

Asymmetric cyclic interactions lead to stable mean-field equilibria while demographic noise induces sustained oscillations

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(Received 21 January 2026; revised 6 July 2026; accepted 24 September 2026; published 16 October 2026)

The dynamics of symmetric cyclically competing subpopulations is well known to lead to oscillatory dynamics. However, in some situations the competition is fundamentally asymmetric in that it is better modeled by binary interactions all of which involve the same subpopulation. Here we analyze a conceptual model of this second kind, motivated by design principles for communications networks where binary interactions inform decisions made by individual transmitters according to their competing operating policies, which may be to reduce overall network energy consumption, to improve coverage, or to limit interference between transmitters. We show that the mean-field dynamics in the asymmetric case always converges to equilibrium and cannot generate periodic oscillations. However, for a large but finite population of interacting agents, the intrinsic noise generated by the stochastic dynamics contains a dominant positive frequency component. We also analyze the robustness of the system to cyclically competing switching.

DOI: [10.1103/9tjx-cw7q](https://doi.org/10.1103/9tjx-cw7q)

I. INTRODUCTION

The existence of novel macroscopic system-scale dynamical phenomena that emerge as a result of microscale competition between individuals in a population is a key theme in complexity science and there is a substantial body of literature in applied mathematics, stretching from quantum phenomena, to mathematical ecology and epidemiology, to mathematical models for social science problems; for example, the voter model [1–4]. In many cases the dynamics can be well described by a set of state-dependent probabilistic rules, which in the simplest case are independent of past history and therefore are Markovian in character. Further, if individuals have only a discrete set of possible states, and the system is well mixed and interactions between any pair of individuals are equally likely, then the state of the system overall can be characterized completely in terms of the numbers of individuals in each state, and there is a substantial body of theory available to analyze the dynamics. Such systems behave in ways similar to chemical reactions and the (continuous time, discrete space) Markov process can be written out explicitly, starting from a “chemical master equation” that frames the dynamics as a set of hypothetical chemical reactions between different “chemical species.”

In terms of the states of individuals, these interactions between pairs of individuals in different states are often

hierarchical (e.g., in predator-prey networks or food webs) but there are intriguing examples where no such hierarchy can be defined. The most well-known example of this is the rock-paper-scissors (RPS) game and its variants, which, perhaps surprisingly, also has ecological relevance for particular species [5–7]. Very similar cyclical (also referred to as “competitive,” “nontransitive,” or “intransitive”) interactions have been found to arise in fluid dynamics [8–10] and have been the subject of substantial mathematical interest [11,12]. Mathematical analysis has revealed how the RPS game, and extensions of it, lead to a range of oscillatory dynamics, including the formation of robust heteroclinic cycles (RHCs) [13–17]. RHCs have mostly been studied in mean-field models, i.e., analysis of ordinary differential equations, but the influence of demographic (or “intrinsic”) noise that is present in large but finite populations is also of considerable interest, as is spatial variation and the generation of traveling waves and spiral patterns that results from spatially extended versions of mean-field models [18,19].

Here we bring this mathematical approach to bear on an example of cyclical interactions in another context: interactions between wireless communication transmitters. This is motivated by modern developments in wireless communication networks, in particular the broad characteristics of *self-organizing networks* (SONs) in which individual base stations sense their radio environment and adjust their behavior in response. The decision-making algorithms employed by an individual base station are referred to as their policies. Policies may differ between manufacturers and these details may be kept commercially confidential; as a result a key requirement in the design of 5G (and now 6G) communication networks is to allow other network components to be able to respond

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flexibly without requiring detailed knowledge of the operating characteristics of all other possible devices.

The SON design paradigm [20–22] has the aim of allowing a communications network to operate in a way that is robust to failures, and self-adaptive to current traffic conditions. It may be summarized as having three distinct parts, corresponding to separate phases of network operation: (1) self-configuration when a network is switched on, upgraded, or otherwise reconfigured; (2) self-optimization during the regular operation of the network, as the network adjusts to its radio environment and decides how to allocate the available resources; and (3) self-healing when unexpected external events trigger a requirement for a substantial resource reallocation.

A SON is a broad set of principles, under which heading many different procedures may be classified. But an essential common feature is that they are all *distributed heuristics*; that is, there is no single central control node, but instead many separate local decisions are taken by individual agents operating in different physical parts of the network. This ensures scalability to large systems, but comes with the risk of conflicting decisions by distinct agents, or other types of instability. This motivates a mathematical analysis of the dynamics of the interacting agents.

A key quantity in evaluating the quality of a network's performance is the signal to interference and noise ratio (SINR). The SINR at receiver i , receiving signal S_i from its paired transmitter and unwanted signals of strength I_j along with noise of variance σ^2 is defined to be $\text{SINR} = S_i/(\sigma^2 + \sum_j I_j)$. Low SINR causes the signal that is received to be undecodable and hence disruption to communication occurs; hence, in spatially extended settings the problem of matching transmitters to individual devices becomes important [23]; in contrast here we consider a dense region where the overall network load is (potentially) distributed across every base station.

We consider a wireless communication network consisting of N transmitters, where we take N to be large. The dynamics are assumed independent from the spatial configuration or distribution of transmitter locations. For simplicity, each transmitter has a choice of three available power settings: high, H ; low, L ; and off, O . We examine the consequences of three common competing policy aims:

- (i) provide sufficient network capacity (coverage) to the underlying region,
- (ii) reduce total power consumption by the network, and
- (iii) increase SINR as far as possible.

A higher power setting increases coverage yet may also reduce the SINR for other communication channels operating in this same dense region. Reducing the power setting from high to low may improve the SINR but may reduce the network capacity such that the energy-efficient consequence would be to switch such a low-power transmitter off completely, since there will be a fixed energy cost to operating a transmitter even at low power. Further, transmitters that are switched off do not contribute at all to the ability of the network to provide sufficient coverage, and this may lead them to be switched back on, which here we assume for simplicity takes place at full power.

These interactions therefore correspond to one of the many possible challenges of the self-optimization aspect of SON.

An overriding need is to improve energy efficiency, not only because network traffic is constantly increasing, but also because new features currently being proposed for 6G threaten to require yet more energy [24,25].

These various changes of state of an individual transmitter are driven by information about the local radio environment that is collected from paired transmitters and receivers, typically individual mobile phones or Internet of Things devices, often termed “user equipment” (UE), and then, as described by our conceptual model, shared between pairs of transmitters in the network. Here we assume that there is a sufficiently large amount of UE that its mean-field characteristics are all that is required, and it can be modeled as driving switches of base station state at constant rates, as the base stations test out the competing requirements (i)–(iii) set out above.

Our focus here is on the possible mean-field dynamics for the set of transmitters (also often referred to as base stations), and the fluctuations around that in the case of a large but finite number of transmitters. Our “design characteristics” for a set of binary interactions between transmitters in different states is to describe a set of binary interactions with the following characteristics: (1) the interactions should drive the mean-field dynamics towards equilibrium, (2) the interactions should allow for substantial flexibility and controllability of the numbers of transmitters operating at different power levels, through variation of the system's policies which set the effective switching rates between states, in order to be able to settle into efficient system configurations over a wide range of required network capacities, and (3) the fluctuations around the equilibrium state should be relatively small.

While it is possible that all transmitters might settle to the same power level, this seems unlikely to be an optimal configuration in practice, and selection of a system state in which positive proportions of transmitters sit in different states implies that the system overall is continually probing the underlying distribution of UE (which will change over time) and so is able to dynamically adjust to it. System stability involves individual transmitters continuing to switch between states and relearning their environment even when the system as a whole is near equilibrium; we examine conditions under which such a robust configuration is achieved.

The outline of the paper is as follows. In Sec. II we introduce the individual-based stochastic model that describes the competition between the policy aims (i)–(iii), and argue for a minimal set of interaction rules that describe nontransitive but asymmetric interactions. We derive the resulting mean-field ordinary differential equation (ODE) and relevant stochastic differential equation for a large but finite collection of base stations. Section III analyzes the mean-field ODE behavior: all trajectories starting with a positive number of base stations on high power tend towards a unique equilibrium in the interior of the phase space. Section IV presents details of the stochastic differential equation (SDE) approximation to the individual-based model and describes the computation of the power spectral density (PSD), which characterizes the frequency content of fluctuations around the mean-field equilibrium. In Sec. IV A we analyze the location of the peaks of the PSD and show, intriguingly, that the PSD may have peaks at positive frequencies even when the linearization at the equilibrium has only real eigenvalues (and so in the mean-field

approximation there are no damped oscillations). In Sec. IV B we consider the probability distribution of the system state in phase space and navigate around the constraint that the total number of transmitters N is fixed, which introduces unwanted features into the analysis. In Sec. V we extend the model to include five additional binary interactions between different pairs of transmitters and analyze their effect on the dynamics induced by the minimal set of interactions. These interactions are competitive rather than all being catalyzed by the high-power transmitters, and so relax some of the assumptions of the minimal model. Our analysis here is at the level of the mean-field ODEs and qualitatively indicates which additional interactions would reinforce, or destabilize, the equilibrium reached by the minimal model. This enables us to say more about the robustness of our results and provide additional interpretation. Section VI concludes and suggests directions for future work, including other contexts where models of this kind could be appropriate.

II. INDIVIDUAL-BASED MODEL

Consider a collection of N identical transmitters, each of which is in one of three states given by the transmission power settings, as above: high, H ; low, L ; and off, O . We assume for simplicity that the transmitters are concentrated in an area that is small compared to the area that each could individually cover, so that every transmitter is influenced by the behavior of every other one and so spatial effects can be ignored. The state of the system is then fully described by the numbers $n_j \geq 0$ of transmitters at each power setting $j \in \{H, L, O\}$. Hence the system state is given by $\mathbf{n} := (n_H, n_L, n_O)$ and $n_H + n_L + n_O = N$ is constant.

We introduce stochastic dynamics of the simplest kind: a continuous-time Markov chain on this discrete state space, as follows [26]. We select at random a pair of transmitters that are supposed to have inferred each other's power settings and consider how they interact, causing one of the transmitters to change state. There are six possible pairs of states: $\{O O\}$, $\{O L\}$, $\{O H\}$, $\{L L\}$, $\{L H\}$, and $\{H H\}$. However, if neither transmitter is in state H , then we consider that no interaction takes place, noting that (a) if no transmitter is in state H , then the coverage objective [objective (i) above] is unlikely to be met, and the network is not operating in a regime in which there is competition between the three objectives, and (b) that transmitters in state H are assumed to be able to reach every other transmitter and therefore initiate the binary interactions that the model considers. We therefore consider only the three interaction rules involving H , where one of the policies drives a specific state transition. An interaction aiming to increase coverage or network capacity would switch a transmitter from off to high, following objective (i):

$$OH \xrightarrow{\hat{\alpha}} HH, \quad (1)$$

while an interaction aiming to reduce energy consumption would switch a transmitter from low to off, following objective (ii):

$$LH \xrightarrow{\hat{\beta}} OH, \quad (2)$$

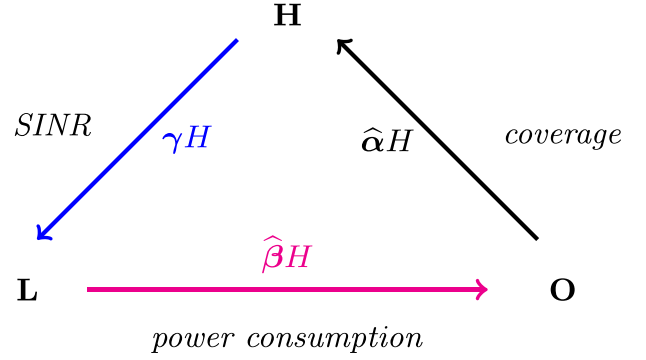


FIG. 1. State transitions driven by interactions due to competing policy objectives: coverage, power consumption, and SINR (i.e., channel quality), driven by interactions with transmitters on high power. $\hat{\alpha}$, $\hat{\beta}$, and γ denote the respective reaction rates.

and an interaction aiming to increase SINR would switch a transmitter from high to low, in line with objective (iii):

$$HH \xrightarrow{\gamma} LH, \quad (3)$$

where $\hat{\alpha}$, $\hat{\beta}$, and γ denote the “reaction rates” for these interactions. The $\hat{\cdot}$ marks indicate that we will later rescale $\hat{\alpha}$ and $\hat{\beta}$ for convenience. Figure 1 illustrates the interaction of these three optimization objectives on pairs of transmitters. The rate parameters $\hat{\alpha}$, $\hat{\beta}$, γ describe jump rates per unit time and per pair of transmitters. These rate parameters then are the variables that the transmitters are able to adjust in order to implement the different policies described in the previous section.

For a large class of density-dependent stochastic processes of this kind, we can describe the evolution of the system state $\mathbf{n}(t)$ as a Markov jump process in the form

$$\mathbf{n}(t) = \mathbf{n}(0) + \sum_k \mathbf{s}_k \mathbf{P}^{(k)} \left(N \int_0^t \tilde{r}_k(\mathbf{n}; N) dt' \right), \quad (4)$$

where $\{\mathbf{P}^{(k)}(\xi) : \xi \geq 0\}$ is a collection of independent rate-1 Poisson processes, $\mathbf{s}_k$ is the k th column of the stoichiometric matrix S , and $\tilde{r}_k(\mathbf{n}; N)$ is the corresponding state-dependent reaction rate parameter for the k th reaction. In thinking about the large N limit, we assume that the concentrations $\mathbf{x}(t)_N = \mathbf{n}(t)/N$ and the rates $\tilde{r}_k(\mathbf{n}; N) = r_k(\mathbf{x}_N)$, remain well defined in the limit $N \rightarrow \infty$. Well-known theory [27] provides rigorous support to the intuition that the behavior of the Poisson processes at large N will resemble a mean plus a sum of Brownian motions.

