

Finite-size corrections to the crosscap overlap in the two-dimensional Ising model

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We analyze the finite-size corrections to the crosscap overlap in the two-dimensional classical Ising model along its self-dual critical line. Using a fermionic formulation, we express the lattice crosscap overlap in terms of Bogoliubov angles and develop a contour-integral approach by analytically continuing the lattice momentum to the complex plane. This leads to a remarkably simple expression for the crosscap overlap, which demonstrates that the finite-size corrections decay exponentially with system size. We further derive an exact analytical formula for the corresponding decay constant and show that it is determined by the complex singularity structure of the Bogoliubov angle.

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Introduction. Boundary effects are fundamental to the understanding of critical phenomena in low-dimensional systems. At critical points, the low-energy physics of many systems is described by conformal field theories (CFTs), for which conformal boundary conditions give rise to universal quantities characterizing the underlying critical theory [1–5]. A paradigmatic example is the boundary entropy introduced by Affleck and Ludwig [6,7], which quantifies the effective ground-state degeneracy associated with a conformal boundary and serves as an important characteristic quantity for boundary critical phenomena.

In two dimensions, boundary critical phenomena are far more analytically tractable due to the infinite-dimensional conformal symmetry of the underlying CFT. In the path integral formulation, a boundary of the two-dimensional CFT corresponds geometrically to an edge of the space-time manifold, forming a circle, to which an associated boundary condition is assigned. Alongside conventional boundaries, nonorientable structures provide additional probes of CFT data. A crosscap, constructed by identifying antipodal points of the circle, serves as a fundamental building block for defining CFTs on nonorientable manifolds, such as the Klein bottle and the real projective plane [8,9]. When the circle, equipped with the crosscap boundary condition, is taken as the spatial direction, the quantization procedure yields the crosscap state, and the crosscap overlap is defined as its overlap with the

CFT vacuum [9–12]. The logarithm of the squared norm of this overlap, known as the Klein bottle entropy [13], is a universal quantity that encodes essential CFT data and can be regarded as the nonorientable analog of the Affleck-Ludwig boundary entropy in conventional boundary CFTs. The study of crosscap states and the associated Klein bottle entropy has seen considerable recent development across condensed matter and statistical physics, as well as high-energy theory and mathematical physics, with these concepts extended both at and near conformal critical points and applied to various one-dimensional (1D) quantum and two-dimensional (2D) classical models, in both equilibrium and nonequilibrium contexts [14–38]. An important ongoing effort is to construct lattice crosscap states in concrete models and to establish their identification with the corresponding CFT counterparts [27,28].

It is useful to contrast the crosscap boundary condition with more familiar boundary choices. Periodic and antiperiodic boundary conditions preserve orientability and define, respectively, the ordinary and $\mathbb{Z}_2$ -twisted torus partition functions. Their finite-size corrections are typically organized as algebraic expansions in the inverse system size (see, e.g., Ref. [39]). Open (free) and fixed boundaries, by contrast, are described by conformal boundary states [2,3]. Their free energies contain nonuniversal surface and, when applicable, corner contributions, together with the universal Affleck-Ludwig boundary entropy [6], as well as algebraic finite-size corrections [40]. The crosscap condition instead implements an orientation-reversing identification and produces a nonorientable geometry without a physical boundary. The corresponding universal constant is the Klein bottle entropy $\ln |\langle 0|C\rangle|^2$ [13], where $|C\rangle$ and $|0\rangle$ denote the crosscap state and the CFT vacuum, respectively. It therefore probes universal information associated with orientation reversal that is not captured by conventional conformal boundaries. While finite-size corrections have been extensively studied for conventional boundary conditions, the finite-size

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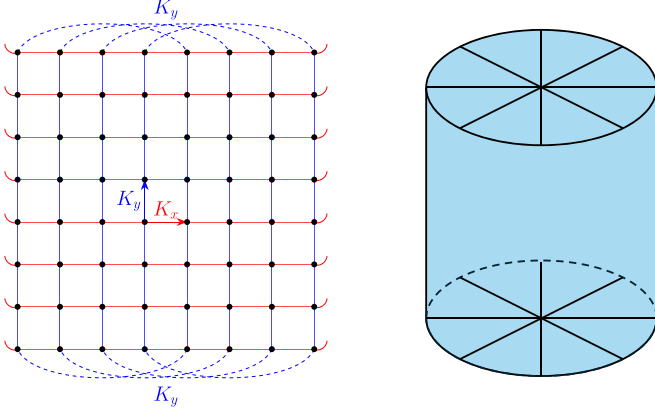


FIG. 1. Left: schematic of the 2D classical Ising model with anisotropic couplings and periodic (crosscap) boundary conditions along the x (y) direction. Right: geometric depiction of the cylinder with crosscap boundaries.

corrections governing how the lattice crosscap overlap approaches its universal CFT value $\langle 0|\mathcal{C}\rangle$, and hence the corresponding corrections to the Klein bottle entropy, remain much less explored.

