

Megastable quantization in generalized pilot-wave hydrodynamicsÁlvaro G. López^{1,*} and Rahil N. Valani^{2,†}¹*Nonlinear Dynamics, Chaos and Complex Systems Group, Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n, Móstoles 28933, Madrid, Spain*²*Rudolf Peierls Centre for Theoretical Physics, Parks Road, University of Oxford, Oxford OX1 3PU, United Kingdom*

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A classical particle in a harmonic potential gives rise to a continuous energy spectra, whereas the corresponding quantum particle exhibits countably infinite quantized energy levels. In recent years, classical non-Markovian wave-particle entities that materialize as walking droplets have been shown to exhibit various hydrodynamic quantum analogs, including quantization in a harmonic potential by displaying few coexisting limit cycle orbits. By considering a truncated-memory stroboscopic pilot-wave model of the system in the low dissipation regime, we obtain a classical harmonic oscillator perturbed by oscillatory nonconservative forces that display countably infinite coexisting limit-cycle states, also known as *megastability*. Using averaging techniques in the low-memory regime, we derive analytical approximations of the orbital radii, orbital frequency and Lyapunov energy function of this megastable spectrum, and further show average energy conservation along these quantized states. Our formalism extends to a general class of self-excited oscillators and can be used to construct megastable spectrum with different energy-frequency relations.

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Introduction. Orbit quantization is largely considered an exclusive feature of microscopic quantum systems. However, recent studies with a classical hydrodynamic system of millimeter-sized walking oil droplets [1] have demonstrated several hydrodynamic quantum analogs [2–4], including an analog of quantization [5]. In experiments, the droplet walks, while bouncing periodically, on a vibrating bath of the same liquid. Each bounce of the droplet imprints a localized standing wave on the liquid surface that decays slowly in time. The droplet in turn interacts with these self-generated waves on subsequent bounces resulting in horizontal self-propulsion. Three key features that make this system dynamically rich: (i) the droplet and its wave coexist as a *wave-particle entity* (WPE); without the droplet, surface waves decay completely, (ii) the system is *non-Markovian* since the droplet retains *path memory* from the slowly decaying self-generated waves that sculpt its complex dynamical landscape, and (iii) the system is *active* in the sense of active matter [6] because the droplet locally extracts energy from the vibrating bath and converts it into horizontal motion.

The hydrodynamic analog of quantization is experimentally realized by confining the WPE in a two-dimensional harmonic potential, which results in a discrete but finite set of periodic orbits such as circles, ovals, lemniscates, and

trefoils, as the width of the harmonic potential is varied [5,7–9]. The time-delayed feedback produced by the slowly decaying particle-generated waves constrain the particle's trajectory, resulting in quantum-like eigenmodes. In both experimental and theoretical studies of a WPE in a harmonic potential (or in general confining potentials), one observes at most a few coexisting quantized periodic orbits [5,7,8,10–15], whereas the corresponding quantum particle in a harmonic potential has an infinite spectrum of eigenstates.

In dynamical systems literature, the notion of a countably infinite structure of stable periodic orbits exists and has been dubbed *megastability*. It was first numerically reported for periodically driven nonlinear oscillators [16]. In this letter, we investigate the megastable structure of stable limit cycles emerging in the very low dissipation regime of a truncated-memory pilot-wave model of a walking droplet in a harmonic potential.

Truncated-memory pilot-wave model. By averaging over the fast vertical periodic bouncing motion of the walking droplet, Oza *et al.* [18] developed a stroboscopic model that represents the walking droplet as a continuous emitter of damped standing waves and results in an integro-differential trajectory equation for the horizontal walking motion. We consider a one-dimensional setup as illustrated schematically in Fig. 1. Consider a drop (particle) of mass m located at x_d at time t and moving horizontally in one-dimension with velocity $\dot{x}_d$ in the presence of an external harmonic potential $V(x) = kx^2/2$ with spring constant k . At each instance of time, the particle generates a standing wave with spatial form $W(x)$ centered at the particle position, and the wave decays exponentially in time. As opposed to the standard stroboscopic model that takes into account the effects of wave memory up to the present time t [18], here we consider a modified

