

q -deformed evolutionary dynamics in simple matrix gamesChristopher R. Kitching^{1,*} and Tobias Galla^{2,†}¹*Department of Physics and Astronomy, School of Natural Sciences, The University of Manchester, Manchester M13 9PL, United Kingdom*²*Instituto de Física Interdisciplinar y Sistemas Complejos, IFISC (CSIC-UIB), Campus Universitat Illes Balears, E-07122 Palma de Mallorca, Spain*

(Received 23 July 2024; accepted 27 November 2024; published 27 December 2024)

We consider evolutionary games in which the agent selected for update compares strategies with q neighbors, rather than a single neighbor as in standard evolutionary game theory. If all q sampled neighbors agree, then agents update their strategy using a smoothed best-response rule. Following ideas from social impact theory, we also study mathematical generalizations of the dynamics to noninteger $q > 0$. Looking at fixed point stability and fixation times for 2×2 games with all-to-all interactions, we find that the flow changes significantly as a function of q . Further, we develop the pair approximation for the q -deformed dynamics on uncorrelated graphs. We also study games with more than two strategies, such as the rock-paper-scissors game where we show that changing q leads to the emergence of new types of flow within the simplex.

DOI: [10.1103/PhysRevE.110.064319](https://doi.org/10.1103/PhysRevE.110.064319)**I. INTRODUCTION**

Modern game theory was pioneered by mathematician John von Neumann in the 1920s to mathematically model strategic interactions among agents whose decisions are rational [1–3]. In the 1970s John Maynard Smith and George Price built on this idea by introducing evolutionary game theory [4,5]. Here, agents do not act rationally, instead each individual carries a particular strategy that is passed on from parent to offspring. Reproduction occurs in proportion to fitness. This was initially conceived as a model of Darwinian evolution in biology, but has also found applications in the social sciences and in economics [6]. Originally, evolutionary game dynamics in populations were described mostly by deterministic differential equations, such as the commonly used replicator equations [7]. These equations are formally valid for infinite populations.

However, it is now well established that stochastic effects can considerably affect the outcome of evolution (see, e.g., Refs. [8–10]). There is thus a continuously growing body of work on evolutionary game theory in finite populations. One particular focus is also on networked populations, that is, populations in which any one individual can only interact and compete with its immediate neighbors. This has sometimes been referred to as “evolutionary graph theory” [11]. This should not be misunderstood as a theory of evolving

graphs. Despite the terminology, the underlying graph does often not change in time.

The starting point for studies of evolutionary game theory in finite populations and networks is usually an individual-based model. This is a set of rules by which the agents interact and the population evolves. A number of different interaction models have been proposed, see, e.g., Refs. [12,13]; a summary can also be found in Ref. [14]. For analytical convenience the total size of the population is often kept fixed. Here, we focus on the so-called pairwise comparison process mediated via “Fermi functions” [15]. Detailed definitions will be given in Sec. II.

The so-called voter model is a related but different model of interacting individuals. It was originally introduced by Holley and Liggett in 1975 to model interacting particle systems [16]. In the most basic voter model, the population consists of voters which are of binary types (“opinions”), 0 or 1, connected in some way through a network. In each step of the dynamics one randomly chosen individual adopts the opinion of a randomly chosen neighbor. Many extensions and variations of this model have been proposed and studied [17–21]. One such variation is the so-called “nonlinear voter model,” in which q neighbors need to agree with one another in order for a voter to change opinion [22–24]. We note that this is to be distinguished from the “nonlinear q -voter model” in which there is a separate probability for change if the q neighbors do not all agree with one another [25].

In this paper we seek to combine the ideas of evolutionary games and the nonlinear voter model into “ q -deformed” evolutionary game dynamics. We consider a population of individuals who each carry a particular strategy, and the population then evolves following rules similar to those in conventional models of evolutionary game theory, in particular the probability for an agent to change state depends on payoff. However, before a change can occur, an agent must consult with q of its neighbors. As in the nonlinear voter

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model a change of that agent can then only occur if all those q neighbors are of the same type. As we will show this can significantly affect the evolutionary flow in strategy space, and quantities such as fixation probabilities and times in finite populations. Motivated by results in social impact theory [26], previous work on language dynamics [27] and on the statistical physics of nonlinear and q -voter models [22,24,25,28], we include generalizations of the model to arbitrary real-valued $q > 0$.

The remainder of the paper is organized as follows. In Sec. II we introduce the model of q -deformed evolutionary game dynamics for 2×2 games (two-strategy two-player games). In Sec. III we investigate the rate equations for the q -deformed dynamics in infinite all-to-all populations. We study the fixed points and their stability, and show that the type of evolutionary flow changes as a function of q . In Sec. IV we look at fixation times and probabilities in finite all-to-all systems. In Sec. V we study the effects of changing the underlying topology from a complete graph to an uncorrelated graph. Analytical results for q -deformed dynamics are obtained within the pair approximation. In Sec. VI we study games with more than two strategies. We investigate cyclic games (motivated by the familiar rock-paper-scissors game) as an example and show that q -deformation can induce new behavior, in particular the emergence of stable limit cycles. In Sec. VII we look at a modified version of the model, where we select the q neighbors without replacement. We summarize our results and give an outlook on possible future work in Sec. VIII.

II. MODEL DEFINITIONS

A. Payoff matrix

We consider a population of N agents, who are each of type A or B . For the time being we always consider a population with all-to-all interaction.

The game is defined by the 2×2 payoff matrix

$$\begin{array}{c|cc} & A & B \\ \hline A & a & b \\ B & c & d \end{array}. \quad (1)$$

An A agent interacting with another A agent thus receives payoff a , and it receives b if interacting with an agent of type B . Likewise, a B agent receives c when interacting with an A , and d when interacting with a B .

In finite systems we characterize the state by the number of type A agents i , or, equivalently, by the fraction of A agents in the population, $x = i/N$. The expected payoff for type A and B agents, respectively, is then

$$\begin{aligned} \pi_A(x) &= ax + b(1-x), \\ \pi_B(x) &= cx + d(1-x). \end{aligned} \quad (2)$$

We can define the payoff difference as

$$\Delta\pi(x) = \pi_A(x) - \pi_B(x) = ux + v, \quad (3)$$

where

$$\begin{aligned} u &= (a+d) - (b+c), \\ v &= b-d. \end{aligned} \quad (4)$$

For evolutionary processes relying only on payoff differences it is thus possible to represent the payoff matrix in Eq. (1) with only two variables: u and v .

B. q -deformed evolutionary dynamics

At each time step we randomly choose an agent for update, say it is of type A . We then choose q agents from the population (with replacement). If all those agents are of type B , then the A switches to B , with a probability g^- that depends on the payoffs of type A and B agents. Similarly, we write g^+ for the probability for a B to switch to A if q other randomly selected agents also are of type A .

There are many possible choices for the functions $g^\pm$. One such choice is to make $g^\pm$ proportional to the payoff of the type that is being switched to, i.e.,

$$g^\pm(x) = \frac{\pi_{A/B}(x)}{\max\{a, b, c, d\}}. \quad (5)$$

This choice is only valid if all entries in the payoff matrix are positive. The denominator ensures that $g^\pm(x) \leq 1$. In this way, if type B agents have a large average payoff, then type A agents are likely to switch during interactions with them. But it is still possible for an agent to switch to a type with a lower expected payoff than its current type. A second popular choice involves nonlinear Fermi functions $g^\pm$ [15,29],

$$g^\pm(x) = \frac{1}{1 + e^{\mp\beta\Delta\pi(x)}}. \quad (6)$$

These depend on the payoff difference. The parameter $\beta \geq 0$, known as the intensity of selection, reflects the uncertainty in decision-making processes. In the strong selection limit, $\beta \rightarrow \infty$ an individual will always switch to the type with the higher expected payoff. For any finite β the reverse process occurs with nonzero probability. For $\beta = 0$ one has $g^\pm = \frac{1}{2}$, i.e., neutral selection. Payoffs then have no effect. Since the parameter β only appears in combination with u and v , we will absorb it into those parameters, effectively setting $\beta = 1$.

We stress the two-step setup of the dynamics, as commonly used in evolutionary game theory; see, e.g., Refs. [9,10,30,31]. Every agent plays the game against all of their neighbors, and from this every agent establishes an expected payoff. This is assumed to occur on a fast time scale. Evolution of the population proceeds at a slower pace. One agent is selected for potential update as described above. This proceeds based on the payoffs established in the earlier step.

In total, the rates at which the type A and B agents switch are given by

$$\begin{aligned} T^+(x) &\equiv T_{B \rightarrow A}(x) = N(1-x)x^q g^+(x), \\ T^-(x) &\equiv T_{A \rightarrow B}(x) = Nx(1-x)^q g^-(x), \end{aligned} \quad (7)$$

respectively. The first expression can be understood as follows. The agent chosen for reproduction is of type B with probability $1-x$. The probability that q randomly chosen neighbors (with replacement) in this all-to-all population are all of type A is x^q . The type B agent switches to type A with probability $g^+(x)$. We will sometimes write $T_i^\pm$ instead of $T^\pm(x)$, noting that $x = i/N$.

We measure time in units of generations, the rates are therefore proportional to the size N of the population. While

the above motivation of the model implicitly assumes that q is a positive integer, the rates in Eq. (7) define a continuous-time Markov process for all $q > 0$. This process can be simulated using the celebrated Gillespie algorithm [32]. Analyses of the nonlinear voter model have shown that the dynamics are very different in the regimes $q < 1$, $q = 1$, and $q > 1$, respectively [22,28]. We note that empirical evidence in social impact theory suggests that impact is a sublinear function, $\propto x^q$, with $q < 1$, in human social dynamics [26]. Additionally, we refer to Ref. [27], which reports a noninteger exponent of 1.31 in the context of language dynamics. We thus treat $q > 0$ as a real-valued model parameter, similar to what is done in the literature on nonlinear and q -voter models [22,24,25,33–37]. For $q = 1$ the setup reduces to that of conventional evolutionary game theory in finite populations (see, e.g., Refs. [9,10,14,38]). We will refer to choices $q \neq 1$ as a " q -deformation" of the dynamics.

It is easy to understand that q -deformation favors strategies that are in the minority when $q < 1$, or in the majority, respectively, for $q > 1$. To see this it is useful to focus on the case with no selection ($g^\pm \equiv 1/2$ in our notation). If the frequency of strategy A is x , then the rate of change from A to B is proportional to $x(1-x)^q$ and that for the reverse change proportional to $x^q(1-x)$. For $q = 1$ these terms balance for all x . This is the case of the conventional voter model, where there is no deterministic drift. For $q < 1$ we have $x(1-x)^q < (1-x)x^q$ when $x < 1/2$. Thus, for $q < 1$ strategy A is favored when it is in the minority, and disfavored when more than half of population is of type A. The direction of the flow reverses for $q > 1$.

III. RATE EQUATIONS FOR INFINITE POPULATIONS

A. q -deformed rate equations

For large populations the average dynamics of the population is approximated by a q -deformed version of the conventional rate equations in an infinite population. This is explained further in the Appendix; see, in particular, Appendix A. We have

$$\dot{x} = (1-x)x^q g^+(x) - x(1-x)^q g^-(x). \quad (8)$$

The quantity x in this equation is really $\langle x \rangle$, an average over individual realisations of the dynamics. However to ease notation we will write x throughout. We also note that if we choose $g^\pm$ as in Eq. (5), then after absorbing the constant by rescaling time we find

$$\dot{x} = (1-x)x^q \pi_A(x) - x(1-x)^q \pi_B(x). \quad (9)$$

This is a q -deformed variant of the conventional replicator equation. Setting $q = 1$ reduces this to the standard replicator equation for two-player two-strategy games [39],

$$\dot{x} = x(1-x)[\pi_A(x) - \pi_B(x)]. \quad (10)$$

Given an initial distribution of type A and B agents in a population, Eq. (8) determines how the composition of the population changes with time. The system will eventually reach a fixed point. The location of the fixed point depends on u and v , and (potentially) on the initial condition. Our first step is therefore an analysis of the fixed points of the q -deformed rate equations and of the stability of these fixed points. As a

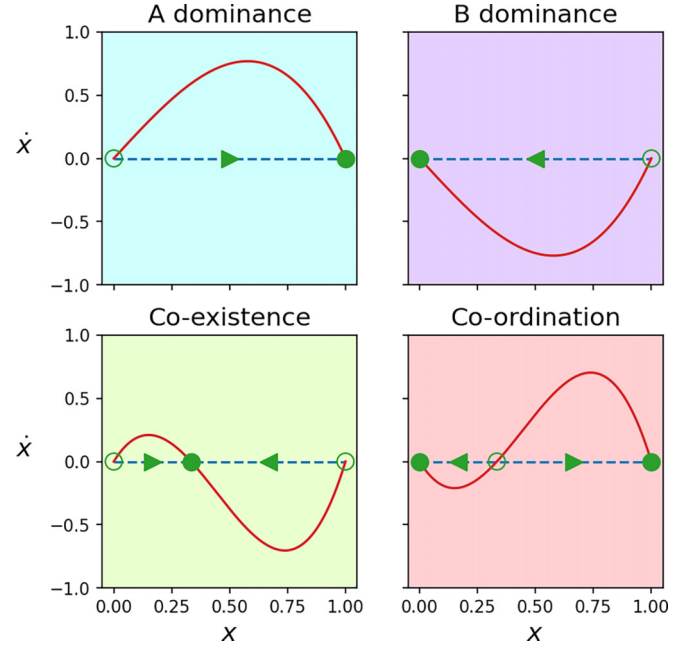


FIG. 1. Illustration of the different types of deterministic flow for $q = 1$, Eq. (11). Each classification has an associated color for easy identification in other plots. Arrows indicate the direction of the flow. Filled/empty green circles are stable/unstable fixed points.

benchmark we first summarize the well-known results for the case $q = 1$.

B. The case $q = 1$

The simplest case is that of $q = 1$, where Eq. (8) reduces to

$$\dot{x} = x(1-x)[g^+(x) - g^-(x)]. \quad (11)$$

In this case, there are always two fixed points at the boundaries, $x^* = 0, 1$. There is also potentially a third, interior, fixed point given by the solution of $g^+(x^*) = g^-(x^*)$. For the functions in Eqs. (5) and (6), this equates to solving $\Delta\pi(x^*) = 0$, so

$$x^* = \frac{-v}{u} = \frac{b-d}{c-a+b-d}. \quad (12)$$

For some values of u and v this solution is unphysical ($x^* < 0$ or $x^* > 1$). We then only have the fixed points at $x^* = 0$ and $x^* = 1$. In the special case $a = c$ and $b = d$ (neutral selection) we have $\dot{x} = 0$, in which case every point x is a fixed point.

Given values for u and v we can characterize the type of flow based on the number of fixed points and their linear stability. From Eq. (11), if $g^+(0) > g^-(0)$, then $x^* = 0$ is unstable. If $g^+(1) > g^-(1)$, then $x^* = 1$ is a stable fixed point. If an interior fixed point x^* exists, then it is stable if

$$\frac{d}{dx}[g^+(x) - g^-(x)]|_{x^*} < 0. \quad (13)$$

These scenarios are summarised in Fig. 1. They are conventionally referred to as the dominance of A or B, "coexistence" or "coordination," respectively [10].

We can create a phase diagram in the (u, v) space to demonstrate how the evolutionary flow changes when we alter

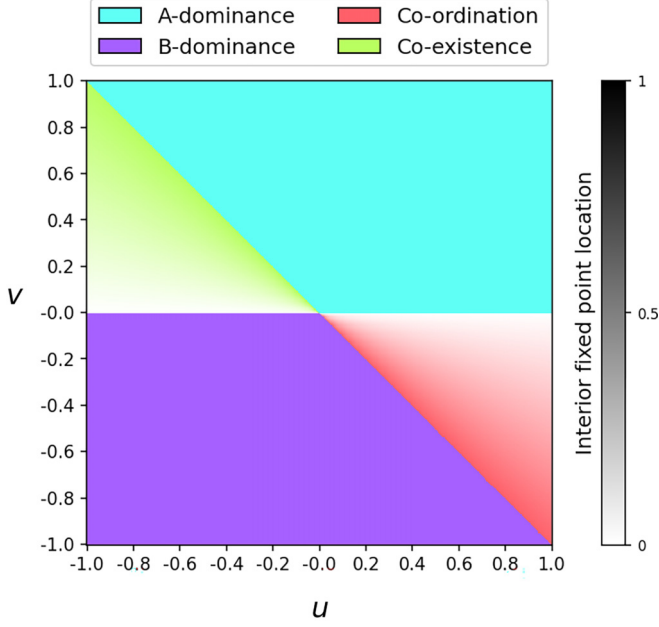


FIG. 2. Phase diagram in (u, v) space showing the classification regions for evolutionary flow for $q = 1$, Eq. (11). The four colors represent the four possible classifications. The coexistence and coordination regions have varying opacity to represent the position of the interior fixed point on the interval $(0, 1)$.

the payoff matrix. Figure 2 shows how this space divides into four regions with the four types of flow. The diagram is valid for the choices of $g^\pm$ in Eqs. (5) and (6). In the case of coexistence or coordination, the interior fixed point can vary on the interval $(0, 1)$ which is represented by the opacity of those regions. The shape of these regions can be explained as follows. We have one interior fixed point when $0 < x^* < 1$ and from Eq. (12) this implies $0 < \frac{-v}{u} < 1$. When $u > 0$ we find that $v < 0$ and $v > -u$, this defines the red region in Fig. 2. We then evaluate Eq. (13) at $x^* = \frac{-v}{u}$, to find that $u > 0$ means that the interior fixed point is unstable, hence one has coordination type flow (Fig. 1). Similar analyses can be carried out for the other regions. We note here how the coexistence and coordination games blend into dominance games as we change u and v , since their interior fixed points move closer to the boundaries, eventually being absorbed.

