

Localized and delocalized modes on random geometric graphs in one dimension

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We perform an extensive investigation of the localization properties of the eigenmodes of the Laplace and adjacency matrices for one-dimensional random geometric graphs. We evaluate the density of states, the probability distribution of the participation ratio, and its relation to the eigenvalues. By disentangling the influence of system size, graph component size distribution, mean degree of nodes, network motifs, and degeneracy, we provide a comprehensive understanding of this system. We compare our findings to ordered graphs with the same mean degree and to one-dimensional tight-binding models.

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I. INTRODUCTION

The transport of charges in a semiconductor, the propagation of light in a crystal, and the dispersal of species in a patchy landscape are all affected by the presence of disorder. Starting from the seminal work by Anderson in 1958 [1], it has become clear that a sufficiently strong disorder leads to the localization of eigenvectors of such systems [2]. Anderson considered a tight-binding model for electrons with a random on-site potential. The localization of eigenvectors implies destructive interference of the wave function beyond a correlation length, preventing transport of electrons through the system. Subsequent work generalized these findings to tight-binding models with disorder also in the hopping terms, and to other wave equations that can be mapped on the same type of eigenvalue problem [2–6]. Generally, all states are localized in one dimension [7], with two being the critical dimension for a disorder-induced metal-insulator phase transition [8].

When, however, site and bond disorder are correlated so that the Hamiltonian of the Anderson model becomes a Laplace matrix where the sum of all entries in a row vanishes, there is a conserved quantity, leading to system-spanning eigenvectors in finite systems even in the presence of strong disorder and even in one dimension [9,10]. Such models apply, for instance, to classical harmonic chains and to the diffusion of a substance or a biological population through a set of coupled sites [4,6,11,12]. For one-dimensional systems, it was shown that the number of system-spanning eigenvectors scales as $\sqrt{N}$ with N being the number of sites [6,9,10,13,14]. In the limit of infinitely large systems, this is a vanishing fraction of all modes.

More recently, research on the localization of eigenvectors was extended to complex networks. Complex networks

incorporate disorder in their connection pattern and have “localized” modes that are concentrated on a small number of network nodes, in addition to delocalized modes. The structure of a complex network is characterized by its adjacency matrix A_{ij} , with A_{ij} being 1 if the nodes i and j are connected and 0 otherwise. Evaluating the spectrum of the adjacency matrix of a network yields information on its structure [15,16]. The spectrum of the Laplace matrix is relevant when studying the diffusion of a substance on the network or when studying the synchronizability of a set of coupled oscillators, the couplings being represented by the edges of the network [17,18]. Laplacian spectra are also relevant for the linear stability analysis of dynamical processes on networks (see, for example, [12]). Often, such studies are based on Erdős-Rényi and Barabási-Albert networks [19,20]. However, these two types of networks do not include the concept of spatial proximity. In contrast, many real-world networks, such as transportation networks, wireless networks, social networks, or ecological networks, are embedded in space (see [21] for a review of spatial networks). A generic model for networks embedded in space is given by random geometric graphs (RGGs) [22–25]. Here, nodes are randomly distributed in space, and pairs of nodes with a distance smaller than a cutoff (connectivity radius) are connected. In one dimension, RGGs are a special case of interval graphs if the length of the intervals is constant, or circular-arc graphs for periodic boundary conditions and constant arc lengths.

The local connection rules of RGGs lead to motifs, that is, recurrent substructures of the network [24,26]. A subset of those motifs is orbits that result from two nodes sharing the same neighborhood. Orbits lead to eigenvectors that are fully localized on the nodes of the orbits and have the value zero elsewhere [27]. The associated eigenvalues of the Laplacian are positive integers, and the associated eigenvalues of the adjacency matrix are -1 and 0 , depending on whether the nodes sharing the same neighborhood are connected or not (Type-I and Type-II orbits, respectively). Nyberg *et al.* have found a closed mathematical expression for the number of Type-I orbits in 1D RGGs with periodic boundary conditions [26,27]. From this, they conclude that approximately one-third of the

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FIG. 1. A sample graph visualizing the periodic boundary conditions and the shifting of the node positions such that the first node of one of the components is at 0, with all its adjacent nodes having larger position values—unless there are no consecutive nodes farther than r apart, in which case the graph has no gaps. This reordering leads to the adjacency matrix being block diagonal.

eigenvalues stem from Type-I orbits in the limit of large N and fixed r . The same holds in the thermodynamic limit when the mean degree is fixed and much larger than two. Hamidouche *et al.* [28] compared the spectra of normalized Laplacians (which are rescaled such that the diagonal elements are 1) with those of a lattice model where all nodes have the same number of neighbors, and they found upper bounds for the deviation between the two in the limit of large node numbers. These bounds decrease with increasing mean number of neighbors per node. Similar results were obtained when the adjacency matrices were used instead of the normalized Laplacians [29]. The moments of the eigenvalue distribution of the adjacency matrix can be expressed in terms of the node density and connectivity radius (henceforth called r), which in turn affects spreading processes, for instance, of viruses [30].

In this work, we complement the cited studies of one-dimensional RGGs with an extensive investigation of the spectra of their adjacency and Laplace matrix. We focus on the participation ratio as a measure of localization and investigate its probability distribution and the relation between the participation ratios and the eigenvalues. We will see that the participation ratio is not a good measure of localization for modes with integer eigenvalues, since orbits inevitably lead to degeneracy. Although the Laplace and adjacency matrices encode the same information about the structure of the RGG, the localization properties of their eigenmodes are qualitatively different due to the conservation law encoded in the Laplace matrix. By disentangling the influence of system size, mean degree, graph components, and orbits, we can provide an intuitive understanding of many observed features of the data. Comparison with ordered graphs where each node has the same number of neighbors, and with the 1D tight-binding model yields additional insights.

II. MODELS

We generate one-dimensional random geometric graphs (RGG) of N nodes by drawing N random values from a uniform distribution on the interval $[0,1)$, which serve as coordinates of the nodes. Two nodes are connected if their distance is smaller than the connectivity radius, r . In the following, we assume periodic boundary conditions, that is, we distribute the nodes on a circle with circumference equal to 1. There are $N!$ ways to label the N nodes of the graph. We choose the labeling such that it preserves the ordering of the positions of the nodes. Furthermore, we shift the node positions so that the first node of one of the network components is at position 0, as illustrated in Fig. 1. This has the effect that the adjacency matrix and the Laplace matrix become block diagonal. The connectivity of the graph is determined by the

parameter $z := Nr$ [27], with the mean degree $\langle k \rangle$ being equal to $2z$. Smaller z results in less connected graphs.

We consider two versions of the model, the adjacency model (AM) and the Laplace model (LM), i.e., we evaluate the spectra of the adjacency matrix (here: the negative adjacency matrix) and the Laplace matrix. Both of these spectra can be related to dynamical processes by considering the tight-binding Schrödinger equation (see, for instance, [1,2])

$$E\psi_n = - \sum_{m \in N(n)} t_{nm} \psi_m, \quad (1)$$

for the AM and

$$E\psi_n = \sum_{m \in N(n)} t_{nm} (\psi_n - \psi_m), \quad (2)$$

for the LM. Here, ψ_n denotes the amplitude of the wavefunction on node n , E is the total energy of the system, $N(n)$ denotes the set of nodes that are connected to node n , and t_{nm} is the transition rate from node m to node n , which we set to 1 if the two nodes are connected and to 0 otherwise. The LM is a model that has a conserved quantity $\sum_n \psi_n$, where the sum runs over the amplitudes on the lattice sites of the state ψ . This conserved quantity is not realistic for quantum mechanical systems where ψ_n represents a wave function, but the model applies, for instance, to a classical system where a substance with local concentration ψ_n diffuses between connected sites. The full time-dependent dynamics of such a diffusion system corresponds formally to a Schrödinger equation with imaginary time, since it is given by

$$\frac{d}{dt} \Psi_n(t) = - \sum_{m \in N(n)} t_{nm} (\Psi_n(t) - \Psi_m(t)), \quad (3)$$

and results in Eq. (2) when the ansatz $\Psi_n(t) = e^{-\lambda t} \psi_n$ is made [4]. In this case, E corresponds to the relaxation constant λ , and the eigenmodes are not excited states (as they would be for the Schrödinger equation), but exponentially relaxing modes describing the approach to equilibrium, implying that all λ are nonnegative. The LM is an RGG analog of the diffusion model (DM) on 1D lattices with random bonds t_{ij} and with on-site energies ϵ_i such that they are equal to the negative sum of the coupling terms, $\epsilon_i = -\sum_j t_{ij}$ with j being the nearest neighbors of i [10]. The conservation law of these models leads to system-spanning modes, since an imbalance in the distribution of the diffusing substance can only be equilibrated by transport over the scale of the imbalance [10]. When Eq. (1) is written in matrix form, the matrix is minus the adjacency matrix of the RGG. When Eq. (2) is written in matrix form, the matrix is the Laplace matrix of the RGG, for which the sum of all elements in a row is 0.