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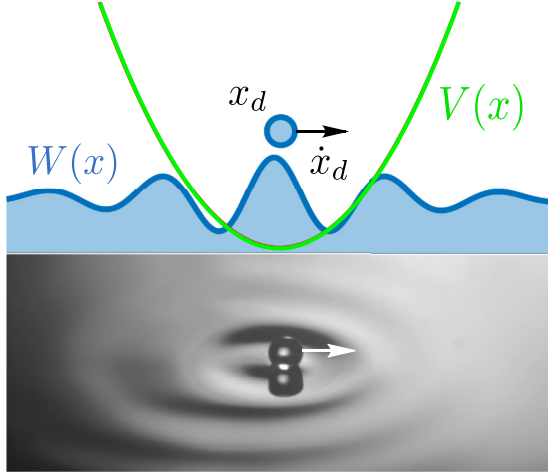


FIG. 1. (Top) Schematic of the hydrodynamic pilot-wave system. A particle located at position x_d at time t moves horizontally in one-dimension with velocity $\dot{x}_d$ while continuously generating waves of spatial form $W(x)$ that decay exponentially in time. The particle experiences a force from external harmonic potential $V(x)$, a dissipative drag force and a propulsive force from its past self-generated waves up to time $t - t_c$. (Bottom) Experimental image of the hydrodynamic pilot-wave system showing a superwalking droplet [17].

stroboscopic model where the particle experiences a wave-memory force from its self-generated waves up to time $t - t_c$ with t_c as the cut-off memory time. The integro-differential trajectory equation of this one-dimensional WPE is given by [18–20]:

$$m\ddot{x}_d + D\dot{x}_d + kx_d = -\frac{F}{T_F} \int_{-\infty}^{t-t_c} W'(k_F(x_d(t) - x_d(s))) \times e^{-\frac{(t-s)}{T_F \text{Me}}} ds. \quad (1)$$

On the left side of Eq. (1), there are three terms: the first representing the inertia of the particle, the second representing the effective drag force on the particle, and the third term is the force from the harmonic potential; where dots denote derivatives with respect to time. The term on the right side of the equation represents the force exerted on the particle by its self-generated wave field. The wave field is a superposition of the individual exponentially time-decaying waves of spatial form $W(x)$ that are generated by the particle continuously along its trajectory. This force is proportional to the gradient of the underlying wave field. The model in Eq. (1) reduces to the standard stroboscopic model [18] for $t_c = 0$. Other physical parameters in Eq. (1) are as follows: D is an effective drag coefficient, k_F is the Faraday wavenumber, $F = mgAk_F$ is a non-negative wave-memory force coefficient where g is the gravitational acceleration and A is the amplitude of droplet-generated surface waves, Me is the memory parameter that describes the proximity to the Faraday instability, and T_F is the Faraday period of droplet-generated standing waves and also the bouncing period of the walking droplet (see Oza *et al.* [18] for more details).

Low-memory self-excited oscillator model. Nondimensionalizing Eq. (1) by rescaling time with T_F and space with k_F , and considering the low-memory regime where the particle

is only influenced by its most recent wave at time $t - t_c$, we get (see Supplemental Material Sec. I [21] for derivation, including [22])

$$\ddot{x}_d + \mu\dot{x}_d + Kx_d = -\epsilon W'(\tau\dot{x}_d). \quad (2)$$

Here the dimensionless parameters are given by $\mu = DT_F/m$ as the drag coefficient, $K = kT_F^2/m$ as the spring constant, $\tau = t_c/T_F$ as the cut-off memory time and $\epsilon = e^{-\tau/\text{Me}} F k_F T_F^2/m$ as the wave force coefficient. For analytical tractability and clarity of presentation, we consider this low-memory model with a simple cosine wave form $W(x) = \cos(x)$ [19,20,23,24] for the remainder of the letter. However, we highlight that the existence of megastability is robust in stroboscopic models with truncated memory and extends to larger wave-memory (see Supplemental Material Secs. II and III [21]) and also alternate wave forms $W(x)$ with both oscillations and spatial decay (see Supplemental Material Secs. IV and V [21], including [25]). We also fix $\tau = 1$, which corresponds to cut-off memory time duration equivalent to that of a single bounce of the droplet.

In phase space, we rewrite our dynamical system in Eq. (2) as

$$\begin{aligned} \dot{x}_d &= v_d, \\ \dot{v}_d &= -x_d - \mu v_d + \epsilon \sin v_d, \end{aligned} \quad (3)$$

where we have rescaled $(x_d, t, \epsilon, \mu) \rightarrow (x_d/\sqrt{K}, t/\sqrt{K}, \epsilon\sqrt{K}, \mu\sqrt{K})$ to simplify the equations further.

