Letter

Relation between critical exponent of the conductivity and the morphological exponents of percolation theory

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A central unsolved problem in percolation theory over the past five decades has been whether there is a direct relationship between the critical exponents that characterize the power-law behavior of the transport properties near the percolation threshold, particularly the effective electrical conductivity σ_e, and the exponents that describe the morphology of percolation clusters. The problem is also relevant to the relation between the static exponents of percolation clusters and the critical dynamics of spin waves in dilute ferromagnets, the elasticity of gels and composite solids, hopping conductivity in semiconductors, solute transport in porous media, and many others. We propose an approach to address the problem by showing that the contributions to σ_e can be decomposed into several groups representing the structure of percolation networks, including their mass and tortuosity, as well as constrictivity that describes the fluctuations in the driving potential gradient along the transport paths. The decomposition leads to a relationship between the critical exponent t of σ_e and other percolation exponents in d dimensions, $t/\nu = (d - D_{\rm bb}) + 2(D_{\rm op} - 1) + d_C$, where ν, $D_{\rm bb}$, $D_{\rm op}$, and d_C are, respectively, the correlation length exponent, the fractal dimensions of the backbones and the optimal paths, and the exponent that characterizes the constrictivity. Numerical simulations in two and three dimensions, as well as analytical results in $d = 1$ and $d = 6$, the upper critical dimension of percolation, validate the relationship. We, therefore, believe that the solution to the 50-year-old problem has been derived.

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Transport processes in heterogeneous materials occur in a wide range of processes and have spurred extensive research to understand their macroscopic properties [1,2]. As a model of heterogeneous media, random conductance networks have played a key role [3] and have been used for over half a century [4] for computing the electrical and thermal conductivity, diffusivity, and magnetic properties. The use of such networks began with the work of Fatt [5] who utilized them for predicting the hydraulic conductivity of porous media, but it became widespread beginning with the work of Kirkpatrick [6].

Strongly heterogeneous materials and media are represented by random conductance networks in which a fraction p of the bonds have a finite conductivity, while the rest do not allow for the transport process to occur. Such networks are characterized by a percolation threshold p_c [1,7] below which the flow and transport properties, as well as the elastic moduli [8,9], vanish. Near p_c, the effective conductivity σ_e of the network follows a power law,

$$\sigma_e \sim (p - p_c)^t, \tag{1}$$

where t is mostly universal and depends only on the dimensionality of the network, except when the conductances are very broadly distributed [10,11], with its universal values being $t \simeq 1.3$ and 2 in two and three dimensions, respectively. A central unsolved problem is whether the exponent t is related to the critical exponents that describe the power-law behavior of the morphology of percolation clusters near p_c, including the correlation length ξ_p that follows, $\xi_p \sim (p - p_c)^{-\nu}$, and the fractions $P(p)$ and $X_B(p)$ of the conducting bonds that are, respectively, in the sample-spanning cluster and its backbone (the current-carrying part of the cluster), which follow the power laws, $P(p) \sim (p - p_c)^{\beta}$ and $X_B(p) \sim (p - p_c)^{\beta_{\rm bb}}$, where the exponents ν, β, and $\beta_{\rm bb}$ are universal. The problem has been studied for over 50 years because (i) the exponent t is also relevant to critical dynamics of spin waves in dilute ferromagnets, scaling of elasticity of gels near the gel point, hopping conductivity in semiconductors, solute transport in heterogeneous porous media [12], and many other problems, and (ii) more fundamentally, whether the critical dynamics of a heterogeneous medium is related to its static properties is an open question. One relation between t and the other percolation exponents was conjectured by Alexander and Orbach

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[13]; an effort was made in Ref. [14] to directly derive the relation, while a Flory approximation was also proposed [15], none of which was exact and, therefore, the problem remains unsolved.