On a regular 1D lattice, where each node has exactly $2z$ neighbors, the dispersion relation can be evaluated

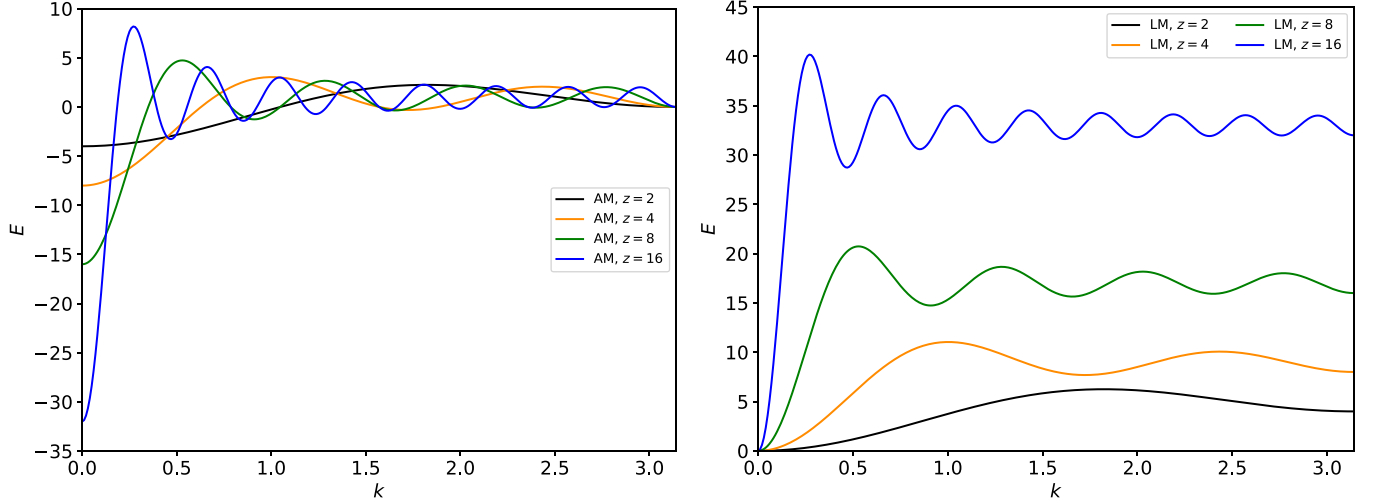


FIG. 2. Dispersion relation of the adjacency model (left) and the Laplace model (right) for different values of z for a regular lattice where each site has exactly $2z$ neighbors. To make the individual curves more easily recognizable, $z = 32$ has been omitted.

analytically using the ansatz of a plane wave $\psi_n = e^{ikn}$, resulting in

$$E(k) = -2 \sum_{n=1}^z \cos(nk), \quad (4)$$

for the AM and

$$E(k) = 4 \sum_{n=1}^z \sin^2\left(n \frac{k}{2}\right), \quad (5)$$

for the LM. Figure 2 depicts these dispersion relations. They differ by an energy shift of $2z$, as is evident from the comparison of Eqs. (1) and (2). For $z \geq 2$, the dispersion relation is not invertible on the interval $[0, \pi]$, so that there exist z modes that have the same energy, resulting in several terms that contribute to the density of states at a given value of E . The typical energy value $2z$ of the Laplace model will manifest itself below when the Laplace spectrum of the RGG is evaluated, and the large weight at negative energies (compared to positive energies) of the AM will show below when the adjacency spectrum of the RGG is evaluated.

III. METHODS

In order to quantify the localization behavior of the models, we calculate the eigenvectors and associated energies of Eqs. (1) and (2) by diagonalizing the Hamiltonian using the Julia library `LinearAlgebra`. We then determine their participation ratio P and evaluate the probability distribution of P and the relation between P and the energy E . The features found in these data will be explained by considering the component size distribution of the RGGs, the orbits, and the density of states (DOS), and by comparing to the model without disorder and to the lattice model with random couplings.

A. Participation ratio

The participation ratio [2,3] of the i -th eigenstate ψ_i is defined as

$$P = \frac{(\sum_n |\psi_{i,n}|^2)^2}{\sum_n |\psi_{i,n}|^4}, \quad (6)$$

where the numerator is 1 when the wave function is normalized. This definition has the properties we require for a localization measure. That is, $P = 1$ for a state that sits at only one site, and $P = N$ for a state with the same amplitude at every site. A special case, which we will encounter later, is the participation ratio of a sine function, for which we have $P = 2N/3$ in one dimension. For large N , the evaluation of the probability distribution of P requires large computational resources as all N eigenvectors of the system must be determined. Therefore, we limit our study mostly to system sizes up to $N = 10^4$.

B. Component size distribution (CSD)

An RGG is constructed by placing N nodes randomly on the unit interval and connecting two nodes if their distance is smaller than r . In the limit of large N , the random placement of N nodes becomes equivalent to occupying each small interval, dx , with probability Ndx by a node, independently of the other intervals. Therefore, the probability that the interval dx to the right of a given node is not occupied by a node is $(1 - Ndx)$. The probability that there is no other node within a distance x to the right of a given node is $(1 - Ndx)^{x/dx} \simeq e^{-Nx}$. The probability that the distance to the next node is larger than the connectivity radius, r , is therefore $P(x > r) = e^{-rN}$. The size distribution of the components $P(n)$ is given by the probability that, starting from the first node of a component, the next $(n-1)$ nodes each have a distance smaller than r to the previous node and the n -th node has a distance larger than r ,

$$P(n) = (1 - e^{-rN})^{n-1} e^{-rN}, \quad (7)$$

$$= \frac{1}{e^{rN} - 1} e^{n \ln(1 - e^{-rN})}. \quad (8)$$

This is an exponential distribution in n . For $z = 2, 4, 8, 16, 32$ (i.e., $r = 2/N, 4/N, 8/N, 16/N, 32/N$), we obtain approximately $P(n) \sim e^{-n/7}, e^{-n/54}, e^{-n/2980}, e^{-n/(8.89 \times 10^6)}, e^{-n/(7.9 \times 10^{13})}$. For the system sizes $N \leq 10000$ used in our simulations, the RGGs are composed of several unconnected



FIG. 3. A sample graph visualizing Type-I (left) and Type-II orbits (right). In both cases, the black nodes have the same neighborhood (red nodes), but for Type-I orbits, the black nodes are also connected. The left configuration leads to the eigenvalue 4 and the right to the eigenvalue 2 of the Laplace matrix. In 1D RGGs, Type-II orbits are always isolated.

components for $z = 2$ and 4. For $z = 8$, the RGG might have a system-spanning component, and for $z \geq 16$, most of the RGG realizations span the entire system. The properties of the eigenvectors of the system will therefore change qualitatively with z , since the average size of the components on which the modes are located changes so strongly with z .

C. Orbits

In contrast to lattice graphs, RGGs contain so-called orbits [26]. These are motifs, that is, recurring substructures, in which two nodes share the same neighborhood (see Fig. 3), leading to eigenvectors that are fully localized on these nodes. In the spectrum of the Laplace matrix, such motifs appear as integer eigenvalues and contribute to its discrete part [26]. These eigenvalues are directly related to the degree k of the nodes [Eq. (9)]. If the nodes themselves are not connected, the eigenvalue is their degree (Type-II orbit). If they are connected, the eigenvalue is their degree plus 1 (Type-I orbit). The corresponding eigenvector $\vec{v}$ has the value 1 on one node (node 1) and -1 on the other (node 2), and 0 on all remaining nodes [26]. Let L be the Laplace matrix and $\vec{e}_i$ the i -th canonical basis vector. Then

$$\begin{aligned} L\vec{v} &= L(\vec{e}_1 - \vec{e}_2) \\ &= \sum_{i,j=1}^N \vec{e}_i L_{ij} v_j \\ &= \sum_{i,j=1}^N \vec{e}_i L_{ij} (\delta_{j1} - \delta_{j2}) \\ &= \sum_{i=1}^N \vec{e}_i (L_{i1} - L_{i2}) \\ &= (k - L_{12})(\vec{e}_1 - \vec{e}_2). \end{aligned} \quad (9)$$

If $n > 2$ nodes share the same neighborhood, they all have the same degree, and the above considerations apply to each subset of two nodes. Since it is impossible that more than two nodes of a one-dimensional RGG share the same neighborhood but are not connected among each other, the eigenvalues of the Laplace matrix must be $k + 1$ in this case. Linear combinations of the eigenvector pairs of nodes are also eigenvectors of the eigenvalue $k + 1$, with the degeneracy being $n - 1$.

