

Reservoir-induced stabilization of a periodically driven many-body systemThomas Veness and Kay Brandner *School of Physics and Astronomy, University of Nottingham, Nottingham NG7 2RD, United Kingdom*

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Exploiting the rich phenomenology of periodically driven many-body systems is notoriously hindered by persistent heating in both the classical and the quantum realm. Here, we investigate to what extent coupling to a large thermal reservoir makes stabilization of a nontrivial steady state possible. To this end, we model both the system and the reservoir as classical spin chains where driving is applied through a rotating magnetic field, and we simulate the Hamiltonian dynamics of this setup. We find that the intuitive limits of infinite frequency and vanishing frequency, where the system dynamics is governed by the average and the instantaneous Hamiltonian, respectively, can be smoothly extended into entire regimes separated only by a small crossover region. At high frequencies, the driven system stroboscopically attains a Floquet-type Gibbs state at the reservoir temperature. At low frequencies, a global synchronized Gibbs state emerges, whose temperature may depart significantly from the initial temperature of the reservoir. Although our analysis in some parts relies on the specific properties of our setup, we argue that much of its phenomenology could be generic.

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Introduction. Our understanding of periodically driven quantum systems is largely facilitated by *Floquet's theorem* [1–3], which implies that the explicit time-dependence of Schrödinger's equation can be removed by means of a unitary basis transformation with the same periodicity as the drive [4,5]. Hence, the stroboscopic time evolution of such a system is generated by a time-independent Hermitian operator, the Floquet Hamiltonian, whose properties can be tailored through the applied driving protocol. This result opened the field of Floquet engineering [6] and led to the discovery of novel phenomena with no static counterpart such as anomalous Floquet topological insulators [7,8] or so-called time crystals [9–11]. The Floquet Hamiltonian of a many-body system is generally a complicated object, which in most cases can be determined only via approximate methods. Its eigenstates, which govern the long-time behavior of the system, are generically superpositions of its instantaneous energy eigenstates with quasihomogeneously distributed coefficients [12,13]. Combining this observation with the notion that energy eigenstates typically display thermal behavior, i.e., the *eigenstate thermalization hypothesis* [14–17], leads to an ensemble with no conservation laws [18,19]. That is, a generic many-body system continuously absorbs energy until the statistics of all observables are described by a structureless *infinite-temperature ensemble* [20–23]. In practice, this behavior confines Floquet engineering to the high-frequency regime, where heating is exponentially suppressed, or to special systems where thermalization is hindered by other means such as many-body localization [24–28]. Floquet theory does

not apply to systems obeying nonlinear dynamics. A variety of techniques used in analyzing periodically driven quantum systems such as high-frequency expansions may, however, be naturally adopted in a classical Hamiltonian framework [4] and much phenomenology may persist. Nonetheless, systematic investigations of classical systems have begun only recently, despite greater numerical accessibility of larger system sizes and longer evolution times than in the quantum case [4,29–33].

Here, we exploit this advantage to explore *reservoir-induced thermalization* as an approach to avoid overheating in periodically driven many-body systems. For concreteness, we consider a classical spin chain with weak disorder and a magnetic drive applied to a finite region, which we refer to as the *system proper*, while the remainder of the system acts as a *reservoir* (see Fig. 1). At high driving frequencies, the system proper is well-described by the time-averaged Hamiltonian, up to perturbative corrections in the inverse driving frequency; this behavior is qualitatively insensitive to the presence of the reservoir. In the quasistatic limit, the system relaxes to the instantaneous Hamiltonian before it changes appreciably. At any finite frequency, however, the existence and nature of a stroboscopic steady state are *a priori* unclear. Our main finding is that the reservoir stabilizes a nontrivial steady state over a wide range of frequencies and on an intermediate timescale, which is determined by the strength of the disorder and can in principle be arbitrarily long [34]. This steady state smoothly extends into the quasistatic limit and is connected to the high-frequency regime by a sharp crossover, whose width is set by the interaction strength between neighboring spins (see Fig. 1). As we show, this state is described by an effective Gibbs ensemble, whose temperature can deviate significantly from the initial temperature of the reservoir. It emerges through synchronization between the drive and the system proper, while synchronization with the reservoir leads