Factors that contribute to the flow and transport properties of a material, or a network model of it, can be partitioned into several groups that represent its morphology [16] and any nonlinear relationship between the potential drop driving the process and the mass of network elements [17,18]. When the influencing groups are combined, one obtains a comprehensive description of flow and transport in terms of structure-related parameters. In this Letter, we combine percolation theory with the structural description of transport processes, and in particular conduction, in order to derive a direct relationship between the exponent t and other percolation exponents. As we show below, this allows us to decompose the power law in Eq. (1) into several components that represent the morphology of the network, hence not only establishing a direct relationship between the exponent t and the morphological exponents of percolation but also gaining a fuller understanding of factors that control the power laws for the transport properties. Moreover, we show that, in networks with a broad distribution of conductances, it is the constrictions in the paths that are responsible for the nonuniversality of t when the conductances are broadly distributed [10,11].

Decomposing critical exponents. Here we decompose the critical exponent t into the sum of other percolation exponents, including exponents for the tortuosity and constrictivity. Consider a lattice of side length L, where each bond has a probability p of having unit conductance or a zero conductance with probability $(1-p)$. Assume that a potential drop $\Delta\phi$ over the network yields a total current I. The backbone is divided into the transport paths as follows: Let $\mathbb{S}=\{\mathcal{S}\}$ be the set of transport paths $\mathcal{S}$ with associated currents $I_\mathcal{S}$. If $I(e)$ is the current carried by bond e, then $I(e)=\sum I_\mathcal{S}$, where the sum runs over all transport paths $\mathcal{S}$ containing the bond e in their path. The parameter τ represents the tortuosity that describes the relative length of the transport paths, defined as [16]

$$\tau^2=\frac{1}{\sum_{\mathcal{S}\in\mathbb{S}}M_\mathcal{S}}\sum_{\mathcal{S}\in\mathbb{S}}\tau_\mathcal{S}^2M_\mathcal{S}=\frac{1}{M_{\rm bb}}\sum_{\mathcal{S}\in\mathbb{S}}\tau_\mathcal{S}^2M_\mathcal{S}, \tag{2}$$

where $\tau_\mathcal{S}=L/L_\mathcal{S}$ is the tortuosity of $\mathcal{S}$ with $L_\mathcal{S}$ being the length of $\mathcal{S}$ (i.e., the number of bonds traversed by $\mathcal{S}$), $M_{\rm bb}$ is the number of bonds in the backbone (the backbone's mass), and $M_\mathcal{S}=\sum_{e\in\mathcal{S}}I_\mathcal{S}/\Delta\phi(e)$ is the mass associated with the transport path $\mathcal{S}$. Here, $\Delta\phi(e)$ is the potential drop over bond e. The second equality in Eq. (2) is given by

$$\begin{aligned}\sum_{\mathcal{S}\in\mathbb{S}}M_\mathcal{S}&=\sum_{\mathcal{S}\in\mathbb{S}}\sum_{e\in\mathcal{S}}\frac{I_\mathcal{S}}{\Delta\phi(e)}=\sum_{e\in E_{\rm bb}}\sum_{\mathcal{S}\in\mathbb{S}}\frac{I_\mathcal{S}}{\Delta\phi(e)}\\&=\sum_{e\in E_{\rm bb}}\frac{I(e)}{\Delta\phi(e)}=M_{\rm bb},\end{aligned} \tag{3}$$

where $E_{\rm bb}$ is the set of bonds in the backbone, and $I(e)$ is the current carried by bond e. This definition of tortuosity follows [19,20], with the sum running over squared tortuosities dating back to Ref. [21].