The 2D classical Ising model provides an ideal platform for examining these concepts. Its critical point is described by the $c = 1/2$ Ising CFT [41,42], and its exact solvability permits rigorous analytical treatment [43–49]. Recent work [28] established an explicit lattice construction of crosscap states in the quantum transverse-field Ising chain and derived their field-theoretical counterparts. A notable finding is that the crosscap overlap in the quantum transverse-field Ising chain at criticality is free of finite-size corrections [28,50]. This result raises a natural question: away from the anisotropic limit—where the 2D classical Ising model maps onto the 1D quantum Ising chain—what is the scaling form of the finite-size corrections to the crosscap overlap along the critical line?

In this work, we address this question by analyzing the crosscap overlap in the 2D classical Ising model along its critical self-dual line, using the exact solution. We find that finite-size corrections exhibit an exponential decay as a function of system size, except at the anisotropic limit, where corrections vanish. This exponential scaling is quite unusual and stands in sharp contrast to the power-law finite-size corrections [39,40] typically generated by irrelevant operators in conformal perturbation theory [51]. We derive an exact finite-size correction formula using contour integral techniques and trace the origin of the exponential decay to branch point singularities in the complex plane.

Crosscap overlap. We consider the 2D classical Ising model on an $M \times N$ square lattice with spin variables $s_{ij} = \pm 1$, where $i = 1, \dots, M$ and $j = 1, \dots, N$ label the row and column indices, respectively. Throughout this work we impose periodic boundary conditions along the horizontal (x) direction and crosscap boundary conditions along the vertical (y) direction; see Fig. 1. More explicitly, the partition function is written as

$$Z = \sum_{\{s_{ij}\}} Z_{\text{cross}}(\{s_{1,j}\}) Z_{\text{bulk}}(\{s_{ij}\}) Z_{\text{cross}}(\{s_{M,j}\}), \quad (1)$$

where Z_{bulk} contains the Boltzmann weights associated with bulk nearest-neighbor interactions,

$$Z_{\text{bulk}}(\{s_{ij}\}) = e^{K_x \sum_{i=1}^M \sum_{j=1}^N s_{ij} s_{i,j+1}} e^{K_y \sum_{i=1}^{M-1} \sum_{j=1}^N s_{ij} s_{i+1,j}}, \quad (2)$$

with K_x and K_y denoting the horizontal and vertical couplings, respectively. The factors Z_{cross} encode crosscap boundary interactions,

$$Z_{\text{cross}}(\{s_{l,j}\}) = e^{K_y \sum_{j=1}^{N/2} s_{l,j} s_{l,j+N/2}}, \quad l = 1, M. \quad (3)$$

To formulate the partition function in the transfer-matrix formalism, we identify each Ising spin $|s\rangle$ ($s = \pm 1$) with the eigenbasis of the Pauli matrix σ^x , and introduce row-to-row transfer matrices [48,52]:

$$\begin{aligned} V_1 &= [2 \sinh(2K_y)]^{N/2} e^{K_y^* \sum_{j=1}^N \sigma_j^z}, \\ V_2 &= e^{K_x \sum_{j=1}^N \sigma_j^x \sigma_{j+1}^x}, \end{aligned} \quad (4)$$

with $K_y^* = \frac{1}{2} \ln[\coth(K_y)]$. Equivalently, relative to the more common convention in which the Ising spin is represented by σ^z , this choice amounts to a uniform spin rotation $\sigma^x \rightarrow \sigma^z$, $\sigma^z \rightarrow -\sigma^x$, which leaves the spectrum unchanged but gives the transfer matrix the form used below. The factor Z_{cross} can be rewritten as

$$Z_{\text{cross}}(\{s_{l,j}\}) = \langle s_{l,j} | \sqrt{V_1} | \mathcal{C}_{\text{latt}} \rangle, \quad (5)$$

where the lattice crosscap state is defined by

$$| \mathcal{C}_{\text{latt}} \rangle = \prod_{j=1}^{N/2} (|+1\rangle_j | +1\rangle_{j+N/2} + |-1\rangle_j | -1\rangle_{j+N/2}). \quad (6)$$