Dividing (4) by N yields the equation

$$\mathbf{x}_N(t) = \mathbf{x}(0) + \frac{1}{N} \sum_k \mathbf{s}_k \mathbf{P}^{(k)} \left(N \int_0^t \tilde{r}_k(\mathbf{x}_N) dt' \right). \quad (5)$$

Kurtz [27] shows that, firstly, the mean-field deterministic behavior satisfies

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \sum_k \mathbf{s}_k r_k[\mathbf{x}(t')] dt'$$

and that $\lim_{N \rightarrow \infty} \mathbf{x}_N(t) = \mathbf{x}(t)$ almost surely. Then, secondly, the process $\mathbf{x}_N(t)$ remains within a distance of $O[(\log N)/N]$

to a solution of the diffusion approximation to (5), i.e., to the process $\mathbf{z}_N(t)$, which solves the equation

$$\mathbf{z}_N(t) = \mathbf{x}_0 + \frac{1}{N} \sum_k s_k \mathbf{B}^{(k)} \left(N \int_0^t \tilde{r}_k(\mathbf{z}_N) dt' \right), \quad (6)$$

where $\{\mathbf{B}^{(k)}(\xi)\}$ denotes a collection of independent Brownian motions. Equation (6) is equivalent to writing

$$\begin{aligned} \mathbf{z}_N(t) = \mathbf{x}_0 + \int_0^t \sum_k s_k r_k[\mathbf{z}_N(t')] dt' \\ + \frac{1}{\sqrt{N}} \sum_k s_k \int_0^t \sqrt{r_k(\mathbf{z}_N)} dW_t^{(k)}(t'), \end{aligned} \quad (7)$$

where $W_t^{(k)}$ denotes a standard Wiener process with zero mean and unit variance. Written in differential form, this justifies considering solutions $\mathbf{x}(t)$ of the stochastic differential equation

$$d\mathbf{x}(t) = \left(\sum_k s_k r_k(\mathbf{x}) \right) dt + \frac{1}{\sqrt{N}} \sum_k s_k \sqrt{r_k} dW_t^{(k)}, \quad (8)$$

as providing asymptotically correct approximations to solutions of the Markov jump process (4).

Following [28–30], for the specific set of reactions (1)–(3) that we consider here, the three transition rates from state $\mathbf{n}$ to state $\mathbf{n}'$ are given by

$$\begin{aligned} T_1((n_H + 1, n_L, n_O - 1) | \mathbf{n}) &= \hat{\alpha} \frac{2n_O n_H}{N(N-1)}, \\ T_2((n_H, n_L - 1, n_O + 1) | \mathbf{n}) &= \hat{\beta} \frac{2n_L n_H}{N(N-1)}, \\ T_3((n_H - 1, n_L + 1, n_O) | \mathbf{n}) &= \gamma \frac{n_H(n_H - 1)}{N(N-1)}. \end{aligned} \quad (9)$$

Here the right-hand side of the first equation of Eq. (9) arises from there being $N(N-1)/2$ pairs of transmitters, of which $n_O n_H$ pairs participate in the type of interaction in Eq. (1). The right-hand side of the second equation of Eq. (9) is obtained similarly. In the third equation of Eq. (9), there are $n_H(n_H - 1)/2$ pairs participating in the interaction in Eq. (3), and hence the factors of 2 cancel out.

From (9) we deduce the vector of reaction rates $\mathbf{r}(\mathbf{x})$,

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix} 2\hat{\alpha}x_H x_O \\ 2\hat{\beta}x_H x_L \\ \gamma x_H x_H \end{pmatrix} \equiv \begin{pmatrix} \alpha x_O \\ \beta x_L \\ \gamma x_H \end{pmatrix}_{x_H}, \quad (10)$$

where we omit terms with factors of $1/N$ (which could formally be absorbed into the parameters), and introduce scaled parameters α and β to absorb factors of 2. From the transitions (9) we also deduce the stoichiometric matrix S :

$$S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}. \quad (11)$$

The k th column of S , denoted s_k , corresponds to the k th element of the vector $\mathbf{r}(\mathbf{x})$. We see that $\mathbf{x}(t)$ satisfies the SDE

$$d\mathbf{x}(t) = \mathbf{u}(\mathbf{x})dt + \frac{1}{\sqrt{N}} M(\mathbf{x})d\mathbf{W}_t, \quad (12)$$

where

$$u_j(\mathbf{x}) := \left(\sum_k s_k r_k(\mathbf{x}) \right)_j = \sum_k s_{jk} r_k(\mathbf{x}), \quad (13)$$

and the elements of the covariance matrix $B(\mathbf{x}) = M(\mathbf{x})M(\mathbf{x})^T$ are defined by

$$B_{ij}(\mathbf{x}) := \sum_k s_{ik} s_{jk} r_k(\mathbf{x}). \quad (14)$$

We deduce that

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \alpha x_H x_O - \gamma x_H x_H \\ \gamma x_H x_H - \beta x_H x_L \\ \beta x_H x_L - \alpha x_H x_O \end{pmatrix} \quad (15)$$

and

$$B(\mathbf{x}) = x_H \begin{pmatrix} \alpha x_O + \gamma x_H & -\gamma x_H & -\alpha x_O \\ -\gamma x_H & \gamma x_H + \beta x_L & -\beta x_L \\ -\alpha x_O & -\beta x_L & \alpha x_O + \beta x_L \end{pmatrix}. \quad (16)$$

Clearly $\mathbf{u}$ and B vanish if $x_H = 0$ as we might expect from the way the model is constructed.

Finally, in this section, we check the covariance matrix B , assuming $x_H > 0$. Since its row sums are zero, B has a zero eigenvalue with eigenvector $(1 \ 1 \ 1)^T$. The remaining eigenvalues of B are

$$\begin{aligned} \lambda_{\pm} &= r_1 + r_2 + r_3 \\ &\pm \sqrt{\frac{1}{2}(r_1 - r_2)^2 + \frac{1}{2}(r_2 - r_3)^2 + \frac{1}{2}(r_3 - r_1)^2}, \end{aligned}$$

which are clearly both real, and can be easily checked to be positive. So B is non-negative semidefinite for all combinations of parameter values.

Finally in this section, Fig. 2 compares solutions of the mean-field ODEs $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$ with typical solutions of the individual-based model computed using the Gillespie stochastic simulation algorithm (SSA) [31,32], for two different system sizes N ; at larger N the fluctuations around the mean field are much smaller in amplitude.

Figure 2 shows simulations of the SSA for a total simulation time of $10N$ in order to account for the timescale rescaling required to match the ODE trajectories for a time $t = 10$ with SSA trajectories.

III. ANALYSIS OF THE MEAN-FIELD ODES

The constraint $n_H + n_L + n_O = N$ implies $x_H + x_L + x_O = 1$ and so we can reduce the ODE dynamics to two variables, writing $x_O = 1 - x_H - x_L$, so that

$$\begin{aligned} \dot{x}_H &= x_H[\alpha(1 - x_H - x_L) - \gamma x_H], \\ \dot{x}_L &= x_H(\gamma x_H - \beta x_L). \end{aligned} \quad (17)$$

For all positive values of α , β , and γ there exists a unique interior equilibrium point at $(x_H^*, x_L^*) = (c/\gamma, c/\beta)$ where we define the parameter combination c via $c^{-1} = \alpha^{-1} + \beta^{-1} + \gamma^{-1}$; note that $3c$ is the harmonic mean of the rates α, β, γ .

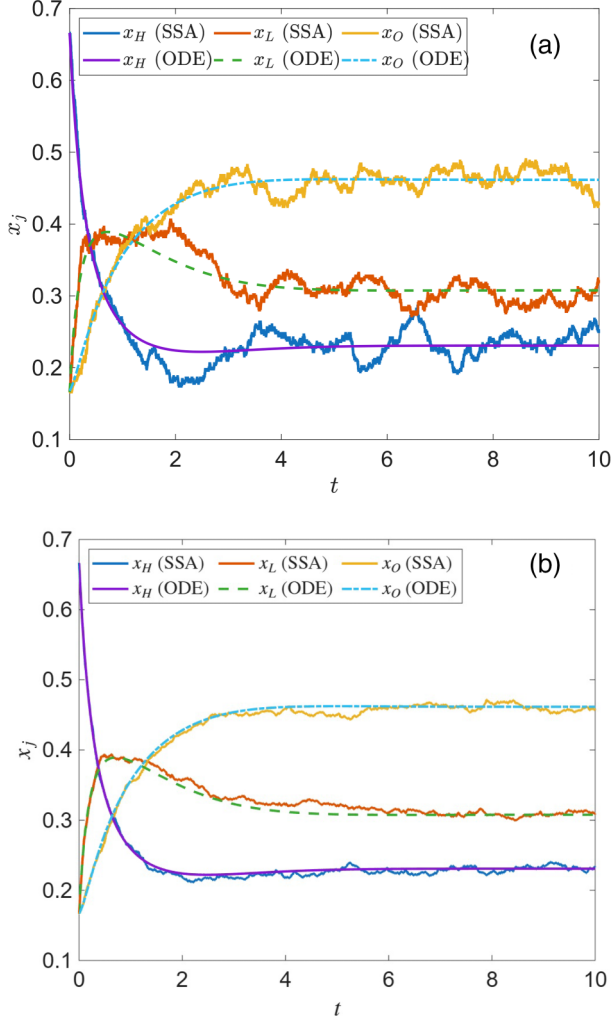


FIG. 2. Comparisons between the solution of the individual-based stochastic model, using the stochastic simulation algorithm (SSA), and the mean-field ODEs for population sizes (a) $N = 500$; (b) $N = 5000$. Parameter values: $\alpha = 2$, $\beta = 3$, $\gamma = 4$.

The Jacobian matrix A at (x_H^*, x_L^*) is

$$A = \begin{bmatrix} -\alpha - \gamma & -\alpha \\ \gamma & -\beta \end{bmatrix} \frac{c}{\gamma},$$

which has a negative trace $\text{tr}(A) = -(\alpha + \beta + \gamma)c/\gamma$ and a positive determinant $\det(A) = \alpha\beta c/\gamma$, so (x_H^*, x_L^*) is an asymptotically stable equilibrium point, either a stable focus if

$$\text{tr}(A)^2 - 4\det(A) = (\alpha + \beta + \gamma)^2 \frac{c^2}{\gamma^2} - 4\frac{\alpha\beta c}{\gamma} < 0, \quad (18)$$

or a stable node if the reverse inequality holds. That the equilibrium point (x_H^*, x_L^*, x_O^*) is a global attractor can be seen easily by considering the Lyapunov function $V(\mathbf{x}) = \gamma x_H^2 + \beta x_L^2 + \alpha x_O^2$ for which

$$\begin{aligned} \dot{V} &= -(\gamma x_H - \alpha x_O)^2 x_H - (\beta x_L - \gamma x_H)^2 x_H \\ &\quad - (\alpha x_O - \beta x_L)^2 x_H \leq 0, \end{aligned}$$

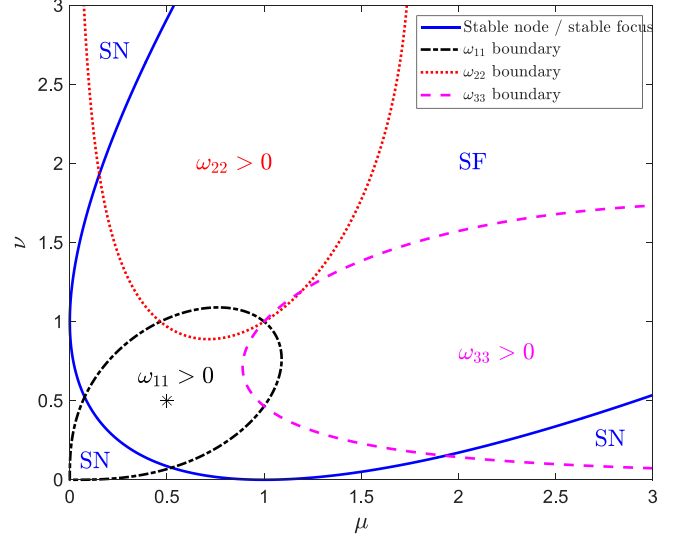


FIG. 3. The (μ, ν) plane showing (solid blue curve) the parabola (19) inside which the equilibrium point (x_H^*, x_L^*) is a stable focus (SF) and outside which it is a stable node (SN). Also shown are the three regions where the diagonal entries of the power spectral density matrix $\mathcal{P}(\omega)$ achieve maximum values at positive frequencies. The point $\mu = \nu = 1/2$ indicated by the asterisk (*) corresponds to the parameter values chosen for the numerical simulations shown in Figs. 7–9.

so that V is decreasing and bounded below by zero, with $V = 0$ only if $x_H = 0$ or $\mathbf{x} = \mathbf{x}^*$.

The inequality (18) can be visualized more easily by defining new parameter combinations $(\mu, \nu) = (\alpha/\gamma, \beta/\gamma)$ so that (18) becomes

$$\text{tr}(A)^2 - 4\det(A) = c^2[(\mu + \nu + 1)^2 - 4(\mu\nu + \mu + \nu)],$$

and then defining $u = \nu + \mu$ and $v = \nu - \mu$. Then the boundary between stable focus and stable node behavior is the parabola

$$u = (v^2 + 1)/2, \quad (19)$$

which passes through the points $(\mu, \nu) = (1, 0)$ and $(0, 1)$, sketched in Fig. 3, and the equilibrium is a stable focus when $u > (v^2 + 1)/2$.

All points with $x_H = 0$ are also equilibria; these are non-hyperbolic and unstable since for small x_H we see from (17) that $\dot{x}_H = \alpha(1 - x_L)x_H + O(x_H^2)$ and so x_H increases along trajectories if it starts at a positive value. This is confirmed by typical phase portraits of the mean-field model; for example, those shown in Fig. 4, which illustrate how trajectories approach the interior equilibrium point (x_H^*, x_L^*) .

The eigenvalues of A are

$$\lambda_{\pm}^{(A)} = -\frac{c}{2}(u + 1) \pm \frac{c}{2}\sqrt{v^2 + 1 - 2u},$$

showing that the linear frequency of the damped oscillations near (x_H^*, x_L^*) in the stable focus regime increases as we move further towards the top right of Fig. 3 as μ and ν increase.