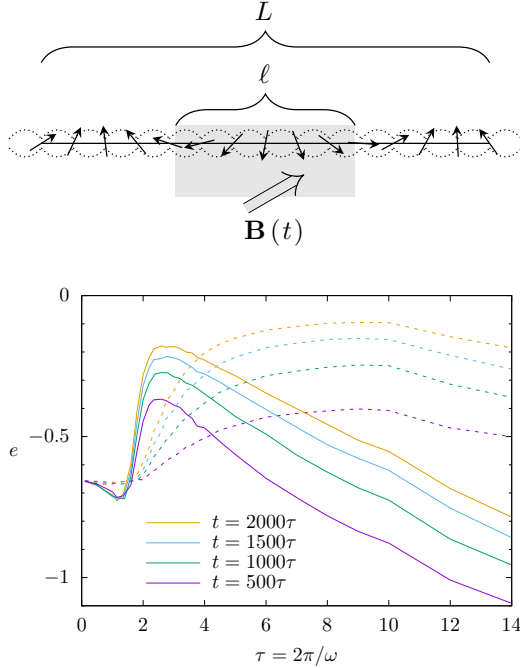


FIG. 1. Top: Sketch of the system. A classical spin chain with nearest-neighbor interactions, finite disorder, and total number of sites L , ℓ of which are exposed to a time-periodic magnetic field $\mathbf{B}(t)$. Bottom: Mean energy density of the driven system as a function of the period τ after the exposure time t . While nearly no energy is absorbed for $\tau \lesssim 2$, a reservoir-stabilized steady state with significantly reduced energy density emerges for $\tau \gtrsim 4$, before disorder-induced heating prevails (solid, $L = 2000$). The isolated system rapidly heats up for $\tau \gtrsim 2$ (dashed, $L = \ell$). For all plots we have set $\ell = 20$, $\delta J = 2 \times 10^{-2}$, $e_{\text{initial}} = -0.66$, and averaged over $\mathcal{N} = 8000$ initial states and disorder realization.

to a significant suppression of overall energy absorption [34]. This behavior is in clear contrast with the intuitive expectation of the reservoir acting as a mere energy sink, whose state is invariant up to finite-size corrections. In the following, we explicitly construct both high- and low-frequency ensembles, which describe all local observables of the driven system, leaving only the crossover region unexplained by a steady state.

Setup. We consider the setup of Fig. 1. The ℓ driven sites form the system proper, and the remaining $L - \ell$ sites form the reservoir, where we are particularly interested in the regime $\ell \ll L$. The dynamical variables are classical spin vectors normalized such that $|\mathbf{S}_j|^2 = 1$. The Hamiltonian of the entire system is

$$\mathcal{H}(t) = - \sum_{j=1}^L \mathbf{S}_j^\top J_j \mathbf{S}_{j+1} + \sum_{j=1}^{\ell} \mathbf{B}(t) \cdot \mathbf{S}_j. \quad (1)$$

We assume periodic boundary conditions, i.e., $\mathbf{S}_{L+1} = \mathbf{S}_1$. The J_j are diagonal matrices whose entries J_j^α are independently and identically distributed and drawn from a normal distribution with mean $J = 1$, which we use as our energy scale throughout, and variance δJ . The driving $\mathbf{B}(t) = J(\cos \omega t, \sin \omega t, 0)^\top$ is a rotating planar magnetic field with the period $\tau = 2\pi/\omega$. We choose δJ to be small but nonzero

to ensure that there are no conserved quantities, which would constrain the dynamics. The time evolution of the system is determined by Hamilton's equations of motion, $\frac{d\mathbf{f}}{dt} = \frac{\partial \mathbf{f}}{\partial t} + \{\mathbf{f}, \mathcal{H}\}$, where the Poisson bracket for spin degrees of freedom is determined by the Lie algebra of $\text{SO}(3)$, $\{S_j^\alpha, S_k^\beta\} = \delta_{jk} \varepsilon^{\alpha\beta\gamma} S_j^\gamma$. Accordingly, the microscopic equations of motion are

$$\begin{aligned} \frac{d\mathbf{S}_j}{dt} &= -\boldsymbol{\Omega}_j \times \mathbf{S}_j, \\ \boldsymbol{\Omega}_j &= J_{j-1} \mathbf{S}_{j-1} + J_j \mathbf{S}_{j+1} - \begin{cases} \mathbf{B}(t), & 1 \leq j \leq \ell, \\ 0, & \ell < j \leq L. \end{cases} \end{aligned} \quad (2)$$