The constriction factor $C_\mathcal{S}$ describes how the mass $I_\mathcal{S}/\Delta\phi(e)$ that carries the current $I_\mathcal{S}$ varies along the transport path $\mathcal{S}$ [16]:

$$\begin{aligned}C_\mathcal{S}&=\frac{1}{L_\mathcal{S}^2}\left(\sum_{e\in\mathcal{S}}\frac{\Delta\phi(e)}{I_\mathcal{S}}\right)\left(\sum_{e\in\mathcal{S}}\frac{I_\mathcal{S}}{\Delta\phi(e)}\right)\\&=\frac{1}{L_\mathcal{S}^2}\Delta\phi\left(\sum_{e\in\mathcal{S}}\frac{1}{\Delta\phi(e)}\right)\\&=L^{-2}\tau_\mathcal{S}^2\Delta\phi M_\mathcal{S}/I_\mathcal{S},\end{aligned} \tag{4}$$

where $\Delta\phi=\sum_{e\in\mathcal{S}}\Delta\phi(e)$ is the total potential drop driving the current. A sum over all transport paths weighted by the transport path current $I_\mathcal{S}$ defines the constriction factor as

$$C=\frac{1}{I}\sum_{\mathcal{S}\in\mathbb{S}}C_\mathcal{S}I_\mathcal{S}. \tag{5}$$

Invoking Eq. (4) and $I=\sigma_e\Delta\phi L^{d-2}$, we obtain the equality

$$\frac{\sigma_eC}{m_{\rm bb}}=\frac{L^2}{M_{\rm bb}\Delta\phi}\sum_{\mathcal{S}\in\mathbb{S}}C_\mathcal{S}I_\mathcal{S}=\frac{1}{M_{\rm bb}}\sum_{\mathcal{S}\in\mathbb{S}}\tau_\mathcal{S}^2M_\mathcal{S}=\tau^2, \tag{6}$$

where $m_{\rm bb}=X_B=M_{\rm bb}/L^d$ is the fraction of bonds in the backbone.

Rearranging the above equation, the effective conductivity is decomposed into four parts [16,17]:

$$\sigma_e=\frac{m_{\rm bb}\tau^2}{C}. \tag{7}$$

Let an overline indicate a geometric average for an ensemble of conductance networks. Both $\overline{\sigma}_e$ and $\overline{m}_{\rm bb}$ follow power laws [7] and, thus, the fraction $\overline{\tau}^2/\overline{C}$ must also follow a power law. If, as postulated in Refs. [20,22–24], the tortuosity $\overline{\tau}$ follows a power law, then the constriction factor $\overline{C}$ must also follow a power law. Therefore, if d_τ describes the power-law behavior of $\overline{\tau}$ at p_c, $\overline{\tau}\propto L^{d_\tau}$, a critical exponent d_C does the same for $\overline{C}$, i.e., $\overline{C}\propto L^{d_C}$.

For the combined structural properties we have

$$\overline{\sigma}_e=\overline{\left(\frac{m_{\rm bb}\tau^2}{C}\right)}=\frac{\overline{m}_{\rm bb}\overline{\tau}^2}{\overline{C}}, \tag{8}$$

$$\propto\frac{L^{d_{\rm bb}}L^{2d_\tau}}{L^{d_C}}=L^{d_{\rm bb}+2d_\tau-d_C}, \tag{9}$$

where $\overline{m}_{\rm bb}\propto L^{d_{\rm bb}}$, and where we use the fact that the geometric averages can be separated. Since the conductivity σ_e scales as $\overline{\sigma}_e\propto L^{-t/\nu}$ at p_c, we have

$$\zeta=-t/\nu=d_{\rm bb}+2d_\tau-d_C, \tag{10}$$

giving the exponent t in terms of the exponents that characterize the network structure.

Let f_b be the fraction of bonds on the percolating cluster that are in the backbone, then $f_b\propto L^{d_{\rm bb}-d_p}=L^{d_f}$, where $d_p=-\beta/\nu$ is the exponent for the fraction of bonds in the percolating cluster $m_p=P$. As $d_{\rm bb}=d_p+d_f$, the structural decomposition can also be written as $\zeta=d_p+d_f+2d_\tau-d_C$. The fraction $f_b\tau^2/C$ equals the diffusion coefficient [7]. Thus, we have

$$\frac{\beta-t}{\nu}=d_f+2d_\tau-d_C, \tag{11}$$

where $(\beta - t)$ characterizes the universal power law of diffusivity in the sample-spanning cluster. Measuring the diffusion coefficient, and thereby the combined effect of the morphological factors, is a routine macroscopic measurement [25]. The backbone fraction of real percolating materials has been measured [26], as have the tortuosity [27–29] and constriction factors [18,30–32], all measurements involving detailed description of the microstructure.