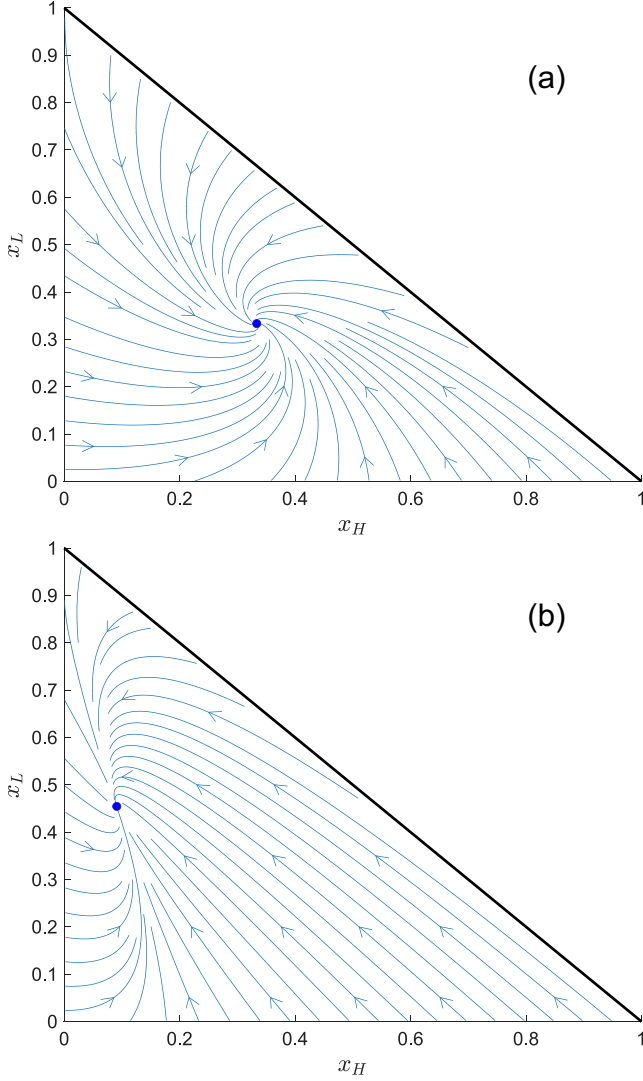


FIG. 4. Phase portraits in the (x_H, x_L) plane showing the convergence to (x_H^*, x_L^*) of all trajectories starting in the interior. The equilibrium point (x_H^*, x_L^*) is indicated by the blue dot. (a) $(\alpha, \beta, \gamma) = (2, 2, 2)$ for which $(x_H^*, x_L^*) = (1/3, 1/3)$. (b) $(\alpha, \beta, \gamma) = (2, 2, 10)$ for which $(x_H^*, x_L^*) = (1/11, 5/11)$.

IV. ANALYSIS OF THE STOCHASTIC MODEL

As all trajectories of the mean-field ODE model for which $x_H > 0$ initially tend to the equilibrium point, it is of interest to examine the influence of the intrinsic noise (also known as demographic noise) on the system in the case of a large but finite number of transmitters [33,34].

The stochastic behavior of trajectories near the equilibrium point may generate fluctuations with dominant frequency components generated by the interplay between the linearization at the equilibrium point and the covariance between the variables. We explore this through computing the PSD.

Near the equilibrium point we approximate the nonlinear SDE (12)

$$d\mathbf{x} = \mathbf{u}(\mathbf{x})dt + \frac{1}{\sqrt{N}}M(\mathbf{x})d\mathbf{W}_t, \quad (20)$$

by the linear SDE

$$d\mathbf{x} = A\mathbf{x}dt + \frac{1}{\sqrt{N}}M(\mathbf{x}_*)d\mathbf{W}_t, \quad (21)$$

where

$$A \equiv A(\mathbf{x}^*) = c \begin{pmatrix} -1 & 0 & \alpha/\gamma \\ 1 & -\beta/\gamma & 0 \\ 0 & \beta/\gamma & -\alpha/\gamma \end{pmatrix}, \quad (22)$$

and $M = M(\mathbf{x}^*)$ with $MM^T = B$ from which we deduce

$$B = B(\mathbf{x}^*) = \frac{c^2}{\gamma} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (23)$$

Equation (21) describes a multivariate Ornstein-Uhlenbeck process, and its power spectral density $\mathcal{P}(\omega)$ is given by the matrix

$$\mathcal{P}(\omega) = \frac{1}{2\pi N} (A - i\omega I)^{-1} B (A - i\omega I)^{-\dagger}, \quad (24)$$

where $Q^\dagger$ denotes the complex conjugate transpose of a matrix Q , and so $Q^{-\dagger}$ denotes the inverse of the complex conjugate transpose of Q . Although A is not invertible, for $\omega \neq 0$ we can compute $\mathcal{P}(\omega)$ without difficulty. The diagonal elements of $\mathcal{P}(\omega)$ then yield the PSD for each component of $\mathbf{x} = (x_H, x_L, x_O)$ in turn. We obtain

$$\mathcal{P}(\omega)_{11} = \frac{1}{2\pi N} \frac{2c^4\gamma(\alpha^2 + \alpha\beta + \beta^2) + 2c^2\gamma^3\omega^2}{c^2(\alpha\beta\gamma)^2 + c^2\gamma^2(\alpha^2 + \beta^2 + \gamma^2)\omega^2 + \gamma^4\omega^4}, \quad (25)$$

$$\mathcal{P}(\omega)_{22} = \frac{1}{2\pi N} \frac{2c^4\gamma(\alpha^2 + \alpha\gamma + \gamma^2) + 2c^2\gamma^3\omega^2}{c^2(\alpha\beta\gamma)^2 + c^2\gamma^2(\alpha^2 + \beta^2 + \gamma^2)\omega^2 + \gamma^4\omega^4}, \quad (26)$$

$$\mathcal{P}(\omega)_{33} = \frac{1}{2\pi N} \frac{2c^4\gamma(\beta^2 + \beta\gamma + \gamma^2) + 2c^2\gamma^3\omega^2}{c^2(\alpha\beta\gamma)^2 + c^2\gamma^2(\alpha^2 + \beta^2 + \gamma^2)\omega^2 + \gamma^4\omega^4}, \quad (27)$$

which describe the PSD for x_H , x_L , and x_O , respectively. Despite the noninvertibility of A noted above, it is reassuring that $\mathcal{P}(\omega)_{jj}$ remains finite at $\omega = 0$. We note further that the PSDs for x_L and x_O are related by interchanging α and β .

A. Frequency peaks in the PSD

Due to their form as rational functions of ω^2 with coefficients that are all positive, these power spectral densities either attain their maximum value at $\omega = 0$, in which case the PSD is then monotonically decreasing with ω , or have a single local maximum at some positive value of ω , in which case the PSD is maximized there. We denote the value of ω that maximizes $\mathcal{P}(\omega)_{jj}$ by ω_{jj} for $j = 1, 2, 3$. It is simplest to present the following results in terms of the coordinates $(\mu, \nu) = (\alpha/\gamma, \beta/\gamma)$ introduced previously. For $\mathcal{P}(\omega)_{11}$, we find that the PSD is maximized in the interior, at $\omega_{11} > 0$, when

$$\mu\nu(2\mu + 2\nu + 1) > \mu^4 + \mu^3\nu + \mu^2\nu^2 + \mu\nu^3 + \nu^4, \quad (28)$$

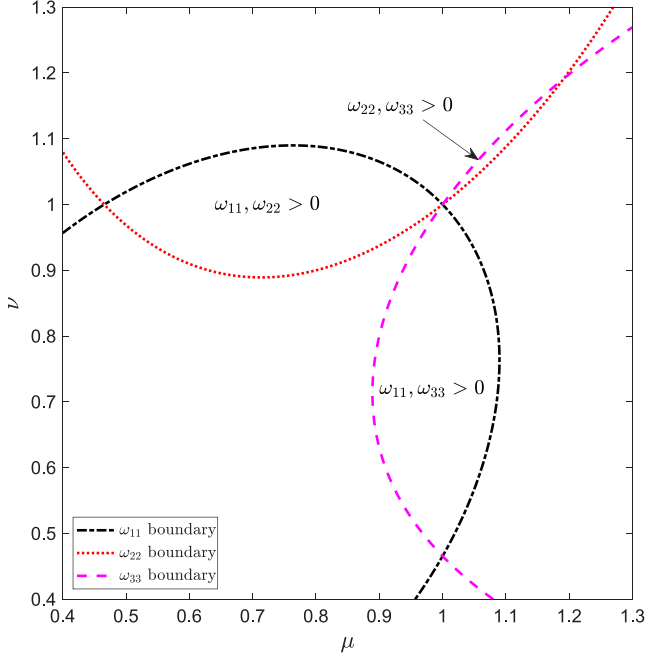


FIG. 5. Enlargement of Fig. 3 showing the intersections near $\mu = \nu = 1$ of the boundaries of the regions in which two of the peak frequencies ω_{jj} ($j = 1, 2, 3$) are positive. Note that there is no region in which all three are simultaneously positive.

at which the maximizing frequency ω_{11} is given by

$$\omega_{11}^2 = c^2[(\mu + \nu + 1)\sqrt{\mu\nu} - \mu^2 - \mu\nu - \nu^2]. \quad (29)$$

The region defined by (28) is shown as the black dot-dashed curve in Fig. 3; its boundary passes through both $\mu = \nu = 1$ and $\mu = \nu = 0$. For reference the parameter combination c can be expressed as $c = \gamma\mu\nu/(\mu + \nu + \mu\nu)$ showing that ω_{11} can become arbitrarily large if α, β , and γ all increase in such a way that α/γ and β/γ remain constant. Similarly, $\mathcal{P}(\omega)_{22}$ has a maximum at $\omega_{22} > 0$ in the region

$$\mu\nu(2\mu + \nu + 2) > \mu^4 + \mu^3 + \mu^2\nu^2 + \mu + 1, \quad (30)$$

where the frequency ω_{22} that maximizes the PSD is given by

$$\omega_{22}^2 = c^2[(\mu + \nu + 1)\sqrt{\mu} - \mu^2 - \mu - 1]. \quad (31)$$

Similarly, using the interchange symmetry $(\mu, \nu) \rightarrow (\nu, \mu)$, which enables us to deduce the corresponding results for $\mathcal{P}(\omega)_{33}$, we have that $\mathcal{P}(\omega)_{33}$ has a maximum at $\omega_{33} > 0$ in the region

$$\mu\nu(\mu + 2\nu + 2) > \nu^4 + \nu^3 + \nu^2\mu^2 + \nu + 1, \quad (32)$$

where the frequency ω_{33} that maximizes the PSD is given by

$$\omega_{33}^2 = c^2[(\mu + \nu + 1)\sqrt{\nu} - \nu^2 - \nu - 1]. \quad (33)$$

The regions corresponding to (30) and (32) are shown in Figs. 3 and 5 by the red dotted and magenta dashed lines, respectively. It is interesting to note that all three boundaries (28), (30), and (32) pass through (1,1). Analytic confirmation of the organization of these boundary curves is algebraically complicated in general as they are fourth-order polynomials in μ and ν . Local to (1,1) we can deduce sufficient information from computing the first and second derivatives of the

boundary curves and evaluating them at $(\mu, \nu) = (1, 1)$. For $\mathcal{P}(\omega)_{11}$, treating (28) as an equality that locally defines the boundary in the form $\nu = \nu(\mu)$ we have $d\nu/d\mu = -1$ and $d^2\nu/d\mu^2 = -26/3$ so the ω_{11} boundary is negatively sloped and concave, agreeing with Fig. 5. Similarly, for the ω_{22} boundary (30) we have $d\nu/d\mu = 1/2$ and $d^2\nu/d\mu^2 = 1/4$. Hence, using the interchange symmetry, for ω_{33} we have $d\nu/d\mu = 2$ and $d^2\nu/d\mu^2 = -1/4$. Finally, the second intersection point (μ_2, ν_2) between the ω_{22} and ω_{33} boundaries lies on the diagonal $\mu = \nu$ and is the only real root of the polynomial $2\mu^3 - 2\mu - 1 = 0$, which turns out to be

$$\begin{aligned} \mu_2 = \nu_2 &= \left(\frac{9 + \sqrt{33}}{36}\right)^{1/3} + \left(\frac{9 - \sqrt{33}}{36}\right)^{1/3} \\ &= 1.19148788\dots \end{aligned}$$

Figure 6 illustrates the variation in the shape of the PSDs for x_H and x_O by plotting the matrix components $\mathcal{P}(\omega)_{11}$ and $\mathcal{P}(\omega)_{33}$. These are normalized by dividing by the value at $\omega = 0$ for convenience, in order to highlight that the maxima of each component may occur at $\omega = 0$ or for positive ω , in line with the boundary curves computed in (28) and (32) for $\mathcal{P}(\omega)_{11}$ and $\mathcal{P}(\omega)_{33}$, respectively. For the values of (μ, ν) used in Fig. 6 we find, as expected, that $\mathcal{P}(\omega)_{22}$ always attains its maximum at $\omega = 0$.

Figures 7–9 compare the theoretical predictions for the diagonal elements of $\mathcal{P}(\omega)$ with numerical results for long-time (and therefore high resolution in the time domain) simulations. For the parameter values $\alpha = \beta = 2, \gamma = 4$ we see that $\mathcal{P}(\omega)_{11}$ has a peak at a positive value of ω whereas $\mathcal{P}(\omega)_{22}$ and $\mathcal{P}(\omega)_{33}$ achieve their maximum value at $\omega = 0$. Figure 8 is a log-log plot to confirm the decay of $\mathcal{P}(\omega) \propto \omega^{-2}$ at large ω , which is the theoretical power-law behavior for all diagonal elements of $\mathcal{P}(\omega)$.

B. Stationary distribution in phase space

The stationary distribution of sample paths of the multi-variable Ornstein-Uhlenbeck process (21) can be computed explicitly [35,36], which sheds additional light on the typical behavior expected for the dynamics in large but finite populations. For an SDE of the form (21) in $\mathbb{R}^n$ for which the constant matrix A has all its eigenvalues λ satisfying $\text{Re}(\lambda) < 0$ the stationary density is

$$P_{\text{st}}(\mathbf{x}) = (2\pi)^{-n/2} [\det(G)]^{-1/2} \exp(-\frac{1}{2}\mathbf{x}^T G^{-1} \mathbf{x}), \quad (34)$$

where G is the $n \times n$ matrix that solves the Lyapunov equation

$$AG + GA^T + \frac{1}{N}B = 0, \quad (35)$$

where, as previously, $B = MM^T$ is the covariance matrix. The solution to (35) can be computed explicitly and is given by

$$G = \frac{1}{N} \int_0^\infty e^{tA} B e^{tA^T} dt. \quad (36)$$

Since e^{tA} is the evolution operator of the linear ODE $\dot{\mathbf{x}} = A\mathbf{x}$, it is always invertible (and so also is e^{tA^T}). However, in the case at hand we know that the covariance matrix B is only non-negative semidefinite, and from (23) it clearly has zero determinant. Hence (34) cannot be used directly.