C. The case $q \neq 1$

We now focus on the case $q \neq 1$ in Eq. (8). Much like the $q = 1$ case we always have fixed points at the boundaries, $x^* = 0$ and $x^* = 1$. The number of interior fixed points depends on the functions $g^\pm$.

All results in this section are for the Fermi function Eq. (6). We find that there can be between one and three interior fixed points. For $q > 1$, we only find coordination type flows (i.e., flows with stable fixed points at the boundaries), and for $q < 1$ we only find coexistence type flows (i.e., flows with unstable fixed points at the boundaries) (see Appendix B). This is because the q -deformation favors majority strategies for $q > 1$, while the minority strategy has an advantage for $q < 1$. The possible types of flow of Eq. (8) include standard coordination

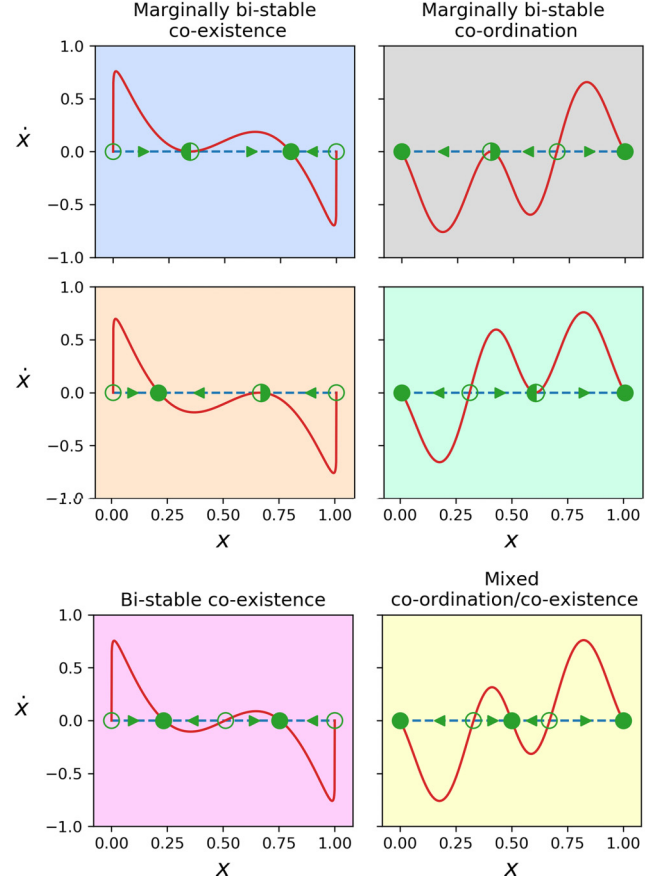


FIG. 3. Illustration of the possible classifications of the $q \neq 1$ game dynamical flow in Eq. (8), with $g^\pm$ Fermi functions as in Eq. (6). The left column shows $q < 1$ coexistence type games, and the right column shows $q > 1$ coordination type games. Each classification has an associated color for easy identification in other plots.

and coexistence, as shown in Fig. 1, and the additional types shown in Fig. 3. The left column shows coexistence type behavior. The right column shows coordination type dynamics.

Even though mathematically we might classify a flow as coordination, coexistence or one of the new types, the interior fixed point(s) are often close to $x = 0$ or $x = 1$. The behavior is then more like a dominance scenario. It is often insightful to plot $\dot{x}$ as a function of x for specific parameters u , v and q .

For the remainder of this section we focus on $q > 1$ (we study the case $q < 1$ in the Appendix; see, for example, Fig. 11). Figure 4 is analogous to Fig. 2, except now $q = 2$. What was previously a coexistence region for $q = 1$ is now bistable coordination. What was A/B-dominance for $q = 1$ is now classified as coordination so there is one large coordination region where the position of the interior fixed point varies continuously. The upper and lower boundaries between the mixed coordination/coexistence and coordination regions are where marginally bistable coordination occurs.

As we increase q , the shapes of these regions change. We can choose particular values of u and v , and investigate how the fixed points move as q is varied. This leads to a bifurcation diagram, as shown in Fig. 5. For $q \leq 1$ and this particular choice of u and v we would have a coexistence type flow,

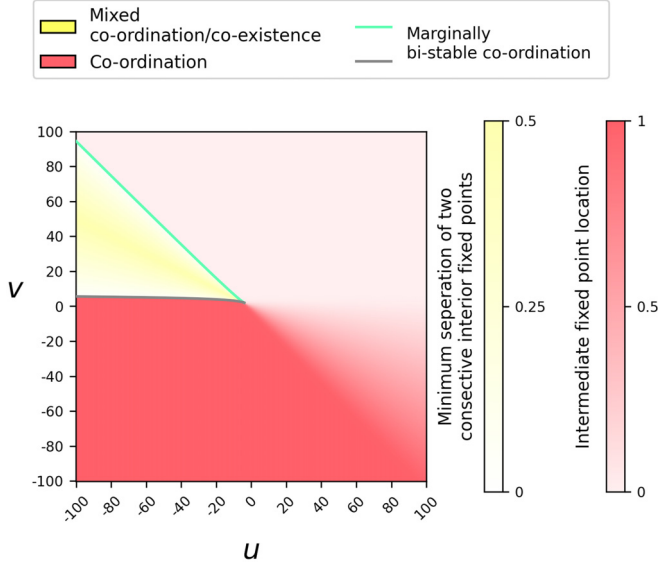


FIG. 4. A phase diagram in (u, v) space showing the classification regions for the dynamics in Eq. (8) for $q = 2$, and with $g^\pm$ Fermi functions as in Eq. (6). The opacity of the red region represents the position of the single unstable interior fixed point present in coordination type flows on the interval $(0,1)$. The opacity of the yellow region represents the minimum distance between two consecutive interior fixed points in mixed coordination/coexistence type flows.

the single interior fixed point is stable. For $q > 1$ we move into a mixed coordination/coexistence regime, and two new interior fixed points spawn at the boundaries, both of which are unstable. As q increases further the stable interior fixed point and one of the unstable interior fixed points move closer to one another, and meet at $q \approx 2.15$. Beyond this point, we find one interior fixed point, which is now unstable. This

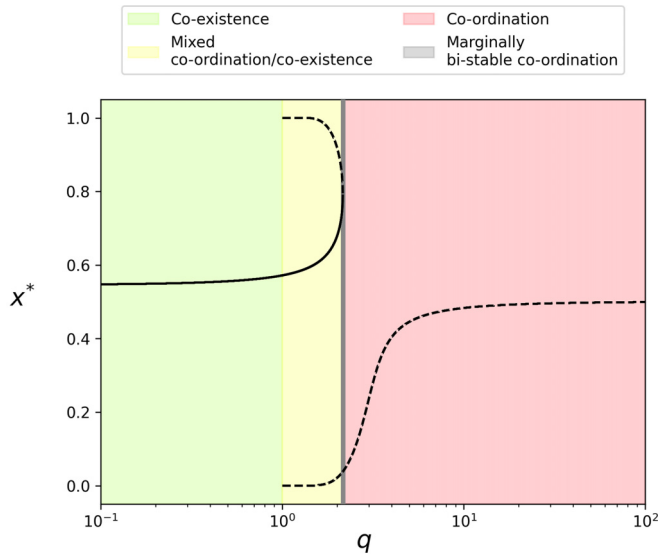


FIG. 5. Bifurcation diagram showing how the interior fixed points, x^* , change as the value of q changes. Solid lines show stable fixed points, dashed lines indicate unstable fixed points. The figure is for $u = -7$, $v = 4$. Colored regions classify the type of game (see Fig. 1).

means that the flow is of the coordination type. The diagram confirms that small values of q promote the minority strategy, i.e., if only a small fraction of individuals are of type A , then x will increase, and if only a small fraction of individuals are of type B , then x will decrease. This produces a stable internal fixed point (coexistence). Conversely, large values of q promote the majority strategy, leading to coordination type flows.

In Appendix B 3 we show two additional bifurcation diagrams: one for the cases where the flow is dominance type at $q = 1$ (Fig. 12), and the other where the flow is coordination type at $q = 1$ (Fig. 13).

IV. FIXATION PROBABILITY FOR A SINGLE INVADER AND FIXATION TIMES

Having analyzed the q -deformed deterministic dynamics in infinite populations, we now look at the fixation probabilities and times in finite populations. We will focus on the Fermi update in Eq. (6), noting that the agent chosen for update uses a smoothed best-response rule [7]. That is to say the quantity $\Delta\pi(x)$ used in the Fermi update is the difference between the payoff to the focal agent upon using strategy A , minus that upon using strategy B , given that a fraction x of the agent's neighbors are playing strategy A . Our analysis follows the steps established, for example, in Ref. [38].

The fixation probability ϕ_i is the probability that the evolutionary process ends in a state in which the entire population is of type A (as opposed to entirely of type B), if there are initially i type A individuals (and $N - i$ type B individuals). The conditional average time to fixation t_1^A is the expected time a single invader of type A needs to take over a population consisting initially of $N - 1$ individuals of type B , given such a takeover happens. The unconditional average time to fixation t_1 is the expectation value of the time until the population is monomorphic ($i = 0$ or $i = N$) if there is initially one individual of type A and $N - 1$ individuals of type B .

For the rates in Eq. (7), the probability ϕ_i is given by (see Appendix C 1)

$$\phi_i = \frac{1 + \sum_{k=1}^{i-1} f_k^{q-1} e^{-H_k}}{1 + \sum_{k=1}^{N-1} f_k^{q-1} e^{-H_k}}. \quad (14)$$

Here, f_k is defined as

$$f_k \equiv f(k; N) = \frac{(-1)^k (1 - N)_k}{k!}, \quad (15)$$

where $(\cdot)_k$ is the Pochhammer symbol [40]. The object H_k is defined as

$$H_k \equiv H(k; u, v, N) = k^2 \frac{u}{2N} + k \left(\frac{u}{2N} + v \right). \quad (16)$$

The fixation times can then be written (see Appendix C 2) as

$$t_1 = \phi_1 \sum_{k=1}^{N-1} \sum_{l=1}^k \frac{1}{T_l^+} \left(\frac{f_k}{f_l} \right)^{q-1} e^{-(H_k - H_l)}, \quad (17a)$$

$$t_1^A = \sum_{k=1}^{N-1} \sum_{l=1}^k \frac{\phi_l}{T_l^+} \left(\frac{f_k}{f_l} \right)^{q-1} e^{-(H_k - H_l)}. \quad (17b)$$

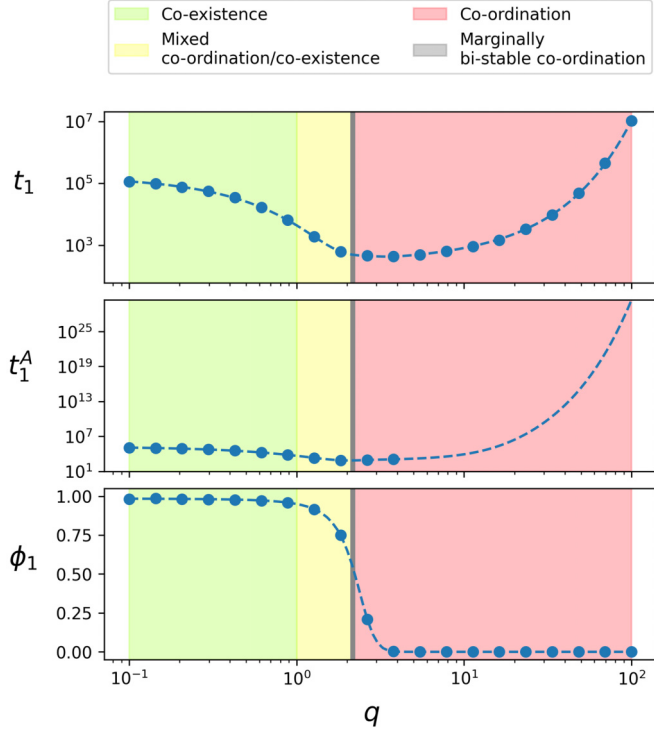


FIG. 6. Unconditional fixation time t_1 (upper panel), conditional fixation time t_1^A (middle panel) and fixation probability ϕ_1 (lower panel) varying with q for $u = -7$, $v = 4$, and $N = 10$ [Fig. 5 shows the corresponding bifurcation diagram for the same game (same u and v)]. Markers are results from averaging over 1×10^3 independent simulations of a population evolving via the process described by Eqs. (6) and (7). Dashed lines are from the theory [Eqs. (14), (17a), and (17b)]. Colored regions indicate the type of flow (see Fig. 1). Additionally, in Fig. 15 in the Appendix we plot t_1^A on a linear scale.

In Fig. 6 we plot the unconditional and conditional fixation times [Eqs. (17a) and (17b)] as well as the fixation probability [Eq. (14)] as functions of q , for the same game as in Fig. 5, and for a population of size $N = 10$. The type of dynamics in the deterministic system is indicated by the colored background shading in both figures. The population size in Fig. 6 is deliberately small. As seen in the figure the fixation times can be extremely large. This would become even more pronounced for larger populations, and verification of the theory in simulations would then be impossible.

For $q < 1$ this particular system is of the coexistence type (green region in Figs. 5 and 6). Thus, we expect trajectories of the stochastic dynamics for finite N to move toward the coexistence fixed point, and then to remain near this meta-stable state until a large fluctuation drives the mutant to extinction or fixation. The escape time depends on the population size (the larger the population, the more difficult the escape) and on the strength of the attraction of the fixed point. In the region $q < 1$ the coexistence fixed points becomes less attractive as q is increased (this is because for small q , the deformation favors the minority strategy). This is in-line with the decrease of fixation times in Fig. 6. Consistent with this, the fixation probability decreases. This behavior is not materially changed by the appearance of two additional unstable fixed points at $q = 0$ (yellow region in Figs. 5 and 6).

For $q \gtrsim 2.15$ the deterministic system is of the coordination type (red region in Figs. 5 and 6). We then find that the evolutionary flow becomes smaller in absolute terms as q is increased. This flow is given by the right-hand side of Eq. (8), and describes the net change of the system per unit time. For large q , actual events only occur very rarely in time, as changes in the population only happen when a sample of q players all have the same type. Mathematically, the factors x^q and $(1-x)^q$ become small when $x \neq 0, 1$. Thus, the population dynamics becomes slower as q is increased, in-line with the increase of the fixation times in the red region of Fig. 6.

The fixation probability is not affected by time scales, but instead by the ratio of transition rates $T^-(x)/T^+(x)$, as detailed in Eq. (C1) in the Appendix. Using Eqs. (6) and (7) we have

$$\frac{T^-(x)}{T^+(x)} = \left(\frac{1-x}{x} \right)^{q-1} e^{-(ux+v)}. \quad (18)$$

For large q , the prefactor $[(1-x)/x]^{q-1}$ dominates this expression. If x is near zero, then the prefactor is large, and thus, with overwhelming probability the next event in the population will lead to a decrease in mutant numbers. Similarly, when x is near one, the prefactor is small, and the next event in the population is very likely to lead to an increase of the number of mutants. In essence, q -deformation favors the majority type for $q > 1$, and promotes coordination. As illustrated in Fig. 14 in the Appendix this effect becomes stronger as q is increased. Thus, it becomes more difficult for a single mutant to take over the population, and ϕ_1 decreases with q in the coordination regime, as seen in the lower panel of Fig. 6.

V. q -DEFORMED DYNAMICS ON GRAPHS

So far we have only considered populations with all-to-all connectivity, i.e., the dynamics runs on a complete graph. We now extend the model to more general networks.