We choose our initial state as a global equilibrium state with zero magnetic field. Initial conditions are sampled using standard Metropolis-Hastings Monte Carlo (MC) methods from the Gibbs distribution

$$\mathcal{P}_0 = e^{-\beta \mathcal{H}_0} / \mathcal{Z}_0, \quad (3)$$

with $H_0 = - \sum_{j=1}^L \mathbf{S}_j^\top J_j \mathbf{S}_{j+1}$ [35]. Throughout this article, we set $\beta = 2.888$, which corresponds to the initial energy density $\langle \mathcal{H}_0 \rangle_0 / L = -0.66$. The dynamics are numerically realized via symplectic integration which manifestly conserves spin normalization [36]. Unless otherwise stated, presented results are averages over a large number of initial states and disorder realizations. We confirm in Ref. [34] that, in the absence of driving and with small $\delta J = 10^{-3}$, time and ensemble averages are indeed equivalent; i.e., the free system is ergodic.

We now explore the existence of a steady-state through observables of the system proper, specifically focusing on magnetization and energy density, given by

$$\begin{aligned} \mathbf{m} &= \frac{1}{\ell} \sum_{j=1}^{\ell} \mathbf{S}_j, \\ e &= - \frac{1}{\ell - 1} \sum_{j=1}^{\ell-1} \mathbf{S}_j^\top J_j \mathbf{S}_{j+1} + \mathbf{m} \cdot \mathbf{B}(t). \end{aligned} \quad (4)$$

In Fig. 1, we plot the stroboscopic energy density $e(t = n\tau)$ after a large number of cycles n , where we set $\delta J = 2 \times 10^{-2}$ such that heating is observable on the timescales considered. To gain a clearer theoretical picture of the different regimes, we henceforth take $\delta J = 10^{-3}$ in order to increase the separation of timescales between heating and relaxation to the intermediate steady state.

High-frequency regime. When the system is driven at a frequency of $\omega \gg J$, we expect its physics to be well approximated by the time-averaged Hamiltonian, with corrections scaling as powers of ω^{-1} , given by the classical Floquet-Magnus expansion [4, 37–39]. Note that we formally set $\hbar = 1$ such that the spins are dimensionless and energies and inverse timescales have the same dimension. Such expansions are generally asymptotic in ω^{-1} [43]. More precisely, rigorous recent results on the classical limit of quantum spin chains demonstrate that the Floquet-Magnus Hamiltonian approximates a conserved quantity as $\omega \rightarrow \infty$, although