The (negative) adjacency matrix has the same eigenvectors, and the eigenvalues are 1 if the nodes of the motif are connected and 0 otherwise.

D. Data handling

The simulations were run in Julia 1.10.1. The algorithm used for the exact diagonalization is `eigen()` from the `LinearAlgebra` library. The random positions of the nodes were generated with `rand()`, which returns pseudorandom numbers (Xoshiro256++) distributed uniformly in the interval $[0,1]$. We sorted the nodes by position, with position zero coinciding with the first node of a component. This means that the Hamiltonian becomes a block diagonal matrix, and the eigenvectors produced by the algorithm are confined to a single component each, even if there is degeneracy.

The data shown in the figures is based on the average of an ensemble of 1000 realizations for the system sizes 625, 2500, and 10000. Due to hardware limitations, only eight and 12 realizations were simulated for $N = 40000$ for the DM and the AM, respectively. If not stated otherwise, bins with less than five counts were excluded.

IV. RESULTS

A. Density of states

In the spectrum of the AM [Fig. 4(a)], there is a discrete peak resulting from the Type-I orbits at $E = 1$. Around this discrete peak, the spectrum shows a funnel-shaped plunge, which must be due to points moving exactly to the value 1. A corresponding funnel around $E = 0$ due to Type-II orbits is visible for $z = 2$. Since Type-II orbits are disconnected from the rest of the RGG, they occur mostly for small z where small graph components are frequent. For higher absolute values of the energy, the DOS of the AM exhibits step-like features for $z \geq 8$, where the graph has system-spanning components. Compared with the ordered case where each node has exactly $2z$ neighbors (Fig. 5), it is apparent that these steps occur at the band edges of the ordered system where the DOS diverges. Similarly, the flat parts of the DOS of the AM can be explained by comparison with the ordered case: In the vicinity of its local extrema, the function $E(k)$ is nonzero [see Fig. 2(a)], leading to a constant density of states in this energy region. Such connections between the DOS of the disordered and regular system were also observed in the 1D random coupling model [10]. Beyond the maximum values of $|E|$ of the ordered model, the density of eigenstates of the RGG decreases rapidly.

In the LM, the spectrum has a discrete part and a continuous part [shown separately in Figs. 5(c) and 5(d)], with the discrete peaks at integer values of E corresponding to orbits in the RGG (Fig. 4) [26]. The continuous and the discrete parts show a maximum around the mean degree $2z$, which determines the order of magnitude of the majority of eigenvalues. For small E , the density of states, shown in Fig. 5(b), follows

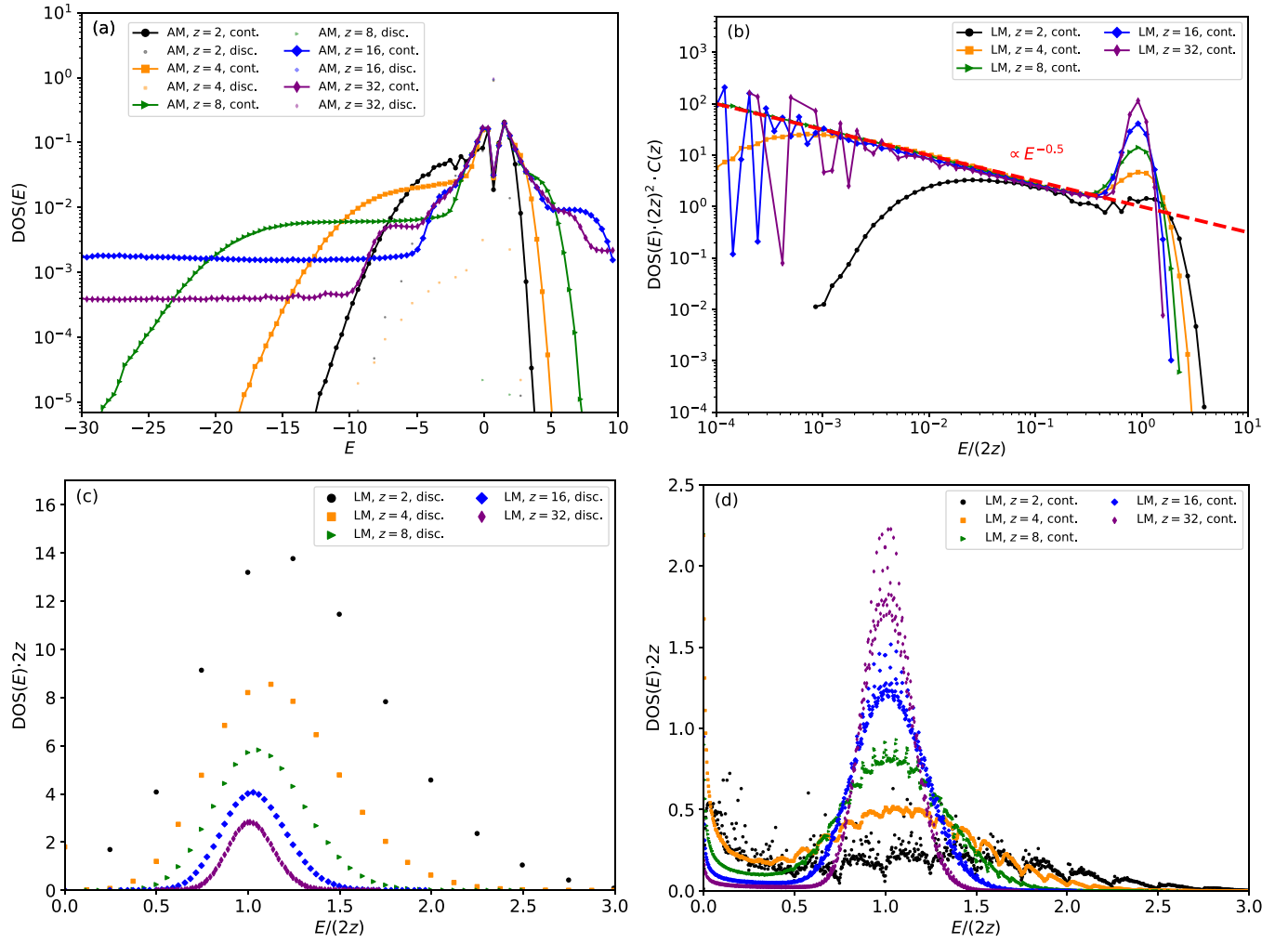


FIG. 4. Density of states for the (a) AM and (b) LM for systems of size 10^4 . The y-axis of (b) is scaled by $(2z)^2 \cdot C(z)$ where $C(z) = \sqrt{(1 + 1/z)(1 + 1/(2z))}$ [see Eq. (11)]. Graphs (c) and (d) show for the LM the discrete and continuous part separately (i.e., the part for integer and noninteger energy values).