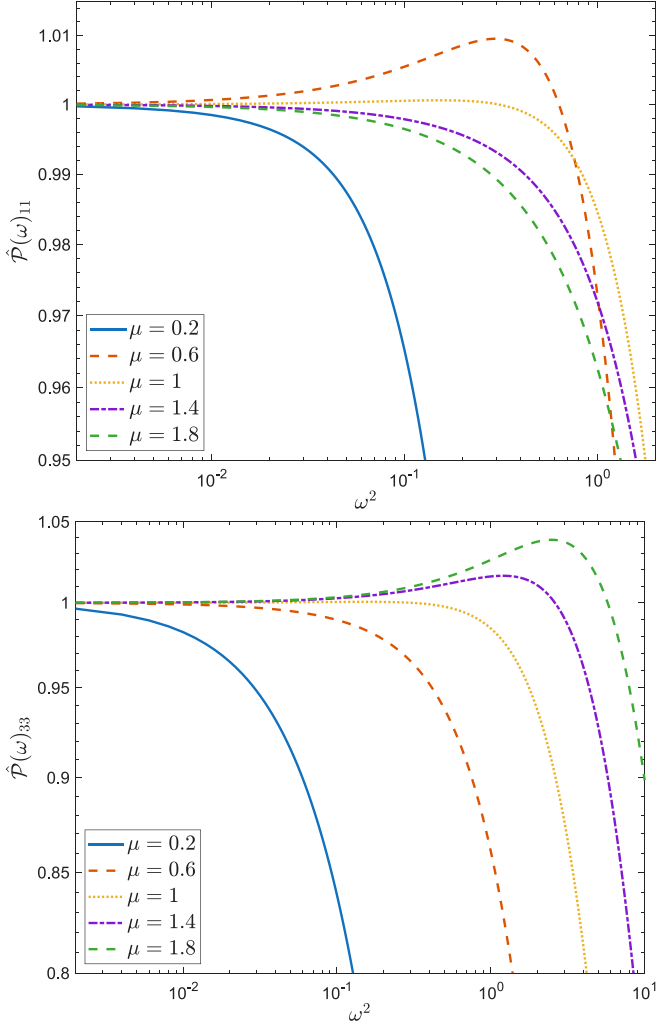


FIG. 6. Power spectral densities for $\nu = 0.8$ and increasing values of μ , fixing $\gamma = 5$. (a) $\hat{\mathcal{P}}(\omega)_{11}$ for five values of μ , showing that the PSD is maximized at $\omega = 0$ when $\mu = 0.2, 1.4$, and 1.8 but has a peak at $\omega > 0$ when $\mu = 0.6, 1$ corresponding to parameter values inside the enclosed region of the (μ, ν) plane shown in Fig. 3. The vertical axis is normalized by dividing by $\mathcal{P}(0)_{11}$: $\hat{\mathcal{P}}(\omega)_{11} := \mathcal{P}(\omega)_{11}/\mathcal{P}(0)_{11}$. (b) $\hat{\mathcal{P}}(\omega)_{33}$ at the same five values of μ , showing that this PSD is maximized at a positive value of ω in the cases $\mu = 1, 1.4, 1.8$ which lie inside the magenta dashed region in Fig. 3, but $\hat{\mathcal{P}}(\omega)_{33}$ is maximized at $\omega = 0$ for $\mu = 0.2, 0.6$. The vertical axis is normalized by dividing by $\mathcal{P}(0)_{33}$: $\hat{\mathcal{P}}(\omega)_{33} := \mathcal{P}(\omega)_{33}/\mathcal{P}(0)_{33}$.

We overcome this issue by returning to the two-dimensional (2D) formulation in (x_H, x_L) variables, extending (17) to include stochastic terms and changing to variables in which the stochastic terms always preserve the constraint $x_H + x_L + x_O = 1$, as we now describe. Let $\mathbf{y} \equiv (y_1, y_2) = (2x_H - x_L - x_O, x_L - x_O)$, so that the region where all three x_i coordinates lie in $[0, 1]$ corresponds to the region $-1 \leq y_1 \leq 2$ and $y_1 - 2 \leq 3y_2 \leq 2 - y_1$. Applying this linear change of variables to (12), we obtain

$$d\mathbf{y} = \mathbf{v}(\mathbf{y})dt + \frac{1}{\sqrt{N}}\tilde{\mathbf{M}}d\mathbf{W}_t,$$

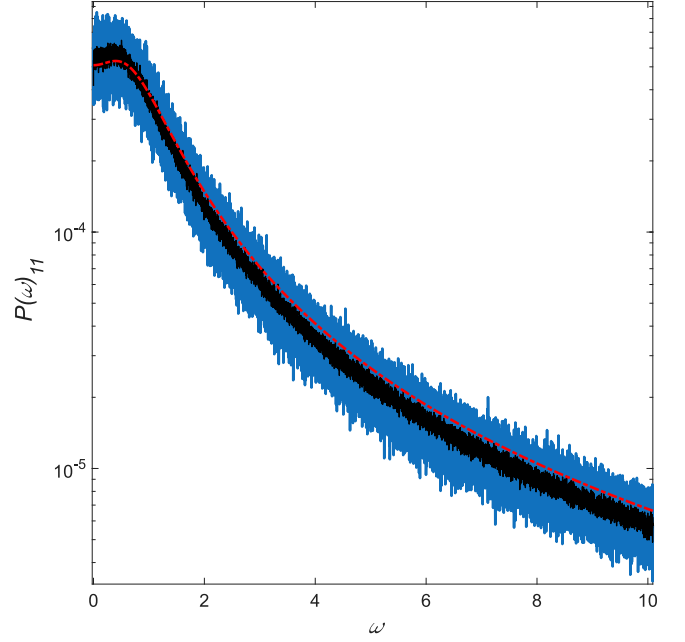


FIG. 7. Comparison of numerical simulations (blue lines) for $\mathcal{P}(\omega)_{11}$ (averaged over 50 simulations) with the theoretical result (25) (red dash-dotted curve). The black line is the moving average over a window of 11 data points, to reduce the variation in the PSD. Note the maximum of $\mathcal{P}(\omega)_{11}$ occurs at a positive value of ω . Parameter values: $\alpha = \beta = 2, \gamma = 4$ (corresponding to point * in Fig. 3), $N = 1000, T = 50\,000N$.

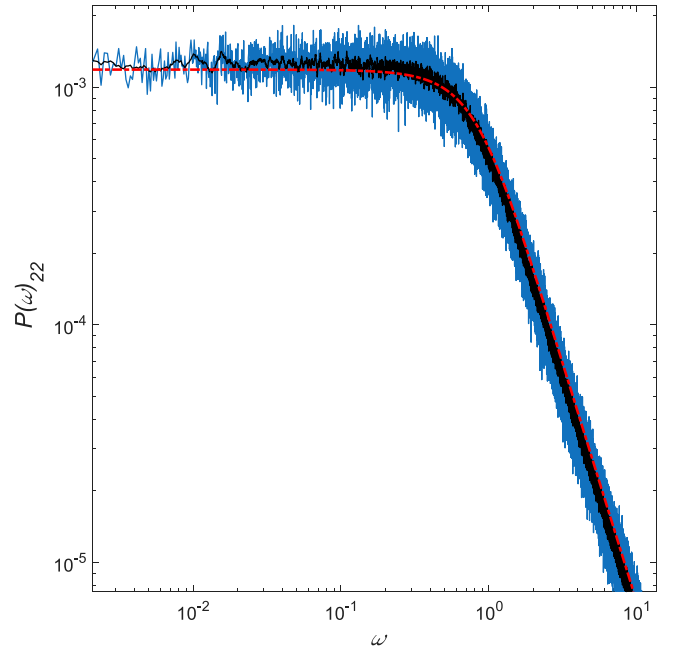


FIG. 8. Comparison of numerical simulations (blue lines) for $\mathcal{P}(\omega)_{22}$ (averaged over 50 simulations) with the theoretical result (26) (red dash-dotted curve). The black line is the moving average over a window of 11 data points, to reduce the variation in the PSD. Note the maximum of $\mathcal{P}(\omega)_{22}$ occurs at $\omega = 0$. The log-log plot here illustrates the ω^{-2} scaling at large ω . Parameter values: $\alpha = \beta = 2, \gamma = 4$ (corresponding to point * in Fig. 3), $N = 1000, T = 50\,000N$.

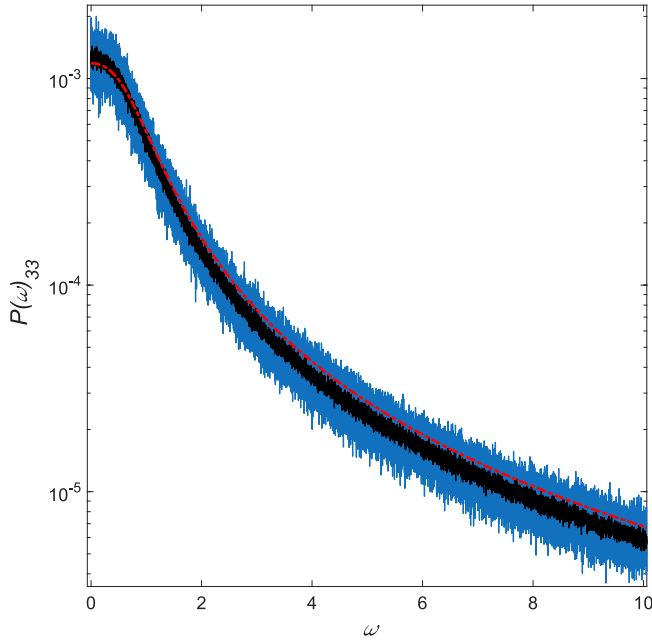


FIG. 9. Comparison of numerical simulations (blue lines) for $\mathcal{P}(\omega)_{33}$ (averaged over 50 simulations) with the theoretical result (27) (red dash-dotted curve). The black line is the moving average over a window of 11 data points, to reduce the variation in the PSD. Note the maximum of $\mathcal{P}(\omega)_{33}$ occurs at $\omega = 0$. Parameter values: $\alpha = \beta = 2$, $\gamma = 4$ (corresponding to point * in Fig. 3), $N = 1000$, $T = 50\,000N$.

where the drift term is

$$\mathbf{v} = \begin{pmatrix} \alpha - \gamma - (\frac{1}{2}\alpha + \gamma)y_1 - \frac{3}{2}\alpha y_2 \\ \frac{1}{3}(\alpha - 2\beta + \gamma) - \frac{1}{6}(\alpha - 2\beta - 2\gamma)y_1 - (\frac{1}{2}\alpha + \beta)y_2 \end{pmatrix},$$

and the 2×3 diffusion matrix $\tilde{M}$ is given by

$$\tilde{M} = \begin{pmatrix} 3\sqrt{\alpha z_1/6} & 0 & -3\sqrt{\gamma z_3/3} \\ \sqrt{\alpha z_1/6} & -2\sqrt{\beta z_2/6} & \sqrt{\gamma z_3/3} \end{pmatrix},$$

where $z_1 = 2 - y_1 - 3y_2$, $z_2 = 2 - y_1 + 3y_2$, and $z_3 = 1 + y_1$. In the $\mathbf{y}$ coordinates the (unique) interior equilibrium point is at $\mathbf{y}_* = [3c/\gamma - 1, c(1/\beta - 1/\alpha)]$. Evaluating the Jacobian matrix $\tilde{A}$ and diffusion matrix $\tilde{M}$ at $\mathbf{y}_*$ we obtain the linear approximation

$$d\mathbf{y} = \tilde{A}\mathbf{y} dt + \frac{1}{\sqrt{N}}\tilde{M} d\mathbf{W}_t,$$

where

$$\tilde{A} = \begin{pmatrix} -\frac{1}{2}\alpha - \gamma & -\frac{3}{2}\alpha \\ -\frac{1}{6}(\alpha - 2\beta - 2\gamma) & -\frac{1}{2}\alpha - \beta \end{pmatrix}$$

and

$$\tilde{M} = \sqrt{c} \begin{pmatrix} 3 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix},$$

so that

$$\tilde{M}\tilde{M}^T \equiv \tilde{B} = c \begin{pmatrix} 18 & 0 \\ 0 & 6 \end{pmatrix}, \quad (37)$$

which is clearly positive definite, and so the solution $\tilde{G}$ to the Lyapunov equation (35) is invertible:

$$\begin{aligned} \tilde{G} &= \frac{1}{N} \int_0^\infty e^{t\tilde{A}} \tilde{B} e^{t\tilde{A}^T} dt \\ &= \frac{c^2}{N\alpha\beta\gamma} \begin{pmatrix} 9(\alpha + \beta) & 3(\beta - \alpha) \\ 3(\beta - \alpha) & \alpha + \beta + 4\gamma \end{pmatrix}. \end{aligned} \quad (38)$$

We find that $\det(\tilde{G}) = 36c^3/(\alpha\beta\gamma N^2)$ and

$$\tilde{G}^{-1} = \frac{N}{36c} \begin{pmatrix} \alpha + \beta + 4\gamma & 3(\alpha - \beta) \\ 3(\alpha - \beta) & 9(\alpha + \beta) \end{pmatrix}.$$

Using this $\tilde{G}^{-1}$ we obtain the stationary distribution in $\mathbf{y}$ coordinates to be

$$P_{\text{st}}(\mathbf{y}) = (2\pi)^{-1} \left(\frac{36c^3}{\alpha\beta\gamma N^2} \right)^{-1/2} \exp\left(-\frac{1}{2}\mathbf{y}^T \tilde{G}^{-1} \mathbf{y}\right), \quad (39)$$

which can now be reexpressed in (x_H, x_L) coordinates writing $y_1 = 3x_H - 1$, $y_2 = x_H + 2x_L - 1$.

In Figs. 10–13 we present a series of comparisons between the stationary distribution (39) and individual-based simulations, which show qualitative agreement between the two, especially in the orientation of the principal axes of the bivariate Gaussian distribution and its overall extent. The stationary distribution is estimated from stochastic simulations by interpolating the output of simulations from the Gillespie algorithm onto a regularly spaced grid of time points and computing a histogram in (x_H, x_L) coordinates using equally spaced bins in $[0, 1]^2$. Darker colors indicate a higher value of the probability density and the same color scale is used in every figure to enable qualitative comparison between them. White areas indicate empty bins. Generically there is no relation between the principal axes of the stationary distribution (i.e., the eigenvectors of $\tilde{G}^{-1}$) and the eigenvectors of the linearization A as given in Sec. III, not least because the eigenvalues of A may be a complex conjugate pair whereas those of $\tilde{G}^{-1}$ are always real since $\tilde{G}^{-1}$ is symmetric. Each part of Figs. 10–13 shows five concentric ellipses, which are contours at equally spaced values of the stationary distribution (39), which are therefore at values of $\{1/6, 1/3, 1/2, 2/3, 5/6\}$. The ellipses are colored using the same color scheme as the simulation data, darkened slightly so that they are visible in the plots; there is very good agreement between the colors of the ellipses and the histogram in every case.