Duda *et al.* [20], and later Ghanbarian *et al.* [23], pointed out that $d_\tau = 1 - D_{\rm op}$, where $D_{\rm op}$ is the scaling exponent at p_c of the most probable, or optimal, travel path [33]. Using $d_\tau = 1 - D_{\rm op}$, Eq. (10) for $\zeta = -t/\nu$ can alternatively be written as

$$\zeta = -\frac{t}{\nu} = d_{\rm bb} + 2(1 - D_{\rm op}) - d_C. \tag{12}$$

Numerical and analytical verification. We carried out extensive numerical simulations to confirm the scaling relation in two and three dimensions, while the limiting cases of one and six dimensions are considered analytically. The numerical simulations include solving Kirchhoff's equations to obtain the local potentials and the associated currents $I(e)$. We used a conjugate gradient method to get the local potentials, and we tracked transport paths along the currents. The transport network is viewed as an abstraction of transport in a continuous system, implying that we do not allow the transport paths to cross in two dimensions, yielding a unique set $\mathbb{S}$. In contrast to a continuous system where transport paths are separated, full mixing at the network sites leads to nonuniqueness in three-dimensional networks; however, this has a limited impact on the tortuosity and constriction factor values [18].

In Fig. 1 we present the various structural properties and the effective conductance versus network size for $d = 2$, together with slopes describing their scaling exponents. To highlight the match of the slopes, we also plotted the geometric average divided by the network size raised to the scaling exponent in Fig. 2. This plot shows that all the considered structure components are well described by their scaling exponents.

For the scaling exponents, we obtain $t/\nu = -\zeta = 0.975 \pm 0.011$, in agreement with the literature value of 0.975 in two dimensions. We also note that $D_p = d_p + 2 = 1.901$ and $D_{\rm bb} = d_{\rm bb} + d = 1.652$ are in agreement with the literature values of 1.89 and 1.64 [1]. For the structural exponents that we have introduced, we have $d_{\rm bb} + 2d_\tau - d_C = -0.976 = \zeta$, as given by the decomposition in Eq. (10). Moreover, $\beta/\nu = -d_p = 0.099 \pm 0.014$ is in agreement with the literature value $5/48 \simeq 0.104$. The structural parameters yield $d_f + 2d_\tau - d_C = -0.249 - 0.392 - 0.236 = -0.877$, which is in agreement with $(\beta - t)/\nu = \zeta - d_p = -0.876$ as given by Eq. (11). Duda *et al.* [20] calculated the tortuosity for two-dimensional (2D) pore structures and found $d_\tau \simeq 0.19$, in agreement with what we obtained from our simulations. Lee *et al.* [33] found $D_{\rm op} \simeq 1.21 \pm 0.02$ in 2D networks, which gives $d_\tau = 1 - D_{\rm op} = -0.21 \pm 0.02$, also in agreement with the values for d_τ obtained from our simulations.

Our simulations for $d = 3$ yielded the following values, where the parenthesis gives the literature value [1]: $\zeta = -2.27\,(-2.27)$, $d_p = -0.48\,(-0.47)$, $d_{\rm bb} = -1.09\,(-1.13)$, $d_\tau = -0.43$, and $-d_c = -0.31$. Thus, the decomposition given by Eq. (10) also holds for $d = 3$. Sheppard *et al.* [34] found $D_{\rm op} \simeq 1.43 \pm 0.01$ in three dimensions, which gives $d_\tau = 1 - D_{\rm op} = -0.43 \pm 0.01$, in perfect agreement with our obtained value for d_τ.