Megastable spectrum. The system in Eq. (3) displays an unbound set of nested limit cycles and we use the Krylov-Bogoliubov averaging method to obtain analytical expressions for this spectrum. Physically, the megastable quantization results from a dynamical balance of forces on the particle that arise from the external harmonic potential and the self-generated waves. The Krylov-Bogoliubov method [26] proposes the phase space ansatz $(x_d(t), v_d(t)) = (r(t) \sin(t + \varphi(t)), r(t) \cos(t + \varphi(t)))$, and averages over the phase variable $\theta(t) = t + \varphi(t)$, using the fact that for small μ and ϵ , there is a time-scale separation between the variations of the phase $\theta(t)$ and the amplitude $r(t)$ of the oscillation. The error of this approximation is bounded and can be made arbitrarily small by decreasing the size of $\max(\epsilon, \mu)$ [26]. Substituting the ansatz in Eq. (3), we obtain the following system of differential equations for the amplitude and the phase of the orbit

$$\dot{r} = -(\mu r \cos \theta - \epsilon \sin(r \cos \theta)) \cos \theta, \quad (4)$$

$$r\dot{\varphi} = (\mu r \cos \theta - \epsilon \sin(r \cos \theta)) \sin \theta. \quad (5)$$

Averaging Eqs. (4) and (5) over the period $\theta \in [0, 2\pi]$ of the unperturbed ($\mu = \epsilon = 0$) harmonic oscillator yields $\dot{\varphi} = 0$ and

$$\dot{r} = -\bar{\mu}r + \epsilon J_1(r), \quad (6)$$

where $J_1(r)$ is the first-order Bessel function of the first kind, and we have defined the parameter $\bar{\mu} = \mu/2$, to simplify the equation further. In the particular case of no dissipation ($\bar{\mu} = 0$) the fixed points of Eq. (6), and hence the amplitudes of the limit cycles are given by the infinite roots of

the Bessel function, which for large r have the asymptotic relation $J_1(r) \approx \sqrt{\pi/2r} \cos(r - 3\pi/4)$. Thus, we have the existence of a countably infinite sequence of limit cycles with alternating stability, with the amplitude sequence converging asymptotically to the zeros of the cosine function, $r_n \approx \pi(n + 5/4)$, where r_n denotes the n th root of Eq. (6). We note that the dynamical system in Eq. (3) with $\bar{\mu} = 0$ has been previously explored as a mathematical curiosity [27], and attempts have been made to make vague connections with quantum eigenstates [28]. However, these were not rooted in a physical system, whereas in our pilot-wave walking-droplet system, these equations emerge in the low-memory limit of the droplet's integro-differential equation of motion.

In the more general case for $\bar{\mu} > 0$, there is only a finite number N of limit cycles. Solving for $\dot{r} = 0$, and using again the asymptotic expansion of the Bessel function, we obtain the transcendental equation $r^{3/2} = \epsilon/\bar{\mu} \sqrt{\pi/2} \cos(r - 3\pi/4)$. For $r \gg 1$, the maximum number of limit cycles is restricted to $N = \lfloor \delta/2\pi - 3/8 \rfloor$, where $\delta = (\pi/2)^{1/3} (\epsilon/\bar{\mu})^{2/3}$ and $\lfloor \cdot \rfloor$ is the floor function. This gives us a scaling relation for the number of stable limit cycles in the form $N(\epsilon, \mu) \propto (\epsilon/\mu)^{2/3}$. Among this spectrum of limit cycle states, the fundamental (first) level corresponds to a self-excited attractor since it shares its basin of attraction with the unstable equilibrium point $(x_d^*, v_d^*) = (0, 0)$, while the remaining higher quantized levels are hidden attractors [29], since they do not share their basin of attraction with any equilibrium point. As the dissipation coefficient μ increases, an infinite cascade of subcritical saddle-node bifurcations take place disintegrating the megastable structure of periodic orbits in decreasing order of their size.

In Fig. 2, we have numerically computed the limit cycles of Eq. (3) beyond the first ten levels, using constant initial conditions $(x_d(0), v_d(0)) = (A, 0)$, with increasing values of A and fixed values of $\mu = 0$ and $\epsilon = 0.5$. Numerical simulations confirm the prediction of a megastable structure and for $n \gg 1$, the limit-cycle size grows following an asymptotically linear trend with n .