The partition function then takes a compact form

$$\begin{aligned} Z &= \langle \mathcal{C}_{\text{latt}} | \sqrt{V_1} (V_2 V_1)^{M-1} V_2 \sqrt{V_1} | \mathcal{C}_{\text{latt}} \rangle \\ &\equiv \langle \mathcal{C}_{\text{latt}} | V^M | \mathcal{C}_{\text{latt}} \rangle, \end{aligned} \quad (7)$$

where $V = \sqrt{V_1} V_2 \sqrt{V_1}$ is the transfer matrix of the Ising model. In the limit $M \gg N$, only the leading eigenvalue and its associated eigenvector of the transfer matrix V contribute in Eq. (7), yielding the asymptotic expansion

$$\ln Z \simeq M \ln \Lambda_0 + \ln |\langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle|^2 + \dots, \quad (8)$$

where Λ_0 is the leading eigenvalue of V , and $|\psi_0\rangle$ is the corresponding eigenvector. The first term represents the bulk free energy, while the second term, expressed as the logarithm of the squared norm of the crosscap overlap, is known as the Klein bottle entropy [13].

The 2D classical Ising model is critical on the self-dual line $K_x = K_y^*$ [48,53]. We will focus on this case in the present work. Its continuum limit is described by the Ising CFT. Then, the lattice crosscap overlap in Eq. (8) is expected to approach a universal constant in the thermodynamic limit,

$$\lim_{N \rightarrow \infty} \langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle = \sqrt{\frac{2 + \sqrt{2}}{2}}, \quad (9)$$

a value determined by the quantum dimensions of the Ising CFT [13]. The purpose of this work is to derive the finite-size corrections to this universal value.

Before proceeding to the derivation, it is worth noting that the transfer matrix V in Eq. (7) commutes with the Hamiltonian of an anisotropic spin-1/2 quantum XY chain [54,55],

$$H = - \sum_{j=1}^N \left(\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z \right), \quad (10)$$

where the parameters γ and h are related to the Ising couplings via $\gamma = 1/\cosh(2K_y^*)$ and $h = \tanh(2K_y^*)/\tanh(2K_x)$. The transfer matrix V and the Hamiltonian H can be simultaneously diagonalized [48,54], and the leading eigenvector $|\psi_0\rangle$ of V is the ground state of H [54]. Thus, they share the same crosscap overlap results, even for finite-size cases.

The transfer matrix V , as well as the Hamiltonian (10), possesses a global $\mathbb{Z}_2$ symmetry $[V, Q] = 0$ with $Q = \prod_{j=1}^N \sigma_j^z$. Both the leading eigenvector $|\psi_0\rangle$ and the lattice crosscap state $|\mathcal{C}_{\text{latt}}\rangle$ [Eq. (6)] live in the $\mathbb{Z}_2$ even ($Q = 1$) sector, which, following the CFT convention, is called the Neveu-Schwarz (NS) sector. Utilizing the Jordan-Wigner transformation $\sigma_j^z = 2c_j^\dagger c_j - 1$ and $\sigma_j^x = (c_j^\dagger + c_j)e^{i\pi \sum_{l=1}^{j-1} c_l^\dagger c_l}$, the Fourier transform $c_j^\dagger = \frac{1}{\sqrt{N}} \sum_{k \in \text{NS}} e^{-ikj} c_k^\dagger$ ($k \in \text{NS}$ denotes allowed lattice momenta in the NS sector $k = \pm[\pi - \frac{2\pi}{N}(n_k - \frac{1}{2})]$ with $n_k = 1, 2, \dots, N/2$), and a Bogoliubov transformation, the transfer matrix V in the NS sector can be diagonalized [48,56]. Along the critical line $K_x = K_y^*$, the leading eigenvector $|\psi_0\rangle$ reads

$$|\psi_0\rangle = \prod_{k>0} \left[\sin \frac{\theta(k)}{2} + i \cos \frac{\theta(k)}{2} c_k^\dagger c_{-k}^\dagger \right] |0\rangle_c, \quad (11)$$