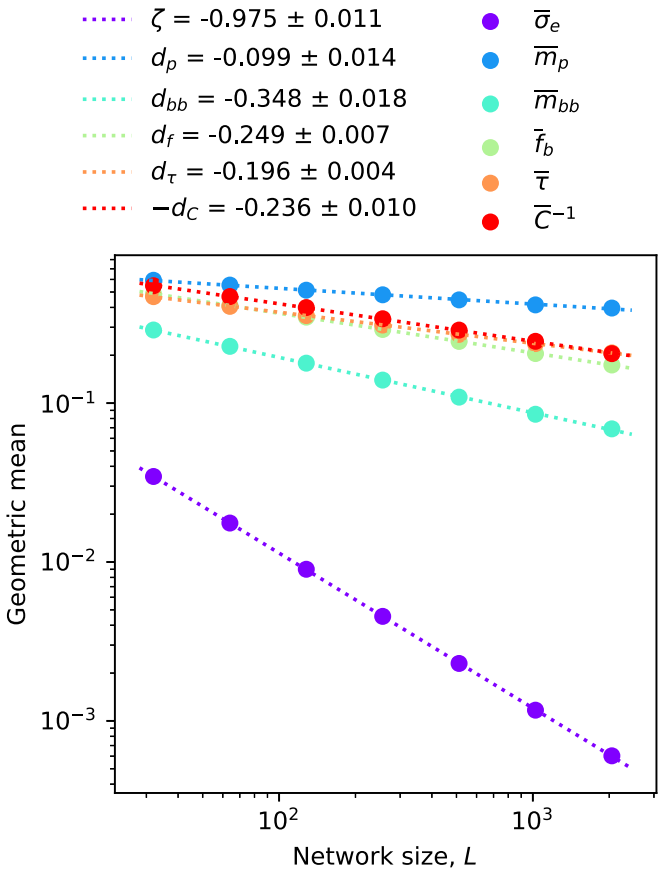


FIG. 1. The geometric mean of the various structure parameters, including slopes, for the scaling exponents. We generated approximately 12 000 realizations of the models up to a size of $L = 512$, while 514 realizations were used for the largest model size, $L = 2048$. The given uncertainty is the standard deviation of the slopes between consecutive points.

The limiting cases of $d = 1$ and $d \geqslant 6$ are analyzed analytically, with $d = 6$ representing the upper critical dimension of percolation, where the mean-field theory is exact. For $d = 1$, we have $p_c = 1$, $d_{bb} = 0$, and $\tau = 1$, so $d_\tau = 0 = 1 - D_{\rm op}$. Since $C = 1$, we have $d_C = 0$. Since $\nu = 1$, Eq. (10) gives $t = -\zeta = 0$, as expected. At the upper critical dimension of percolation, $d = 6$, the critical exponents equal those for the Bethe lattice [7,35], $d_{\rm bb} = -4$ and $D_{\rm op} = 2$ [7]. This gives $d_\tau = -1$. In high dimensions, $d \geqslant 6$, there are few loops, so we will again have $d_C = 0$. From Eq. (10) we then have $\zeta = -4 - 2 + 0 = -6$. Since $\nu = 1/2$, this gives $t = 3$, which agrees with the literature values [1,7].