The dynamics on graphs are as follows. A random node is selected. In a second step, q neighbors of that node are chosen at random with replacement. If all q neighbors are in the opposite state to that of the node, then it will change its state with a specific probability. This probability is as in Eq. (6), but the payoff difference that enters into the functions $g^\pm$ is now

$$\Delta\pi_{n,k} = u \frac{n}{k} + v. \quad (19)$$

Here n is the number of A type neighbors of the node chosen for potential update, and k is its degree. The quantities u and v are as in Eq. (4). Thus, $\Delta\pi_{n,k}$ is the change in the expected payoff of the focal node if it switches from B to A . The payoffs in Eq. (19) have been normalized by the degree of the agent in question. This guarantees consistency with Eq. (3) in the limit of infinite connectivity (k and n both to infinity, at fixed $x = n/k$). We note that some literature uses total payoff [41,42], while others, such as Refs. [43,44], use average payoff. Additionally we point to the review [45].

Using the pair approximation we can derive differential equations that can be numerically integrated to approximately describe the density of type A agents, x , and the density of

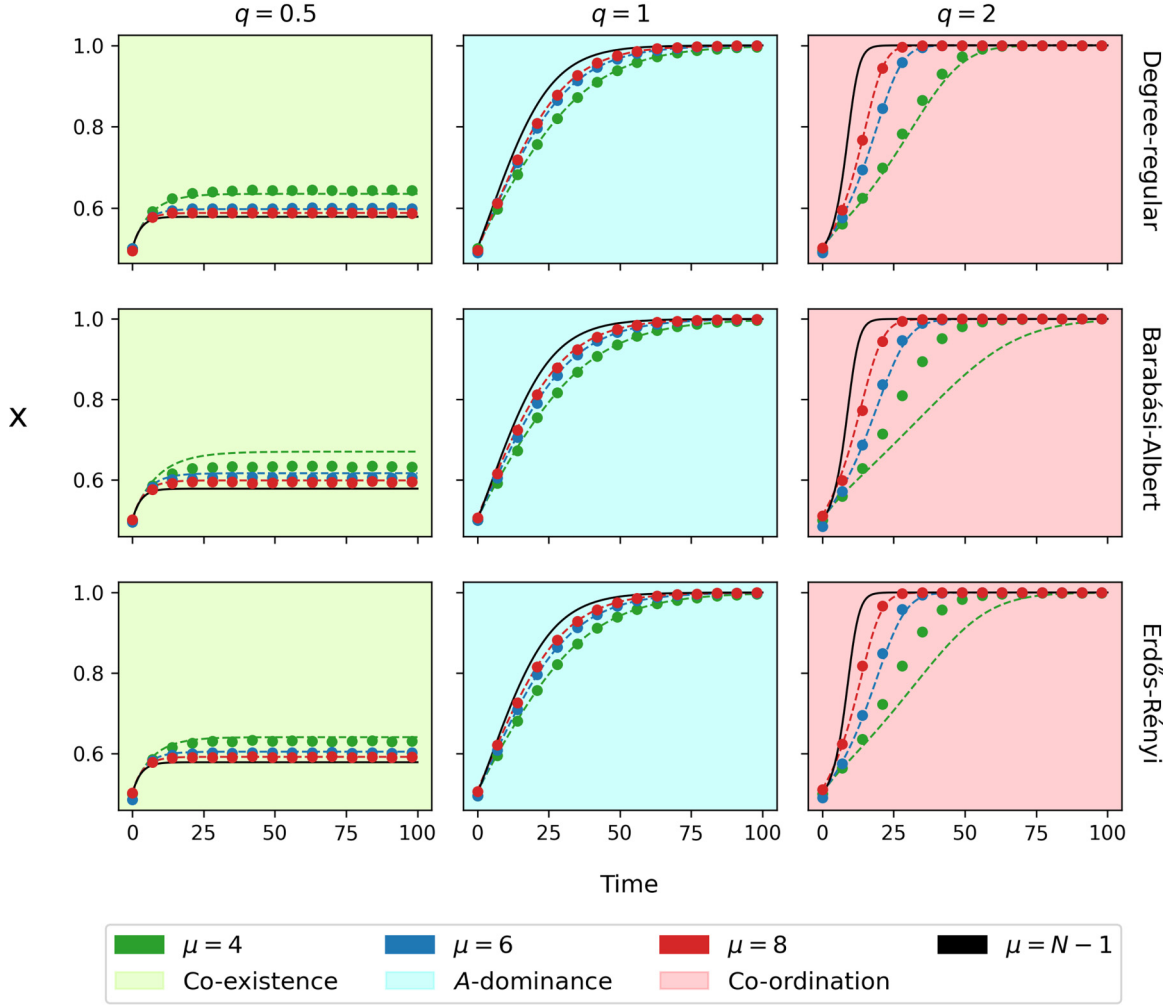


FIG. 7. Density of type A agents, x , as a function of time on various graphs with different average degree, μ , for different values of q . Dashed lines are the analytical results from numerically integrating Eqs. (D12a) and (D12b). Markers are results from averaging over 100 independent trajectories. The graphs in our simulations have size $N = 10000$, and average degree as shown. The indicated average degrees can be generated exactly for random degree-regular graphs (graphs in which all nodes have the same degree), and Barabási–Albert graphs but only approximately for Erdős–Rényi graphs. The game parameters are $u = 0.1$ and $v = 0.1$. The background colors indicate the type of flow (see Fig. 1).

"active links," σ , on infinite uncorrelated graphs (see Appendix D). A link in the network is said to be active when it connects two individuals of different types. These equations are of the form

$$\frac{dx}{dt} = \sum_k P_k \sum_{n=0}^k \left[(1-x) \left(\frac{n}{k} \right)^q g_{n,k}^+ B_-(n|k) - x \left(\frac{k-n}{k} \right)^q g_{n,k}^- B_+(n|k) \right] \quad (20)$$

and

$$\frac{d\sigma}{dt} = \frac{2}{\mu} \sum_k P_k \sum_{n=0}^k \left[(1-x) \left(\frac{n}{k} \right)^q g_{n,k}^+ B_-(n|k) - x \left(\frac{k-n}{k} \right)^q g_{n,k}^- B_+(n|k) \right] (k-2n). \quad (21)$$

Here P_k is the degree distribution of the graph, and μ is the mean degree. The quantity $B_{\pm}(n|k)$ is the probability that a type A/B node, which has degree k , has n neighbors of type A. Under the pair approximation this is a binomial distribution (see Appendix D).

In Fig. 7 we show the average fraction of agents of type A as a function of time for different graphs of varying average degree and for different values of q . The upper three panels are for random degree-regular graphs, and as the data shows the pair approximation captures simulation results well, even quantitatively. Deviations are seen for Barabási–Albert and Erdős–Rényi graphs, in particular for $q = 2$ (the largest value of q shown in the figure). The pair approximation correctly predicts the convergence to $x = 1$, but the speed of the approach is underestimated, in particular for smaller mean degrees, where the pair approximation is known to breakdown [46].

Overall, Fig. 7 demonstrates that the structure of the network affects the dynamics. In the examples shown, the effects

of the mean degree are limited to intermediate times for $q \geq 1$ for all graphs we have tested. This is because in the long-run the system fixates $x = 1$, and the stability of this state is not affected in the range of mean degrees tested in Fig. 7. It is interesting to note that the dynamics in the pair-approximation can have multiple stable fixed points. We find this to be the case for $q > 2$ (see Fig. 17 in the Appendix, indicating coordination-type behavior). Convergence to $x = 1$ is then only seen for some initial conditions.

The long-term average density of type A agents itself is affected for $q < 1$ (left-hand panels in Fig. 7). We note that the stationary density is here strictly between zero and one. Changes of the mean degree then directly alter the location of this fixed point of Eqs. (20) and (21).

The main purpose of this section was to describe how the pair approximation can be extended to q -deformed game dynamics. We stress again that the focal agent in our setup compares its current expected payoff to that it would receive if it were to change strategy. This is at variance with some existing studies of game dynamics on networks, where interaction only occurs with one single neighbor ($q = 1$) [47]. If there is only one interaction partner, then it is perhaps natural to compare the payoffs of the focal node and that of the interaction partner. However, for $q > 1$, the focal node interacts with multiple other agents, so that payoff comparison with one single neighbor does not seem sensible. As an alternative to our dynamics one could consider a model in which the focal agent compares its current payoff to the average payoff of q neighbors (but again changes can only occur provided these neighbors are all of the same type). While we expect that the pair approximation can be developed also in such a scenario, we have not pursued this here. The main reason is that the resulting theory would become more cumbersome, as the evaluation of the functions $g_{n,k}^{\pm}$ is then no longer local, but would also have to be based on the payoffs of the neighbors of the focal node, which in turn will depend on the degrees of those neighbors, and the states of the neighbors of the neighbors.

VI. MULTISTRATEGY CYCLIC GAMES

A. q -deformed dynamics for multistrategy games

We now consider games with $S > 2$ strategies, writing A_{ab} for the payoff that an agent playing strategy a receives when playing against an agent using strategy b . We also write x_a for the proportion of individuals in the population playing strategy a , and introduce the column vector $\mathbf{x} = (x_1, \dots, x_S)^T$, where T stands for transpose. We have $\sum_{a=1}^S x_a = 1$. The average payoff of strategy a is then

$$\pi_a(\mathbf{x}) = \sum_b A_{ab} x_b. \quad (22)$$

We focus on populations with all-to-all interactions. The individual-based process is as before. We choose one individual at random for potential update, say this individual is of type a . In a second step, q individuals are sampled at random from the population (with replacement). Only if all of these individuals are of the same type (which we will call b) can a change of the original type a individual occur. If this is

the case, then the update from a to b is implemented with probability $g_{a \rightarrow b}(\mathbf{x})$.

The rate for changes from a to b is then

$$T_{a \rightarrow b} = N x_a x_b^q g_{a \rightarrow b}(\mathbf{x}), \quad (23)$$

where we use the Fermi function and define, similar to Eq. (6),

$$g_{a \rightarrow b}(\mathbf{x}) = \frac{1}{1 + e^{-\beta[\pi_a(\mathbf{x}) - \pi_b(\mathbf{x})]}}. \quad (24)$$

As in Sec. III we use these rates to obtain equations which govern the dynamics of the average density of type a agents in infinite populations. We find

$$\dot{x}_a = x_a^q \sum_{b \neq a} x_b g_{b \rightarrow a}(\mathbf{x}) - x_a \sum_{b \neq a} x_b^q g_{a \rightarrow b}(\mathbf{x}). \quad (25)$$

In the case of two strategies, we recover Eq. (8).

B. Three-strategy cyclic games

As an example we consider cyclic games with three pure strategies. This generalizes the well-known rock-paper-scissors (RPS) game. Following Refs. [48,49] we use the payoff matrix

$$\begin{array}{c|cc} & R & P & S \\ \hline R & 0 & -1 & 1 + \delta \\ P & 1 + \delta & 0 & -1 \\ S & -1 & 1 + \delta & 0 \end{array}, \quad (26)$$

where δ is real-valued. There are then two main model parameters, δ and q . Our notation follows that of Ref. [48], we note that a different parametrization is used, for example, in Ref. [49]. In Appendix E we also analyze a more general two-parameter family of payoff matrices.

We focus mostly on the case $\delta > -1$, such that the payoff to strategy "scissors," for example, is positive when playing against "paper." Each pure strategy then beats one other pure strategy, and is beaten by the remaining pure strategy. The case $\delta \leq -1$ can also be analyzed, but the game is then not a *bona fide* cyclic game.

The center of the strategy simplex $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a fixed point of the dynamics for all q and δ , indicating coexistence of all three strategies. The monomorphic states at the corners are also fixed points (only one type of strategy survives). Performing a linear stability analysis (see Appendix E) we find that the coexistence fixed point is linearly stable if and only if

$$q < q_c(\delta) \equiv 1 + \frac{\delta}{6}. \quad (27)$$

The eigenvalues are complex for all $\delta \neq -1$, and thus the fixed point is a spiral sink for $q < q_c$, and a spiral source for $q > q_c$. When $q = q_c$ the fixed point is a center.

A linear stability analysis of the corners show that these are stable (with two real-valued eigenvalues) for $q > 1$. For $q < 1$ the leading order-terms in the rate equations are sub-linear near the corners, and linear stability analysis does not apply. Nonetheless, the corners can be seen to be sources. When $q = 1$ the corners are saddle points for all $\delta > -1$. Appendix E contains further details.

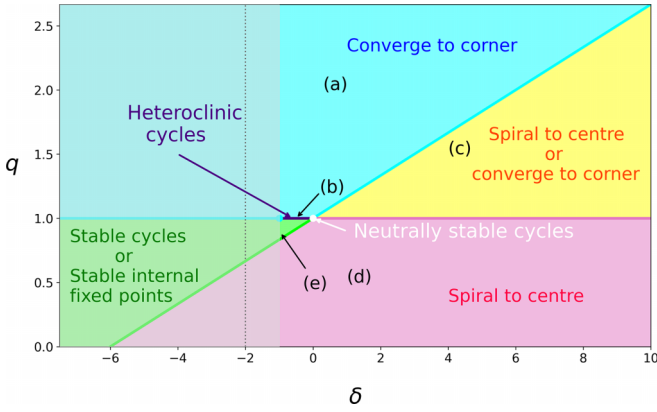


FIG. 8. Types of behavior of the deterministic rate equations for the three-strategy cyclic game using the payoff matrix in Eq. (26) as a function of q and δ . We mostly consider $\delta > -1$. The region $\delta \leq -1$ is also shown but greyed out. Blue: trajectories converge to one of the corners depending on initial conditions. Pink: trajectories spiral to center point for all initial conditions. Green: trajectories converge to the same limit cycle [one for each (q, δ) pair] for all initial conditions, or trajectories can converge to stable internal fixed points within the simplex that are not the center or corners. Indigo: trajectories converge to heteroclinic cycles on the edges of the simplex for all initial conditions. Yellow: depending on the initial conditions, trajectories will either spiral to the center or converge to one of the corners. White: trajectories perform neutrally stable cycles for all initial conditions. By initial conditions we mean that the trajectories start from a point in the simplex that is not the center or one of the corners. We include a feint dotted grey line at $\delta = -2$ for reference. The annotations in the plot correspond to the ternary diagrams in Fig. 9. $(q, \delta) = [(2, 0.3), (1, -0.5), (1.5, 4), (0.5, 1), (0.87, -0.95)]$ for panels (a–e), respectively. The diagram is constructed from analyzing the stability of the central fixed points and the corners, together with numerical integration of the rate equations. As a caveat we add that we cannot exclude the existence of further fixed points (see also Appendix E).

For $q = 1$ the above payoff matrix is known to show different dynamics depending on the value of δ [48,49]. For $\delta > 0$ trajectories spiral to the center of the strategy simplex, i.e., the densities of all strategies tend to $\frac{1}{3}$. When $\delta = 0$ we have neutrally stable cyclic orbits around the central fixed point. For $-1 < \delta < 0$ one finds heteroclinic cycles, where the trajectories orbit near the edge of the simplex. For completeness we remark that the corners of the strategy simplex are stable fixed points when $\delta \leq -1$ and $q = 1$. One then finds convergence to the corners.

In Fig. 8 we show a phase diagram in the (q, δ) plane highlighting the different types of outcome for the cyclic three-strategy game for different q and δ . We emphasize that the behavior in the different phases is determined from the stability analysis of the corners and the center, combined with numerical exploration of the q -deformed rate equations. We cannot exclude the possibility of further fixed points in the interior, or on the edges of the simplex (see also Appendix E4).

Subject to this disclaimer, we find that q -deformation can generate new types of flow, which we will now describe in turn. When $\delta > 0$ and $1 < q < q_c(\delta)$ (yellow region in Fig. 8) the dynamics either spirals to the center or converges to one of the corners, depending on initial conditions. For $q > \max\{1, q_c(\delta)\}$, $q = q_c(\delta)$ with $\delta > 0$, or $q = 1$ with $\delta \leq -1$ (cyan region, including the cyan lines), we find convergence to one of the corners, which corner depends again on initial conditions. For $\delta < 0$ and $q_c(\delta) \leq q < 1$ (green region, including the green line), we either have stable limit cycles or trajectories converge to stable internal fixed points that are not the center or the corners. The latter tends to occur for $\delta \approx -2$, an example is given in Appendix E4. In the indigo region ($q = 1, -1 < \delta < 0$) the system shows heteroclinic cycles. For $q = 1, \delta = 0$ (white point in the diagram) one finds neutrally stable cycles. Finally for $q < \min\{1, q_c(\delta)\}$, or $q = 1$ with $\delta > 0$, (pink region, including the pink line) the dynamics spirals to the center. For completeness we have included some range of $\delta \leq -1$ in Fig. 8, even if the payoff matrix there does not describe a *bona fide* cyclic game. $\delta = -2$ is a special case where the center is a stable/unstable star (see Appendix E4).

In Fig. 9 we illustrate these different types of flow in the strategy simplex. The choice for (q, δ) are as annotated in Fig. 8.