where $k > 0$ denotes positive momenta in the NS sector, and $\theta(k)$ is the Bogoliubov angle:

$$\theta(k) = \arctan \left[\frac{\tan(k/2)}{\cosh(2K_x)} \right], \quad (12)$$

and $|0\rangle_c$ is the vacuum of the Jordan-Wigner fermions. Using the results in Ref. [28], the lattice crosscap state in Eq. (6) is rewritten as a superposition of two fermionic Gaussian states in the convention of Ref. [28], which is recovered by rewriting Eq. (6) in the σ^z eigenbasis:

$$\begin{aligned} |\mathcal{C}_{\text{latt}}\rangle &= \frac{1-i}{2} \prod_{j=1}^{N/2} (1 + i c_j^\dagger c_{j+N/2}^\dagger) |0\rangle_c \\ &+ \frac{1+i}{2} \prod_{j=1}^{N/2} (1 - i c_j^\dagger c_{j+N/2}^\dagger) |0\rangle_c. \end{aligned} \quad (13)$$

The lattice crosscap overlap of the critical 2D classical Ising model can hence be calculated [28], yielding

$$\langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle = \sqrt{2} \cos \Theta, \quad (14)$$

with $\Theta = \frac{\pi}{4} + \frac{1}{2} \sum_{k>0} (-1)^{n_k} \theta(k)$.

In the thermodynamic limit, the alternating sum in Θ becomes an integral and can be analytically calculated,

$$\lim_{N \rightarrow \infty} \sum_{k>0} (-1)^{n_k} \theta(k) = -\frac{1}{2} \int_0^\pi \theta'(k) dk = -\frac{\pi}{4}, \quad (15)$$

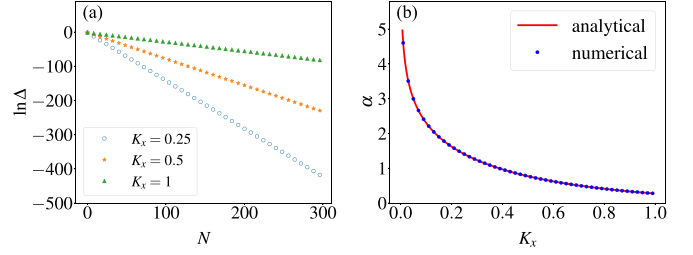


FIG. 2. Finite-size corrections to the crosscap overlap for the 2D Ising model along the self-dual line ($K_x = K_y^*$). (a) Logarithm of the deviation $\ln \Delta$ [Eq. (16)] as a function of the system size N for several values of the coupling K_x . (b) Decay constant α as a function of K_x : numerical estimates (blue dots) extracted from the slopes in (a), compared with the analytical result (red solid line) in Eq. (20).

where we used $\theta(k + \frac{2\pi}{N}) - \theta(k) \approx \frac{1}{2} \theta'(k) dk$ with $dk = \frac{4\pi}{N}$. This leads to $\lim_{N \rightarrow \infty} \langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle = \sqrt{2} \cos \frac{\pi}{8}$, which agrees with the CFT prediction in Eq. (9).

Finite-size corrections. We now analyze the finite-size corrections to the crosscap overlap using Eq. (14). We define the deviation from the CFT prediction as

$$\Delta := \left| \langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle - \sqrt{\frac{2+\sqrt{2}}{2}} \right|. \quad (16)$$

Figure 2(a) shows $\ln \Delta$ as a function of the system size N for several Ising couplings K_x along the self-dual line ($K_x = K_y^*$). We find that the finite-size corrections decay exponentially with system size $\Delta \sim e^{-\alpha N}$, where α is a coupling-dependent decay constant. Moreover, α increases and diverges as the coupling approaches the anisotropic limit $K_x \rightarrow 0$, indicating that the finite-size corrections vanish in this limit. Indeed, on the self-dual line $K_x = K_y^*$ [corresponding to $h = 1$ in Eq. (10)], the anisotropic limit $K_x \rightarrow 0$ yields $\gamma = 1/\cosh(2K_y^*) \rightarrow 1$. The Hamiltonian H therefore reduces to the critical transverse-field Ising chain, whose crosscap overlap is known to be free of finite-size corrections [28,50].