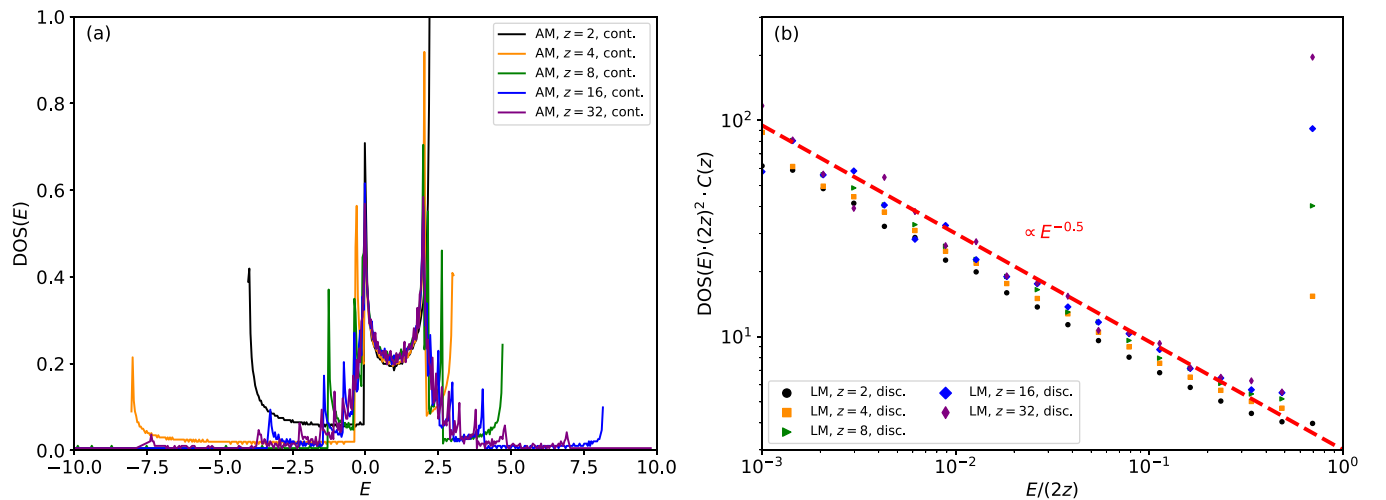


FIG. 5. Density of states of a regular lattice for the AM in (a) and the LM in (b), where each site has exactly $2z$ neighbors. For the LM, we chose logarithmic axes so that the power law $-1/2$ for small energies becomes visible. In fact, the DOS of the LM and AM are identical, apart from a shift of the energy values by $2z$, which follows directly from the corresponding shift for the dispersion relations (see Fig. 2).

a power law $\sim 1/\sqrt{E}$ (red dashed line), just as for the ordered system with exactly $2z$ neighbors per site [see Fig. 5(b)]. These are the modes with a large wavelength and thus a large extension. For $z = 2$ and 4 , the cutoff in the component size distribution leads to a cutoff for the power law. For $z = 16$ and $z = 32$, we see a scattered set of data below the power law, which we ascribe to rare components of a size smaller than the system size. On the other hand, we see data above the power law, which are due to finite-size effects since the sine function modes must have at least one wavelength, and the corresponding energy is $E \simeq 5.9 \times 10^{-4}$ for $z = 16$ and $E \simeq 4.5 \times 10^{-3}$ for $z = 32$ [compare Eq. (10)].

The dependence of the DOS on E for small E of the ordered system can be determined analytically from the dispersion relation (5): We have

$$E(k) \simeq 4 \sum_{n=1}^z \frac{n^2 k^2}{4} = k^2 \frac{z(z+1)(2z+1)}{6}, \quad (10)$$

and from the density $N/2\pi$ of k values, we obtain the density of E values as

$$\text{DOS}(E) = \frac{N}{2\pi} \frac{1}{dE/dk} = \frac{N}{2\pi} \frac{3}{\sqrt{6Ez(z+1)(2z+1)}}. \quad (11)$$

This means that if we plot $\text{DOS}(E) \cdot 4z^2 \sqrt{(1+1/z)(1+1/2z)}$ versus $E/2z$, the curves for different z coincide for small E . From this data collapse, it follows also that the depth of the minimum, which is located between the power-law part and the peak around $2z$, scales as $1/z^{1.5}$ for large z , leading to an increasing separation between these two parts of the dispersion relation with increasing z .

The continuous part of the density of states shows a series of bows, which is best visible for the smaller z due to the scaling of the x -axis by $2z$. The minima between the bows occur at integer values of E , where the discrete spectrum has its data points (see Fig. 4). This seems to be the same effect as for the AM, where there is a funnel-shaped plunge around 1. For the LM, the discrete spectrum has points at all integer values of E , and thus there are funnels around each of them.

Another striking feature of the DOS is the fuzziness of the data visible in Fig. 5(d), which is most pronounced for the main peak and for larger z . The data do not become sharper when the sample size is increased. This means that nearby energy values occur with different frequencies. We explain this by the limited number of different connection patterns for subsets of several neighboring nodes. These connection patterns occur with different frequencies. Eigenvectors localized on such subsets have a discrete set of possible eigenvalues with their associated frequency of occurrence in the statistical ensemble.

B. Selected eigenvectors

Since the Hamiltonian has a block-diagonal form, each eigenvector is confined to one component. However, when there is degeneracy of modes within one component, the algorithm gives linear combinations of these modes. For the AM, all orbits have the same energy $E = 1$, and therefore the

degeneracy is greater, manifested in larger participation ratios [compare Fig. 6(d) with Fig. 7(d)].

The modes belonging to the continuous spectrum change qualitatively as their energy changes. For the AM, the degree of localization changes in a nonmonotonic way with the energy, see Fig. 6. Negative eigenvalues with a large absolute value require the ψ_n values of the sites that contribute to the eigenvector to preferably have the same sign, see Eq. (1), leading to localized modes with a wavelength that is determined by the persistence length of the sign of ψ_n [Fig. 6(a)]. With increasing energy, the wavelength becomes shorter [compare Fig. 2(b)] and the participation ratio becomes larger, with some eigenvectors extending over the entire system [Fig. 6(b)]. As the eigenvalue $E = 1$ is approached, the eigenvectors become increasingly localized [Figs. 6(c) and 6(e)]. At $E = 1$, the eigenvectors are superpositions of orbits, all of which have the same eigenvalue [Fig. 6(d)]. This nonmonotonous behavior of the extension of eigenvectors with changing energy leads to the curious feature that eigenvectors of vastly different energies can have similar participation ratios, see also Fig. 10 below, where the relation between energy and participation ratio is plotted.

For the LM, the smallest energies belong to modes that resemble the sine functions of the ordered system. For larger energies, modes become increasingly localized; see Figs. 7(a)–7(c) and 7(e).

C. Probability distribution of the participation ratio

1. AM

Figure 8 shows the probability distribution of the participation ratio, separating the data belonging to the continuous part of the spectrum (i.e., noninteger energy values, large data points connected by lines) from those belonging to the discrete part (small data points of the same color with a lighter shade). We also separate the data belonging to modes with positive energy from the ones belonging to negative energy, since they differ qualitatively due to the occurrence of large negative energy values in the dispersion relation [see Fig. 2(a)], and since the orbits have nonnegative energies.

For the AM, the continuous part of the data is fully localized, and finite-size effects occur only if the system size is in or below the cutoff region of the probability distribution of the participation ratio [see Figs. 8(b) and 8(d)]. The cutoff is larger for the data associated with negative energies, which means that the occurrence of larger $|E|$ is correlated with the occurrence of larger P . The discrete part of the data occurs only for positive energies, as all orbits have the energy value 1 (or 0). Large components can contain several of these orbits, and obviously, the algorithm that determines the eigenvectors yields linear combinations of the minimal modes that are fully localized, see Fig. 6(d). The position of the peak due to orbits is almost independent of z [see Fig. 8(a)], indicating that the proportion of orbit eigenmodes among all eigenmodes remains of the same order as z is changed. This agrees with the finding of Nyberg *et al.* [26,27] that the proportion of eigenvalues due to Type-I orbits is of the order of $1/3$ if $z \gg 1$.