VII. q -DEFORMED DYNAMICS WITHOUT REPLACEMENT

Until now we have always assumed that the q neighbors of the focal agent are chosen with replacement (i.e., the same individual can be chosen more than once). In this section, we focus on the case without replacement. It now only makes sense to consider integer values of q . It is useful to note that the distinction between sampling with and without replacement makes no difference in populations in which every individual has an infinite number of neighbors, as the sampling then occurs from an infinite set. This is the case, for example, in infinite populations with all-to-all interaction. We focus on two-strategy two-player games and populations with all-to-all interaction. The state of the population is then fully described by the number i of individuals playing strategy A.

The rates for the events $i \rightarrow i \pm 1$ in the model without replacement are

$$T_i^+ = \begin{cases} 0, & 0 \leq i < q \\ (N-i)g_i^+ \prod_{k=1}^q \left(\frac{i-k+1}{N-k}\right), & q \leq i \leq N \end{cases} \quad (28a)$$

$$T_i^- = \begin{cases} ig_i^- \prod_{k=1}^q \left(\frac{N-i-k+1}{N-k}\right), & 0 \leq i \leq N-q \\ 0, & N-q < i \leq N \end{cases} \quad (28b)$$

respectively, where we have defined $g_i^\pm \equiv g^\pm(i/N)$. We have $T_i^+ = 0$ for $i < q$, this is because there must be at least q type A agents in the system in order for a type B agent to select q type A other agents without replacement. Similarly $T_i^- = 0$ for $i > N - q$. More detail on these rates can be found in Appendix F.

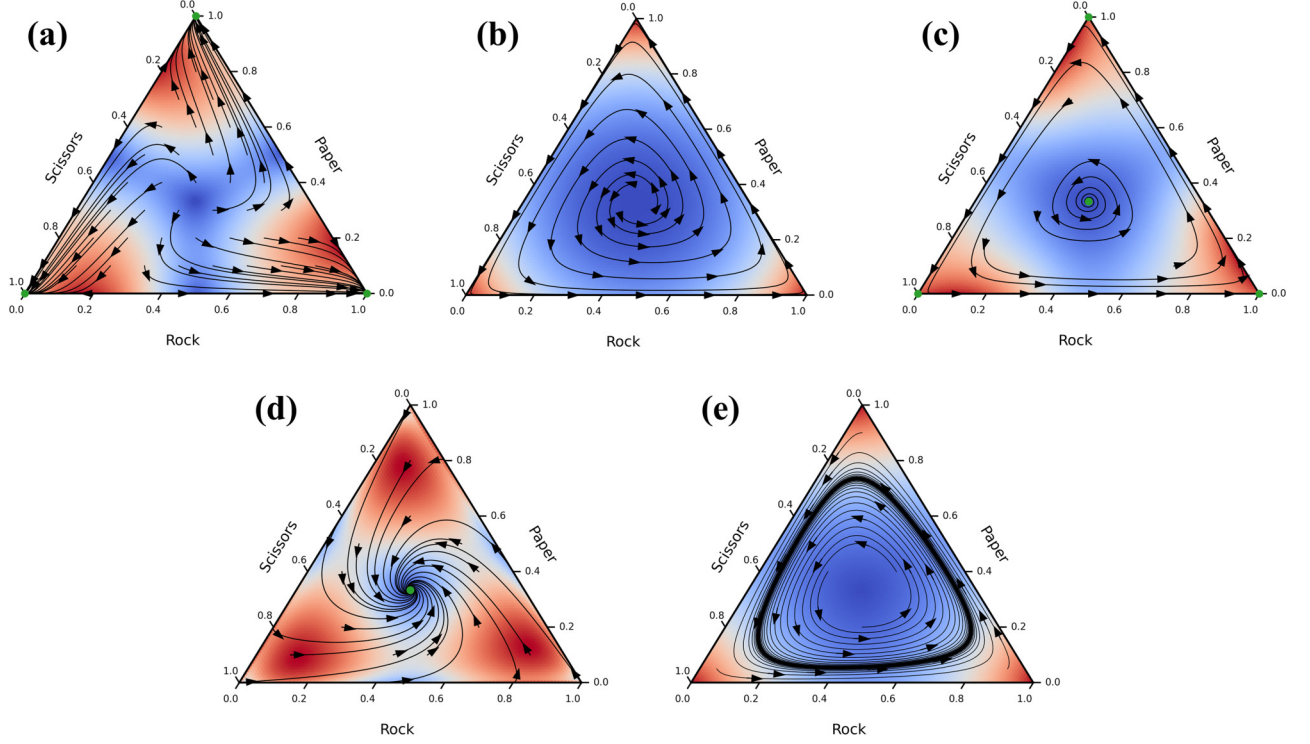


FIG. 9. Ternary plots for the three-strategy cyclic game defined by the payoff matrix in Eq. (26) for different values of (q, δ) as annotated in Fig. 8. Green points are sinks. The background color indicates the speed with which the trajectories move in the simplex, hotter colors are faster. The lines of constant x_i are parallel to the tick marks on edge labeled $i \in \{\text{Rock, Paper, Scissors}\}$. For example, lines of constant x_{paper} are horizontal.

Given the rates in Eqs. (28a) and (28b) we derive the fixation probability to be (see again Appendix F):

$$\phi_i = \begin{cases} 0, & (q \leq \frac{N}{2} \text{ and } 0 \leq i < q) \text{ or } (q > \frac{N}{2} \text{ and } 0 \leq i < q), \\ \frac{\sum_{k=q}^i \prod_{j=q}^{k-1} \gamma_j}{\sum_{k=q}^{N-q+1} \prod_{j=q}^{k-1} \gamma_j}, & q \leq \frac{N}{2} \text{ and } q \leq i \leq N-q, \\ 1, & (q \leq \frac{N}{2} \text{ and } N-q < i \leq N) \text{ or } (q > \frac{N}{2} \text{ and } q \leq i \leq N). \end{cases} \quad (29)$$

Here $\gamma_j = T_j^-/T_j^+$ with $T_j^\pm$ as in Eqs. (28a) and (28b). The fixation probabilities are better illustrated in (q, i) space, as shown in Fig. 10 for different choices of u and v . We here choose a population size of $N = 100$. This allows for a sensible heat-map plot in the figure, while also keeping the computing time required to generate the figure reasonable. The interesting region is that defined by the middle condition of Eq. (29) as this is the only region where the fixation probability depends on the game (in the other two regions, the fixation probability is zero or one, respectively, independent of u and v). We highlight this in Fig. 10, the region of interest is that in the triangle on the left of each panel.

In Fig. 10(a) the fixation probability is very close to one in this area. Since u and v are large in panel (a) this would typically be an A -dominance game (see Fig. 2). At the other extreme (u and v sufficiently negative) the flow is mostly of the B -dominance type [Fig. 10(d)], and in the region of interest the fixation probability is very close to zero. For intermediate values of u and v the region of interest divides nontrivially into two areas, one in which ϕ_1 is close

to one, and another in which its close to zero [Figs. 10(b) and 10(c)].

VIII. DISCUSSION

In summary, we have combined ideas from the nonlinear voter model with dynamics in evolutionary game theory. The nonlinear voter model was originally introduced as an extension of the conventional voter model, and with a view toward understanding how nonlinearity affects the statistical physics of simple systems with absorbing states. Evolutionary game dynamics is nonlinear by itself, thus the goal of this work was to study how additional q -deformation affects the outcome both in infinite and in finite populations. For integer q there is a clear interpretation of the dynamics: an agent in the population consults with q other agents. If all those other agents are in a state which is different from that of the focal agent, then that latter agent considers a change of strategy. This is then implemented with a probability based on payoff gain. Mathematically, the model can be studied for any real-valued q .

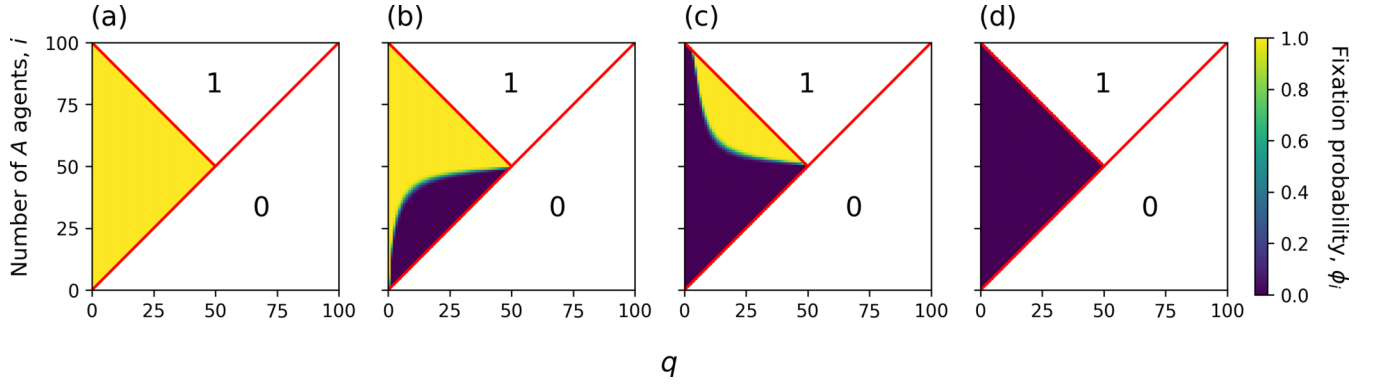


FIG. 10. Fixation probability for the q -deformed dynamics without replacement, Eq. (29), in the (q, i) plane. This is for a population with $N = 100$ for different game parameters, namely, $(u, v) = [(10, 10), (0.1, -0.03), (-0.1, -0.1), (-1, -1)]$ from left to right. There are two white regions, annotated with 0 and 1, where the fixation probability is given by those values. These regions correspond to cases one and three of Eq. (29). In the colored region [case two of Eq. (29)], the fixation probability is often extremely close to 0 (purple) or 1 (yellow), but we stress that it does not take these values exactly.

As we have seen, the combination of q -deformation and selection because of the underlying game can produce a number of new types of flow, not seen in conventional game dynamics ($q = 1$). In 2×2 games multiple internal fixed points become possible, thus generating scenarios such as bistable coexistence or mixed coordination/coexistence. We have systematically studied where in the space of games these different cases occur, as q is varied. In cyclic games with three strategies one also finds dynamics which are not possible without q -deformation, notably stable limit cycles and deterministic flow converging to pure-strategy points. We also find cases with multiple attracting fixed points in the interior of the strategy simplex (Appendix E 4). Finally, we have shown how fixation times and probabilities can be calculated for q -deformed dynamics in finite populations, and we have extended pair approximation methods for interacting agents on networks to q -deformed processes.

The main purpose of this paper is to introduce the general idea of q -deformation in game dynamics, to study a number of basic scenarios and to put the relevant tools in place. There are a number of lines which could be pursued in future work. For example, it might be interesting to see how q -deformation affects the ordering dynamics of evolutionary games on regular lattices. One might expect departures from the universality class of directed percolation, for example [47]. Similarly, one could study more systematically how the system departs from the nonlinear voter model in the limit of weak selection.

More generally, changing the strength of selection in an evolving population can have surprising effects. For example, so-called "stochastic slowdown" has been reported [50], that is the conditional fixation time of a mutant can decrease even if the selective advantage of the mutant is increased. Nonmonotonic behavior of the conditional fixation time as a function of selection strength has also been observed [51]. Our analysis has shown that q -deformation also affects fixation times. It would be interesting to investigate if effects similar to stochastic slowdown also occur. We note that q -deformation not only changes the strength of evolutionary flow but also the location of coexistence fixed points. This is not the case in Refs. [50,51], where the position of fixed points remain the same as the strength of selection is changed.

Further aspects for future work might include multiplayer games (e.g., public-good games), threshold or majority-type update rules [52,53], or asking if and how q -deformation affects the tendency of specific graphs to act as amplifiers or suppressors of selection [11,54]. It would also be interesting to study q -deformation with alternative sampling methods, such as those in Ref. [55], and to compare lattice models with random matching models. In this context, one could also investigate the coarsening dynamics of q -deformation on regular lattices. Finally, extensions of the pair approximation to the model without replacement could be developed [56,57].

ACKNOWLEDGMENTS

This work was supported by the Agencia Estatal de Investigación and Fondo Europeo de Desarrollo Regional (FEDER, UE) under Project APASOS (Grants No. PID2021-122256NB-C21 and No. PID2021-122256NB-C22), the María de Maeztu programme for Units of Excellence, CEX2021-001164-M, funded by MCIN/AEI/10.13039/501100011033. We also acknowledge a studentship by the Engineering and Physical Sciences Research Council (EPSRC, UK), Reference No. EP/T517823/1.

APPENDIX A: DERIVING THE q -DEFORMED RATE EQUATIONS

We focus on a well-mixed population of size N , and a game with two pure strategies. Define $P(x; t)$ as the probability for the system to be in state $x = i/N$ at time t , where i is the number of type A individuals. Given that the dynamics can be described via the rates $T^+(x)$ and $T^-(x)$, which increase/decrease x by $\frac{1}{N}$, respectively, the master equation can be written

$$\begin{aligned} \frac{d}{dt} P(x; t) = & P\left(x - \frac{1}{N}, t\right) T^+\left(x - \frac{1}{N}\right) \\ & + P\left(x + \frac{1}{N}, t\right) T^-\left(x + \frac{1}{N}\right) \\ & - P(x, t) T^+(x) - P(x, t) T^-(x). \end{aligned} \quad (\text{A1})$$

From this, the first moment $\langle x(t) \rangle$, which is the average density of type A agents, has the following differential equation:

$$\begin{aligned} \frac{d\langle x(t) \rangle}{dt} &= \sum_x x \left[P\left(x - \frac{1}{N}, t\right) T^+\left(x - \frac{1}{N}\right) \right. \\ &\quad \left. + P\left(x + \frac{1}{N}, t\right) T^-\left(x + \frac{1}{N}\right) \right] \\ &\quad - \sum_x x P(x, t) [T^+(x) + T^-(x)] \\ &= \sum_x \left(x - \frac{1}{N}\right) P(x, t) T^+(x) \\ &\quad + \sum_x \left(x + \frac{1}{N}\right) P(x, t) T^-(x) \\ &\quad - \langle x T^+(x) \rangle - \langle x T^-(x) \rangle \\ &= \langle x T^+(x) \rangle - \frac{1}{N} \langle T^+(x) \rangle + \langle x T^-(x) \rangle \\ &\quad - \frac{1}{N} \langle T^-(x) \rangle - \langle x T^+(x) \rangle - \langle x T^-(x) \rangle \\ &= \frac{1}{N} [\langle T^+(x) \rangle + \langle T^-(x) \rangle]. \end{aligned} \quad (\text{A2})$$

Assuming the population is of infinite size, we can ignore fluctuations. This means that the probability distribution $P(x; t)$, i.e., the solution to the master equation in Eq. (A1), is concentrated on its mean,

$$P(x, t) \rightarrow P(x, t) = \delta(x - \langle x(t) \rangle). \quad (\text{A3})$$

With this we can simplify the average rates that appear in Eq. (A2),

$$\begin{aligned} \langle T^\pm(x) \rangle &= \int T^\pm(x) P(x, t) dx \\ &= \int T^\pm(x) \delta(x - \langle x(t) \rangle) dx \\ &= T^\pm(\langle x(t) \rangle). \end{aligned} \quad (\text{A4})$$

We ease the notation by writing $\langle x(t) \rangle$ as just x , thus we have

$$\frac{dx}{dt} = \frac{1}{N} [T^+(x) - T^-(x)]. \quad (\text{A5})$$

We can then use the rates from Eq. (7) to get the " q -deformed" dynamics in Eq. (8).

APPENDIX B: CLASSIFYING 2×2 q -DEFORMED EVOLUTIONARY DYNAMICS

To classify the outcome of q -deformed dynamics for 2×2 games we determine all fixed points, x^* , and their stability.

1. Fixed points and stability

The fixed points are obtained by setting the right-hand side of Eq. (8) to zero,

$$(1 - x^*)(x^*)^q g^+(x^*) - x^*(1 - x^*)^q g^-(x^*) = 0. \quad (\text{B1})$$

Of course $x^* = 0, 1$ are trivial solutions corresponding to fixed points on the boundaries. We will determine their stability

below. To find the interior fixed points we define

$$f(x) = \ln\left(\frac{x}{1-x}\right) - \frac{1}{q-1} \frac{g^+(x)}{g^-(x)}, \quad (\text{B2})$$

assuming $x \neq 0, 1$. The solutions to $f(x^*) = 0$ would give all interior fixed points. However, this is a nonlinear equation with no analytic solution in general. Despite this, it is still possible to determine the dynamics for some given function g .