To characterize the nature of the energy spectrum, we compute the quantization relation for both the energy and the frequency of quantized states. Motivated by recent connections between self-excited oscillators and the quantum potential [30,31], we define the energy of our oscillator using the Lyapunov function associated with Eq. (2), when $\epsilon = 0$. This function describes the conservative energy content of the oscillator [32,33], and can be written as

$$E(x_d, v_d) = \frac{1}{2}v_d^2 + \frac{1}{2}x_d^2. \quad (7)$$

As opposed to Hamiltonian dynamical systems, for which energy E is a constant of motion and corresponds to fine-tuned parameters $\mu = 0$ and $\epsilon = 0$, self-excited oscillators gain and lose mechanical energy during different parts of the oscillation, forming a thermodynamic engine [34]. The nonconservative self-force on the right hand side of Eq. (2) implies that energy is gained by the oscillator during some part of the limit cycle, while energy is released during the complementary part, even when linear dissipation is absent (*i.e.*, $\mu = 0$). The addition of linear dissipation disrupts this balance and ceases the existence of larger orbits. These nonconservative forces are related to the time-reversal

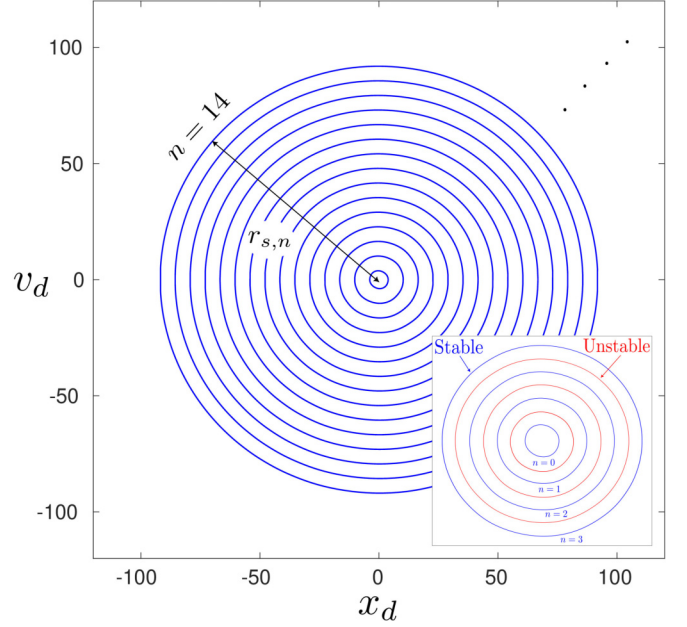


FIG. 2. Orbit spectrum of the WPE in the low-memory self-oscillator model in Eq. (3) (dots indicate that the sequence continues). A series of fifteen concentric limit cycles for $\mu = 0$ and $\epsilon = 0.5$ displaying a countably infinite megastable set of quantized orbits in the phase space formed by the dimensionless position x_d and the dimensionless velocity v_d of the particle. The inset shows the alternating structure of stable (blue) and unstable (red) limit cycles with the first four shown. The unstable limit cycles also form smooth basin boundaries separating the basin of attraction of stable limit cycles.

asymmetry of Eqs. (2) and (3), and, more generally, of self-excited systems [35,36]. The resulting dissipative structure [37] constitutes an active, open dynamical system, which *conserves its mechanical energy on average*. This can be shown by computing the Melnikov function [38] for Eq. (3) along the megastable quantized orbits, which corresponds to an averaging of Eq. (4) over θ , and showing that it takes a zero value (see Supplemental Material Sec. VII [21]).

Consequently, we assign an energy eigenvalue for each stable limit cycle, $E_n = (1/T) \int_0^T E(x_{d,n}(t), v_{d,n}(t)) dt$, where $(x_{d,n}(t), v_{d,n}(t))$ represent the trajectory of the n th stable limit cycle with period T . These energy levels within the external harmonic potential are depicted in Fig. 3. Since the stable limit cycles are approximately circular, with radius $r_{s,n} \approx \pi(2n + 5/4)$, we can estimate the energy spectrum of Eq. (3) as $E_n \approx r_{s,n}^2/2 = 2\pi^2(n + 5/8)^2$, obtaining a quadratic relation for the energy eigenvalues as a function of the orbit number n . This is shown in Fig. 4, where a least squares fitting to the quantization rule $E_n = an^2 + bn + c$ gives the parameter values $a = 19.74$, $b = 24.70$, and $c = 7.42$. These values are in excellent agreement with the theoretical approximations of $a = 2\pi^2$, $b = 5\pi^2/2$, and $c = 25\pi^2/32$, with c corresponding to the zero-point energy of the oscillator. The excellent agreement of analytical approximation is a consequence of the quasi-harmonic approximation of the limit cycles, due to the smallness of the parameter ϵ .