The exponential decay of the finite-size corrections implies that they are intrinsically nonperturbative, in the sense that they do not admit an expansion in powers of $1/N$. This motivates a nonperturbative analysis of the factor Θ , which can be achieved by rewriting the alternating sum as a contour integral:

$$\sum_{k>0} (-1)^{n_k} \theta(k) = \frac{-1}{2\pi i} \oint_C \theta(z) d \ln \varphi(z), \quad (17)$$

where $\varphi(z) = (e^{iNz/2} + i)/(e^{iNz/2} - i)$ is an auxiliary function. Its zeros are at $z_\ell^{(0)} = \frac{2\pi}{N}(2\ell - \frac{1}{2})$, while its poles are at $z_\ell^{(\infty)} = \frac{2\pi}{N}(2\ell + \frac{1}{2})$, with $\ell \in \mathbb{Z}$. Together, these points form the set $z_n = \frac{2\pi}{N}(n - \frac{1}{2})$, $n \in \mathbb{Z}$, which gives the logarithmic branch points of $\ln \varphi(z)$. Moreover, $d \ln \varphi(z) = [\varphi'(z)/\varphi(z)] dz$, with $\varphi'(z)/\varphi(z) = N e^{iNz/2}/(e^{iNz} + 1)$, so Eq. (17) follows by collecting the corresponding residues.

The contour C encircles these branch points on the real axis for $n = 1, \dots, N/2$.

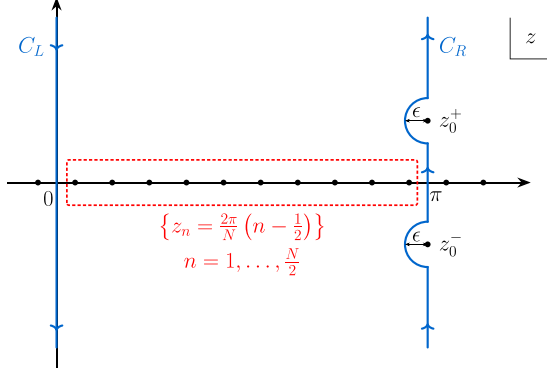


FIG. 3. Contour integration for analyzing finite-size corrections. The contour C is chosen to be a rectangle path extending infinitely along the imaginary axis, with small arcs around the branch points. It encloses the real-axis singularities of $d \ln \varphi(z)$ corresponding to the allowed $k > 0$ momenta in the NS sector.

Since $\ln \varphi(z)$ is multivalued, its value must be fixed consistently along C by analytic continuation. To do so, we write $\varphi(z) = \varphi_+(z)/\varphi_-(z)$ with $\varphi_{\pm}(z) := e^{iNz/2} \pm i$, and express each factor in polar form, $\varphi_{\pm}(z) = \rho_{\pm}(z)e^{i\phi_{\pm}(z)}$ with $\phi_{\pm} \in (-\pi, \pi)$. The local section of $\ln \varphi(z)$ along C is then defined as

$$\ln \varphi(z) := \ln[\rho_+(z)/\rho_-(z)] + i[\phi_+(z) - \phi_-(z) + 2\pi m], \quad (18)$$

where the integer $m \in \mathbb{Z}$ is the winding number that labels the section. The choice of m fixes a local branch of $\ln \varphi(z)$ along the contour. The intermediate quantities $\ln(-1)$ and $\ln \varphi(z_0^+)$ below must be evaluated in the same section; changing m shifts them consistently and leaves the final combination in Eq. (27) unchanged. In the explicit evaluation below, we use the section $m = 0$. With this prescription, $\ln \varphi(z)$ is single-valued along the closed contour C when $\text{mod}(N, 4) = 0$, as the phase contributions from the branch points associated with the zeros and poles of $\varphi(z)$ cancel exactly. In contrast, for $\text{mod}(N, 4) = 2$, traversing the contour accumulates an additional phase shift of 2π .