We point out that, since conduction occurs in the backbone, any scaling relation between t and other percolation exponents should involve only the backbone exponents or, if it involves those of the sample-spanning cluster, the exponent(s) should be invariant under the transformation of the sample-spanning cluster → the backbone. Previous attempts to relate t to the other exponents failed the latter condition, whereas Eq. (12) relates t to only the exponents that define the transport paths

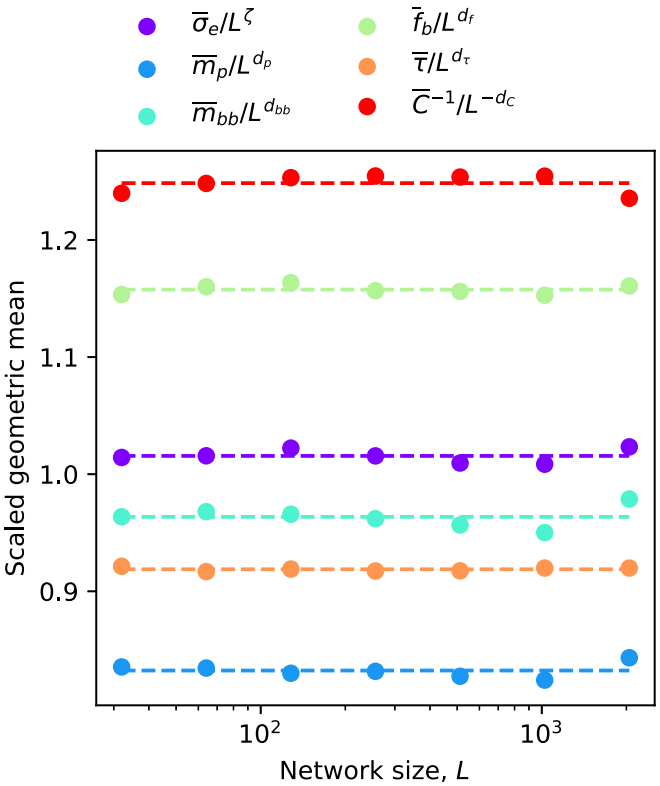


FIG. 2. To further investigate the slopes shown in Fig. 1, this plot shows the geometric average data divided by the network size raised to the corresponding scaling exponent. The dashed lines show the average of the point values.

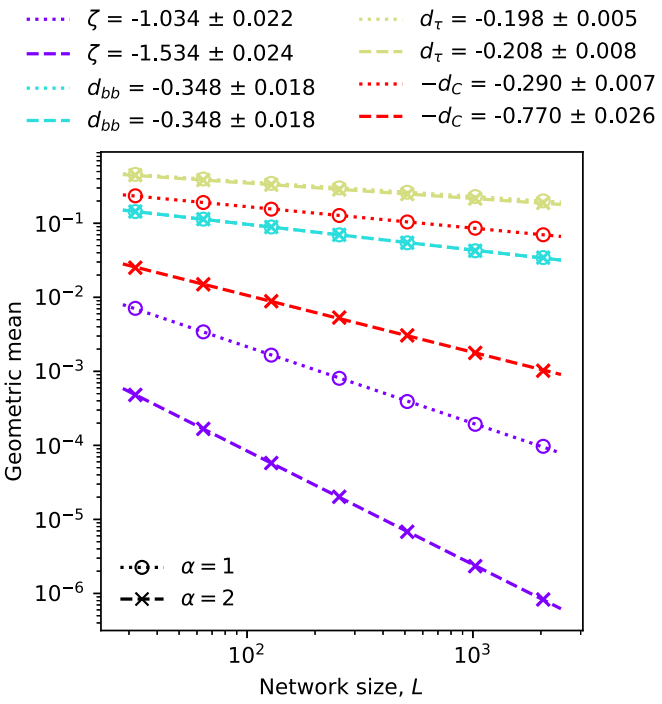


FIG. 3. Various exponents for $\alpha = 1$ and $\alpha = 2$. The points are the geometric mean for the same set of models as considered in Fig. 1.

on the backbone, and ν is the same for the sample-spanning cluster and its backbone. To our knowledge, our proposed equation is the only scaling relation that is valid for all $1 \leqslant d \leqslant 6$.