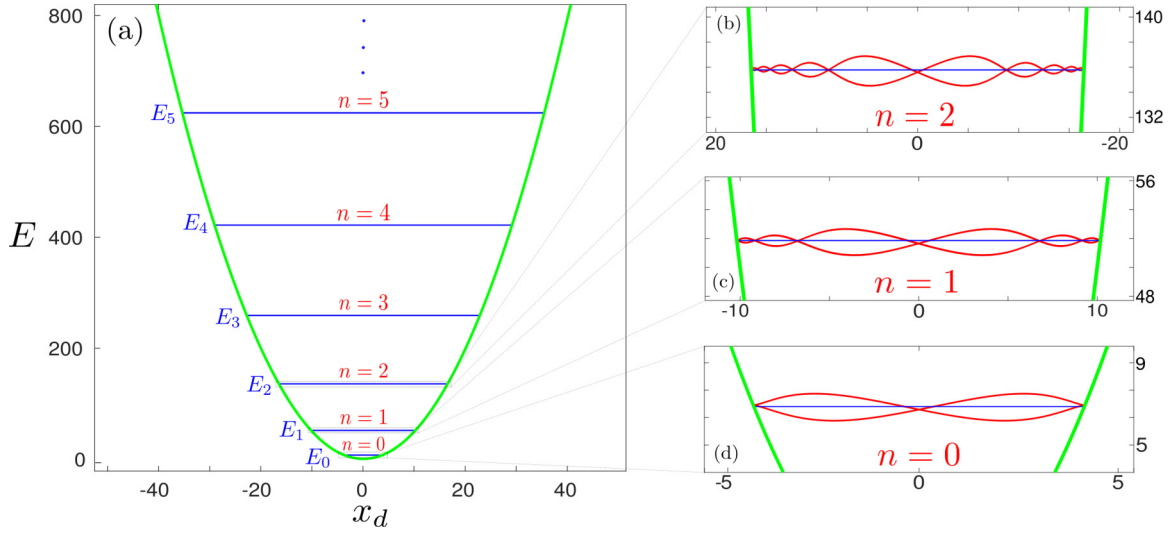


FIG. 3. Energy spectrum of the WPE in the low-memory self-oscillator model (dots indicate that the sequence continues). (a) The first six mean Lyapunov energy levels E_n (blue) in the external harmonic well (green), together with the time evolution of the Lyapunov energy function $E(x_d, v_d)$ (b–d) for orbits in the zero-point energy level ($n = 0$) and the two first excited levels ($n = 1$ and $n = 2$). The energy oscillations with the dimensionless particle position x_d along quantized orbits (red) appear to obey a mathematical relation for the number of intersections, i.e., nodes, with $4n + 3$ nodes for n th energy level.

The period or frequency of the limit cycles remains almost constant for the entire megastable set and does not vary significantly with the orbit number n . The Fourier analysis of the periodicity of the spectrum of the limit cycles set is quasi-monochromatic with an average asymptotic value of $\omega_n \approx 0.995$, nearly coinciding with the predicted value of $\omega_n = 1$ for the unperturbed harmonic oscillator. This constitutes a fundamental difference with respect to a quantum harmonic oscillator, for which both the energy and the frequency spectra

are proportional to each other. Nevertheless, we suggest the possibility that different energy-frequency quantum-like functional relations may arise in our system for other nonlinear confining potentials $V(x)$ and/or different wave forms $W(x)$ of the WPE.