We first focus on the case $\text{mod}(N, 4) = 0$, and the case $\text{mod}(N, 4) = 2$ will be discussed afterward. For $\text{mod}(N, 4) = 0$, the monodromy of $\ln \varphi(z)$ allows integration by parts without boundary contributions, yielding

$$\frac{-1}{2\pi i} \oint_C \theta(z) d \ln \varphi(z) = \frac{1}{2\pi i} \oint_C \theta'(z) \ln \varphi(z) dz. \quad (19)$$

For practical evaluation, we take the contour to be the boundary of the infinite strip $\{z : 0 \leq \text{Re}(z) \leq \pi\}$, deformed to pass through the poles of $\theta'(z)$ at $z_0^{\pm} = \pi \pm 2i\alpha$, with

$$\alpha = \frac{1}{2} \ln \left[\frac{\cosh(2K_x) + 1}{\cosh(2K_x) - 1} \right], \quad (20)$$

as illustrated in Fig. 3. We will show that the constant α defined in Eq. (20) indeed governs the exponential decay of the finite-size corrections, in agreement with the numerical results [see Fig. 2(b)].

The contour can be divided into the left and right parts $C = C_L + C_R$, with the right part further split as $C_R = \tilde{C}_R + C_{\epsilon}^+ + C_{\epsilon}^-$. The integrals over these segments can then be evaluated

term by term. The contributions from $C_{\epsilon}^{\pm}$ are computed via the residue theorem,

$$\frac{1}{2\pi i} \int_{C_{\epsilon}^+} \theta'(z) \ln \varphi(z) dz = -\frac{1}{2} \text{Res}[\theta'(z_0^+)] \ln \varphi(z_0^+), \quad (21)$$

where $\text{Res}[\theta'(z_0^{\pm})] = \mp \frac{i}{2}$ are the residues of $\theta'(z)$ at $z_0^{\pm}$.

The remaining integrals over $\tilde{C}_R$ and C_L can be calculated using the symmetry properties of $\theta'(z)$ and $\varphi(z)$,

$$\theta'(z) = \theta'(\bar{z}), \quad \varphi(z)\varphi(\bar{z}) = -1, \quad (22)$$

for all $z = iy$ and $z = \pi + iy$ with $y \in \mathbb{R}$. For instance, after the substitution $z = \pi + iy$, the integral along $\tilde{C}_R$ becomes

$$\begin{aligned} \frac{1}{2\pi i} \int_{\tilde{C}_R} \theta'(z) \ln \varphi(z) dz \\ = \frac{\ln(-1)}{2\pi} \left(\int_0^{2\alpha-\epsilon} + \int_{2\alpha+\epsilon}^{\infty} \right) \theta'(\pi + iy) dy, \end{aligned} \quad (23)$$

where

$$\ln(-1) := \ln[\varphi(iy)\varphi(-iy)] = \ln[\varphi(\pi + iy)\varphi(\pi - iy)], \quad (24)$$

is a well-defined constant for all $y \in \mathbb{R}$ due to its continuity along the contour: $\{z : \text{Re}(z) = 0\} \cup \{z : \text{Re}(z) = \pi\}$. This can be reexpressed as a contour integral,

$$\begin{aligned} \text{Eq. (23)} &= \frac{\ln(-1)}{2\pi i} \int_{\tilde{C}_R^+} \theta'(z) dz \\ &= \frac{\ln(-1)}{2\pi i} \int_{C_R^+} \theta'(z) dz + \frac{\ln(-1)}{2} \text{Res}[\theta'(z_0^+)], \end{aligned} \quad (25)$$

where $\tilde{C}_R^+$ denotes the upper half of $\tilde{C}_R$, i.e., $C_R^+ = \tilde{C}_R^+ + C_{\epsilon}^+$, and the residue theorem has been applied. Similarly, for the left contour C_L , we have

$$\frac{1}{2\pi i} \int_{C_L} \theta'(z) \ln \varphi(z) dz = \frac{\ln(-1)}{2\pi i} \int_{C_L^+} \theta'(z) dz, \quad (26)$$

where C_L^+ is the upper half part of C_L , and $\ln(-1)$ has been defined in Eq. (24).

Combining the results in Eqs. (21), (25), and (26), the full contour integral over C becomes

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \theta'(z) \ln \varphi(z) dz \\ = \frac{\ln(-1)}{2\pi i} \int_{C^+} \theta'(z) dz - \frac{i}{2} [\ln(-1) - \ln \varphi(z_0^+)], \end{aligned} \quad (27)$$

where $C^+ = C_L^+ + C_R^+$, and the associated integral evaluates to $\int_{C^+} \theta'(z) dz = \theta(0) - \theta(\pi) = -\frac{\pi}{2}$. Although the final result in Eq. (27) is independent of the section chosen for $\ln \varphi(z)$, the explicit values of $\ln(-1)$ and $\ln \varphi(z_0^+)$ do depend on this choice. To obtain a consistent expression, both must be computed in the same section. In a specific section, e.g., $m = 0$ in Eq. (18), we obtain $\ln(-1) = i\pi$ and $\ln \varphi(z_0^+) = i(\pi - 2\delta)$, with $\delta = \arctan(e^{-\alpha N})$. Substituting these expressions into Eq. (27) yields

$$\sum_{k>0} (-1)^k \theta(k) = \frac{1}{2\pi i} \oint_C \theta'(z) \ln \varphi(z) dz = -\frac{\pi}{4} + \delta. \quad (28)$$