Figures 10 and 11 show the symmetric case where $\alpha = \beta = \gamma = 4$ and so the equilibrium point is at $x_H = x_L = 1/3$. As the system size N increases, the distribution becomes much more concentrated near the equilibrium point, but retains its elliptical, rather than circular, shape, due to the asymmetry in $\tilde{G}^{-1}$. Figure 11(a) shows the same plot as Fig. 10(b) to illustrate the increasing concentration of trajectories as N increases further in Fig. 11(b) plotted with the same axis limits.

Figures 12 and 13 show cases in which β or α , respectively, is much larger than the other two parameters. Here the equilibrium point is closer to one of the boundaries of the phase space and the stationary distributions become more elongated in a direction parallel to that boundary. The qualitative agreement between simulations and theory remains very good.

Finally, in this section, we remark that if we compute the PSD using the 2×2 matrices $\tilde{A}$ and $\tilde{B}$, then we recover useful

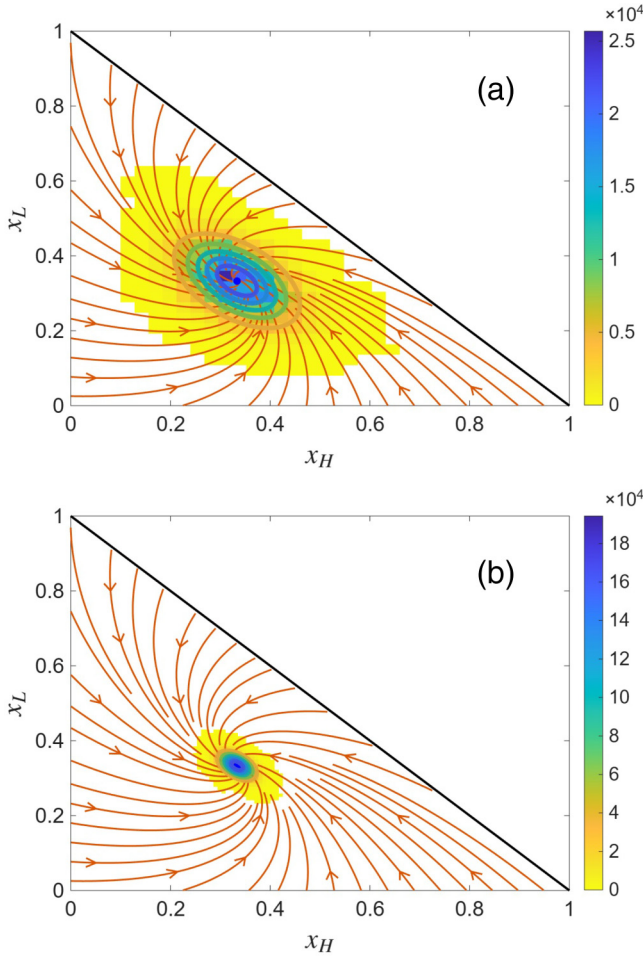


FIG. 10. Dynamics in the (x_H, x_L) plane. Trajectories for the ODEs (17) are shown as the solid red lines, and contours of the stationary distribution P_{st} are indicated by the concentric ellipses. The shaded squares indicate the numerically computed stationary distribution from a stochastic simulation for $t \in [0, 5000N]$. (a) Smaller system size $N = 50$ leads to a broader distribution; (b) larger system size $N = 500$ has a more compact stationary distribution. Parameter values: $\alpha = \beta = \gamma = 4$.

expressions for the PSD for the y variables, similar expressions to those deduced for $\mathcal{P}(\omega)$ in Sec. IV A. The equivalent expression to (24) is

$$\tilde{\mathcal{P}}(\tilde{\omega}) = \frac{1}{2\pi N} (\tilde{A} - i\tilde{\omega}I)^{-1} \tilde{B} (\tilde{A} - i\tilde{\omega}I)^{-\dagger}, \quad (40)$$

where $\tilde{B} = \tilde{M}\tilde{M}^T$ as in (37). For the diagonal elements we obtain

$$\tilde{\mathcal{P}}(\tilde{\omega})_{11} = \frac{9c}{\pi N} \frac{c^4(\alpha^2 + \alpha\beta + \beta^2) + c^4\tilde{\omega}^2}{c^2(\alpha\beta\gamma)^2 + c^4(\alpha^2 + \beta^2 + \gamma^2)\tilde{\omega}^2 + \tilde{\omega}^4} \quad (41)$$

and

$$\tilde{\mathcal{P}}(\tilde{\omega})_{22} = \frac{c^3(\alpha^2 - \alpha\beta + \beta^2 + 2\alpha\gamma + 2\beta\gamma + 4\gamma^2) + 3c^3\tilde{\omega}^2}{\pi N[(\alpha\beta\gamma)^2 + c^2(\alpha^2 + \beta^2 + \gamma^2)\tilde{\omega}^2 + c^2\tilde{\omega}^4]}. \quad (42)$$

If we let $\tilde{\omega} = \omega\gamma/c$, then we see that $\tilde{\mathcal{P}}(\tilde{\omega})_{11} \propto \mathcal{P}(\omega)_{11}$, which shows that the region in which $\tilde{\mathcal{P}}(\tilde{\omega})$ has an inte-

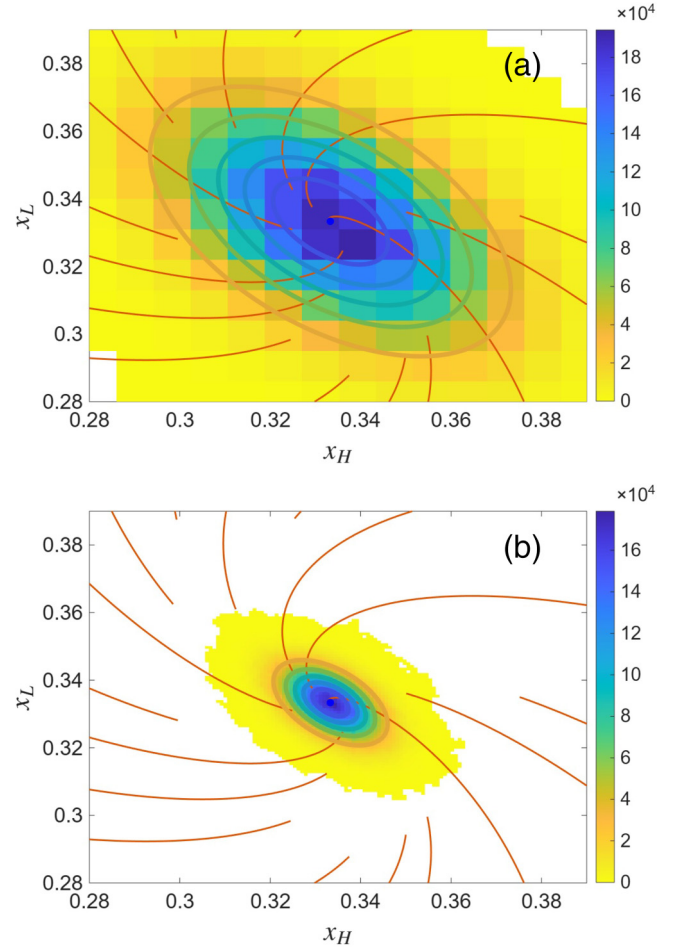


FIG. 11. Enlargements of the dynamics in the (x_H, x_L) plane. As for Fig. 10, trajectories for the ODEs (17) are shown as the solid red lines, and contours of the stationary distribution P_{st} are indicated by the concentric ellipses. The shaded squares indicate the numerically computed stationary distribution from a stochastic simulation for $t \in [0, 5000N]$. The blue dot indicates the equilibrium point. (a) System size $N = 500$ as in Fig. 10(b); (b) even larger system size $N = 5000$ has an even more compact stationary distribution. Parameter values: $\alpha = \beta = \gamma = 4$.

rior maximum is also given by (28) although the frequency corresponding to the maximum differs by a factor of γ/c from (29).

The expression (42) is symmetric in α and β due to the linear combination being $y_2 = x_L - x_O$. $\tilde{\mathcal{P}}(\tilde{\omega})_{22}$ attains its maximum at a positive value of $\tilde{\omega}$ when

$$4\mu^2v + 4\mu v^2 + \mu^2v^2 + \mu^3v + \mu v^3 + 7\mu v > \mu^4 + 2\mu^3 + 2\mu^2 + 2\mu + v^4 + 2v^3 + 2v^2 + 2v + 4.$$

Figure 14 indicates the arrangement in the (μ, v) plane of the regions in which the components of $\tilde{\mathcal{P}}(\tilde{\omega})$ attain their maxima at positive values of $\tilde{\omega}$.

V. STRUCTURAL STABILITY

In Sec. III we observed that trajectories of the mean-field ODEs for the set of reactions (1)-(3) always converge to the unique interior equilibrium point, when starting from

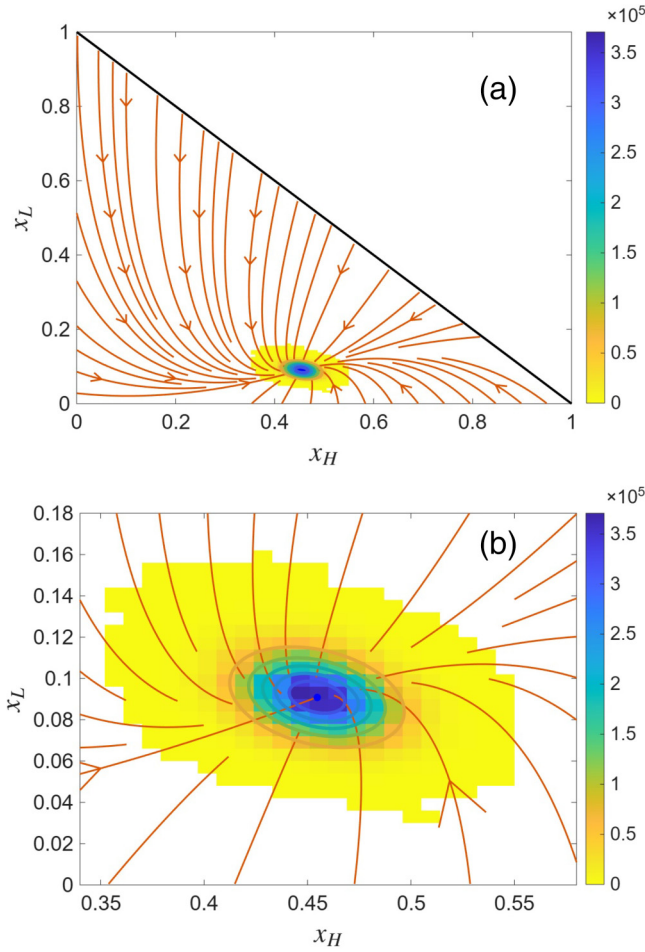


FIG. 12. Dynamics in the (x_H, x_L) plane. Trajectories for the ODEs (17) are shown as the solid red lines, and contours of the stationary distribution P_{st} are indicated by the concentric ellipses. The shaded squares indicate the numerically computed stationary distribution from a stochastic simulation for $t \in [0, 5000N]$, $N = 500$ in both cases. (a) Full phase space; (b) enlargement near the equilibrium point $(x_H^*, x_L^*) = (5/11, 1/11)$. The blue dot indicates the equilibrium point. Parameter values: $\alpha = 4$, $\beta = 20$, $\gamma = 4$.

initial conditions with all components positive. This is an important characteristic since, despite the nonlinearities, it demonstrates the controllability of the system described by the model. In this section we examine how this convergence to equilibrium changes when further possible interactions are added that describe intransitive sets of interactions. Such intransitive interactions are often associated with oscillatory behavior and are motivated, for example, by population models of the rock-paper-scissors game. Of the many possible additional interactions that could be added, we chose these sets of intransitive interactions as an “extreme case” in our attempt to make the dynamics more complex and less controllable.

For the conceptual model with three states, there are only two possible sets of intransitive (or “winnerless”) interactions. In each set the binary interactions describe the recruitment by one transmitter state of transmitters of one other type; each state is also itself recruited by the other remaining type. We

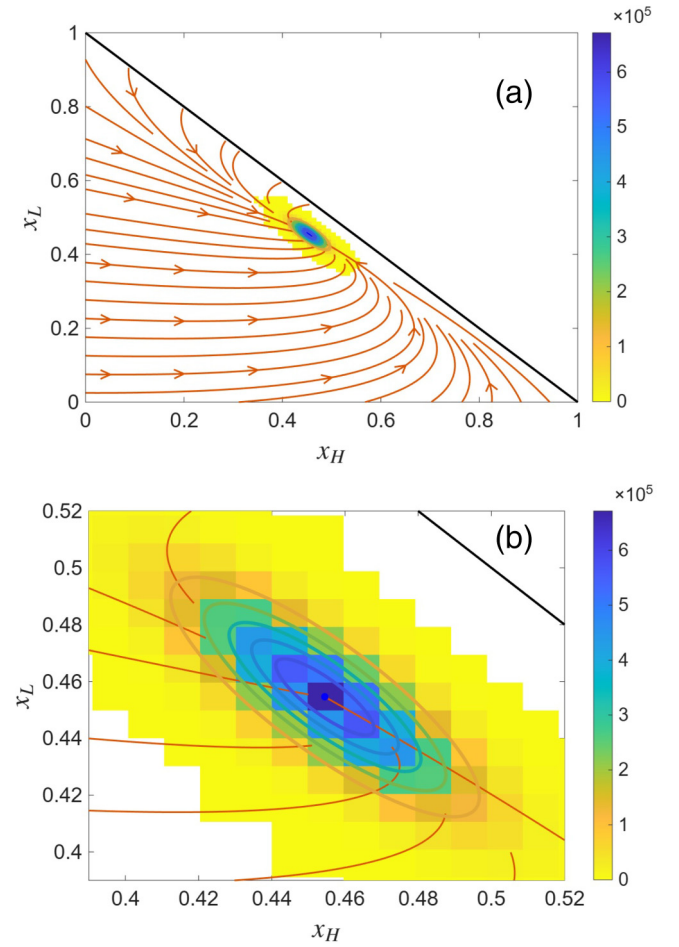


FIG. 13. Dynamics in the (x_H, x_L) plane. Trajectories for the ODEs (17) are shown as the solid red lines and contours of the stationary distribution P_{st} are indicated by the concentric ellipses. The shaded squares indicate the numerically computed stationary distribution from a stochastic simulation for $t \in [0, 5000N]$, $N = 500$ in both cases. (a) Full phase space; (b) enlargement near the equilibrium point $(x_H^*, x_L^*) = (5/11, 5/11)$. The blue dot indicates the equilibrium point. Parameter values: $\alpha = 20$, $\beta = 4$, $\gamma = 4$.

refer to the set that contains the $OH \rightarrow HH$ interaction, which was part of the initial model definition, as the “forward cycle” and the other set as the “reverse cycle” since the reaction rate for the $OH \rightarrow OO$ interaction acts in the reverse sense by increasing the number of transmitters in state O and reducing the number in state H . These are summarized schematically in Fig. 15, which is an extension of Fig. 1.