Discussion and conclusions. Experimental and theoretical works with the walking-droplet system have shown the existence of a finite spectrum of quantized states with at most a few coexisting limit cycle states [5,7,8,10–15]. Our results indicate that an extended set of quantized orbits might be achievable in a confining one-dimensional potential for very small dissipation. In addition to the simplified model presented here, we obtain megastability in both the continuous stroboscopic model with truncated memory of duration equal to one bouncing period, i.e., $\tau = 1$ (see Supplemental Material Secs. II and III [21]), as well as the discrete model (see Supplemental Material Secs. I and III [21]) of Oza *et al.* [18] with one bounce truncated. In the standard stroboscopic model of Oza *et al.* [18] with $\tau = 0$, we do not obtain megastable orbits, and the particle continuously climbs up the harmonic potential. Although the relevance of our truncated-memory pilot-wave model to the experimental walking-droplet system is presently unclear, nevertheless, it would be interesting to perform experiments with a walking droplet in a one-dimensional harmonic potential in the very low-dissipation regime to either confirm or rule out megastability. It might be possible to generate the 1D harmonic potential in experiments by injecting the droplet with ferrofluid and applying an external magnetic field as demonstrated by Perrard *et al.* [5], whereas the low-dissipation regime might be difficult to achieve. For typical experimental set-up with silicone oil droplets, we get the value of dimensionless wave force coefficient $\epsilon \sim 0.01 - 0.1$ and the dimensionless drag coefficient $\mu \sim 0.1$. One would need to reduce this drag coefficient by two or three orders of magnitude to obtain an appreciable number of megastable orbits. These limitations might perhaps

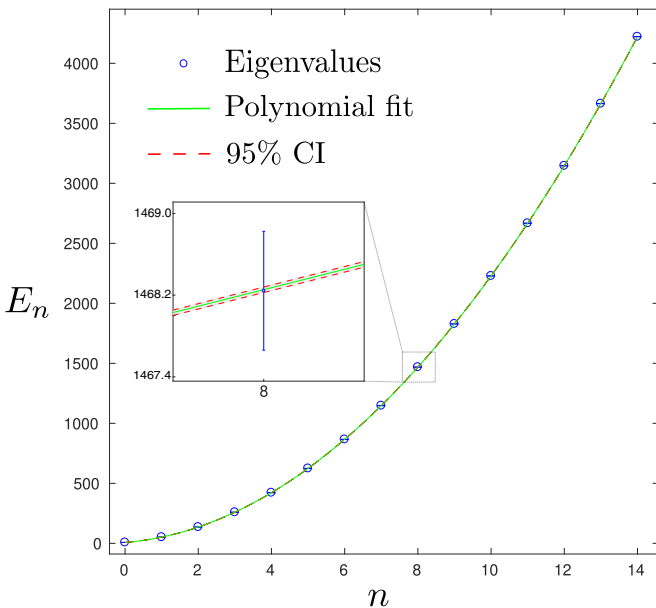


FIG. 4. The energy levels of the discrete spectrum follow a quadratic trend. The least-squares fitting is very accurate, with tight 95% confidence intervals (CI). The error bars (blue) represent the variation in energy along the quantized orbit, computed as the variance of the Lyapunov energy function.

be circumvented using alternate fluids, reducing the parameter μ to almost zero. For example, the Faraday instability has been observed in superfluids [39], suggesting the possibility of constructing a walking droplet setup for superfluids where the low-dissipation regime might be realized. Independent of experiments, our work contributes to the framework of generalized pilot-wave hydrodynamics [40], which has been formulated to encourage the exploration of hydrodynamic quantum analogs in a wider class of dynamical systems and parameter-space that are not restricted by experimental constraints. In this context, we show that a walking-droplet inspired pilot-wave model with truncated memory can exhibit megastable quantization in a confining potential.

We suspect that the megastable structure observed in our system is not unique, and it may be possible to obtain megastability more ubiquitously for a harmonic oscillator perturbed by nonlinear and/or time-delayed non-conservative forces. For example, we have also demonstrated the existence of a megastable structure of orbits in a time-delayed system describing a particle in a harmonic

potential with a state-dependent time-delayed force, inspired from a retarded self-oscillator recently found in the context of classical electrodynamics of extended bodies [32] (see Supplemental Material Sec. VIII [21], including [41,42]). Further, the formalism presented here can be extended more generally to perturbed harmonic oscillators of the form

$$\ddot{x} - \epsilon f(\dot{x}) + x = 0. \quad (8)$$

Any function $f(\dot{x})$ that can be expressed as the product of an even periodic function and an odd polynomial (or vice-versa) can provide megastable dissipative structures (see Supplemental Material Sec. VI [21]). The Krylov-Bogoliubov method can be applied in general to such near-Hamiltonian systems [43], allowing one to compute the maximum number of stable limit cycles and determine the existence of megastable structures [44].

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