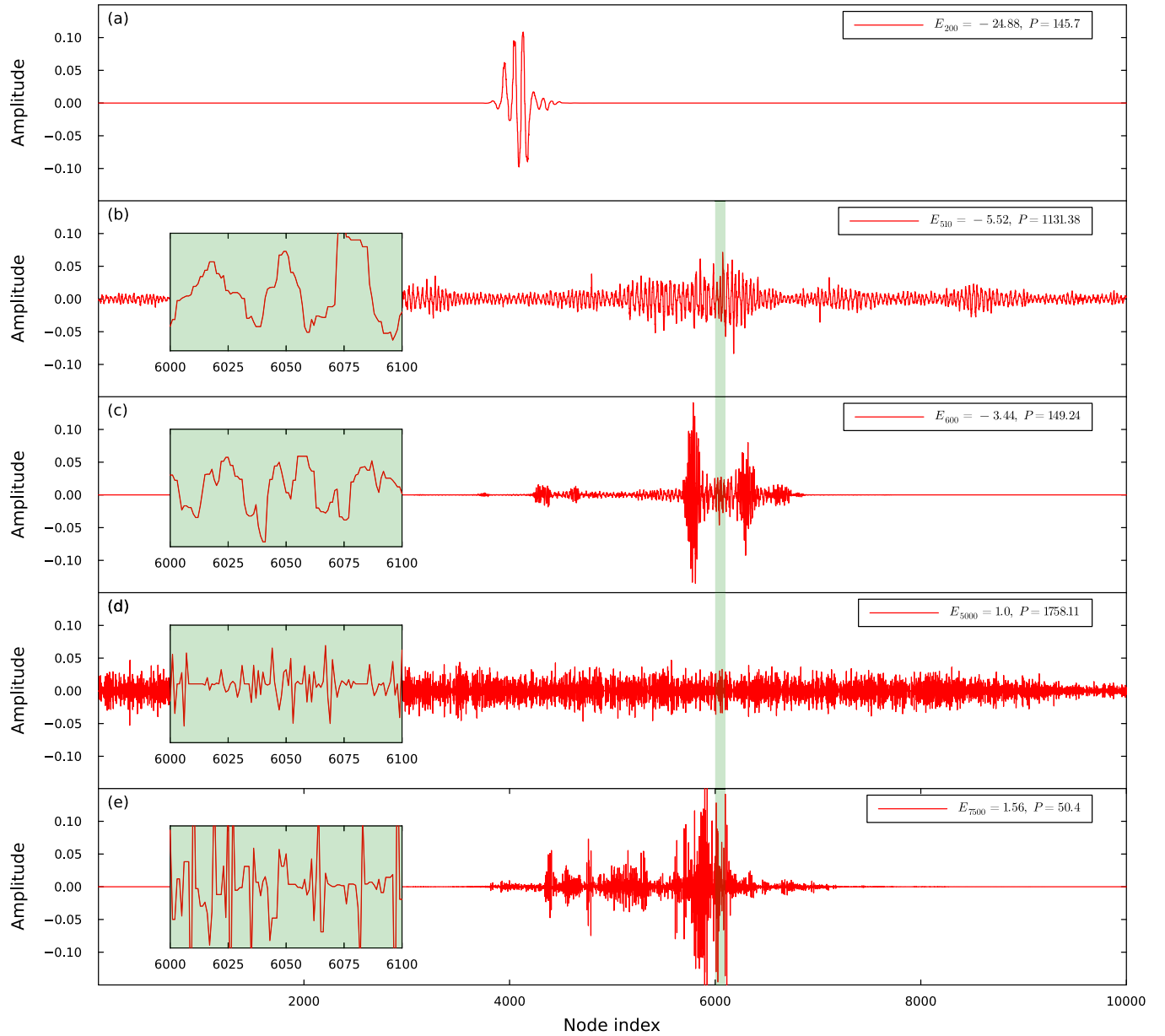


FIG. 6. Selected eigenvectors of the AM for $z = 16$ and $N = 10^4$: (a) eigenvector close to $E = 0$, (b) superposition of the degenerate eigenvalue $E = 1$ due to orbits, (c) localized eigenvector close to $E = 1$, and (d) localized eigenvector.

2. LM

The distribution of the participation ratio (PPDF) changes in a nontrivial way with z and system size, see Figs. 8(e) and 8(f). The features of the distribution result from the interplay of the CSD, of the discrete and continuous parts of the spectrum, and of localized modes and system-spanning modes. For $z = 2$ and $z = 4$, the PPDF has a cutoff much smaller than the system size, and this is due to the CSD having a cutoff much smaller than the system size. (The CSD for $z = 2$ is shown in Fig. 8(e) as a gray dashed line.) For larger z , there exist system-spanning components and associated system-spanning eigenvectors that resemble the sine-function modes of the ordered system [see Fig. 7(a)], which manifest themselves in the final increase at the right end of the PPDFs. Apart from this peak, the PPDFs for $z = 8, 16, 32$ show four additional

features: At small participation ratios, there are localized modes, as can be seen from Fig. 8(f), where the left part of the curve does not change anymore for the larger system sizes. Next, there is a “nose” near $P = 100$ in the discrete part of the data. With increasing z , the nose moves to the left, since the degeneracy of each version of orbit decreases as more versions of orbits occur. The nose moves to the right with increasing system size, indicating that several identical orbits that are located on the same component contribute to the same eigenvector, see Fig. 7(b). The third feature is a power-law behavior, which is best visible in the $z = 8$ data in the interval $P = 200$ to 3000 . For the $z = 16$ data, this power law becomes visible with increasing N , indicating that the cutoff of the power law depends on the system size. The exponent of the power law is close to -1.5 , which is the value

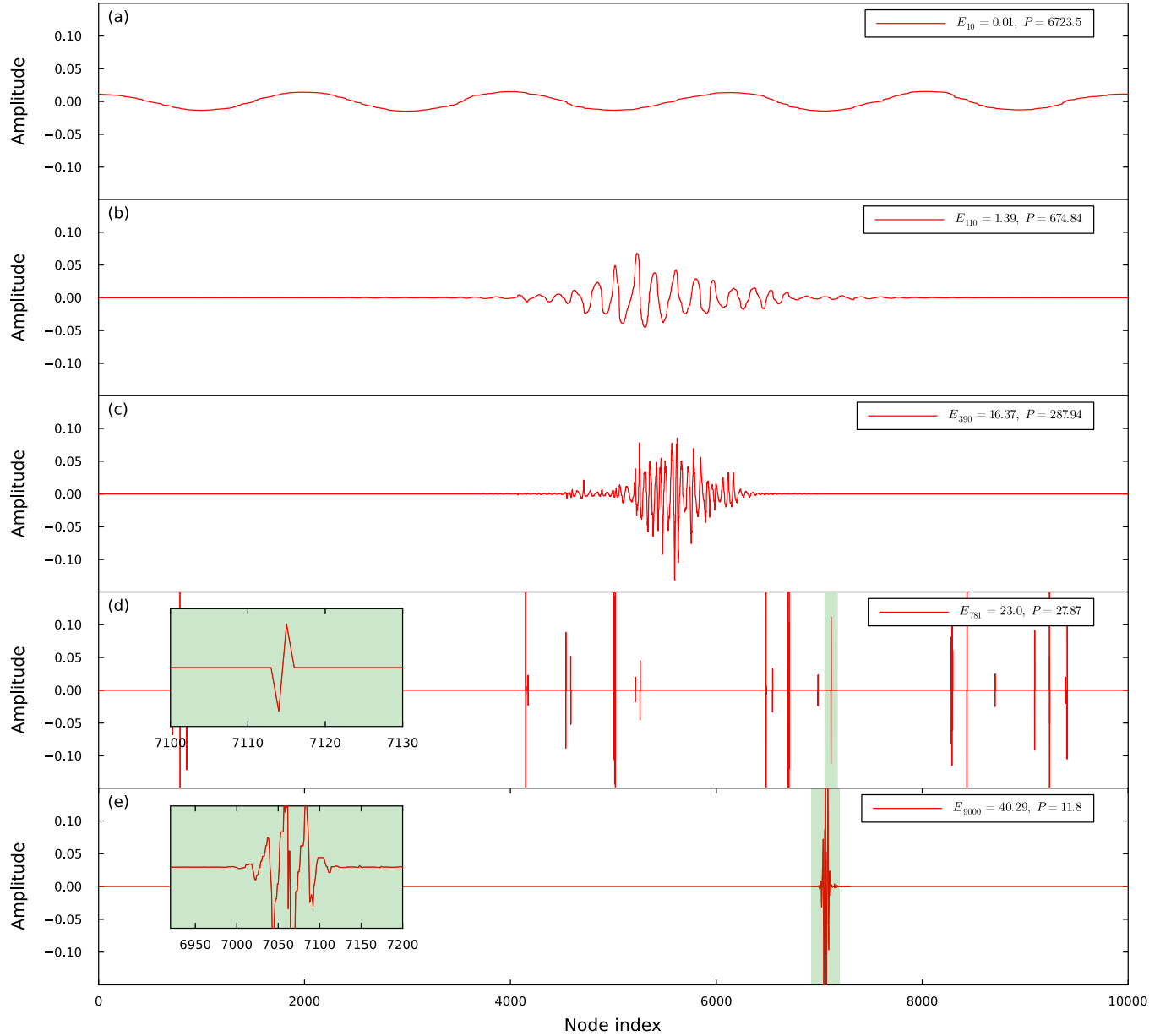


FIG. 7. Selected eigenvectors of the LM for $z = 16$ and $N = 10^4$: (a) system-spanning eigenvector with a large wavelength, (b) localized eigenvector for an energy in the minimum of the DOS between $E = 0$ and $E = 2z$ resembling a sine function with fluctuations of the amplitude due to disorder, (c) localized eigenvector for an energy also in this minimum but closer to the peak at $E = 2z$, leading to a shorter wavelength, (d) eigenvector for an integer energy showing the localization on orbits, and (e) strongly localized eigenvector in the peak of the DOS. The insets show zooms into the shaded region.