We assume that g takes the form of the Fermi function, as in Eq. (6). Equation (B2) then becomes

$$f_{\text{Fermi}}(x) = \ln\left(\frac{x}{1-x}\right) - \frac{1}{1-q}(ux + v). \quad (\text{B3})$$

This function only exists in the interval $x \in (0, 1)$. We find

$$\lim_{x \rightarrow 0^+} f_{\text{Fermi}}(x) \rightarrow -\infty, \quad (\text{B4a})$$

$$\lim_{x \rightarrow 1^-} f_{\text{Fermi}}(x) \rightarrow +\infty. \quad (\text{B4b})$$

Since $f_{\text{Fermi}}(x)$ is a continuous function there must therefore be at least one zero, so there is always at least one interior fixed point. The maximum number of zeros is determined by the number of stationary points (extrema) of $f_{\text{Fermi}}(x)$, alongside the signs of $f_{\text{Fermi}}(x)$ at those stationary points. For example, if there are two stationary points and if $f_{\text{Fermi}}(x)$ takes a positive value at one of these point, and a negative value at the other, then $f_{\text{Fermi}}(x)$ has three zeros, and hence there are three interior fixed points.

Differentiating Eq. (B3) and equating to zero gives the quadratic equation

$$x^2 - x + \frac{1-q}{u} = 0, \quad (\text{B5})$$

which has solutions

$$x_{1,2} = \frac{1}{2} \left[1 \pm \sqrt{1 - 4 \frac{1-q}{u}} \right]. \quad (\text{B6})$$

These are the locations of the stationary points of the function $f_{\text{Fermi}}(x)$ in Eq. (B3). Thus, depending on the values of the parameters q and u , we can have zero, one or two stationary points.

The possible scenarios are as follows:

(1) $\frac{1-q}{u} > \frac{1}{4}$, $f_{\text{Fermi}}(x)$ has no stationary points. Thus, $f_{\text{Fermi}}(x)$ has one zero, meaning there is one interior fixed point.

(2) $\frac{1-q}{u} = \frac{1}{4}$, $f_{\text{Fermi}}(x)$ has one stationary point at $x^* = \frac{1}{2}$, so $f_{\text{Fermi}}(x)$ has one zero, and there is one interior fixed point.

(3) $0 < \frac{1-q}{u} < \frac{1}{4}$, $f_{\text{Fermi}}(x)$ has two stationary points, x_1 and x_2 , both of which are in the interval $x \in (0, 1)$. This leads to the following subscenarios:

(i) The two stationary points have opposite signs, i.e., $f_{\text{Fermi}}(x_1) \cdot f_{\text{Fermi}}(x_2) < 0$, which means $f_{\text{Fermi}}(x)$ has three zeros, hence there are three interior fixed points.

(ii) The two stationary points have the same sign, i.e., $f_{\text{Fermi}}(x_1) \cdot f_{\text{Fermi}}(x_2) > 0$, which means $f_{\text{Fermi}}(x)$ has one zero, and hence there is one interior fixed point.

(iii) One stationary point lies exactly on the axis $f_{\text{Fermi}}(x) = 0$, i.e., either $f_{\text{Fermi}}(x_1) = 0$ or $f_{\text{Fermi}}(x_2) = 0$, which means $f_{\text{Fermi}}(x)$ crosses the horizontal axis once and

TABLE I. Classification of Eq. (8) depending on the various parameters chosen. The first column evaluates the discriminant of Eq. (B5) to determine the number of stationary points of $f_{\text{Fermi}}(x)$, Eq. (B3). The second column evaluates $f_{\text{Fermi}}(x)$ at the stationary points, x_1 and x_2 given by Eq. (B6), where we assume $x_1 < x_2$, to determine the number of x -axis crosses. The third column differentiates between $q > 1$ and $q < 1$. The fourth column then gives the classification of the game.

Discriminant classification	Stationary point value	value of q	Classification
$\frac{1-q}{u} \geq \frac{1}{4}$ or $\frac{1-q}{u} \leq 0$	N/A	$q > 1$	Coordination
		$q < 1$	Coexistence
	$f_{\text{Fermi}}(x_1) \cdot f_{\text{Fermi}}(x_2) > 0$	$q > 1$	Coordination
		$q < 1$	Coexistence
$0 < \frac{1-q}{u} < \frac{1}{4}$	$f_{\text{Fermi}}(x_1) \cdot f_{\text{Fermi}}(x_2) < 0$	$q > 1$	Mixed coordination/coexistence
		$q < 1$	Bistable coexistence
	$f_{\text{Fermi}}(x_1) = 0$	$q > 1$	Marginally bistable coordination
		$q < 1$	Marginally bistable coexistence
	$f_{\text{Fermi}}(x_2) = 0$	$q > 1$	Marginally bistable coordination
		$q < 1$	Marginally bistable coexistence

touches it once at another location. Hence, there are two interior fixed points.

(4) $\frac{1-q}{u} \leq 0$, Eq. (B6) has two solutions but they lie outside of the range $x \in (0, 1)$. Since $f_{\text{Fermi}}(x)$ is bounded on $x \in (0, 1)$ these are not valid solutions. Thus, $f_{\text{Fermi}}(x)$ has no stationary points, which means it crosses the x -axis once, so one interior fixed point.

Next, we determine the stability of the boundary fixed points. If $q > 1$, then $f_{\text{Fermi}}(x)$ and $\dot{x} = \frac{dx}{dt}$ will always be both greater than zero or both less than zero. For $q < 1$, one will be greater than zero and the other less than zero. This can be easily proved by first assuming $\dot{x} > 0$ [Eq. (B1)]. Then manipulate the inequality into the form of Eq. (B3), at which point we find $f_{\text{Fermi}}(x) > 0$ (if $q > 1$) or $f_{\text{Fermi}}(x) < 0$ (if $q < 1$), i.e., the same/opposite sign to $\dot{x}$.

Now, we know from Eq. (B4a) that $f_{\text{Fermi}}(x)$ is negative as $x \rightarrow 0^+$, thus for $q > 1$, $\dot{x}$ will also be negative in this limit. This means that the fixed point at $x^* = 0$ is stable. We can use a similar argument to show that the fixed point at $x^* = 1$ is also stable.

The stability of the interior fixed points can be determined in the same way. For example, if $\dot{x}$ and $f_{\text{Fermi}}(x)$ have the same sign, and $f_{\text{Fermi}}(x)$ only crosses the horizontal axis [$f_{\text{Fermi}}(x) = 0$] once, then there is one interior fixed point, and that fixed point must be unstable.

The above classifications are summarised in Table I. Graphical representations of these classifications are shown in Fig. 3. We note that there are four scenarios that lead to "marginally bistable" flow. These types of flow are rare in comparison to the others. We group the classifications into two pairs: "marginally bistable coexistence" and "marginally bistable coordination." The difference within a pair is the ordering of the two interior fixed points.

2. Phase diagram

a. General analysis

These classifications are better illustrated by a phase plot in the (u, v) plane, which will also help to highlight how the

q -deformed flow changes as we alter the payoff matrix. We first want to determine all points in the (u, v) plane that would give marginally bistable flow. To do this we substitute the stationary point solutions x_1 and x_2 , given by Eq. (B6), into $f_{\text{Fermi}}(x)$ [Eq. (B3)], then set this equal to zero and solve. This is equivalent to solving $f_{\text{Fermi}}(x_{1,2}) = 0$ which, as seen from Table I, is what defines marginally bistable flow. We find

$$v^{(1)} = (1 - q) \ln \left(\frac{x_1}{1 - x_1} \right) - ux_1, \quad (\text{B7a})$$

$$v^{(2)} = (1 - q) \ln \left(\frac{x_2}{1 - x_2} \right) - ux_2. \quad (\text{B7b})$$

Recall from Eq. (B6) that $x_{1,2}$ are functions of q and u , thus for a given value of q , Eqs. (B7a) and (B7b) are lines in the (u, v) plane. These equations are only valid for $q > 1$ when $-\infty < u \leq \tilde{u}$, and for $q < 1$ when $\tilde{u} \leq u < \infty$, where

$$\tilde{u} = 4(1 - q). \quad (\text{B8})$$

Otherwise, Eq. (B6) does not have real solutions.

Along the line defined by $v^{(1)}$ one has $f_{\text{Fermi}}(x_1) = 0$, i.e., marginally bistable coordination and marginally bistable coexistence for $q > 1$ and $q < 1$, respectively. Similarly, sitting on the $v^{(2)}$ line corresponds to $f_{\text{Fermi}}(x_2) = 0$, again marginally bistable coordination and marginally bistable coexistence for $q > 1$ and $q < 1$, respectively. The region between these lines corresponds to $f_{\text{Fermi}}(x_1) \cdot f_{\text{Fermi}}(x_2) < 0$, i.e., mixed coordination/coexistence or bistable coexistence for $q > 1$ and $q < 1$, respectively. The region outside of these lines is simply standard coordination or coexistence for $q > 1$ and $q < 1$, respectively.

Figure 11 demonstrates the idea for different values of q . This figure highlights the fact that marginally bistable coordination and marginally bistable coexistence type flow are rare, as they are only seen on well defined lines in the (u, v) plane. Most of the time we see either coordination, coexistence, or mixed coordination/coexistence ($q > 1$) and bistable coexistence ($q < 1$).

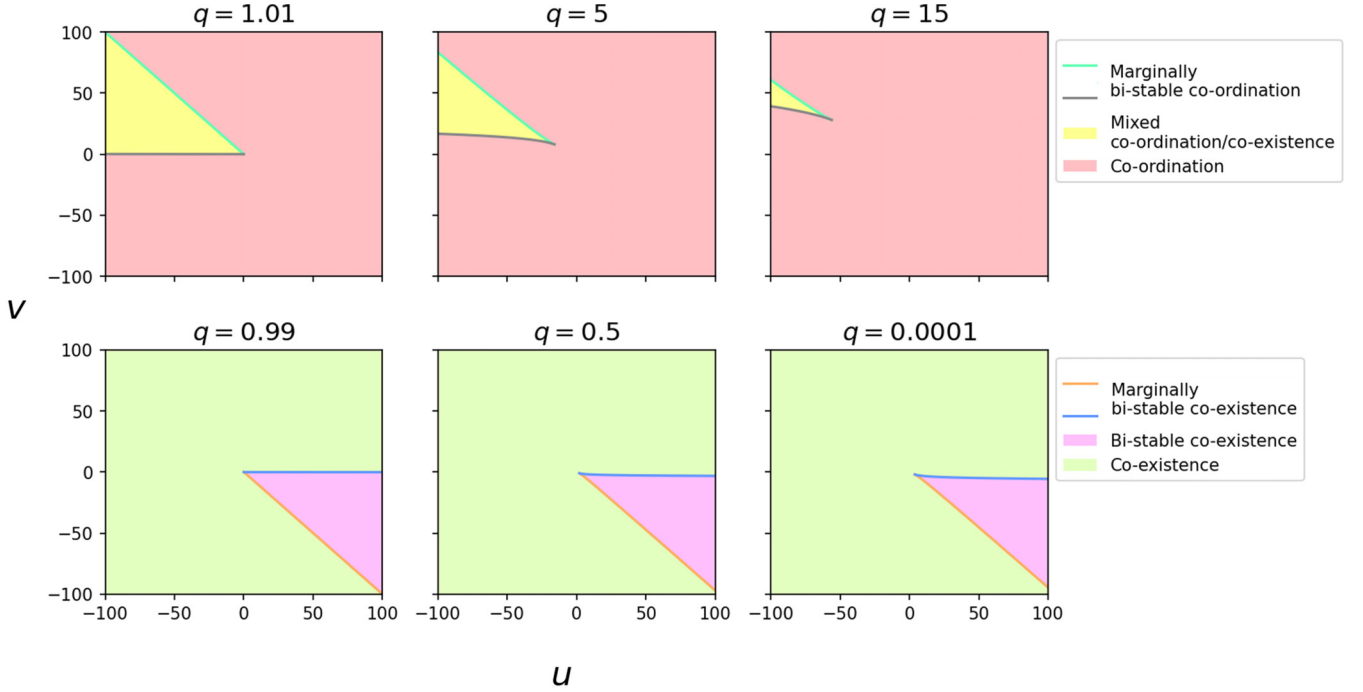


FIG. 11. Phase plots in (u, v) space showing how the classification regions of the q -deformed replicator equations change as we alter the payoff matrix for different q . Regions are colored in line with Figs. 1 and 3.

b. The limit $q \rightarrow \infty$

In the limit $q \rightarrow \infty$ both $(1-x)^q$ and x^q go to zero for any $0 < x < 1$. Thus, Eq. (B1) is satisfied for all values of $x \in (0, 1)$, meaning all such points are fixed points. This is because for $q \rightarrow \infty$ no events can occur in a population containing individuals both of type A and type B.

c. Large but finite q

For large but finite q , the mixed coordination/coexistence region shrinks and standard coordination dominates due to the fact that $\tilde{u}$ becomes large and negative [see Eq. (B8)], which is the upper bound for u for the mixed coordination/coexistence region. The interior fixed point of the dominating coordination region is $x^* \approx \frac{1}{2}$ at all points in the space classified as coordination. This can be seen by noting that as q becomes large,

$$f_{\text{Fermi}}(x) \approx \ln\left(\frac{x}{1-x}\right), \quad (\text{B9})$$

and $\ln(\frac{x^*}{1-x^*}) = 0$ has one solution, namely, $x^* = \frac{1}{2}$. So for large positive q we only get coordination type flow with $x^* \approx \frac{1}{2}$, and we conclude that $q > 1$ dynamics promotes the majority, i.e., if there are less A types than B types, then the dynamics drives the population of A types down to zero.

Since q is only large but still finite, there always exists a region of mixed coordination/coexistence (yellow region in Fig. 11), but only for large negative u , and $v > 0$. This is because in the standard $q = 1$ case these mixed coordination/coexistence regions are coexistence regions (see Fig. 2). The dynamics promotes the minority, i.e., if there are less A types, then the dynamics attempts to drive the number upwards. In this way, there is a balance between the the

game itself favoring the minority, and the q -deformation for $q > 1$ promoting the majority. As we increase q , the influence of the q -deformation dominates, and the mixed coordination/coexistence region shrinks.

d. The limit $q \rightarrow 0$

In the limit $q \rightarrow 0$, the q -deformation favors coexistence. This time, however, $\tilde{u} \rightarrow 4$ is a finite limit [see Eq. (B8)]. Thus, there always exists a region of bistable coexistence (even for $q \rightarrow 0$), as seen in the lower panels of Fig. 11. The reasoning for this is analogous to that for large finite q : the q -deformed dynamics promotes the minority while the standard replicator flow ($q = 1$) in this region would promote the majority (see Fig. 2). Decreasing q initially means the influence of the q -deformation increases, so coexistence starts to take over. However, past a certain point this effect diminishes, and decreasing q further results in no change.

3. Further bifurcation diagrams

Here we show two further bifurcation diagrams which are analogous to Fig. 5. In Figs. 12 and 13 the flow at $q = 1$ is dominance and coordination type, respectively.

APPENDIX C: FIXATION PROBABILITY AND FIXATION TIME FOR 2×2 GAMES WITH q -DEFORMED DYNAMICS

1. Fixation probability

For an initial number of type A agents, i , we want to calculate the fixation probability ϕ_i , which is the probability for the system to reach the state $i = N$, i.e., all agents in the population are of type A.

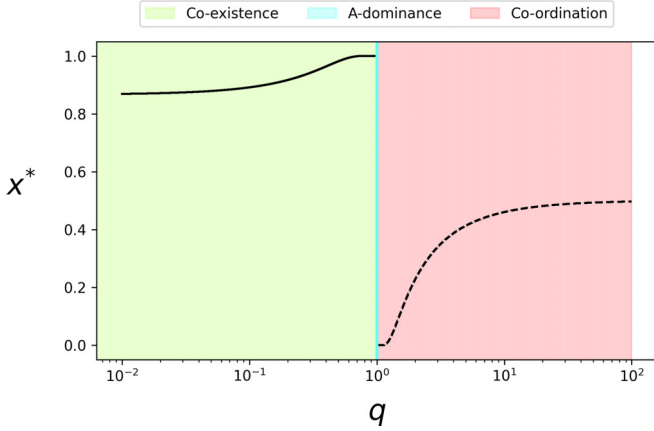


FIG. 12. Bifurcation diagram showing how the interior fixed points, x^* , change as the value of q changes. Solid lines show stable fixed points, dashed lines show unstable fixed points. This figure is for $u = 1$, $v = 1$. Colored regions classify the type of flow (see Fig. 1).

