial F_x}{\partial \dot{\delta}}(V, 0, \delta_{\text{eq}})(\dot{\delta} - 0) + \frac{\partial F_x}{\partial \delta}(V, 0, \delta_{\text{eq}})(\delta - \delta_{\text{eq}}), \quad (\text{A6})$$

where $x_{\text{eq}} = F_{\text{eq}}/K_x$ is the horizontal equilibrium, and $z_{\text{eq}} = -\delta_{\text{eq}}$ is the vertical equilibrium. (See Ref. [40] for analytical solutions of F_{eq} and δ_{eq} .) Therefore, the linearized equations of motion around the equilibrium point are

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{pmatrix} \Delta \ddot{x}_A \\ \Delta \ddot{z}_A \end{pmatrix} + \begin{bmatrix} c_{xx} & c_{xz} \\ c_{zx} & c_{zz} \end{bmatrix} \begin{pmatrix} \Delta \dot{x}_A \\ \Delta \dot{z}_A \end{pmatrix} + \begin{bmatrix} K_x & k_{xz} \\ 0 & k_{zz} \end{bmatrix} \begin{pmatrix} \Delta x_A \\ \Delta z_A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (\text{A7})$$

The parameters in the damping and stiffness matrices are

$$c_{xx} = \frac{\partial F_x}{\partial v_{\text{slip}}}(V, 0, \delta_{\text{eq}}) = L \left(-\frac{2\delta_{\text{eq}}Vt_cC}{R} - \frac{4V^3t_c^3C}{3R^2} + \frac{2\delta_{\text{eq}}C}{3R}\sqrt{2R\delta_{\text{eq}}} + \frac{2R\delta_{\text{eq}}C + 4V^2t_c^2C}{3R^2}\sqrt{2R\delta_{\text{eq}} + V^2t_c^2} \right), \quad (\text{A8})$$

$$c_{xz} = \frac{\partial F_x}{\partial \dot{\delta}}(V, 0, \delta_{\text{eq}}) = L \left(-\frac{V^2t_c^2C}{R} + \frac{Vt_cC}{R}\sqrt{2R\delta_{\text{eq}} + V^2t_c^2} \right), \quad (\text{A9})$$

$$c_{zx} = \frac{\partial F_z}{\partial v_{\text{slip}}}(V, 0, \delta_{\text{eq}}) = L \left(-\frac{V^2t_c^2C}{R} + \frac{Vt_cC}{R}\sqrt{2R\delta_{\text{eq}} + V^2t_c^2} \right), \quad (\text{A10})$$

$$c_{zz} = \frac{\partial F_z}{\partial \dot{\delta}}(V, 0, \delta_{\text{eq}}) = L \left(-Vt_cC + C\sqrt{2R\delta_{\text{eq}}} + C\sqrt{2R\delta_{\text{eq}} + V^2t_c^2} \right), \quad (\text{A11})$$

$$k_{xz} = \frac{\partial F_x}{\partial \delta}(V, 0, \delta_{\text{eq}}) = L \left(-\frac{V^2t_c^2K}{R} + \frac{Vt_cK}{R}\sqrt{2R\delta_{\text{eq}}} + \frac{Vt_cK}{R}\sqrt{2R\delta_{\text{eq}} + V^2t_c^2} \right), \quad (\text{A12})$$

$$k_{zz} = \frac{\partial F_z}{\partial \delta}(V, 0, \delta_{\text{eq}}) = L \left(K\sqrt{2R\delta_{\text{eq}}} + K\sqrt{2R\delta_{\text{eq}} + V^2t_c^2} \right). \quad (\text{A13})$$

Note that the first terms on the right-hand side of Eqs. (A5) and (A6) vanish since $F_z(V, 0, \delta_{\text{eq}}) = Mg$ and $F_x(V, 0, \delta_{\text{eq}}) = K_x x_{\text{eq}}$.

APPENDIX B: DIMENSIONLESS ANALYSIS

We here derive the complete set of dimensionless numbers from the linearized equations of motion (A7). The dimensionless time is $\tau = \omega_{nx}t$, where $\omega_{nx} = (K_x/M)^{1/2}$ is the natural frequency in the x direction. The dimensionless positions are $\tilde{x} = x_A K_x (4LR^2K/W^4)^{1/3}$ and $\tilde{z} = z_A (32L^2RK^2/9W^2)^{1/3}$, so that we can determine the equilibrium points uniquely for a dimensionless speed Ω_1 . Then, we obtain the following dimensionless linearized equations of motion:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \Delta \tilde{x}'' \\ \Delta \tilde{z}'' \end{pmatrix} + \begin{bmatrix} \tilde{c}_{xx} & \tilde{c}_{xz} \\ \tilde{c}_{zx} & \tilde{c}_{zz} \end{bmatrix} \begin{pmatrix} \Delta \tilde{x}' \\ \Delta \tilde{z}' \end{pmatrix} + \begin{bmatrix} 1 & \tilde{k}_{xz} \\ 0 & \tilde{k}_{zz} \end{bmatrix} \begin{pmatrix} \Delta \tilde{x} \\ \Delta \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (\text{B1})$$