Decomposing nonuniversal power laws for the conductivity. We now show how the decomposition of the critical exponent t improves our understanding of nonuniversal power laws when the bonds' conductances are broadly distributed. In the Supplemental Material, see Ref. [36], we show how the decomposition of the structural properties, derived above, can be extended to lattices with varying bond conductance values. We associate a mass $m(e)$ with each bond e, where $m(e)$ is random from a uniform distribution between 0 and 1. The conductance $g(e)$ of bond e is related to the mass by $g(e) = m(e)^\alpha$ via an exponent α. We observe that $\alpha = 0$ gives the traditional case investigated above where all the conducting bonds have a unit conductance, while higher values of α broaden the conductance distribution. The probability distribution of bond conductance values is then given by

$$h(g) = \frac{1}{\alpha} g^{\frac{1}{\alpha}-1}. \tag{13}$$

Let H_n be the harmonic average of n numbers. When the numbers have a probability distribution given by Eq. (13), then $H_n \propto n^{1-\alpha}$ when $n \gg 1$ and $\alpha > 1$, while H_n is bounded for $\alpha \leqslant 1$ [37]. Thus, for $d = 1$ and $L \gg 1$, the constriction factor scales as

$$C^{-1} \propto \begin{cases} 1 & \text{if } \alpha \leqslant 1, \\ L^{1-\alpha} & \text{if } \alpha > 1. \end{cases} \tag{14}$$

This gives $-d_C = 1 - \alpha$ for $\alpha > 1$, and 0 elsewhere. Since $d_{\text{bb}} = 0$, $d_\tau = 0$, and $\nu = 1$ at $d = 1$, Eq. (10) gives $t = -\zeta = d_C$. For $\alpha \leqslant 1$ we have $t = \zeta = 0$, while $t = -\zeta = \alpha - 1$ for $\alpha > 1$. Different α values, i.e., different conductance distributions, lead to different d_C values, and thereby to nonuniversal power laws for ζ and t.

For $d \geqslant 6$ we can again use the derived exponent for the constriction factor, this time for a path length that scales as L^2,

$$-d_C = \begin{cases} 0 & \text{if } \alpha \leqslant 1, \\ 2 - 2\alpha & \text{if } \alpha > 1. \end{cases} \tag{15}$$

As $\nu = 1/2$, we have $-\zeta = t/\nu = 2t$, which gives

$$-2t = \zeta = d_{\text{bb}} + 2d_\tau - d_C = \begin{cases} -6 & \text{if } \alpha \leqslant 1, \\ -4 - 2\alpha & \text{if } \alpha > 1. \end{cases} \tag{16}$$

We again observe nonuniversal power laws for broad conductance distributions.

In Fig. 3 we show the structural parameters for $\alpha = 1$ and $\alpha = 2$. As the exponent α only affects the conductance of the bonds, d_{bb} is independent of α, numerically verified in Fig. 3 and equal to its value obtained in Fig. 1. There should be no difference between the results for $\alpha = 1$ and $\alpha = 0$. It is, however, difficult to obtain equal values for the two cases, due to finite-size effects [38]. While the change in d_τ is within the estimated error, the d_C value has an apparent shift. Since ζ

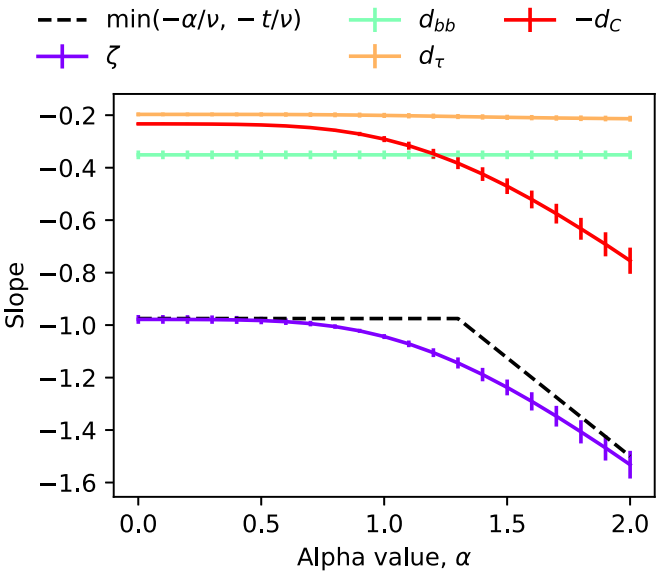


FIG. 4. Change in the exponent with values of α for various structure parameters. Note that the curves for ζ and $-d_C$ are merely shifted. The vertical lines indicate the uncertainty, calculated as in Fig. 1.