It is well known that [10]

$$\phi_i = \frac{1 + \sum_{k=1}^{i-1} \prod_{j=1}^k \gamma_j}{1 + \sum_{k=1}^{N-1} \prod_{j=1}^k \gamma_j}, \quad (\text{C1})$$

where $\gamma_i = \frac{T_i^-}{T_i^+}$. Using the transition rates for our model, Eq. (7), we can write

$$\gamma_i = \left(\frac{N-i}{i} \right)^{q-1} \frac{g_i^-}{g_i^+}. \quad (\text{C2})$$

We note that $g_i^\pm \equiv g^\pm(\frac{i}{N})$. Using the Fermi function as the choice for $g^\pm$ [Eq. (6)], we can write the ratio

$$\frac{g_i^-}{g_i^+} = e^{-\Delta\pi_i}. \quad (\text{C3})$$

Thus, we have

$$\gamma_i = \frac{T_i^-}{T_i^+} = \left(\frac{N-i}{i} \right)^{q-1} e^{-\Delta\pi_i}, \quad (\text{C4})$$

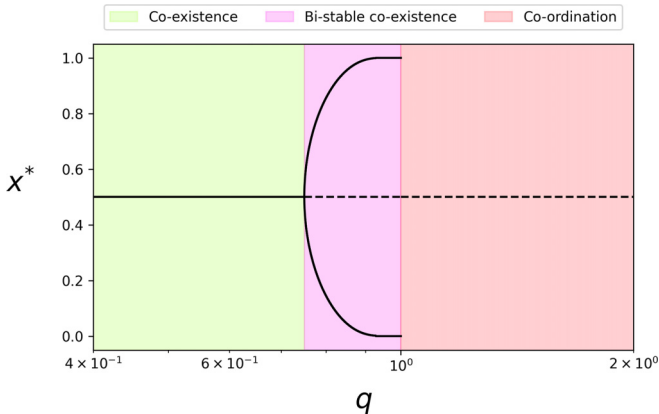


FIG. 13. Bifurcation diagram showing how the interior fixed points, x^* , change as the value of q changes. Solid lines show stable fixed points, dashed lines show unstable fixed points. This figure is for $u = 1$, $v = -0.5$. Colored regions classify the type of flow (see Fig. 1).

which can be interpreted as the tendency for the system to decrease the number of type A agents. Again, we define $\Delta\pi_i \equiv \Delta\pi(\frac{i}{N})$, where $\Delta\pi(\cdot)$ is as in Eq. (3).

We now follow the lines of Ref. [38]. Equation (C1) requires evaluating the product

$$\begin{aligned} \prod_{j=1}^k \gamma_j &= \prod_{j=1}^k \left(\frac{N-j}{j} \right)^{q-1} e^{-\Delta\pi_j} \\ &= \left(\frac{N-1}{1} \right)^{q-1} e^{-\Delta\pi_1} \left(\frac{N-2}{2} \right)^{q-1} e^{-\Delta\pi_2} \\ &\quad \dots \left(\frac{N-k}{k} \right)^{q-1} e^{-\Delta\pi_k} \\ &= \left(\frac{N-1}{1} \right)^{q-1} \left(\frac{N-2}{2} \right)^{q-1} \\ &\quad \dots \left(\frac{N-k}{k} \right)^{q-1} \exp \left\{ - \sum_{j=1}^k \Delta\pi_j \right\} \\ &= \left[\frac{(-1)^k (1-N)_k}{k!} \right]^{q-1} \exp \left\{ - \sum_{j=1}^k \left(u \frac{j}{N} + v \right) \right\}, \end{aligned} \quad (\text{C5})$$

where $(\cdot)_k$ is the Pochhammer symbol [40] and we have used the formula for the payoff difference as in Eq. (3). The summation inside the exponential has a simple closed form which we will write as

$$\sum_{j=1}^k \left(u \frac{j}{N} + v \right) = \frac{k(k+1)}{2N} u + kv = H(k; u, v) \equiv H_k. \quad (\text{C6})$$

We also define the following function which appears in Eq. (C5) as

$$f_k \equiv f(k; N) = \frac{(-1)^k (1-N)_k}{k!}. \quad (\text{C7})$$

With this we can write the fixation probability as

$$\phi_i = \frac{1 + \sum_{k=1}^{i-1} f_k^{q-1} e^{-H_k}}{1 + \sum_{k=1}^{N-1} f_k^{q-1} e^{-H_k}}. \quad (\text{C8})$$

In the limit $q \rightarrow 1$, we recover the known formula for the standard Fermi process (with no deformation) [38, p. 5].

2. Fixation time

We want to calculate the unconditional, t_1 , and conditional, t_1^A , fixation times for q -deformed dynamics. t_1^A is the time it would take type A agents to take over the population, whereas t_1 is the time it would take for the type A agents to take over or become extinct, both starting from a single type A agent.

For general event rates of the one-step process these can be calculated as [10]

$$t_1 = \phi_1 \sum_{k=1}^{N-1} \sum_{\ell=1}^k \frac{1}{T_{\ell}^+} \prod_{m=\ell+1}^k \gamma_m, \quad (\text{C9a})$$

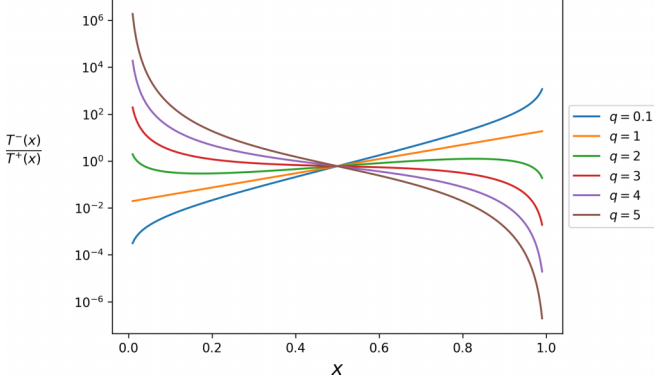


FIG. 14. Ratio $T^-(x)/T^+(x)$ for the system in Figs. 5 and 6. The next event in the population increases the number of mutants (type A) with probability $T^+(x)/[T^+(x) + T^-(x)] = 1/[1 + T^-(x)/T^+(x)]$, or decreases it with probability $1/[1 + T^+(x)/T^-(x)]$.

$$t_1^A = \sum_{k=1}^{N-1} \sum_{\ell=1}^k \frac{\phi_\ell}{T_\ell^+} \prod_{m=\ell+1}^k \gamma_m, \quad (\text{C9b})$$

where γ_k is the ratio of transition probabilities, given by Eq. (C4), and ϕ_ℓ is the probability to fixate from ℓ type A agents, given by Eq. (14). These equations can be written in a simpler form by evaluating the product, analogous to what was done in Appendix C 1,

$$\begin{aligned} \prod_{m=\ell+1}^k \gamma_m &= \frac{\prod_{m=\ell}^k \gamma_m}{\prod_{m=\ell}^{\ell} \gamma_m} \\ &= \frac{f_k^{q-1} e^{-\beta H_k}}{f_\ell^{q-1} e^{-\beta H_\ell}} \\ &= \left(\frac{f_k}{f_\ell} \right)^{q-1} e^{-(H_k - H_\ell)}. \end{aligned} \quad (\text{C10})$$

Thus, the fixation times can be written

$$t_1 = \phi_1 \sum_{k=1}^{N-1} \sum_{\ell=1}^k \frac{1}{T_\ell^+} \left(\frac{f_k}{f_\ell} \right)^{q-1} e^{-(H_k - H_\ell)}, \quad (\text{C11a})$$

$$t_1^A = \sum_{k=1}^{N-1} \sum_{\ell=1}^k \frac{\phi_\ell}{T_\ell^+} \left(\frac{f_k}{f_\ell} \right)^{q-1} e^{-(H_k - H_\ell)}, \quad (\text{C11b})$$

where f_k and H_k are the functions defined in Eqs. (15) and (16), respectively.

3. Further results for the system in Figs. 5 and 6

Further results for the system in Figs. 5 and 6 can be found in Figs. 14 and 15.

For $q \gtrsim 2.15$ the flow is of the coordination type. Figure 14 shows that increasing q in this regime makes it more and more likely that the next event is toward the absorbing monomorphic states at $x = 0$ and $x = 1$. This reduces the probability for a single mutant to take over the population.

In Fig. 15 we show a plot analogous to the middle panel of Fig. 6 except on a linear axis. The analytical solution (dashed line) diverges for large q .

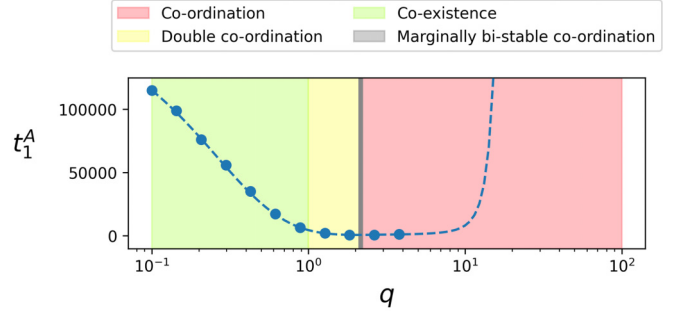


FIG. 15. Conditional fixation time t_1^A varying with q for $u = -7$, $v = 4$ and $N = 10$.

APPENDIX D: PAIR APPROXIMATION ON GRAPHS

1. Notation

We now analyze q -deformed dynamics on graphs. We assign each node a state $s = \pm 1$ if it is an A or B type, respectively. We focus on undirected graphs.

We perform a *homogeneous pair approximation*, similar to Refs. [58,59]. This approximation is known to capture the behavior of the model to good accuracy on infinite uncorrelated graphs [60]. By uncorrelated graphs we mean graphs where nodes have no preference for attaching to nodes of any particular degree [61].

We write the degree distribution of a general graph as P_k . This is the probability that a randomly chosen node in the graph has degree k (i.e., k neighbors). We denote the number of A type neighbors of a node as n . A node in state s and of degree k with n neighbours in the $+1$ state will often be referred to as an (n, k, s) node.

2. Rate of change of the proportion of agents of type A

We wish to determine a differential equation for $\langle x_+ \rangle$, which we define to be the average density of nodes in the $+1$ state. By average we mean average over independent realisations of the dynamics. The angle-bracket notation is dropped to ease notation and keep consistency with the rest of the paper and the literature. We assume a time step of $dt = \frac{1}{N}$, and take $N \rightarrow \infty$ to obtain the continuous-time limit, thus the following now only applies to infinite graphs.

The general rate equation for x_+ is

$$\frac{dx_+}{dt} = \frac{1}{1/N} \sum_{s=\pm 1} T^s \Delta x_+^s, \quad (\text{D1})$$

where T^s is the rate at which $-s$ nodes flip to s nodes, and Δx_+^s is the amount x_+ changes when this happens. The factor $1/N$ in the denominator results from a division by the time step.

It can be seen by inspection that $\Delta x_+^s = -s/N$, therefore

$$\frac{dx_+}{dt} = T^+ - T^-. \quad (\text{D2})$$

3. The homogeneous pair approximation

Determining the rates T^s will require knowing the probability that a node in state s with degree k has n neighbors in

the state $+1$, we will denote this probability $B_s(n|k)$. To derive an expression for $B_s(n|k)$ we will use the *homogeneous pair approximation*. This assumes that the states of the different neighbors are independent, and leads to a binomial distribution of the form

$$B_s(n|k) = \binom{k}{n} p(+1|s)^n [1 - p(+1|s)]^{k-n}, \quad (\text{D3})$$

where $p(+1|s)$ is the single-event probability that a node in state s is connected to a node in state $+1$. This probability can be defined in terms of the density of state- s nodes, x_s , and a quantity σ which is the density of links connecting opposite spin nodes. We refer to σ as the density of active interfaces/links, in-line with standard voter model terminology [58]. The conditional probability $p(+|s)$ can be determined as follows: $p(+|-)$ is the ratio of the number of links connecting opposite-state nodes to the total number of -1 state nodes,

$$p(+|-) = \frac{\sigma \cdot \frac{\mu N}{2}}{x_- \mu N} = \frac{\sigma}{2(1 - x_+)}. \quad (\text{D4})$$

The probability $p(+|+)$ can then be easily determined from

$$\begin{aligned} p(+) &= p(+|-)p(-) + p(+|+)p(+), \\ \Rightarrow p(+|+) &= 1 - \frac{\sigma}{2x_+}. \end{aligned} \quad (\text{D5})$$

Thus, combining Eqs. (D4) and (D5) we have the general expression

$$p(+1|s) = \frac{1+s}{2} - s \frac{\sigma}{2x_s}. \quad (\text{D6})$$

The moments of $B_s(n|k)$ can then be evaluated. As we will see below, only the first and second moments are needed, which are $\langle n \rangle_s = kp$ and $\langle n^2 \rangle_s = k^2 p^2 + kp(1-p)$, respectively, where p is shorthand for $p(+1|s)$ from Eq. (D6).

We emphasize that we have assumed that the probability of selecting a link connecting nodes in opposite states, σ , is independent of the degree of the nodes which it connects. This is a shortcoming of the homogeneous pair approximation. Extensions have been proposed, such as the heterogeneous pair approximation [46], which allows σ to depend on degree. Similarly, we assume an infinite graph. The stochastic pair approximation [62] accounts for finite-size corrections. However, here, we do not consider these extensions.

4. Rate of change of σ

Due to the presence of σ in the binomial moments, the differential equations for x_+ will be coupled to the differential equation for σ . The general form for such an equation has an analogous form to Eq. (D1),

$$\frac{d\sigma}{dt} = \frac{1}{1/N} \sum_{s=\pm 1} T^s \Delta\sigma^s. \quad (\text{D7})$$

Again T^s is the rate at which $-s$ nodes flip to s nodes. The quantity $\Delta\sigma^s$ is the corresponding change in the density of active links. Assuming it is an $(n, k, -s)$ node that flips to a (n, k, s) node, for $s = +1$ there are n active links initially, and $k - n$ active links after flipping, thus a change of $k - 2n$.

There are $\frac{\mu N}{2}$ links overall, so the change in the density of active links is $\frac{2(k-2n)}{\mu N}$. Similar analysis for $s = -1$ gives overall $\Delta\sigma^s = \frac{s(k-2n)}{\mu N}$. Therefore, we can write Eq. (D7) as

$$\frac{d\sigma}{dt} = \frac{2}{\mu} [T^+ - T^-](k - 2n). \quad (\text{D8})$$

5. Transition rates on graphs

The dynamics we have considered so far on complete graphs is a form of pairwise comparison dynamics. A node is chosen at random, q of its neighbors are then chosen. If all q neighbors are of opposite type to the originally selected node, then the node will change type with a probability given by the Fermi function [Eq. (6)]. The argument of this function is the payoff difference $\Delta\pi$, i.e., the difference in the average payoff of the node and one of its neighbors. This is fine to do, as on a complete graph all q neighbors have the same payoff (if they are all in the same state).

On general graphs things are different. A node will select q random neighbors, but, even if they are all in the same state, those neighbors do not necessarily have the same average payoff, as they themselves have different neighborhoods. Instead then on a general graph we replace $\Delta\pi \rightarrow \Delta\pi_{n,k}$, where

$$\begin{aligned} \Delta\pi_{n,k} &\equiv \pi_+(n, k) - \pi_-(n, k) \\ &= a \frac{n}{k} + b \frac{k-n}{k} - c \frac{n}{k} - d \frac{k-n}{k} \\ &= (a - b - c + d) \frac{n}{k} + (b - d) \\ &= u \frac{n}{k} + v. \end{aligned} \quad (\text{D9})$$

Here $\pi_s(n, k)$ is the average payoff of an (n, k, s) node, it is defined analogously to Eq. (2). $\Delta\pi_{n,k}$ then is the increment in the average payoff when an $(n, k, -1)$ node changes to an $(n, k, +1)$ node. In other words, decisions are based on comparing the payoff to one node with the payoff this node would receive if it changed state (and keeping all neighbors in their present state). We thus introduce [Eq. (6)]

$$g_{n,k}^{\pm} = \frac{1}{1 + e^{\mp \Delta\pi_{n,k}}}. \quad (\text{D10})$$

This gives the probability that an $(n, k, -1)$ node changes to an $(n, k, +1)$ node. When $\Delta\pi_{n,k} \rightarrow \infty$, the average payoff of the node being spin $+1$ is much larger than it being spin -1 , $g_{n,k}^+ \rightarrow 1$ accordingly, meaning the node is guaranteed to flip.

We can now form expressions for T^s which are needed for Eqs. (D2) and (D8). We have

$$T^+ = \sum_{k=1}^{N-1} P_k \sum_{n=0}^k B_-(n|k) x_- \left(\frac{n}{k} \right)^q g_{n,k}^+, \quad (\text{D11a})$$

$$T^- = \sum_{k=1}^{N-1} P_k \sum_{n=0}^k B_+(n|k) x_+ \left(\frac{k-n}{k} \right)^q g_{n,k}^-. \quad (\text{D11b})$$

Consider Eq. (D11a), which is the rate at which the density of $+1$ nodes increase under our modified pairwise comparison dynamics. For this to happen an $(n, k, -1)$ node must be picked at random, this gives rise to the first three terms. q random neighbors in the state -1 must then be picked, which

happens with probability $(\frac{n}{k})^q$. The node must then decide to switch its state with probability $g_{n,k}^+$. Equation (D11b) has an analogous structure where an $(n, k, +1)$ node becomes and $(n, k, -1)$ node.