For the forward cycle we have

$$OH \xrightarrow{\hat{\alpha}_1} HH, \quad (43)$$

$$LO \xrightarrow{\hat{\beta}_1} OO, \quad (44)$$

$$HL \xrightarrow{\hat{\gamma}_1} LL, \quad (45)$$

where we note that the first reaction (43) is exactly the same as the one with rate $\hat{\alpha}$ in (1)—we will deal with this duplication

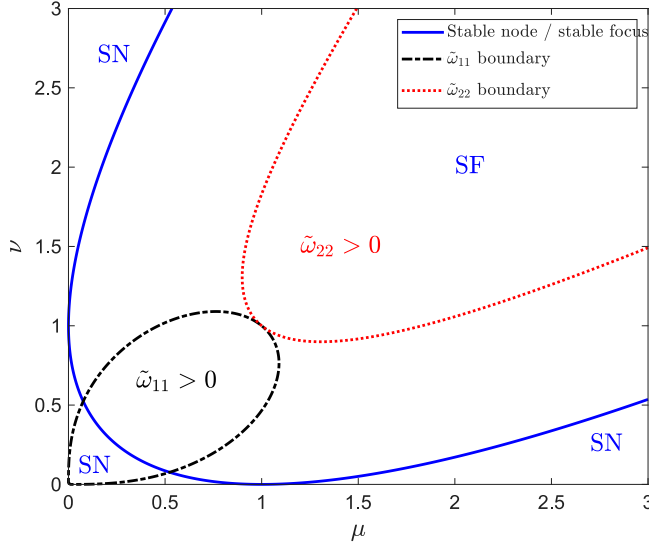


FIG. 14. The (μ, ν) plane showing (solid blue curve) the parabola (19) inside which the equilibrium point (x_H^*, x_L^*) is a stable focus (SF) and outside which it is a stable node (SN), as in Fig. 3. Also shown are the two regions where the diagonal entries of the power spectral density matrix $\tilde{\mathcal{P}}$ achieve maximum values for positive frequencies; the region where $\tilde{\omega}_{11} > 0$ is the same as the region where $\omega_{11} > 0$ in Fig. 3.

presently. For the reverse cycle we have



The stoichiometric matrix for (43)–(45) is exactly the matrix S defined in (11) while the stoichiometric matrix for (46)–(48) is $-S$. But the reaction rates differ since they contain different factors of O , L , and H .

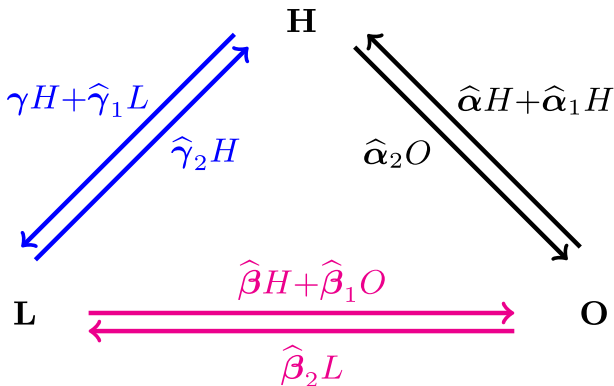


FIG. 15. State transitions including forward and reverse intransitive interactions. $\hat{\beta}_1$ and $\hat{\gamma}_1$ denote reaction rates in the forward cycle (together with $\hat{\alpha}$); $\hat{\alpha}_2$, $\hat{\beta}_2$, and $\hat{\gamma}_2$ denote reaction rates in the reverse cycle.

Extending $\mathbf{r}(\mathbf{x})$ from (10) to include all nine reactions we have

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix} 2\hat{\alpha}x_Hx_O \\ 2\hat{\beta}x_Hx_L \\ \gamma x_Hx_H \\ 2\hat{\alpha}_1x_Hx_O \\ 2\hat{\beta}_1x_Lx_O \\ 2\hat{\gamma}_1x_Hx_L \\ 2\hat{\alpha}_2x_Hx_O \\ 2\hat{\beta}_2x_Lx_O \\ 2\hat{\gamma}_2x_Hx_L \end{pmatrix} \equiv \begin{pmatrix} \alpha x_Hx_O \\ \beta x_Hx_L \\ \gamma x_Hx_H \\ \alpha_1x_Hx_O \\ \beta_1x_Lx_O \\ \gamma_1x_Hx_L \\ \alpha_2x_Hx_O \\ \beta_2x_Lx_O \\ \gamma_2x_Hx_L \end{pmatrix}, \quad (49)$$

where we work as before from the transition rates and introduce scaled parameters α and β to absorb the factors of 2. From (13) the mean-field ODEs $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$ are given by

$$\dot{x}_H = x_H[(\alpha + \alpha_1 - \alpha_2)x_O - \gamma x_H - (\gamma_1 - \gamma_2)x_L], \quad (50)$$

$$\dot{x}_L = x_H[\gamma x_H - \beta x_L] + x_L[(\gamma_1 - \gamma_2)x_H - (\beta_1 - \beta_2)x_O], \quad (51)$$

$$\dot{x}_O = x_H[\beta x_L - \alpha x_O] + x_O[(\beta_1 - \beta_2)x_L - (\alpha_1 - \alpha_2)x_H]. \quad (52)$$

So there are effectively only five parameters: β and γ , which must be non-negative, and the three combinations $\tilde{\alpha} := \alpha + \alpha_1 - \alpha_2$, $p := \beta_1 - \beta_2$, and $q := \gamma_1 - \gamma_2$, which can each take either sign. The diffusion matrix M , however, depends on different combinations of the rate parameters. Writing $x_O = 1 - x_H - x_L$ we have the 2D ODE system

$$\begin{aligned} \dot{x}_H &= x_H[\tilde{\alpha} - (\tilde{\alpha} + \gamma)x_H - (\tilde{\alpha} + q)x_L], \\ \dot{x}_L &= \gamma x_H^2 + p x_L^2 - p x_L + (p + q - \beta)x_H x_L, \end{aligned} \quad (53)$$

which is a direct extension of (17). We observe that the triangular phase-space region $x_H \geq 0$, $x_L \geq 0$, $x_H + x_L \leq 1$ is forward invariant and so must contain an equilibrium point. There are two equilibria on the boundary, at $(x_H, x_L) = (0, 1)$ and $(0, 0)$. The first of these is stable only when $p < 0$ and $q > 0$, independent of $\tilde{\alpha}$; the second is stable only when $p > 0$ and $\tilde{\alpha} < 0$, independent of q .

Although direct analytic calculations are cumbersome, from the above observations and numerical exploration, we find that there are only three qualitatively distinct possibilities: either there are no interior equilibria, and trajectories all tend to a stable boundary equilibrium, or there is a single interior equilibrium that is stable, or there are two equilibria in the interior, one of which is stable. These possibilities are independent of β and γ (which are both non-negative) and are easiest to summarize in the quadrants of the (p, q) plane for the two cases $\tilde{\alpha} > 0$ and $\tilde{\alpha} < 0$, shown in Fig. 16. We find numerically that there are no periodic orbits for any combination of parameter values and we are able to rule out the existence of periodic orbits in six of the eight quadrants shown in Fig. 16, the exceptions being the quadrants where $\tilde{\alpha}$, p , and q all have the same sign. In those cases we are able to demonstrate that, at least for small β and γ , the trace of the Jacobian at the interior equilibrium point will be negative, ensuring that a Hopf bifurcation cannot occur at least for small

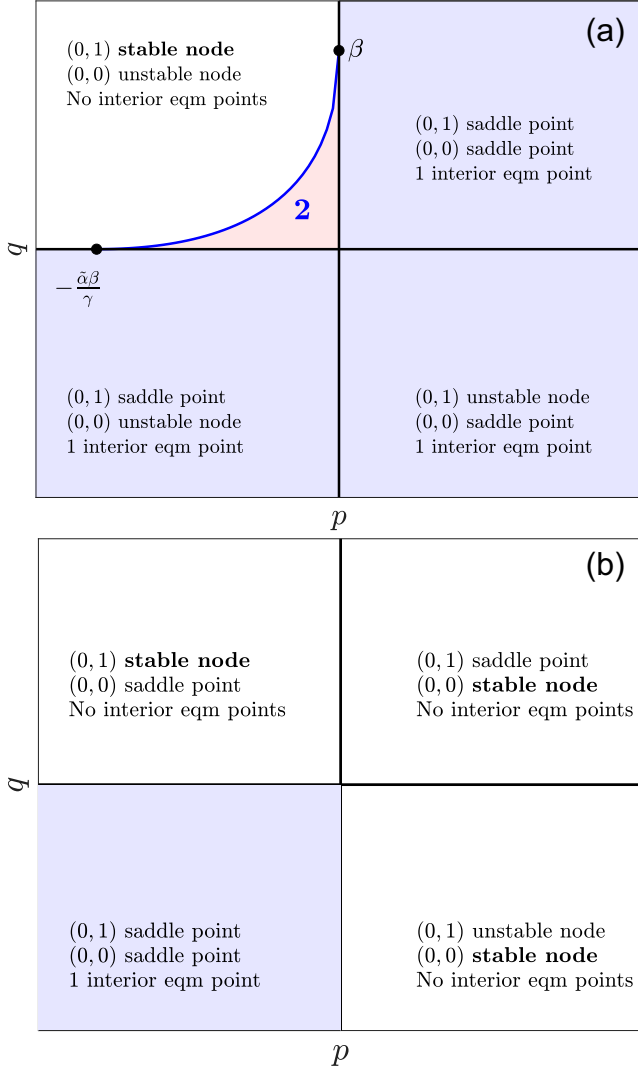


FIG. 16. Equilibria and their stability quadrant by quadrant in the (p, q) plane in the two cases (a) $\tilde{\alpha} > 0$ and (b) $\tilde{\alpha} < 0$. In all cases the existence and stability of equilibria is independent of β and γ . The blue-shaded areas indicate the existence of unique interior equilibrium; this equilibrium is stable when it exists. The pink-shaded region labeled “2” in (a) indicates the region in which two interior equilibria exist, illustrated in Fig. 17(a). These interior equilibria collide and disappear in a saddle-node bifurcation on the blue curve given by the part of (54), leading to no interior equilibria as illustrated in Fig. 17(b).

β and γ . Details of these calculations are contained in the Appendix.

Considering first the case $\tilde{\alpha} > 0$, illustrated in Fig. 16(a), we find that a single interior equilibrium point exists throughout three regions, with the exception being the quadrant $p < 0, q > 0$ in which there are either zero or two equilibria and the boundary equilibrium at $(x_H, x_L) = (0, 1)$ is linearly stable. That an interior equilibrium exists and is stable in all three blue-shaded quadrants is due to the forward invariance of the domain and the unstable nature of all other equilibria. The phase portraits in these blue-shaded regions are topologically the same as those shown in Fig. 4, with the only qualitative

difference being the oscillatory approach to the interior equilibrium in cases in which the eigenvalues there are a complex conjugate pair. In the upper-left quadrant trajectories may converge to the $(0, 1)$ equilibrium point and so the existence of a stable interior equilibrium is no longer guaranteed. The saddle-node bifurcation (blue curve) that bounds the region of existence of two equilibria can be computed by eliminating x_H between the right-hand sides of (53) and computing the discriminant of the resulting quadratic for x_L :

$$a_0 x_L^2 - a_1 x_L + \tilde{\alpha}^2 \gamma = 0,$$

where

$$a_0 = \tilde{\alpha}^2 \gamma - \tilde{\alpha} q^2 + (\tilde{\alpha} + \gamma)(\beta - p)q + a_2,$$

$$a_1 = 2\tilde{\alpha}^2 \gamma + a_2,$$

$$a_2 = \tilde{\alpha} \gamma (p + q + \beta) + \gamma^2 p + \tilde{\alpha}^2 (\beta - q).$$

After lengthy computation we find that the saddle-node bifurcation curve takes the form

$$(\tilde{\alpha} q + \gamma p)^2 - 2\tilde{\alpha} \beta (\tilde{\alpha} q - \gamma p) + (\tilde{\alpha} \beta)^2 = 0. \quad (54)$$

The curve described by (54) is tangent to each axis, at $p = -\tilde{\alpha} \beta / \gamma$ and $q = \beta$, respectively, as shown in Fig. 16(a). In the shaded area, initial conditions tend either to the stable interior equilibrium or to $(x_H, x_L) = (0, 1)$, as shown in Fig. 17(a). Outside the shaded area but within the same quadrant, all interior initial conditions tend to $(x_H, x_L) = (0, 1)$, as illustrated in Fig. 17(b).

In the case $\tilde{\alpha} < 0$ we find that no interior equilibria exist except in the quadrant $p, q < 0$ and in each of the other quadrants one of the boundary equilibria is stable—all interior trajectories tend to this equilibrium, albeit perhaps not monotonically, as illustrated by Fig. 18.

Starting from the original asymmetric cyclic formulation analyzed in Sec. III in which we have $\tilde{\alpha}, \beta, \gamma > 0$ and $p = q = 0$, we see that the desirable behavior of (monotonic) convergence to an interior equilibrium is sensitive to perturbations of two kinds: those that either reduce α , shifting into the cases in which trajectories all tend to a boundary equilibrium (meaning that all transmitters end up in either the **L** or **O** states), or that shift p to become negative (by increasing β_2) and shift q to become positive (by increasing γ_1). Both of the reactions with rates β_2 and γ_1 enhance, autocatalytically, transitions from transmitters in the **O** and **H** states into the **L** state; if the reaction rates β_2 and γ_1 are sufficiently large, then the system is tipped into a state in which all transmitters are in the **L** state. Intriguingly, varying other transition rates does not lead to any additional destabilizing possibilities.

VI. DISCUSSION

In this paper we considered a very simple and highly stylized model of a dense region in a self-organizing wireless network which is determining its behavior through the interaction between three competing system-level objectives: increase network coverage, reduce energy use, and increase the minimum SINR. Our model ignored any spatial structure or arrangement of transmitters and so the system could be described by the total number of transmitters in each of the three states: high power (**H**), low power (**L**), and off (**O**).