that we observed for the regular 1D lattice for the “diffusion model” [10], where the Hamiltonian is a discretized Laplacian for a system with random nearest-neighbor couplings in the interval $[0, 2]$ and corresponding on-site energies. This power law is due to modes that span many lattice sites and several wavelengths, but are still localized, see Fig. 7(b). We expect that for $z = 32$, this power law would show up if the system size were made considerably larger. Our data show instead a pronounced hump at large P for the $z = 32$ data, with the power law just starting to build up at the right-hand slope of this hump. This build-up of the power law at the right-hand side of the hump can be nicely seen in the $z = 16$ data with

increasing system size [Fig. 8(f)]. The hump itself, which is the fourth characteristic feature of the data, is due to a pronounced minimum left of the hump. The location of the minimum on the P axis does not appear to depend on N , and it marks the upper boundary of the well-localized part that is independent of system size, and where the energy is of the order of $2z$, compare Figs. 10(a) and 10(b). Around the energy $E = z$, the DOS shows a minimum, see Fig. 4(d), and this minimum also shows in the PPDF. As explained above, the minimum in the DOS becomes more pronounced when z is larger, explaining the greater height of the drop in the PPDF for increasing z .

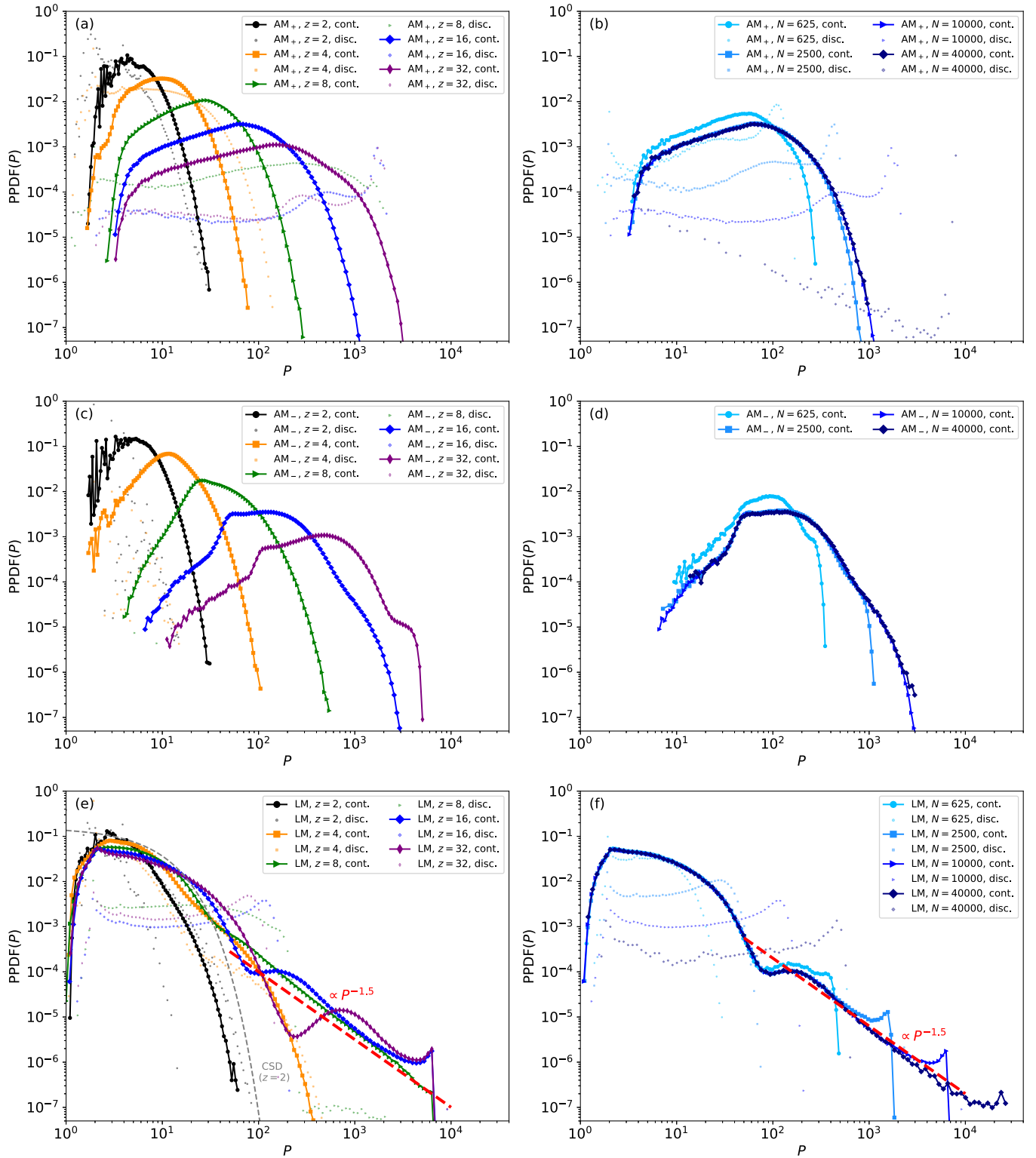


FIG. 8. Distribution of the participation ratio for fixed $N = 10^4$ (left) and fixed $z = 16$ (right). The first two rows show the AM, separating the part belonging to positive energy eigenvalues (first row, legend AM₊) from the one with negative eigenvalues (second row, legend AM₋). The last row shows the LM. The large data points show the data belonging to noninter eigenvalues, the small data points with a lighter shade show the data for the discrete part of the spectrum, which has integer eigenvalues. The gray dashed line in (e) is the CSD of the graph for $z = 2$.

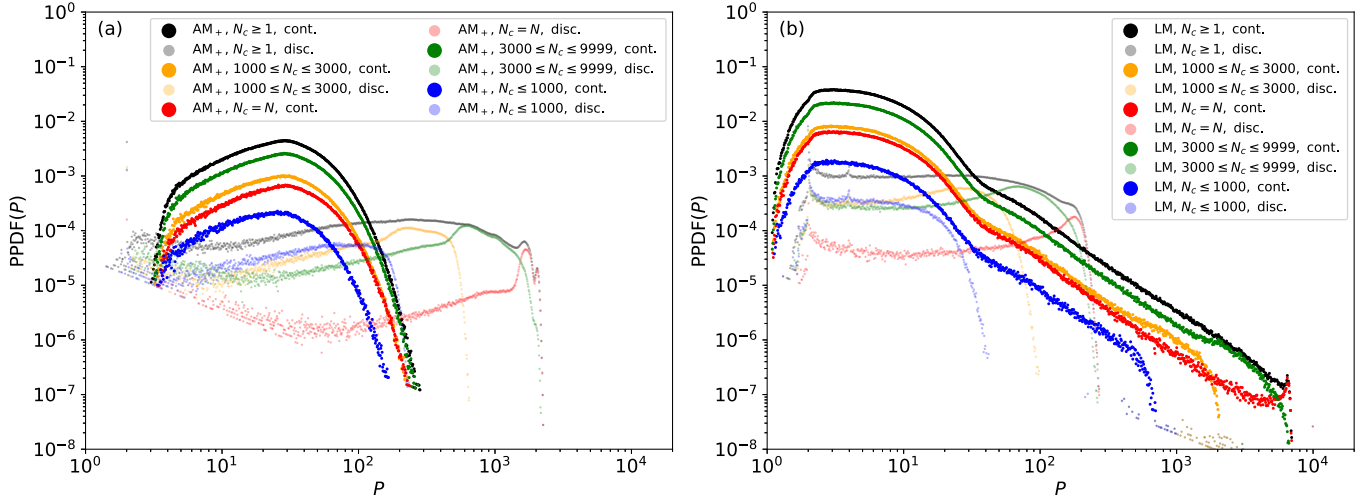


FIG. 9. Contribution of different component sizes to the PPDF for the LM (a) and the AM for positive energies (b), for $N = 10^4$ and $z = 8$. The lighter colors again indicate the modes with noninteger energies.