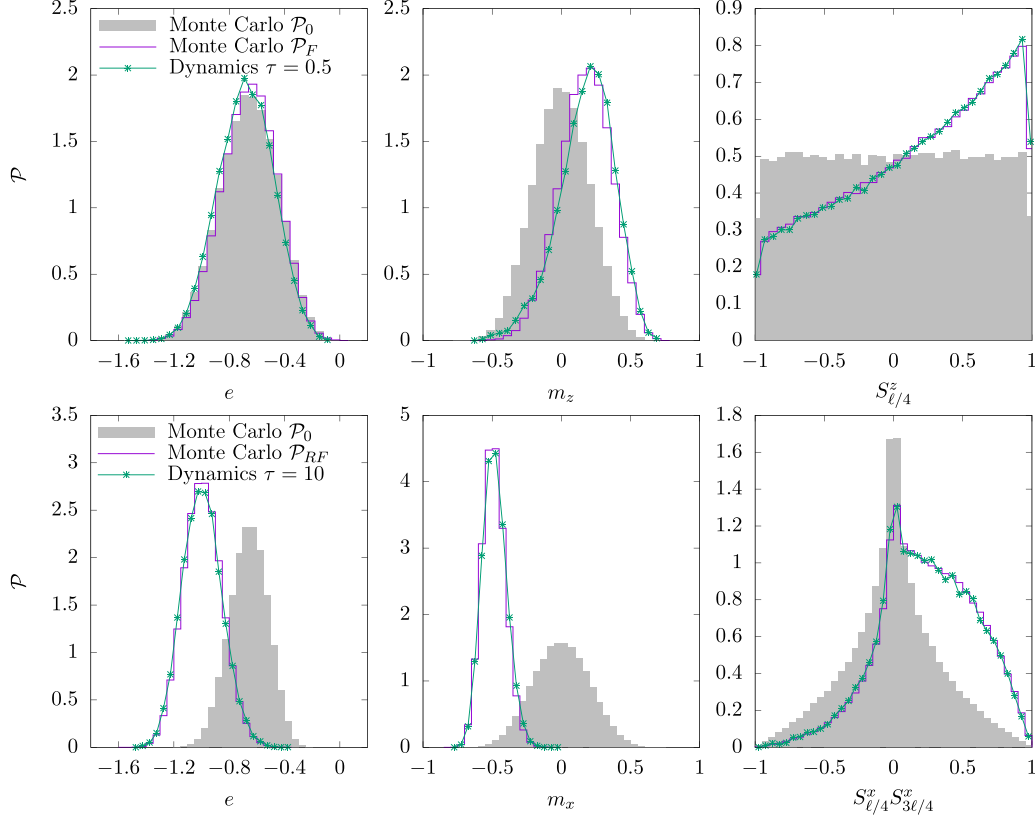


FIG. 2. Probability density histograms of stroboscopic observables for dynamical evolution and statistical descriptions, with $\ell = 20$, $L = 2000$, $\delta J = 10^{-3}$, and β chosen such that $e_{\text{initial}} = -0.66$. Here e , m_x , m_z , and m_x are determined by Eq. (4), $S_{\ell/4}^z$ is the z component of the spin at site $j = \ell/4$, and $S_{\ell/4}^x S_{3\ell/4}^x$ is the product of the x components of the spins at sites $j = \ell/4$ and $j = 3\ell/4$, highlighting that spatial correlations are also captured by the stroboscopic low-frequency ensemble of Eq. (10). For comparison, the initial ensemble $\mathcal{P}_0$ of Eq. (3) is shown in gray. Top panels: Fast-driving regime, $\tau = 0.5$, where purple curves show MC sampling from Eq. (6) with $\beta_F = \beta$ and $M = 128\,000$ samples. Dynamics are sampled at $t = 2000\tau$ from $\mathcal{N} = 180\,000$ trajectories. Bottom panels: Slow-driving regime, $\tau = 10$, where purple curves show MC sampling from Eq. (10) with β_{rot} determined from Eq. (11) and $M = 256\,000$ samples. Dynamics are sampled at $t = 1000\tau$ from $\mathcal{N} = 24\,500$ trajectories.

the stroboscopic evolution of observables generated by $\mathcal{H}_F$ is not guaranteed to converge to those of the true time-dependent evolution [44]. The leading order gives the Floquet Hamiltonian

$$\mathcal{H}_F^{(1)} = - \sum_{j=1}^L \mathbf{S}_j^\top J_j^{\text{HF}} \mathbf{S}_{j+1} - \frac{J^2}{2\omega} \sum_{j=1}^{\ell} \hat{\mathbf{z}} \cdot \mathbf{S}_j + O(\omega^{-2}), \quad (5)$$

with the modified couplings $J_j^{\text{HF}} = Q_j^\top J_j Q_{j+1}$, where $Q_j = R_y(J/\omega)$ represents rotation around the y axis by J/ω for $j = 1, \dots, \ell$ and the identity otherwise [4]. In the absence of other conserved quantities, statistical mechanics leads us to posit a stroboscopic Floquet-Gibbs ensemble of the form