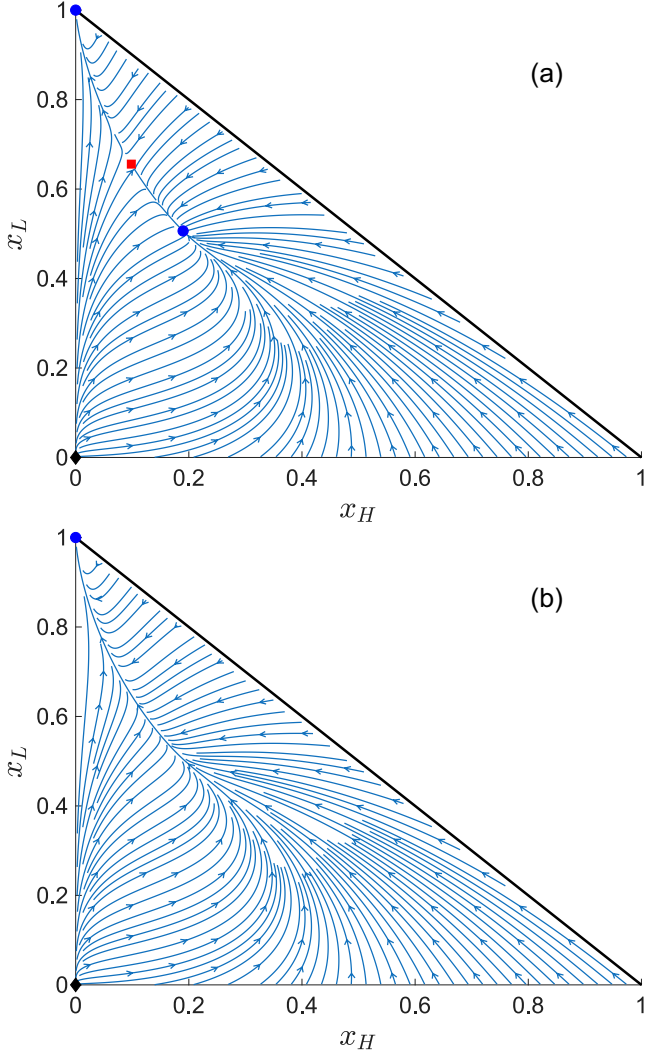


FIG. 17. Dynamics of (53) in the (x_H, x_L) plane. (a) For $q = 0.9$ inside the pink-shaded region in Fig. 16(a), trajectories tend to one of the two stable equilibria shown as blue dots; the red square indicates an interior saddle point. (b) For $q = 1.1$ outside the shaded region all trajectories tend to $(x_H, x_L) = (0, 1)$. The black diamond indicates the unstable node at the origin. Other parameter values: $\alpha = \beta = \gamma = 4$, $p = 1$.

Transmitters interacted in pairs in which at least one transmitter was in the high-power state; the transmitters in state **H** therefore drive the dynamics and cause others to switch stochastically, with this switching assumed to be at given rates α, β, γ .

In the limit of an infinite number of transmitters, the resulting Markov jump process converges to a set of ODEs describing the mean-field limit; these are discussed in Sec. III and show that (as long as the initial state contains at least one transmitter in the high-power state) the system converges to a unique equilibrium $\mathbf{x}^*$ in which the proportion of high-power transmitters is $x_H^* = c/\gamma$ where γ is the rate at which a pair $\{\mathbf{H}, \mathbf{H}\}$ of high power transmitters switches into the state $\{\mathbf{H}, \mathbf{L}\}$, and $c = H(\alpha, \beta, \gamma)/3$ where $H(\cdot)$ denotes the harmonic mean. Trajectories approach the equilibrium either

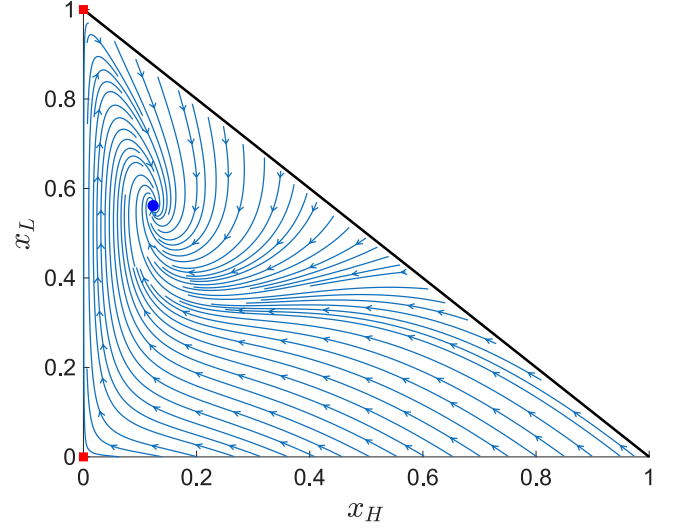


FIG. 18. Dynamics of (53) in the (x_H, x_L) plane in the case $\alpha = p = q = -1$, $\beta = \gamma = 2$, showing the damped oscillatory convergence to equilibrium in the lower-left quadrant of Fig. 16(b). The red squares indicate saddle points at (0,0) and (0,1).

with damped oscillations or monotonically, as indicated in Figs. 4(a) and 4(b), respectively.

For a large but finite number of transmitters N the Markov process can be approximated by a multivariate SDE, analyzed in Sec. IV. The stochastic fluctuations around the equilibrium point are often dominated by frequency components of the same order as the rate parameters and we find that, for a given set of parameter values, the numbers of transmitters in at most two of the three power levels can have a dominant nonzero frequency component, but not all three. Comparisons between the predicted PSD and numerical simulations using the Gillespie stochastic simulation algorithm show excellent agreement (see Figs. 7–9). We derive also the stationary Gaussian probability distribution of the numbers of transmitters (39), which indicates the likely variation in state space around the mean-field equilibrium.

In practice, the switching rates are determined by the environment and the policies of the SON wireless system itself, in order to ensure there is sufficient capacity to handle the amount of UE present (reducing γ , for example, increases x_H at equilibrium, increasing the overall network capacity). The minimal model admits equilibria with all possible proportions x_H, x_L , and x_O and so, through variation of the switching rates, the system can stabilize in the neighborhood of any required combination of proportions, with the caveat that when x_H is very small the likelihood of the stochastic dynamics hitting the absorbing set $x_H = 0$ increases; this can of course be addressed by introducing random switching of transmitters into the **H**, an additional first-order reaction that complicates the analysis but is unlikely to influence the dynamics significantly when the system is further away from the $x_H = 0$ plane.

A. Peaks in the PSD at positive ω

It is of interest to note that the set of parameter values for which the PSD for any of the variables x_H, x_L, x_O has a peak at a positive frequency does not coincide with the set

of parameter values for which the mean-field equilibrium has eigenvalues with nonzero imaginary parts; indeed, while all pairs of these regions intersect, none of these sets is contained in any of the others as Fig. 3 illustrates. The existence of nonzero off-diagonal elements of the drift matrix A (22) and covariance matrix B (23) are clearly crucial since for a one-dimensional Ornstein-Uhlenbeck process the power spectral density always has a peak at zero frequency. So in the case that A and B are diagonal, we have uncoupled one-dimensional Ornstein-Uhlenbeck processes and the peaks in the PSD are always at zero frequency. Since in the two-dimensional analysis of Sec. IV B we find that $\tilde{B}$ is diagonal in these coordinates, clearly having nonzero off-diagonal elements of the drift matrix $\tilde{A}$ is sufficient. Intuitively one can hypothesize that, if trajectories of the linear deterministic approximation are sufficiently curved near the equilibrium, due to nonorthogonal eigenvectors, for example, then this non-normality of the matrix allows transient amplification of stochastic perturbations on a specific timescale, which in turn results in a peak of the PSD at that frequency.

Mathematically we can add some precision to the above intuitive argument as follows. Consider a general stochastic ODE in $\mathbb{R}^2$ with drift matrix A and covariance matrix B in the form

$$A := \begin{pmatrix} -1 & a \\ c & -b \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $B_{12} = B_{21}$, $B_{11} > 0$, $B_{22} > 0$, and after a time rescaling we can set $A_{11} = -1$. a , b , and c are real coefficients and we further assume that $b > 0$. Then, following the standard calculations outlined earlier, the condition for the PSD to attain a peak at a positive value of ω^2 is

$$\begin{aligned} & B_{11}a^2c^2 - ac(2B_{22}a^2 + 2B_{11}b(1+b)) \\ & - 2B_{12}ab(2ac + 1 + b^2) > B_{22}a^2(1 + b^2) + B_{11}b^4. \end{aligned} \quad (55)$$

Here we have not sought to put either A or B into a standard form. Several conclusions can be drawn immediately that support the intuition above, generally expressed as the need for the off-diagonal entries in A and B to be sufficiently negative. In particular, we see that

- (1) if $a = c = 0$, then (55) cannot be satisfied for any choice of B_{12} ;
- (2) if $c = 0$ but $a \neq 0$, then (55) can be satisfied if the product $B_{12}a$ is sufficiently negative; and
- (3) if $B_{12} = 0$ but $ac \neq 0$, then (55) can be satisfied if the product ac is sufficiently negative.

The last of these points is related to, but distinct from, the condition that the ODE $\dot{x} = Ax$ has a stable focus at the origin rather than a stable node: this holds when $-4ac > (1 - b)^2$, which requires $ac < 0$.

To summarize, the existence of coupling terms both through the deterministic dynamics and the covariance of the noise term lead to the generation of an intrinsic timescale within the dynamical relaxation near equilibrium. This has implications for the SON communications scenario that motivated this analysis since this new low-frequency dynamical oscillation timescale, usually at a rate much slower than any

of the switching rates themselves, is likely to make it harder to predict and control the system.

B. Robustness of the minimal model

Our extension of the minimal model and the analysis of Sec. V allows us to draw conclusions about further state switches that would increase the unpredictability of the system as a whole. First, in the case $\tilde{\alpha} > 0$, which is closer to the unperturbed problem, for which $\tilde{\alpha} = \alpha > 0$, the effect of $p = \beta_1 - \beta_2 < 0$ and $q = \gamma_1 - \gamma_2 > 0$, i.e., allowing additional (autocatalytic) recruitment of transmitters in states H and O into state L , is to generate two interior equilibria, as indicated in the upper-left quadrant of Fig. 16(a). For the stochastic version of the problem this is destabilizing since over long timescales a stochastic trajectory will cross the stable manifold of the saddle point and converge on a neighborhood of the equilibrium at $(x_H, x_L) = (0, 1)$, meaning that all transmitters will, probably inefficiently, sit at an intermediate power level, and the H states, which were the drivers of the original dynamics, are lost.

Second, in the case $\tilde{\alpha} < 0$, which implies that α_2 dominates, unsurprisingly we find that transmitters either switch from H to O at a sufficiently high rate to remove the interior equilibrium and stabilize the state in which all transmitters are switched off, or (if $p = \beta_1 - \beta_2 < 0$ and $q = \gamma_1 - \gamma_2 > 0$) then all transmitters move to the low state and again the dynamics fails to stabilize at a controllable, interior equilibrium. In the case in which $\tilde{\alpha}$, p , and q are all negative, we find that an interior equilibrium persists but due to the spiraling of trajectories we expect that the stochastic dynamics are even further subject to dynamics on longer timescales than the switching times for individual transmitters, making it likely to be less controllable than for the original minimal model.

We can conclude, though, that if the additional switching routes described by the forward cycle (43)–(45) and the reverse cycle (46)–(48) are present but at lower rates than the switches (1)–(3) in the minimal model, then the behavior of the minimal model will persist. In particular, the additional switch (44) with rate β_1 appears really to reinforce the minimal dynamics, probably because as long as $\tilde{\alpha}$ is sufficiently positive, the additional transmitters in the O state will be relatively rapidly switched on into the H state, stabilizing the system overall.

C. Future directions

The analysis presented here is likely to generalize in a number of directions. If we consider n power levels rather than three, with x_j denoting the proportion of transmitters in power level j , with $j = 1$ corresponding to H and $j = n$ corresponding to O , then the mean-field ODEs take the form

$$\begin{aligned} \dot{x}_1 &= x_1(\gamma_n x_n - \gamma_1 x_1), \\ \dot{x}_j &= x_j(\gamma_{j-1} x_{j-1} - \gamma_j x_j) \quad \text{for } j = 2 \dots n, \end{aligned}$$

which have a unique equilibrium $x_j = c/\gamma_j$, if $x_1(0) > 0$, where, as before, $cn = H(\gamma_1, \dots, \gamma_n)$ is the harmonic mean of the rates γ_j . Global stability (for initial conditions for which $x_1 > 0$) is again established by consideration of the Lyapunov function $V(x) = \sum_{j=1}^n \gamma_j x_j^2$. Although the analysis

is conducted using fixed switching rates, for slowly varying rates, and a sufficiently large number of transmitters, the results presented here are likely to remain applicable because separation of timescales allowing the system to converge to a quasistationary equilibrium.

The abstract nature of the mathematical model may well make it of use in diverse contexts, for example as a model of social dynamics where an initially homogeneous shared trait spontaneously becomes differentiated between population subgroups, with the emergence of greater heterogeneity being driven by repeated interactions with the original dominant variant of the trait. One particularly intriguing set of connections is with chemical catalysis [37,38] where catalytic loops involving three or more species are able to drive net fluxes of molecules around the loop through the influence of external forcing, in the form of oscillations in time of the energy state of a catalytic surface; these environmental changes mean that the system is not in equilibrium and so does not satisfy detailed balance. Loops comprising first-order reactions have long been discussed in the chemistry literature [39,40] and so the controllability, or not, of such catalytic loops is clearly of interest in a variety of fields.

ACKNOWLEDGMENTS

We acknowledge financial support from the Engineering and Physical Sciences Research Council, part of UKRI, through the Centre for Doctoral Training in Statistical Applied Mathematics at Bath (SAMBa) Grant No. EP/L015684/1, and from BT plc.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: NON-EXISTENCE OF PERIODIC ORBITS

Here we give details of the analysis of (50)–(52), subject to the constraint $x_H + x_L + x_O = 1$, applying Dulac's criterion to rule out the existence of periodic orbits in all cases except those in which $\tilde{\alpha}$, p , and q all have the same sign. We comment in Appendix 4 on these two cases separately.

The existence of periodic orbits in planar polynomial vector fields is a long-studied topic, often referred to as the second part of Hilbert's 16th problem, posed in 1900 [41]. In our context the form (53) demonstrates that there is an invariant line due to the factor of x_H in the first equation. A well-known consequence of this is that the system can have at most one limit cycle [42–44].