We can now substitute Eqs. (D11a) and (D11b) into Eqs. (D2) and (D8) to find

$$\frac{dx_+}{dt} = \sum_k P_k \sum_{n=0}^k \left[x_- \left(\frac{n}{k} \right)^q g_{n,k}^+ B_-(n|k) - x_+ \left(\frac{k-n}{k} \right)^q g_{n,k}^- B_+(n|k) \right], \quad (\text{D12a})$$

$$\frac{d\sigma}{dt} = \frac{2}{\mu} \sum_k P_k \sum_{n=0}^k \left[x_- \left(\frac{n}{k} \right)^q g_{n,k}^+ B_-(n|k) - x_+ \left(\frac{k-n}{k} \right)^q g_{n,k}^- B_+(n|k) \right] (k-2n). \quad (\text{D12b})$$

6. Analytically tractable limits

We will now evaluate Eqs. (D12a) and (D12b) in some analytically tractable limits to check their validity. These are the voter model (Appendix D 6 a) and the complete graph (Appendix D 6 b).

a. Voter model limit

The limit $q = 1$, $\beta = 0$ (since we absorbed β into u and v this means having $u = v = 0$) is the standard voter model and Eqs. (D12a) and (D12b) should reproduce known results from Ref. [58].

Equation (D12a) becomes

$$\begin{aligned} \frac{dx_+}{dt} &= \frac{1}{2} \sum_k P_k \sum_{n=0}^k \left\{ x_- \left(\frac{n}{k} \right) B_-(n|k) - x_+ \left(1 - \frac{n}{k} \right) B_+(n|k) \right\} \\ &= \frac{1}{2} \sum_k P_k \left\{ x_- \frac{\langle n \rangle_-}{k} - x_+ \left(1 - \frac{\langle n \rangle_+}{k} \right) \right\} \\ &= \frac{1}{2} \sum_k P_k \left\{ x_- \left(\frac{\sigma}{2x_-} \right) - x_+ \left(\frac{\sigma}{2x_+} \right) \right\} \\ &= 0, \end{aligned} \quad (\text{D13})$$

where $\langle n \rangle_s$ is the first moment of the binomial distribution $B_s(n|k)$ defined in Eq. (D3). So the average density of nodes in state +1 does not change with time as expected in the standard voter model.

Equation (D12b) becomes

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{1}{\mu} \sum_k P_k \sum_{n=0}^k \left\{ x_- \left(\frac{n}{k} \right) B_-(n|k) - x_+ \left(1 - \frac{n}{k} \right) B_+(n|k) \right\} \\ &\quad \times (k-2n) \\ &= \frac{1}{\mu} \sum_k P_k \left\{ x_- \left(\langle n \rangle_- - 2 \frac{\langle n^2 \rangle_-}{k} \right) - x_+ \left(k - 3 \langle n \rangle_+ + 2 \frac{\langle n^2 \rangle_+}{k} \right) \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\mu} \sum_k P_k \left\{ (k-2)\sigma - \frac{(k-1)\sigma^2}{2x_+(1-x_+)} \right\} \\ &= \frac{1}{\mu} \left\{ (\mu-2)\sigma - \frac{(\mu-1)\sigma^2}{2x_+(1-x_+)} \right\}, \end{aligned} \quad (\text{D14})$$

where $\langle n^2 \rangle_s$ is the second moment of the binomial distribution $B_s(n|k)$ defined in Eq. (D3). With the replacement $x_+ \rightarrow x$, and moving to the steady state we find

$$\sigma_{\text{st}} = 2 \left(\frac{\mu-2}{\mu-1} \right) x_{\text{st}} (1-x_{\text{st}}), \quad (\text{D15})$$

where x_{st} is the long-term stationary fraction of agents of type A. The prefactor describes the long-lived plateau of the density of active links reported in Ref. [58].

b. Complete graph limit

The degree distribution of a complete graph is a delta function peaked around N , i.e., $P_k \rightarrow \delta(k-N)$. Furthermore, the binomial distributions defined in Eq. (D3) are also delta functions peaked around Nx_+ , i.e., $B_s(n|k) \rightarrow \delta(n-Nx_+)$. This is because every node is connected and there are Nx_+ nodes in state +1, so the probability any node has Nx_+ neighbors is 1, and it is 0 for any other number of +1 neighbors.

With this, and using the notation $x_+ \rightarrow x$, Eq. (D12a) becomes

$$\dot{x} = (1-x)x^q g^+(x) - x(1-x)^q g^-(x), \quad (\text{D16})$$

where the standard Fermi function, Eq. (6), has reappeared due to the delta functions acting on $g_{n,k}^\pm$ [Eq. (D10)]. Thus, we recover Eq. (8), which applies for well-mixed populations.

Similarly, Eq. (D12b) becomes

$$\begin{aligned} \frac{d\sigma}{dt} &= 2(1-2x)[(1-x)x^q g^+(x) - x(1-x)^q g^-(x)] \\ &= 2(1-2x)\dot{x}, \end{aligned} \quad (\text{D17})$$

where we have used Eq. (D16). This is exactly as expected. As on an infinite complete graph with a fraction x nodes in state +1, there are $N^2 x(1-x)$ active links and $\frac{N^2}{2}$ links overall, thus $\sigma = 2(x-x^2)$. So we expected $\frac{d}{dt}\sigma = 2(1-2x)\dot{x}$, which is exactly what we see in Eq. (D17).

7. Degree-regular graphs

For degree-regular graphs (graphs in which all nodes have the same degree) Eqs. (D12a) and (D12b) simplify significantly. The degree distribution is simply a delta function peaked at $\mu \in \mathbb{Z}_{\geq 2}$, i.e., $P_k \rightarrow \delta(k-\mu)$, which collapses the outer summations over k , so we have

$$\begin{aligned} \frac{dx_+}{dt} &= \sum_{n=0}^{\mu} \left[x_- \left(\frac{n}{\mu} \right)^q g_{n,\mu}^+ B_-(n|\mu) - x_+ \left(\frac{\mu-n}{\mu} \right)^q g_{n,\mu}^- B_+(n|\mu) \right], \end{aligned} \quad (\text{D18a})$$

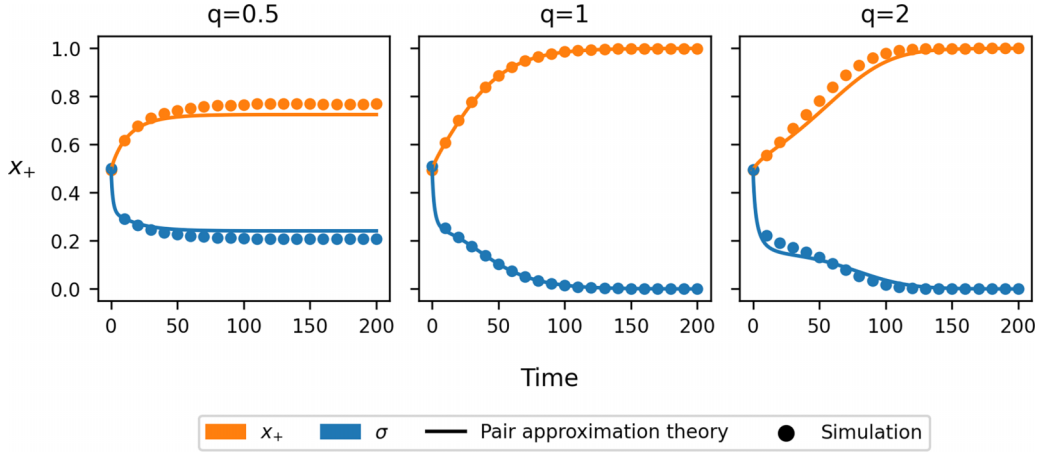


FIG. 16. Plot showing the time evolution of the density of nodes in state +1 x_+ (orange), and density of active links σ (blue) for different values of q , starting from a random state where $x_+ = 0.5$. The game parameters are $u = 0.1$ and $v = 0.1$. The underlying topology is a random degree-regular graph with $N = 10\,000$ and $\mu = 3$. Solid lines are analytical solutions from numerically integrating Eqs. (D18a) and (D18b). Circle markers are from averaging 100 independent Gillespie simulations of the dynamics.

$$\frac{d\sigma}{dt} = \frac{2}{\mu} \sum_{n=0}^{\mu} \left[x_- \left(\frac{n}{\mu} \right)^q g_{n,\mu}^+ B_-(n|\mu) - x_+ \left(\frac{\mu-n}{\mu} \right)^q g_{n,\mu}^- B_+(n|\mu) \right] (\mu - 2n). \quad (\text{D18b})$$

This pair of coupled differential equations can be numerically integrated and in Fig. 16 we show some example trajectories for a single random degree-regular graph, varying the parameter q . The approximation works fairly well given the coarse nature of the pair approximation.

To classify the dynamics for a given set of parameters we can look at multiple pair approximation trajectories for different initial densities of ± 1 type nodes, this is shown in Fig. 17. For $q = 0.5$ there is a stable fixed point at around $x_+ = 0.7$, thus the flow could be classified as coexistence type. This is in-line with the idea that small values of q promote the minority, as discussed for complete graphs in Appendix B. For $q = 1$ (standard game dynamics) there is a stable fixed point at $x_+ = 1$, which is unsurprising as the parameters chosen correspond to an A-dominance type flow for complete graphs (see

Appendix B). For $q = 2$ there appears to be an unstable fixed point just below $x_+ = 0.4$, thus this flow could be classified as coordination type. Again this is in line with Appendix B where we found large values of q promote the majority.

8. Nonregular graphs

The case of graphs which are not regular is more difficult. We use Eqs. (D12a) and (D12b) but we have to perform the summation over the degree distribution P_k .

We show in Figs. 18 and 19 analogous plots to Figs. 16 and 17 but now for Barabási–Albert and Erdős–Rényi graphs, respectively, to demonstrate the validity of the solution for different heterogeneous uncorrelated (or approximately uncorrelated) graphs.

We note that in Fig. 18 the analytical results agree very well with simulation, even more than for degree-regular graphs (Fig. 16). This is likely because in Fig. 16 $\mu = 3$, whereas in Fig. 18 $\mu = 8$. This form of pair approximation is known to breakdown at lower μ [46].

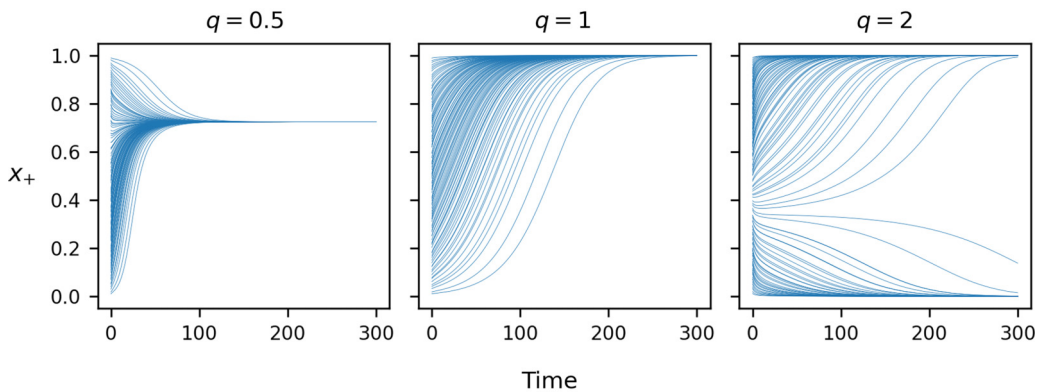


FIG. 17. Pair approximation trajectories from numerically integrating Eqs. (D18a) and (D18b) for different values of q and initial densities of ± 1 type nodes. The model parameters are $u = 0.1$ and $v = 0.1$. The theoretical underlying random degree-regular graph has $\mu = 3$ and is of infinite size.

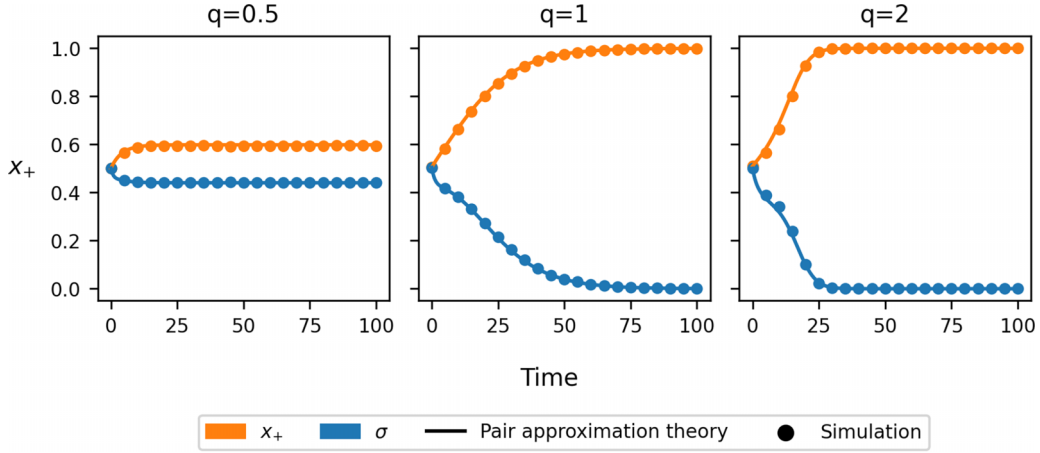


FIG. 18. Plot showing the time evolution of the density of nodes in state +1 x_+ (orange), and density of active links σ (blue) for different values of q , starting from a random state where $x_+ = 0.5$. The game parameters are $u = 0.1$ and $v = 0.1$. The underlying topology is a Barabási–Albert graph with $N = 10\,000$ where each node connects to 4 existing nodes so that $\mu = 8$. Solid lines are analytical solutions from numerically integrating Eqs. (D12a) and (D12b). Circle markers are from averaging 100 independent Gillespie simulations of the dynamics.

APPENDIX E: q -DEFORMED DYNAMICS FOR CYCLIC GAMES

We wish to determine the classification of the q -deformed dynamics for cyclic games [such as the rock-paper-scissors (RPS) game]. In this Appendix we study games defined by the payoff matrix

$$\begin{array}{c|cc} & R & P & S \\ \hline R & 0 & a & b \\ P & b & 0 & a \\ S & a & b & 0 \end{array}. \quad (\text{E1})$$

This is a generalization of Eq. (26). We study the dynamics for general $q \in \mathbb{R}^+$ and $a, b \in \mathbb{R}$.

The center of the strategy simplex $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is always a fixed point, as are the pure strategies at the corners of the simplex. We thus study the linear stability of these fixed points. We note that there are only two degrees of freedom in the system since $\sum_i x_i = 1$.

An accompanying Mathematica notebook with further details of the following calculations can be found at a GitHub repository [63].

1. The center point

Here we analyze the stability of the point $p_{\text{center}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The Jacobian of the dynamics in Eq. (25) (after reduction to two degrees of freedom) evaluated at the center is

$$\mathbf{J}_{\text{center}} = \begin{pmatrix} \frac{1}{2 \times 3^q} (3q - 3 - b) & \frac{1}{2 \times 3^q} (a - b) \\ -\frac{1}{2 \times 3^q} (a - b) & \frac{1}{2 \times 3^q} (3q - 3 - a) \end{pmatrix}. \quad (\text{E2})$$

The eigenvalues of this matrix are,

$$\lambda^{\pm} = \frac{1}{4 \times 3^q} [6q - 6 - (a + b) \pm i|a - b|\sqrt{3}]. \quad (\text{E3})$$

Assuming $a \neq b$, both eigenvalues are complex, and thus the point p_{center} can only be a center, spiral sink, or spiral source [64]. For centers, we require the real part of the eigenvalues to

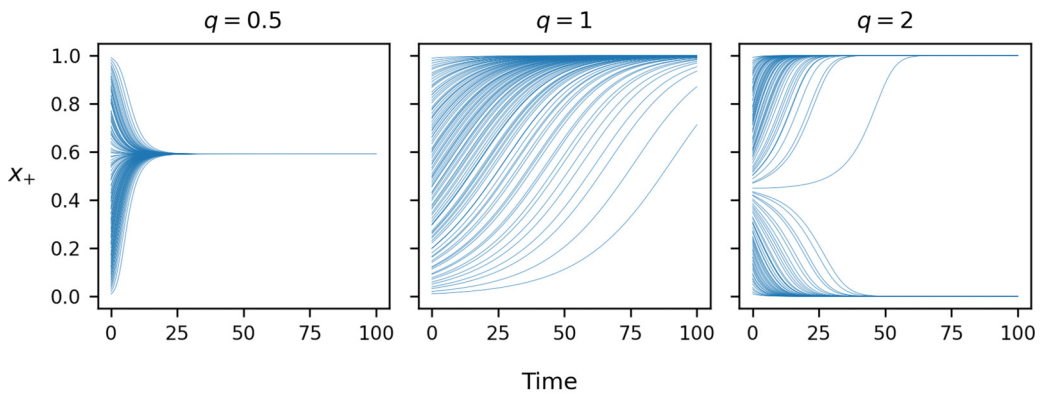


FIG. 19. Pair approximation trajectories from numerically integrating Eqs. (D12a) and (D12b) for different values of q and initial densities of ± 1 type nodes. The model parameters are $u = 0.1$ and $v = 0.1$. The underlying Erdős–Rényi graph has $N = 10\,000$ and $p = 8/N$ so that $\mu \approx 8$.

be zero, this occurs when

$$q = q_c \equiv 1 + \frac{a+b}{6}. \quad (\text{E4})$$

When $q < q_c$, the real part of both eigenvalues is negative, so p_{centre} will be a spiral sink. When $q > q_c$, the real part of both eigenvalues is positive, so p_{centre} will be a spiral source.