D. Contribution of different component sizes to the PPDF

Figure 9 shows the contributions of components of different sizes to the PPDF for $z = 8$, where the ensemble of systems has system-spanning components as well as smaller components. The main data belong again to the continuous part of the spectrum, and the lighter colors to the modes with integer energy values. For the AM, the PPDF for the continuous part of the spectrum has for all component size intervals the same shape (but different weight), with the cutoff for the blue curve (components of size ≤ 1000) being somewhat smaller due to finite-size effects. The discrete part of the data is determined by the high degeneracy of orbits, leading to large P values and a cutoff that scales with component size. For the LM, we see that only the system-spanning component (red) contributes to the peak at the right-hand end of the distribution. The power-law part of the distribution is visible in all component size intervals, with the cutoff depending on component sizes. The P values of the discrete part are considerably smaller than for the AM due to the smaller extent of degeneracy, and the cutoff scales again with component size. At $P = 2$, both models show a distinct peak due to orbits that are located on two nodes.

E. Dependence of the participation ratio on the energy

For the AM, we see an increase of P with decreasing $|E|$ [Fig. 10(a)], at least for the larger z values. The largest P value increases with system size [Fig. 10(b)], as long as there are still some finite-size effects. Near $E = 1$, there are well-localized modes with small participation ratios, due to a funnel centered around $E = 1$, which is the energy value of the orbits.

For the LM, the participation ratio increases with decreasing E , as extended modes are those with long wavelengths and, therefore, small energies [Fig. 10(c)]. The slope becomes steeper as the system size becomes larger, see Fig. 10(d) and we expect the data to eventually approach a power law with slope -1 . This is the slope that we observed in the 1D lattice with random couplings [10]. For even smaller E ,

the P values approach the value $2N/3$, which applies to a sine wave. This value is not seen for the smaller z values 2 and 4, since the cutoff in component size does not allow modes with sufficiently large extension and wavelength. For energy values around $2z$, localization is strongest, as local connectivity patterns determine the eigenmodes. The fig show only the continuous part of the spectrum.

F. Degeneracy of eigenvectors for small z

For $z = 2$ and $z = 4$, the system is made up of many small components. This leads to a high degree of degeneracy of eigenvectors, not only due to orbits, but also due to identical components. The diagonalization algorithm that produced all previous fig was based on an ordering of nodes such that the Hamiltonian had block diagonal form and all eigenvectors were confined to one component. When we do not sort the nodes before determining the eigenvectors, the algorithm generates linear combinations of degenerate eigenvectors from different components. Figure 11 shows how the probability distribution of the participation ratio changes when the nodes are not sorted. The participation ratio distributions associated with the continuous part of the spectrum now show a hump for the AM, for both signs of the energy. This must be due to identical components. For the LM, there is a slight shoulder, but the effect is much less pronounced than for the AM. The most striking changes occur for the orbits. The degeneracy of the eigenvalues associated with orbits leads for the AM for positive energies to a pronounced peak at large P , which corresponds to the eigenvalue $E = 1$, and to a second peak at somewhat smaller P , which corresponds to the eigenvalue 0 and occurs in the graphs for positive and negative energies, depending on the sign of the floating-point number that the algorithm yields as eigenvalue. For the LM, there is a series of peaks that are due to the orbits, with the rightmost peak being associated with the eigenvalue 0, and with the eigenvalues increasing from right to left (but not exactly matching the different peaks).

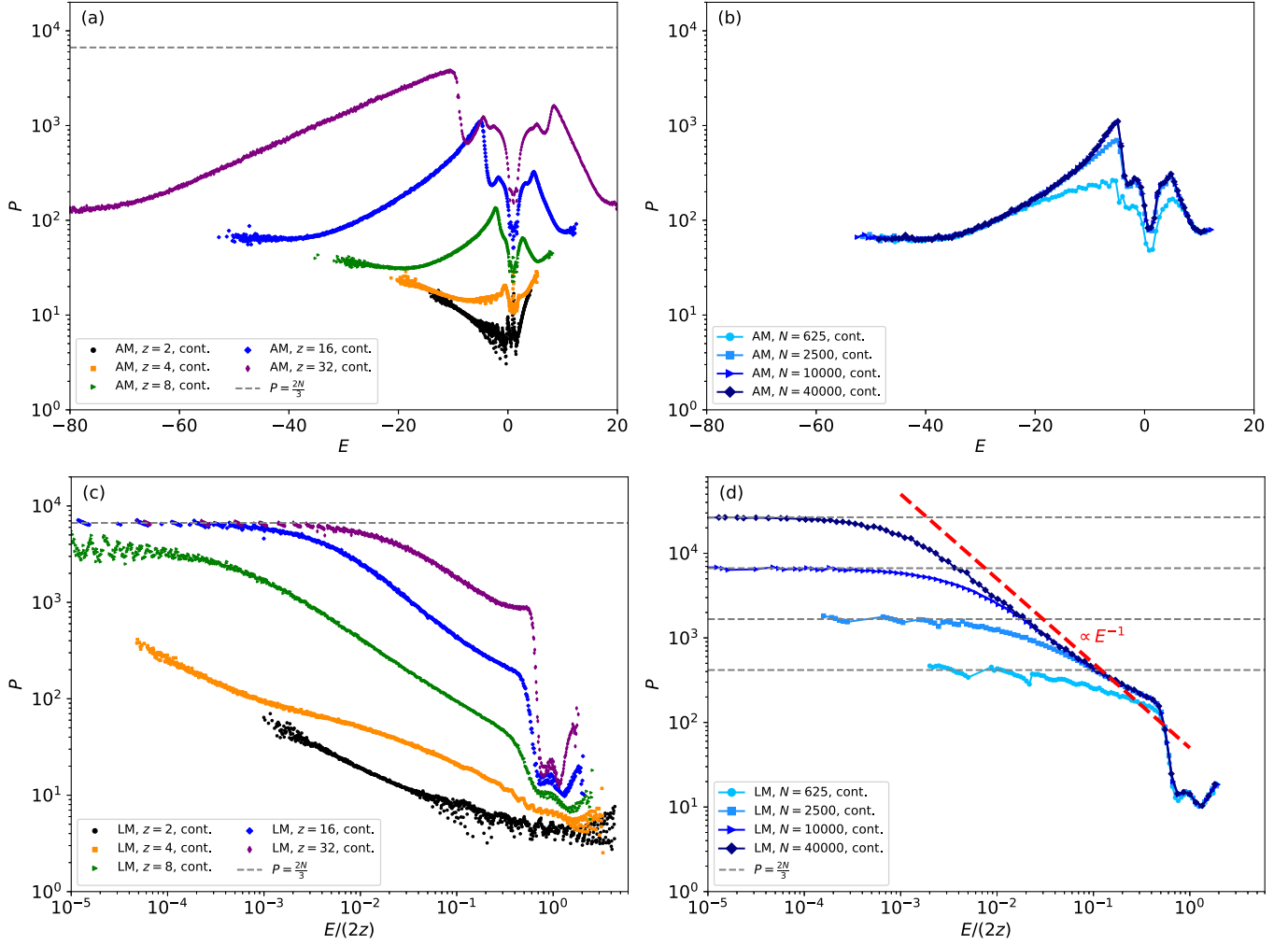


FIG. 10. Dependence of the participation ratio on the energy for (a) the AM with $N = 10^4$, (b) the AM with $z = 16$, (c) the LM with $N = 10^4$, and (d) the LM with $z = 16$. The gray dashed line indicates the participation ratio of sine.

V. DISCUSSION AND CONCLUSION

Our comprehensive evaluation of the localization properties of the eigenmodes of the adjacency and Laplace matrix of one-dimensional random geometric graphs shows how component size distribution, mean degree, network motifs (orbits), and system size affect the distribution of the participation ratio and the extent of localization.