Our approach here in Appendixes 1–3 is direct: we eliminate each of x_O , x_L , and x_H in turn from (50) to (52) and apply Dulac's theorem to the resulting 2D vector field. Dulac's theorem states that if the divergence of a 2D vector field multiplied by any continuously differentiable (weight) function, takes only one sign throughout a simply connected domain, then there are no periodic orbits in that domain.

1. Case 1

Let $x_O = 1 - x_H - x_L$ in (50) and (51). Then the 2D vector field $\mathbf{f}_{HL}$ is defined by

$$\begin{aligned}\dot{x}_H &= x_H[\tilde{\alpha} - (\tilde{\alpha} + \gamma)x_H - (\tilde{\alpha} + q)x_L], \\ \dot{x}_L &= \gamma x_H^2 + p x_L^2 - p x_L + x_H x_L(p + q - \beta).\end{aligned}$$

We compute the divergence using the weight function $1/(x_H x_L)$ and obtain

$$\nabla \cdot \left(\frac{1}{x_H x_L} \mathbf{f}_{HL} \right) = -(\tilde{\alpha} + \gamma) \frac{1}{x_L} - \gamma \frac{x_H}{x_L^2} + p \frac{1}{x_H}, \quad (\text{A1})$$

which is negative when $\tilde{\alpha} > 0$ and $p < 0$, for either sign of q , since $\gamma \geq 0$ and the variables x_j all lie in $[0,1]$.

2. Case 2

Let $x_L = 1 - x_H - x_O$ in (50) and (52). Then the 2D vector field $\mathbf{f}_{HO}$ is defined by

$$\begin{aligned}\dot{x}_H &= x_H[-q + (q - \gamma)x_H + (\tilde{\alpha} + q)x_O], \\ \dot{x}_O &= \beta x_H(1 - x_H) + p x_O(1 - x_O) - (\tilde{\alpha} + \beta + p)x_H x_O.\end{aligned}$$

Computing the divergence using the weight function $1/(x_H x_O)$ we obtain

$$\nabla \cdot \left(\frac{1}{x_H x_O} \mathbf{f}_{HO} \right) = (q - \gamma) \frac{1}{x_O} - \beta \frac{(1 - x_H)}{x_O^2} - p \frac{1}{x_H}, \quad (\text{A2})$$

which is negative when $p > 0$ and $q < 0$, for either sign of $\tilde{\alpha}$, since $\beta, \gamma \geq 0$ and $0 \leq x_O, x_H \leq 1$.

3. Case 3

Let $x_H = 1 - x_L - x_O$ in (50) and (52). Then the 2D vector field $\mathbf{f}_{LO}$ is defined by

$$\begin{aligned}\dot{x}_L &= \gamma + (q - \beta - 2\gamma)x_L + \gamma x_O(x_O - 2) \\ &\quad + (\beta + \gamma - q)x_L^2 + (\beta + 2\gamma - p - q)x_L x_O, \\ \dot{x}_O &= \beta x_L(1 - x_L) - \tilde{\alpha} x_O + (\tilde{\alpha} - \beta + p)x_L x_O + \tilde{\alpha} x_O^2.\end{aligned}$$

We then compute the divergence using the weight function $1/(x_L x_O)$ and obtain

$$\begin{aligned}\nabla \cdot \left(\frac{1}{x_L x_O} \mathbf{f}_{LO} \right) &= - \left[\gamma \frac{1 - x_O + x_L}{x_L^2 x_O} + \frac{\beta}{x_O^2} \right] x_H \\ &\quad + \frac{\tilde{\alpha}}{x_L} - \frac{q}{x_O},\end{aligned} \quad (\text{A3})$$

which is negative when $\tilde{\alpha} < 0$ and $q > 0$, for either sign of p .

Putting the calculations in Appendixes 1–3 together we conclude that as long as the signs of two of $\tilde{\alpha}$, p , and q are different, we can guarantee that no periodic orbits exist.

4. Case 4

The regions of parameter space in which all of $\tilde{\alpha}$, p , and q are positive, or all are negative, are not covered by cases 1–3 and contain a unique interior equilibrium point that is not a saddle point. We were not able to rule out the existence of periodic orbits completely in these regions, but numerical investigations did not indicate the existence of any. While the

case $\beta = \gamma = 0$ leads to all points in the interior of the domain lying on periodic orbits around a center, for small $\beta > 0$ and $\gamma > 0$ we are able to verify that the trace of the linearization at this equilibrium point is always negative in these cases in which $\tilde{\alpha}$, p , and q all have the same sign, implying that no Hopf bifurcation occurs at least for small β and γ , and so no isolated periodic orbits are possible for small β and γ . The calculation proceeds in a perturbative way treating β and γ as small, positive parameters. The interior equilibrium point for (53) is located at

$$x_H^* = \frac{p}{\tilde{\alpha} + p + q} + \frac{p(\tilde{\alpha}^2\beta + \tilde{\alpha}\beta q - \gamma pq)}{\tilde{\alpha}q(\tilde{\alpha} + p + q)^2} + O(|\beta, \gamma|^2),$$

$$x_L^* = \frac{\tilde{\alpha}}{\tilde{\alpha} + p + q} - \frac{p(\tilde{\alpha}\beta + \gamma q)}{q(\tilde{\alpha} + p + q)^2} + O(|\beta, \gamma|^2),$$

where the notation $O(|\beta, \gamma|^2)$ indicates terms that are second order in any combination of β and γ . Hence the trace of the

Jacobian matrix evaluated at (x_H^*, x_L^*) is computed to be

$$\begin{aligned} \text{tr}(J|_{(x_H^*, x_L^*)}) &= \tilde{\alpha} - p - x_H^*(2\tilde{\alpha} + \beta + 2\gamma - p - q) \\ &\quad - x_L^*(\tilde{\alpha} - 2p + q) \\ &= -\frac{\beta\tilde{\alpha}^2p + \gamma pq(\tilde{\alpha} + p)}{\tilde{\alpha}q(\tilde{\alpha} + p + q)} + O(|\beta, \gamma|^2), \end{aligned}$$

which is always negative when $\tilde{\alpha}$, p , and q all take the same sign, since the numerator and denominator both involve only cubic combinations of $\tilde{\alpha}$, p , and q . Note that when $\beta = \gamma = 0$ the trace vanishes, as is well known from the behavior of the model when only the intransitive interactions are included. In the case $\beta = \gamma = 0$ both the coordinate axes $x_L = 0$ and $x_H = 0$ are invariant, and all trajectories in the interior are periodic and lie in level sets of the function $V = \ln(x_H^p x_L^{\tilde{\alpha}} x_O^q)$.

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- [1] T. Rogers and T. Gross, Consensus time and conformity in the adaptive voter model, *Phys. Rev. E* **88**, 030102(R) (2013).
 - [2] H. Silk, G. Demirel, M. Homer, and T. Gross, Exploring the adaptive voter model dynamics with a mathematical triple jump, *New J. Phys.* **16**, 093051 (2014).
 - [3] S. Redner, Reality-inspired voter models: A mini-review, *C. R. Phys.* **20**, 275 (2019).
 - [4] A. Vendeville, Voter model can accurately predict individual opinions in online populations, *Phys. Rev. E* **111**, 064310 (2025).
 - [5] R. M. May and W. J. Leonard, Nonlinear aspects of competition between three species, *SIAM J. Appl. Math.* **29**, 243 (1975).
 - [6] B. Sinervo and C. M. Lively, The rock-paper-scissors game and the evolution of alternative male strategies, *Nature (London)* **380**, 240 (1996).
 - [7] B. Sinervo, B. Heulin, Y. Surget-Groba, J. Clobert, D. B. Miles, A. Corl, A. Chaine, and A. Davis, Models of density-dependent genic selection and a new rock-paper-scissors social system, *Am. Nat.* **170**, 663 (2007).
 - [8] F. H. Busse and K. E. Heikes, Convection in a rotating layer: A simple case of turbulence, *Science* **208**, 173 (1980).
 - [9] M. R. E. Proctor and C. A. Jones, The interaction of two spatially resonant patterns in thermal convection. Part I. Exact 1:2 resonance, *J. Fluid Mech.* **188**, 301 (1988).
 - [10] J. H. P. Dawes, C. M. Postlethwaite, and M. R. E. Proctor, Instabilities induced by a weak breaking of a strong spatial resonance, *Phys. D* **191**, 1 (2004).
 - [11] J. Guckenheimer and P. Holmes, Structurally stable heteroclinic cycles, *Math. Proc. Cambridge Philos. Soc.* **103**, 189 (1988).
 - [12] P. Chossat and R. Lauterbach, Methods in equivariant bifurcations and dynamical systems, *Advanced Series in Nonlinear Dynamics* (World Scientific, Singapore, 2000), Vol. 15.
 - [13] D. C. Groothuizen Dijkema, V. Kirk, and C. M. Postlethwaite, Analysis of dynamics near heteroclinic networks in $\mathbb{R}^4$ with a projected map, *SIAM J. Appl. Dyn. Syst.* **25**, 588 (2026).
 - [14] C. M. Postlethwaite and A. M. Rucklidge, Stability of cycling behaviour near a heteroclinic network model of Rock–Paper–Scissors–Lizard–Spock, *Nonlinearity* **35**, 1702 (2022).
 - [15] Q. Yang, T. Rogers, and J. H. P. Dawes, Demographic noise slows down cycles of dominance, *J. Theor. Biol.* **432**, 157 (2017).
 - [16] C. M. Postlethwaite and J. H. P. Dawes, Regular and irregular cycling near a heteroclinic network, *Nonlinearity* **18**, 1477 (2005).
 - [17] Q. Yang, Oscillations in stochastic multi-species interactions, Ph.D. thesis, University of Bath, 2018.
 - [18] A. Szolnoki, M. Mobilia, L.-L. Jiang, B. Szczesny, A. M. Rucklidge, and M. Perc, Cyclic dominance in evolutionary games: A review, *J. R. Soc. Interface* **11**, 20140735 (2014).
 - [19] Th. Ruijgrok and M. Ruijgrok, A reaction-diffusion equation for a cyclic system with three components, *J. Stat. Phys.* **87**, 1145 (1997).
 - [20] N. Marchetti, N. R. Prasad, J. Johansson, and T. Cai, Self-organizing networks: State-of-the-art, challenges and perspectives, in *2010 8th International Conference on Communications Bucharest, Romania* (IEEE, Piscataway, NJ, 2010), pp. 503–508.
 - [21] J. Moysen and L. Giupponi, From 4G to 5G: Self-organized network management meets machine learning, *Comput. Commun.* **129**, 248 (2018).
 - [22] A. L. L. Middleton, Distributed optimisation of LTE systems. Ph.D. thesis, University of Bath, 2019.
 - [23] A. Ghafourian, O. Georgiou, E. Barter, and T. Gross, Wireless localization with diffusion maps. *Sci. Rep.* **10**, 20655 (2020).
 - [24] P. Bao, Q. Xu, and Y. Yang, Modeling and aggregated control of large-scale 5G base stations and backup energy storage systems towards secondary frequency support, *Appl. Energy* **357**, 122498 (2024).
 - [25] A. Aggarwal, P. Kalita, and D. Selvamuthu, Stochastic modeling of a base station in 5G wireless networks for energy aspects using advanced sleep mechanism, *Methodol. Comput. Appl. Probab.* **27**, 58 (2025).
 - [26] C. W. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences* (Springer, Berlin, 2009).
 - [27] T. G. Kurtz, Strong approximation theorems for density dependent Markov chains, *Stochastic Processes Appl.* **6**, 223 (1978).

- [28] A. J. McKane and T. J. Newman, Predator-prey cycles from resonant amplification of demographic stochasticity, *Phys. Rev. Lett.* **94**, 218102 (2005).
- [29] P. H. Baxendale and P. E. Greenwood, Sustained oscillations for density dependent Markov processes, *J. Math. Biol.* **63**, 433 (2011).
- [30] J. H. P. Dawes and M. Souza, A derivation of Holling's type I, II and III functional responses in predator-prey systems, *J. Theor. Biol.* **327**, 11 (2013).
- [31] D. T. Gillespie, Exact stochastic simulation of coupled chemical reactions, *J. Phys. Chem.* **81**, 2340 (1977).
- [32] N. Abdennur, Gillespie stochastic simulation algorithm, GitHub 2025, <https://github.com/nvictus/Gillespie>.
- [33] F. Di Patti, R. Arbel-Goren, D. Fanelli, and J. Stavans, Demographic noise induced patterns in space and time, *Phys. Rep.* **1179**, 1 (2026).
- [34] L. S. Wang, J. Yu, Y. Liang, and J. Zhang, The breakdown of linear quasi-cycles: Demographic noise and absorbing boundaries in finite predator-prey system, *Electron. Res. Arch.* **34**, 4248 (2026).
- [35] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (Elsevier, Amsterdam, 2007).
- [36] H. Risken and T. Frank, *The Fokker-Planck Equation: Methods of Solutions and Applications*, Springer Series in Synergetics (Springer, New York, 1996).
- [37] M. A. Murphy, S. R. Gathmann, C. J. Bartel, O. A. Abdelrahman, and P. J. Dauenhauer, Catalytic resonance theory: Circumfluence of programmable catalytic loops, *J. Catal.* **430**, 115343 (2024).
- [38] M. A. Murphy, K. Noordhoek, S. R. Gathmann, P. J. Dauenhauer, and C. J. Bartel, Catalytic resonance theory: Forecasting the flow of programmable catalytic loops, *Digital Discovery* **4**, 411 (2025).
- [39] A. G. Fredrickson, Stochastic triangular reactions, *Chem. Eng. Sci.* **21**, 687 (1966).
- [40] L. Onsager, Reciprocal relations in irreversible processes. I., *Phys. Rev.* **37**, 405 (1931).
- [41] M. Alberich-Carramiñana, A. Ferragut, and J. Llibre, Quadratic planar differential systems with algebraic limit cycles via quadratic plane Cremona maps, *Adv. Math.* **389**, 107924 (2021).
- [42] W. A. Coppel, Some quadratic systems with at most one limit cycle, in *Dynamics Reported: A Series in Dynamical Systems and their Applications* (Springer Nature, Berlin, 1989), Vol. 2, p. 61.
- [43] B. Coll and J. Llibre, Limit cycles for a quadratic system with an invariant straight line and some evolution of phase portraits, in *Qualitative Theory of Differential Equations*, Colloquia Mathematica Societatis Janos Bolyai Vol. 53 (Bolyai Institut, Szeged, Hungary, 1988), pp. 111–123.
- [44] R. Bamón, Quadratic vector fields in the plane have a finite number of limit cycles, *Publ. Math. Inst. Hautes Études Sci.* **64**, 111 (1986).