In the special case of $a = b$, we have a single real eigenvalue with algebraic multiplicity 2,

$$\lambda = \frac{1}{2 \times 3^q} (3q - 3 - a). \quad (\text{E5})$$

The eigenspace has geometric multiplicity 2, i.e., there are two linearly independent eigenvectors corresponding to this eigenvalue, namely, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This means that p_{centre} is a star, which is a source when $\lambda > 0$, i.e., $q > 1 + \frac{a}{3}$, and a sink when $\lambda < 0$, i.e., $q < 1 + \frac{a}{3}$. When $q = 1 + \frac{a}{3}$ we have a center once again.

For general a and b then, p_{centre} is some form of sink when $q < q_c$, some form of source when $q > q_c$, and a center when $q = q_c$.

2. The corners

Here we analyze the stability of the corners, p_{corner} , i.e., permutations of the point $(1, 0, 0)$. By symmetry all corners give the same results from stability analysis.

a. $q = 1$

When $q = 1$ the eigenvalues of $\mathbf{J}_{\text{corner}}$ are

$$\lambda_1 = \tanh\left(\frac{a}{2}\right), \quad (\text{E6a})$$

$$\lambda_2 = \tanh\left(\frac{b}{2}\right). \quad (\text{E6b})$$

Thus, both eigenvalues are real. If both $a, b > 0$, then the corners are **sources**. If both $a, b < 0$, then the corners are **sinks**. If a and b have opposite signs, then the corners are **saddle points**.

For $a = b = 0$ all entries in the payoff matrix are zero. There is thus no actual game dynamics, and the flow is solely determined by the q -deformation. For the case $q = 1$ (no deformation), which we are discussing here, this means that there is no dynamics at all ($\dot{x}_i = 0$ for all i).

b. $q > 1$

When $q > 1$ the eigenvalues of $\mathbf{J}_{\text{corner}}$ are

$$\lambda_1 = \frac{-1}{1 + e^a}, \quad (\text{E7a})$$

$$\lambda_2 = \frac{-1}{1 + e^b}. \quad (\text{E7b})$$

Thus, for all a, b both eigenvalues are real and negative, hence the corners are always **sinks**.

c. $0 < q < 1$

For any q the rate equations are (after reduction to two degrees of freedom, which we will call x and y , these are the proportions of two of the strategies):

$$\begin{aligned} \frac{dx}{dt} = x^q & \left[\frac{1-x-y}{1 + \exp\{a(x-y) - b(1-x-2y)\}} \right. \\ & + \frac{y}{1 + \exp\{a(1-x-2y) - b(1-2x-y)\}} \\ & - x \left[\frac{(1-x-y)^q}{1 + \exp\{-a(x-y) + b(1-x-2y)\}} \right. \\ & \left. \left. + \frac{y^q}{1 + \exp\{-a(1-x-2y) + b(1-2x-y)\}} \right] \right], \end{aligned} \quad (\text{E8a})$$

$$\begin{aligned} \frac{dy}{dt} = y^q & \left[\frac{1-x-y}{1 + \exp\{-a(1-2x-y) - b(x-y)\}} \right. \\ & + \frac{x}{1 + \exp\{-a(1-x-2y) + b(1-2x-y)\}} \\ & - y \left[\frac{(1-x-y)^q}{1 + \exp\{a(1-2x-y) + b(x-y)\}} \right. \\ & \left. \left. + \frac{x^q}{1 + \exp\{a(1-x-2y) - b(1-2x-y)\}} \right] \right]. \end{aligned} \quad (\text{E8b})$$

The right-hand sides evaluate to zero at the corners of the strategy simplex ($x = y = 0$, $x = 1$, $y = 0$, and $x = 0$, $y = 1$, respectively). Due to the symmetry with respect to interchange of types, it is sufficient to study the stability of the fixed point at one of the three corners, here we choose $x = y = 0$. For x and y small, but nonzero, and keeping in mind that we focus on $0 < q < 1$, the leading order terms on the right are proportional to x^q or y^q . Therefore, linear stability analysis cannot be used. When both x and y are small, Eqs. (E8a) and (E8b) reduce to

$$\frac{dx}{dt} = \frac{x^q}{1 + e^{-b}}, \quad (\text{E9a})$$

$$\frac{dy}{dt} = \frac{y^q}{1 + e^{-a}}. \quad (\text{E9b})$$

Therefore, both $\dot{x}, \dot{y} > 0$ for x, y small (but positive). This is the case for all a and b , so the corners are always **sources**.

3. Reduction to one-parameter family of payoff matrices

We have shown in Appendix E2 that the stability of the corners is independent of a and b , except when $q = 1$ where only the signs of a and b matter. In Appendix E1 we showed that $a + b$ determines the stability of the center.

In principle, our analysis allows for a complete classification of the flows (as far as linear stability analysis goes) for all values of a, b , and q .

To reduce the number of parameters we now fix $a = -1$ and focus on positive values of b . This reflects the scenario of a cyclic game. Each strategy is beaten by one other strategy, and in turns beats the remaining strategy. It is convenient to set $b = 1 + \delta$ (following Ref. [48]). In this way $\delta = 0$ reflects the

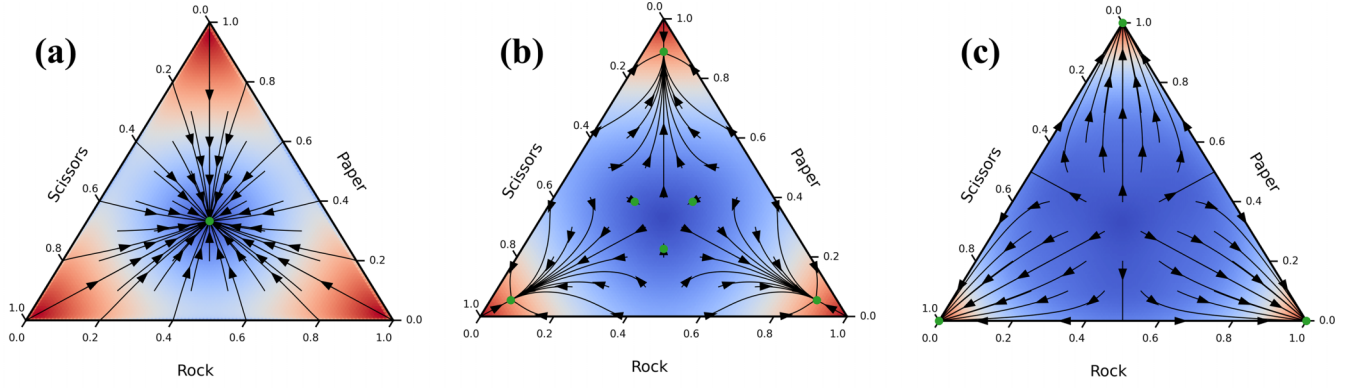


FIG. 20. Ternary plots for the three-strategy q -deformed evolutionary dynamics under the payoff matrix given in Eq. (26) for $\delta = -2$ and $q = [0.2, 0.7, 1.5]$ in panels (a), (b), and (c), respectively. Green markers show stable fixed points.

standard zero-sum rock-paper-scissors game. The constraint $b > 0$ translates into $\delta > -1$. We note that other (equivalent) setups have been used; see, for example, Ref. [49]. For this reduced setup Eq. (E4) becomes

$$q_c(\delta) = 1 + \frac{\delta}{6}. \quad (\text{E10})$$

The center fixed point is a stable spiral for $q < q_c(\delta)$ and an unstable spiral for $q > q_c(\delta)$. It is a center for $q = q_c(\delta)$.

The corners are sources for $q < 1$ and sinks for $q > 1$, as this is independent of δ . For $q = 1$ the corners are saddle points for all $\delta > -1$, otherwise they are sinks.

4. Further ternary plots

As mentioned in Appendix E 1, when $a = b$ (or in the reduced setup $\delta = -2$) the center point is a stable or unstable star depending on the value of q .

For $q < q_c(-2)$ the center point is a stable star. As shown in Appendix E 2 the corners in this region are sources. This leads to trajectories moving directly toward the center point as illustrated in Fig. 20(a)

For $q_c(-2) \leq q < 1$ the center point is now an unstable star. Again the corners in this region are sources. This leads to stable fixed points that are inside the simplex as illustrated by the green markers in Fig. 20(b). We note that stable internal fixed points could arise anywhere in the green region in Fig. 8, not just for $\delta = -2$. However, when δ moves further from -2 the center becomes a stronger spiral sink/source, which forces trajectories into stable orbits rather than to stable fixed points.

For $q \geq 1$ the center point is still an unstable star but the corners become sinks. This leads to trajectories moving directly away from the center point, converging at the corners as illustrated in Fig. 20(c).

APPENDIX F: FIXATION PROBABILITY FOR q -DEFORMED DYNAMICS WITHOUT REPLACEMENT FOR 2×2 GAMES

As in Appendix C 1 we want to calculate the fixation probability ϕ_i , except now for an adapted version of the model where we select an integer number q of neighbors without replacement.

The rates for the model without replacement are defined as follows:

$$T_i^+ = \begin{cases} 0, & 0 \leq i < q \\ (N-i)g_i^+ \prod_{k=1}^q \left(\frac{i-k+1}{N-k}\right), & q \leq i \leq N \end{cases}, \quad (\text{F1a})$$

$$T_i^- = \begin{cases} ig_i^- \prod_{k=1}^q \left(\frac{N-i-k+1}{N-k}\right), & 0 \leq i \leq N-q \\ 0, & N-q < i \leq N \end{cases}. \quad (\text{F1b})$$

Figure 21 illustrates the rates for the cases of $q \leq \frac{N}{2}$ and $q > \frac{N}{2}$. We see that it is often the case that the system is driven to one of the absorbing states at $i = 0$ or $i = N$. For $q \leq \frac{N}{2}$ and $q \leq i \leq N-q$ both T_i^+ and T_i^- are nonzero so the system can move in either direction. For $q > \frac{N}{2}$ and $N-q < i < q$ both T_i^+ and T_i^- are 0 so the system does not move. This is because we require at least q A or B agents in order for a node to choose q neighbors of the opposite type.

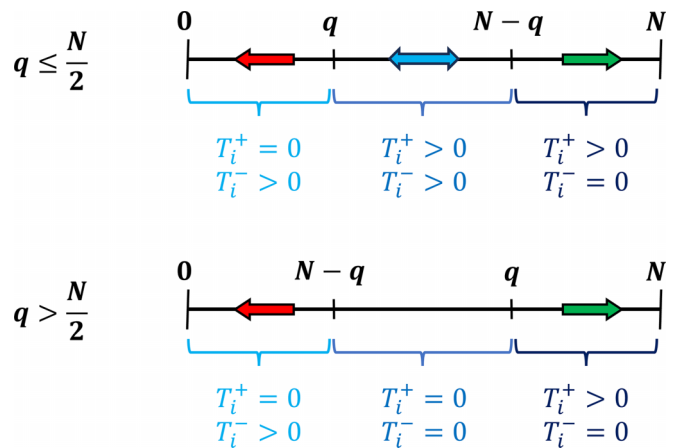


FIG. 21. Demonstration of how the transition rates from Eqs. (F1a) and (F1b) change based on the number of type A agents, i . Top panel shows the case $q \leq \frac{N}{2}$. For $i < q$ or $i > N-q$ the system is always driven to $i = 0$ or $i = N$, respectively. For $q \leq i \leq N-q$ both T_i^+ and T_i^- are finite so the system can move in either direction. The bottom panel shows the case for $q > \frac{N}{2}$. If $i \leq N-q$ or $i \geq q$, then the system is always driven to $i = 0$ or $i = N$, respectively. For $q < i < N-q$ both T_i^+ and T_i^- are 0 so the system does not move.

For the case $q \leq \frac{N}{2}$, the probability to fixate with all agents being A type is trivially 0 if $i < q$ and 1 if $i > N - q$. For $q \leq i \leq N - q$ we write the recursive expression

$$\phi_i = T_i^- \phi_{i-1} + T_i^+ \phi_{i+1} + (1 - T_i^- - T_i^+) \phi_i. \quad (\text{F2})$$

Rearranging this equation, and defining $y_i = \phi_i - \phi_{i-1}$, we find $y_{i+1} = \gamma_i y_i$, where $\gamma_i = T_i^- / T_i^+$ is the ratio of the transmission rates. Unlike in the model with replacement [Eq. (C4)], γ_i has a piecewise definition:

$$\gamma_i = \begin{cases} \infty, & 0 \leq i < q, \\ \frac{i}{N-1} \frac{g^-(i)}{g^+(i)} \prod_{k=1}^q \left(\frac{N-i-k+1}{i-k+1} \right), & q \leq i \leq N - q, \\ 0, & N - q < i \leq N \end{cases} \quad (\text{F3})$$

Noting that ϕ_q is the smallest nonzero ϕ , we can write

$$\begin{aligned} y_q &= \phi_q - \phi_{q-1} = \phi_q, \\ y_{q+1} &= \gamma_q y_q = \gamma_q \phi_q, \\ &\vdots \\ y_k &= \phi_q \prod_{j=q}^{k-1} \gamma_j, \end{aligned} \quad (\text{F4})$$

for $k \geq q$. Now, assuming $i \geq q$ we can manipulate ϕ_i into the following form:

$$\begin{aligned} \phi_i &= (\phi_q - \phi_{q-1}) + (\phi_{q+1} - \phi_q) + \dots + (\phi_i - \phi_{i-1}) \\ &= y_q + y_{q+1} + \dots + y_i \end{aligned}$$

$$\begin{aligned} &= \sum_{k=q}^i y_k \\ &= \phi_q \sum_{k=q}^i \prod_{j=q}^{k-1} \gamma_j, \end{aligned} \quad (\text{F5})$$

noting the first line is possible as all values other than ϕ_{q-1} , which is zero anyway, cancel. In going from the third to fourth line we use Eq. (F4). To determine ϕ_q we use the largest ϕ_i that we know to be equal to one, i.e., ϕ_{N-q+1} ,

$$\begin{aligned} \phi_{N-q+1} &= 1 = \sum_{k=q}^{N-q+1} y_k = \phi_q \sum_{k=q}^{N-q+1} \prod_{j=q}^{k-1} \gamma_j, \\ \Rightarrow \phi_q &= \frac{1}{\sum_{k=q}^{N-q+1} \prod_{j=q}^{k-1} \gamma_j}. \end{aligned} \quad (\text{F6})$$

Substituting Eq. (F6) into Eq. (F5) we get an expression for the fixation probability in the intermediate regime,

$$\phi_i = \frac{\sum_{k=q}^i \prod_{j=q}^{k-1} \gamma_j}{\sum_{k=q}^{N-q+1} \prod_{j=q}^{k-1} \gamma_j}. \quad (\text{F7})$$

When $q > \frac{N}{2}$ all cases are trivial. $i \leq N - q$ and $i \geq q$ give ϕ_i as 0 or 1, respectively, as the system is forced to the absorbing states. The intermediate case of $N - q < i < q$ gives $\phi_i = 0$, as the system cannot move. Overall then for $q > \frac{N}{2}$ we have $\phi_i = 0$ for $0 \leq i < q$, and $\phi_i = 1$ for $q \leq i \leq N$.

Combining all results we have for the fixation probability:

$$\phi_i = \begin{cases} 0, & (q \leq \frac{N}{2} \text{ and } 0 \leq i < q) \text{ or } (q > \frac{N}{2} \text{ and } 0 \leq i < q) \\ \frac{\sum_{k=q}^i \prod_{j=q}^{k-1} \gamma_j}{\sum_{k=q}^{N-q+1} \prod_{j=q}^{k-1} \gamma_j}, & q \leq \frac{N}{2} \text{ and } q \leq i \leq N - q \\ 1, & (q \leq \frac{N}{2} \text{ and } N - q < i \leq N) \text{ or } (q > \frac{N}{2} \text{ and } q \leq i \leq N) \end{cases} \quad (\text{F8})$$

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