The model, which is based on the adjacency matrix (AM), has a cutoff in the distribution of the participation ratio that becomes independent of the system size when the system size is large enough, and the discrete part of the spectrum is ignored [see Figs. 8(a)–(d)]. In contrast, the model which is based on the Laplace matrix (LM) can be interpreted as a dynamical model with a conserved quantity [see Eq. (3)]. This model has therefore system-spanning eigenmodes that have the shape of sine waves [see Fig. 7(a)] and give rise to the power law with exponent -0.5 in the density of states [Fig. 4(b)], just as in the ordered model [Fig. 5(b)]. The associated participation ratio has the value $2N/3$.

The discrete part of the spectrum of both models is mostly due to orbits, i.e., eigenmodes that are fully localized on small network motifs with integer eigenvalues. Due to the high

occurrence of these orbits and the degeneracy of their eigenvalues, they give rise to large values of the participation ratio, in particular for the AM, where all orbits have the eigenvalues 0 or 1 (see Figs. 11 and 8). This means that the participation ratio is not a good measure of localization for the discrete part of the spectrum. Any conclusions on localization should therefore be based on the continuous part of the spectrum.

For $z = 2$ and 4, the size distribution of network components is such that the graph is composed of many components, and the cutoff in the component size distribution is much smaller than the system size and determines also the cutoff in the probability distribution of the participation ratio [see Figs. 8(a) and 8(e)]. The case $z = 8$ is a limit case, where part of the graph realizations consist of a single, system-spanning component, and the probability distribution of the participation ratio has qualitatively different contributions from components of different sizes, as we have shown in Fig. 9. For $z = 16$ and 32, the vast majority of graph realizations consists of one component. For these values of z , the finite-size effects are strongest, since the distribution of the participation ratio becomes shifted to larger P values (Fig. 8). We explain this by the fact that larger values of z lead to larger sets of nearby nodes that are fully connected to each other. With respect to

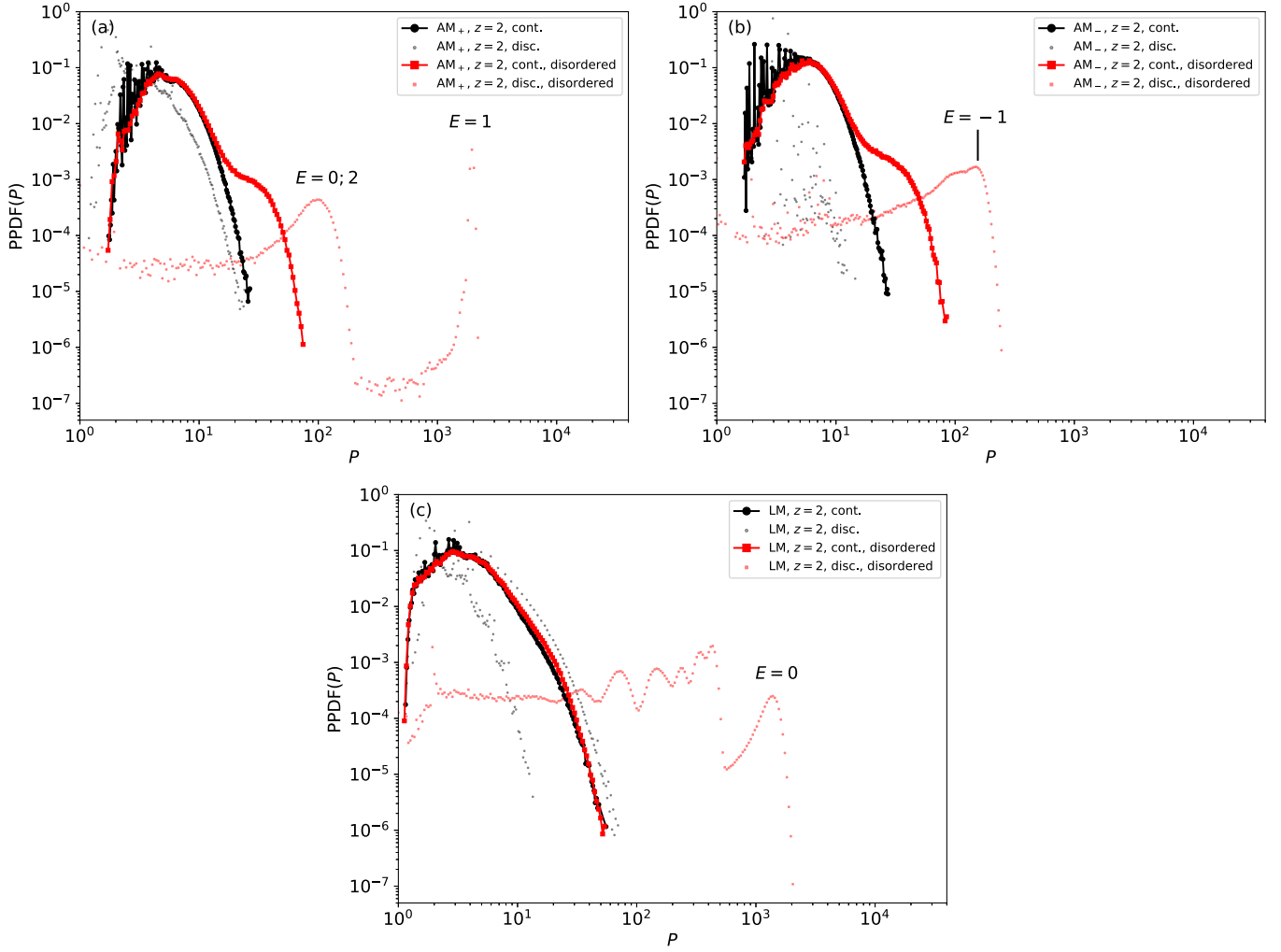


FIG. 11. Distribution of the participation ratio for fixed $N = 10^4$ and $z = 2$. The red data points are obtained from systems in which the node positions have not been ordered before diagonalization, whereas the black ones have been ordered so that eigenvectors are always localized to a single component. (a) shows the PPDF of the AM for positive, (b) for negative eigenvalues, and (c) the PPDF for the LM. The main curve shows the data for the continuous part of the spectrum; the small light data points show the data belonging to discrete eigenvalues.

the disorder in the system, such fully connected groups of nodes act as a single effective node, and the effective system size becomes smaller (when measured in units of these effective nodes). The increase of finite-size effects with increasing z is also visible in Fig. 4(b), where the $z = 8$ data agree with the slope -0.5 over a much larger interval than the data for $z = 16$ or $z = 32$. The finite-size effects for $z = 16$ and $N = 10^4$ lead also to the system-spanning mode shown for the AM [Fig. 6(b)].

Those features that are related to orbits and small components are particular to RGGs and are not present in standard Anderson localization and related disordered tight-binding models. However, on larger scales, the LM and the AM become more similar to one-dimensional tight-binding models with disorder. For these models, the presence or absence of a conservation law determines the large-scale features of eigenmodes and their localization. The random-coupling model (RCM) [10], where each connection of a regular lattice has a random link strength, is the tight-binding model equivalent of the AM: It has no conservation law and shows a

unimodal distribution of participation ratios with a cutoff that becomes independent of the system size when the system size is large enough [compare the green curve in Fig. 3(a) of [10] with Figs. 8(a)–(d)]. The main difference between the RCM and the AM is because the RCM is bipartite, as the one-dimensional regular lattice can be decomposed into two sublattices. This leads to a symmetry of eigenmodes with respect to the sign of the energy, which is no longer present in the data above. For this reason, we show in most plots for the AM the parts for positive and negative energies separately.

The diffusion model (DM) [10] is a version of the tight-binding model where the random on-site energies are chosen to be identical to the negative sum of the link strengths for this site. It therefore has the same conservation law as the LM, and both models have sine-shaped system-spanning modes which resemble those of the ordered system size. Their proportion increases as the square root of the system size (see, e.g., [10]). In addition, both models have extended but localized modes that resemble sine waves with a variable

amplitude, see Fig. 7(b). These modes give rise to the power law with exponent -1.5 in the distribution of the participation ratios [Fig. 8(f)], and -1 in the relation between P and E [Fig. 10(d)], which are the same exponents as reported for the DM [10].

To conclude, while the Laplace and adjacency matrices encode the same structural information of the RGGs, the localization properties of their eigenmodes are qualitatively different, due to the LM having a conserved quantity, while the AM has not. On short scales, the existence of orbits and strongly connected sets of nodes, and the component size distribution determine the features of the eigenmodes, while on large distances, the two models become similar to their

tight-binding counterpart when the mean degree is large enough that the graph consists of only one component.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [31]; embargo periods may apply.

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