$$\mathcal{P}_F = e^{-\beta_F \mathcal{H}_F} / \mathcal{Z}_F, \quad (6)$$

where $\mathcal{H}_F$ is an appropriately truncated Floquet-Magnus expansion and β_F is an effective inverse temperature. For our purposes, we find it sufficient to set $\mathcal{H}_F = \mathcal{H}_F^{(1)}$ [45,46]. As $\omega \gg J$, energy absorption is significantly suppressed, and we expect that $\beta_F \approx \beta$. We note that, as the full system is closed, one would in principle have to use a microcanonical ensemble. However, here we prefer the technically simpler canonical ensemble, which should be equivalent for any local observables

if $L \gg 1$. Figure 2 shows that such an ensemble reproduces the statistics of system proper observables well for $\tau = 0.5$, where the leading correction in ω^{-1} is necessary to capture the asymmetry in observables involving S_j^z and higher-order corrections can be safely neglected [34].

Low-frequency regime. We observe that our model admits an almost-static description in the rotating frame defined by introducing new variables $\tilde{\mathbf{S}}_j = R(t) \mathbf{S}_j$, where

$$R(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

The microscopic equations of motion become

$$\begin{aligned} \frac{d\tilde{\mathbf{S}}_j}{dt} &= -\tilde{\mathbf{\Omega}}_j \times \tilde{\mathbf{S}}_j + O(\delta J) = \{\tilde{\mathbf{S}}_j, \mathcal{H}_{\text{rot}}\} + O(\delta J), \\ \tilde{\mathbf{\Omega}}_j &= J\tilde{\mathbf{S}}_{j-1} + J\tilde{\mathbf{S}}_{j+1} + \begin{cases} \omega\hat{\mathbf{z}} - J\hat{\mathbf{x}}, & 1 \leq j \leq \ell, \\ \omega\hat{\mathbf{z}}, & \ell < j \leq L. \end{cases} \end{aligned} \quad (8)$$

Here, we have neglected time-dependent corrections of order δJ and introduced the rotating-frame Hamiltonian

$$\mathcal{H}_{\text{rot}} = - \sum_{j=1}^L [\tilde{\mathbf{S}}_j \cdot \tilde{\mathbf{S}}_{j+1} + \omega \tilde{\mathbf{z}} \cdot \tilde{\mathbf{S}}_j] + \sum_{j=1}^{\ell} \hat{\mathbf{x}} \cdot \tilde{\mathbf{S}}_j. \quad (9)$$

The autonomous dynamics generated by $\mathcal{H}_{\text{rot}}$ are stroboscopically equivalent to the dynamics of the drive system in the limit $\delta J \ll J$. In this global Floquet picture, statistical mechanics implies that the total system should settle to the canonical ensemble

$$\mathcal{P}_{\text{rot}} = e^{-\beta_{\text{rot}} \mathcal{H}_{\text{rot}}} / \mathcal{Z}_{\text{rot}}, \quad (10)$$

where β_{rot} is an effective inverse temperature.

Two remarks are in order here. First, although we neglect order δJ corrections to the rotating-frame Hamiltonian, which can be calculated systematically by means of a Floquet-Magnus expansion [34], we still assume that the disorder makes the system ergodic.

On a timescale slower than the inverse heating rate, $\mathcal{H}_{\text{rot}}$ is approximately conserved, and so the Lagrange multiplier β_{rot} may be fixed by requiring that the expectation value of $\mathcal{H}_{\text{rot}}$ in the final ensemble of Eq. (10) is the same as in the initial ensemble of Eq. (3), i.e.,

$$\langle \mathcal{H}_{\text{rot}} \rangle_0 = \langle \mathcal{H}_{\text{rot}} \rangle_{\text{rot}}. \quad (11)$$