where $(\cdot)$ is the derivative with respect to dimensionless time τ , and $\Delta \tilde{x} = \Delta x K_x (4LR^2K/W^4)^{1/3}$ and $\Delta \tilde{z} = \Delta z (32L^2RK^2/9W^2)^{1/3}$ are dimensionless disturbances from the equilibrium points. Dimensionless parameters are

$$\tilde{c}_{xx} = \frac{\Omega_1^3 \Omega_3}{\Omega_2} \left(-\frac{4}{3} - \frac{1}{\Gamma^2} + \frac{1}{3\Gamma^3} + \frac{1+4\Gamma^2}{3\Gamma^3} \sqrt{1+\Gamma^2} \right), \quad (\text{B2})$$

$$\tilde{c}_{xz} = \frac{3^{2/3} \Omega_1^2 \Omega_2 \Omega_3}{2} \left(-1 + \frac{1}{\Gamma} \sqrt{1+\Gamma^2} \right), \quad (\text{B3})$$

$$\tilde{c}_{zx} = \frac{2\Omega_1^2 \Omega_3 \Omega_4^2}{3^{2/3} \Omega_2^3} \left(-1 + \frac{1}{\Gamma} \sqrt{1+\Gamma^2} \right), \quad (\text{B4})$$

$$\tilde{c}_{zz} = \frac{\Omega_1 \Omega_3 \Omega_4^2}{\Omega_2} \left(-1 + \frac{1}{\Gamma} + \frac{1}{\Gamma} \sqrt{1+\Gamma^2} \right), \quad (\text{B5})$$

$$\tilde{k}_{xz} = \frac{3^{2/3} \Omega_1^2}{2} \left(-1 + \frac{1}{\Gamma} + \frac{1}{\Gamma} \sqrt{1+\Gamma^2} \right), \quad (\text{B6})$$

$$\tilde{k}_{zz} = \frac{\Omega_1 \Omega_4^2}{\Omega_2^2} \left(\frac{1}{\Gamma} + \frac{1}{\Gamma} \sqrt{1+\Gamma^2} \right), \quad (\text{B7})$$

where

$$\Omega_1 = \frac{CV}{K} \left(\frac{LK}{RW} \right)^{1/3}, \quad \Omega_2 = \left(\frac{RK_x}{W} \right)^{1/2}, \quad \Omega_3 = \frac{C}{K} \left(\frac{W}{RM} \right)^{1/2}, \quad \Omega_4 = \left(\frac{LR^2K}{W} \right)^{1/3}, \quad (\text{B8})$$

$$\Gamma = \frac{CV}{K\sqrt{2R\delta_{eq}}}. \quad (\text{B9})$$

Note that Γ is an implicit function of Ω_1 , determined by solving the following nonlinear equation [40]:

$$\frac{3\Gamma^3}{1 - \Gamma^3 + (1 + \Gamma^2)^{3/2}} = \Omega_1^3. \quad (\text{B10})$$

Therefore, the complete set of independent dimensionless numbers is Ω_1 , Ω_2 , Ω_3 , and Ω_4 .

APPENDIX C: EIGENVALUE ANALYSIS

We here obtain the eigenvalues of the dimensionless linearized equations of motion (B1). Using the state vector

$$\mathbf{X} = (\Delta \tilde{x} \quad \Delta \tilde{z} \quad \Delta \tilde{x}' \quad \Delta \tilde{z}')^T, \quad (\text{C1})$$

Eq. (B1) can be written as

$$\mathbf{X}' = \mathbf{A}\mathbf{X}. \quad (\text{C2})$$

Using zero matrix $\mathbf{O}$ and identity matrix $\mathbf{I}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \tilde{c}_{xx} & \tilde{c}_{xz} \\ \tilde{c}_{zx} & \tilde{c}_{zz} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & \tilde{k}_{xz} \\ 0 & \tilde{k}_{zz} \end{bmatrix}. \quad (\text{C3})$$

The eigenvalue equation is

$$\det(\mathbf{A} - s_i \mathbf{I}) = 0, \quad (\text{C4})$$

where s_i is the i th complex eigenvalue,

$$s_i = \sigma_i + j\omega_i. \quad (\text{C5})$$

The system is stable if every real part σ_i of the eigenvalue is negative. To evaluate the stability around the equilibrium point qualitatively, we use the following minimum modal damping ratio $\zeta_{\min}^*$ [45]:

$$\zeta_{\min}^* = \min(\zeta_i^*) = \min \left(-\frac{\sigma_i}{\sqrt{\sigma_i^2 + \omega_i^2}} \right). \quad (\text{C6})$$

Note that ζ_i^* is invariant for the original equations of motion (A7) and dimensionless equations of motion (B1).

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