is the sum $d_{\rm bb} + 2d_\tau - d_C$, the mismatch for ζ is due to the mismatch for d_C.

For $\alpha = 2$, one has a nonuniversal t [38–40]. Note that the difference between $\alpha = 1$ and $\alpha = 2$ for d_τ is within the uncertainty of the numerical estimates. The constriction d_C has a clear change, which explains the change in $\zeta = -t/\nu$. To highlight this, we show the slopes for the various components versus α in Fig. 4. As $d_{\rm bb}$ is not affected by the conductance distribution, it should not change with α, which is verified in the plot. The shortest path on the backbone $L_{\rm min}$ is known to scale as $L_{\rm min} \propto L^{D_{\rm min}}$, which gives $\tau < L/L_{\rm min} \propto L^{1-D_{\rm min}}$. The longest self-avoiding path on the percolation cluster scales as $L_{\rm max} \propto L^{D_{\rm max}}$, which gives $\tau > L/L_{\rm max} \propto L^{1-D_{\rm max}}$. Thus, the tortuosity is bounded above and below as $L/L_{\rm max} < \tau < L/L_{\rm min}$, and its exponent must be bounded too, $1 - D_{\rm max} < d_\tau < 1 - D_{\rm min}$. Using the literature values $D_{\rm max} \simeq 1.4$ and $D_{\rm min} \simeq 1.13$ [7] gives $-0.4 < d_\tau < -0.13$. As $D_{\rm max}$ and $D_{\rm min}$ are independent of α, this implies that d_τ must be bounded. From our simulations, we note that the changes to d_τ are minor compared to the width between the bounds.

Straley [38], and recently Berg and Sahimi [40], gives a bound, $-\zeta \geqslant \max(\alpha/\nu, t/\nu)$, so that ζ is bounded above by a value that decreases with increasing α. The only network structure components of ζ that can be dependent on α are d_τ and d_C. As d_τ is bounded, as shown above, while ζ is decreasing, we obtain $\zeta(\alpha) \propto -d_C(\alpha)$ for $\alpha \gg 1$ as d_C is the only unbounded component. This is confirmed in Fig. 4.

Final note and summary. As a final note, we point out that the constriction factor, as defined in Eqs. (4) and (5), contains a sum over the first moment of the bonds' voltage gradients. The scaling exponent for the first moment of the voltage distribution for 2D networks was determined to be 0.28 ± 0.02 by Arcangelis *et al.* [41], which is close to our value $d_C\nu = 0.315 \pm 0.013$. Thus, it may very well be that d_C is, in fact, related to the exponent that characterizes the power-law behavior of the first moment of voltage distribution at or near p_c, but this requires further investigation. We do note that the moments of current and voltage fluctuations in real percolating materials have also been measured [42,43].

Summarizing, we derived a scaling relation between the critical exponent t of the conductivity and three other exponents describing the backbone transport paths. The power laws for the backbone and tortuosity are known, while the constriction factor has been calculated in this Letter. Through the exact decomposition of t, we describe how changes in bond conductance distribution affect values of t, which arise from changes in the network structure parameter for the constrictivity d_C. The proposed relation is confirmed by numerical simulations in $d = 2$ and $d = 3$ and by exact analytical calculations for $d = 1$ and $d = 6$. Thus, the scaling relation is exact for all $d \leqslant 6$.

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