In the limit $L \gg \ell$, both of these averages are dominated by the reservoir and become independent of ℓ [34]. Thus, for any given ω , Eq. (11) defines a one-to-one correspondence between the effective temperature and the initial temperature of the reservoir, $\beta_{\text{rot}} = \beta_{\text{rot}}(\beta, \omega)$, where instantaneous equilibration is recovered in the quasistatic limit, i.e., $\lim_{\omega \rightarrow 0} \beta_{\text{rot}}(\beta, \omega) = \beta$. We show in Fig. 2 that the ensemble of Eq. (10) convincingly reproduces dynamical results after 1000 cycles. These results underpin our main insight here: At low and intermediate frequencies, although being applied only locally, the driving gradually affects the full system. The reservoir thereby synchronizes with the system proper and an equilibrium state with low net energy absorption emerges regardless of the system size [34]. This collective behavior, which is mediated only by short-range interactions, is in stark contrast with the conventional notion of an ideal heat sink, whose state should remain unchanged in the thermodynamic limit. To make this disparity explicit, we show in Fig. 3 that the above phenomenology cannot be reproduced with a generic local-dissipation model, where the state of the reservoir is effectively held fixed.

Perspectives. Our results provide strong evidence that effective equilibrium states may be stroboscopically established in a periodically driven open many-body system over a wide range of frequencies. The limits $\omega \rightarrow 0$ and $\omega \rightarrow \infty$ match physical intuition, and we have demonstrated that even leading-order correction in ω^{-1} and ω , respectively, are sufficient to account for all but a small frequency window around $\tau \approx 2$. This crossover regime may still be described by a nonequilibrium steady state featuring persistent currents. A natural mechanism for generating such currents would be an emerging phase lag in the synchronization that governs

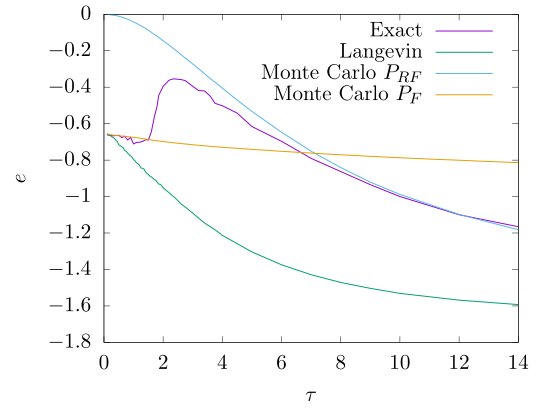


FIG. 3. Energy density of the system proper ($\ell = 20$) after 1000 driving cycles as obtained from microscopic simulations ($L = 2000$) and a local-dissipation model, where the reservoir is described through damping with strength γ_s and Gaussian thermal with strength $\mu_s = 2\gamma_s/\beta$ acting on the boundary spins ($\gamma_s = 1$) (for details, see Ref. [34]). For comparison, we show the energy density of the system proper in the ensembles (6) and (10), where β_{rot} is fixed by the condition (11). We have set $\delta J = 10^{-3}$ throughout. For the dynamical models, averages over $\mathcal{N} = 8000$ trajectories were calculated; ensemble curves show averages over $M = 8000$ MC samples.

the low-frequency regime. Hence, an increasing inability of the reservoir to follow the driven system could be the physical origin for the breakdown of the low-frequency ensemble which, on dimensional grounds, must occur at $\omega/J \sim 1$. The existence of a rotating-frame Hamiltonian, which provides a global conserved quantity and thus ultimately enables the synchronization between the system proper and the reservoir, is contingent on our choice of the drive and the interaction between the spins. We nonetheless generically expect relaxation to an instantaneous Gibbs ensemble in the quasistatic limit $\omega \rightarrow 0$, where the instantaneous Hamiltonian plays the role of an adiabatic invariant. It would be interesting to pursue a perturbative construction of analogous invariants away from this limit. Although numerically accessible, the analytic construction of such ensembles is likely not straightforward. However, by numerically evaluating the response functions entering fluctuation-dissipation relations, it is in principle possible to verify that a system is in equilibrium without specific knowledge of its state. It would also be instructive to explore how the physics that we have uncovered here survives or is modified in higher dimensions, where the same theoretical manipulations are in principle possible.

The source code used for all simulations, and all data used in figures, is freely available from GitHubRef. [47].

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