The analysis for $\text{mod}(N, 4) = 2$ proceeds similarly, although it is slightly more involved: an extra boundary term

appears in the integration-by-parts step [Eq. (19)], and the value of $\ln(-1)$ differs along the left and right contours due to the 2π phase shift accumulated around the full contour. Nevertheless, the final expression remains the same as in Eq. (28).

Equation (28) is the key result of this paper. Substituting it into Eq. (14) yields a remarkably simple formula for the lattice crosscap overlap: $\langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle = \sqrt{2} \cos(\frac{\pi}{8} + \frac{1}{2}\delta)$. For $N \gg 1$, we have $\delta \simeq e^{-\alpha N}$, and the crosscap overlap admits the asymptotic expansion

$$\langle \psi_0 | \mathcal{C}_{\text{latt}} \rangle = \sqrt{\frac{2+\sqrt{2}}{2}} - \frac{1}{2} \sqrt{\frac{2-\sqrt{2}}{2}} e^{-\alpha N} + \dots, \quad (29)$$

which exactly reproduces the exponentially decaying finite-size corrections observed in Fig. 2(a). The decay constant α , given in Eq. (20), depends on the Ising coupling and encodes the singularity of the Bogoliubov angle in the complex plane. In fact, the complex singularity of the Bogoliubov angle is directly related to the Toeplitz determinant kernel of the Ising model under the variable transformation $z' = e^{i(\pi-k)}$ [49,57,58]. This connection provides a more transparent derivation of Eq. (29), in which the exponential finite-size correction is directly associated with the square-root branch-cut structure of the Toeplitz kernel. It also provides a natural framework for extending the analysis to the off-critical region. See Supplemental Material [59] for further details [60].

Finally, we note that the overlaps of the crosscap state with all eigenvectors of the transfer matrix V can be calculated analytically. Actually, only eigenvectors of the form $|\psi_{k_1 \dots k_P}\rangle = \prod_{l=1}^P d_{k_l}^\dagger d_{-k_l}^\dagger |\psi_0\rangle$ have nonvanishing overlaps with $|\mathcal{C}_{\text{latt}}\rangle$, with $d_k = \sin[\theta(k)/2]c_k - i \cos[\theta(k)/2]c_{-k}^\dagger$ being the Bogoliubov quasiparticle operators annihilating $|\psi_0\rangle$ [Eq. (11)]. These excited-state crosscap overlaps are given by

$$|\langle \psi_{k_1 \dots k_P} | \mathcal{C}_{\text{latt}} \rangle| = \begin{cases} \sqrt{2} \cos \Theta, & P \text{ even} \\ \sqrt{2} \sin \Theta, & P \text{ odd} \end{cases}, \quad (30)$$

where Θ is defined below Eq. (14). This immediately shows that all nonvanishing crosscap overlaps in the 2D critical Ising

model exhibit exponentially decaying finite-size corrections, with the same decay constant α given in Eq. (20).

Summary and outlook. To summarize, we have analyzed the finite-size corrections to the crosscap overlap in the 2D classical Ising model along its self-dual critical line. We have demonstrated that the corrections decay exponentially with system size, originating from the branch-point singularities of the Bogoliubov angle treated as a complex function of lattice momentum.

An important open question is whether such exponentially decaying finite-size corrections are a special feature of the 2D Ising model, or arise more generally in other critical systems. While exponential suppression is highly advantageous for numerically locating critical points using crosscap overlaps, power-law finite-size corrections are known to occur in other exactly solvable models, such as the spin-1/2 Haldane-Shastry chain [27,61,62]. Future work should further explore the finite-size scaling of crosscap overlaps in more general microscopic models at criticality, in order to clarify under what conditions exponential versus algebraic behavior emerges. We leave this to future work.

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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