

Thermodynamic cost of random-time protocols

Izaak Neri *Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom*

(Received 2 January 2025; revised 1 July 2025; accepted 19 September 2025; published 9 October 2025)

Systems that are driven by a randomly timed, external protocol can seemingly violate the second law of thermodynamics. We show that this thermodynamic paradox is resolved if the outcome of the random time is stored in a memory device. Specifically, we show that the average work required to erase the memory is always larger than the average work gained from the protocol. We also discuss concrete setups that measure random times directly without continuous monitoring. Taken together, this paper discusses the relationship between temporal information and thermodynamics. This framework is relevant for external protocols employing random times, such as stochastic resetting protocols and cyclically driven heat engines that use randomly timed protocols.

DOI: [10.1103/8dmt-256h](https://doi.org/10.1103/8dmt-256h)

I. INTRODUCTION

In the recent literature there has been interest in thermodynamic setups for which an external agent follows a protocol that begins or ends at times that vary across different realizations of the process. Such random-time protocols can be used to convert heat into work or to speed up search processes.

Examples of randomly timed protocols that convert heat into work, and thus seemingly realize a perpetual mobile of the second kind, include processes that terminate upon the occurrence of a specific event [1–3], often referred to as gambling demons [4–6]; cyclic protocols that incorporate a randomly timed step [7–11]; and stochastic heat engines governed by protocols that contain a randomly time step [1,12]. A simple example of a cyclic protocol that involves a randomly timed step and converts heat into work is shown in Fig. 1. In this setup a demon inserts a movable wall after a particle has reached the right boundary of a bounded domain. Assuming the demon can measure the time when the particle reaches the right boundary without error, the demon can exchange an arbitrary amount of heat into work.

Stochastic resetting is another example of a system that is driven out of equilibrium by a randomly timed protocol. In this case, a diffusing particle is returned to its point of origin at a random time [13,14]. Stochastic resetting protocols can reduce the average time required for the particle to find a target, albeit at the expense of an energetic cost in the form of work done on the particle. Indeed, physically, the resetting procedure can be implemented through a resetting potential that is turned on at a random time and returns the particle to the origin, see Refs. [15–21] for theoretical implementations and Refs. [22–24] for experiments. A thermodynamic analysis of this resetting setup shows that during the protocol work is done on the particle [25,25,26].

In this paper, we identify a thermodynamic cost associated with randomly timed protocols that is informational in nature. Specifically, drawing an analogy with Maxwell's demon in Szilard's engine [27,28], we argue that a nonautonomous demon implementing a random-time protocol must store the time variable in a memory device. The associated thermodynamic cost arises, as with Szilard's engine, from restoring the memory device to its initial state [27,28]. As we show, apparent violations of the second law of thermodynamics are resolved once this additional cost is taken into account.

Furthermore, we introduce in this paper a nonautonomous demon that directly measures first-passage times. Although first-passage demons have been considered before in the literature, see Refs. [7–11], these nonautonomous demons measure the whole trajectory of the system up to the first-passage time through continuous monitoring. For nonautonomous demons, this approach requires a large number of measurements, as the state of the system should be measured at regularly spaced intervals of time. Here, instead, we develop a simpler setup for which a nonautonomous demon directly measures the first-passage time using a single measurement. We present concrete examples of setups that directly measure first-passage times.

The paper is structured as follows. In Sec. II, we introduce the system setup for externally controlled protocols that operate with random times, focusing on the isothermal case. In Sec. III, we argue, by analogy with Szilard's engine, that a nonautonomous demon implementing a random-time protocol requires a memory device to run the protocol. In Sec. IV, we show that storage of the random time is sufficient to derive the second law of thermodynamics for systems controlled by random-time protocols, further confirming that a memory device is needed in these setups. The derivation uses Landauer's erasure principle [27–29] and information thermodynamics for nonautonomous Maxwell demons with a path-dependent protocol [30–32]. Sections V to VII explore a paradigmatic example of a temporal information engine, which is a random-time version of Szilard's engine, as illustrated in Fig. 1. This model is arguably the simplest example of a temporal information engine. It is a simplification of previous setups

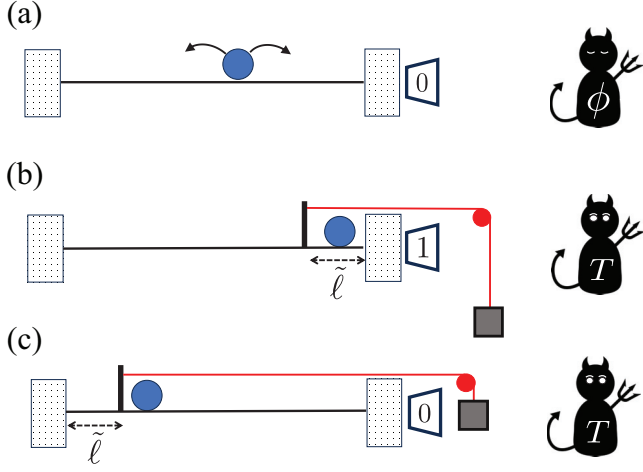


FIG. 1. Random-time Szilard engine: An example of an engine that uses temporal information to convert heat into work. (a) A particle diffuses in an interval with reflecting boundary conditions. At one end of the segment there is a detector (depicted as a trapezoid) that is in the inactive state (0), a nonautonomous Maxwell demon that is dormant, and a memory device in its initial state (ϕ). (b) The particle has hit the right boundary. This activates the detector (brings it into state 1) that sends a signal to Maxwell's demon and awakens it. The demon stores the time T in its memory and initiates a protocol of fixed duration to extract work from the system. In the present example, the demon inserts a movable wall together with an external load at a distance $\tilde{\ell}$ from the right side. (c) The demon quasistatically moves the wall to the left until it reaches a distance $\tilde{\ell}$ from the left boundary. In the process, a weight is lifted and heat is converted into work. The detector has relaxed to its inactive state 0. The demon erases its memory and returns to the sleeping state in (a).

based on continuous Maxwell demons [7–11], as the demon measures the random time directly without continuous monitoring. In Sec. V, we consider the random-time Szilard engine in discrete time and without measurement errors, and in Sec. VI we consider the corresponding model in continuous time. In Sec. VII we develop simple models for random-time detection without continuously monitoring the system, confirming that the model is physically sound. In Sec. VIII we show that stochastic heat engines that use random-time protocols satisfy the Carnot bound on their efficiency, which is further evidence in support of the claim that memory storage is necessary in random-time protocols. Section IX compares the setups of Maxwell demons that continuously monitor a system with those that directly measure first-passage times. We end the paper with a discussion in Sec. X and a few Appendices that contain the derivations of results presented in the main text.

II. SYSTEM SETUP: EXTERNAL PROTOCOLS WITH TEMPORAL INFORMATION

We consider a mesoscopic system whose slow degrees of freedom are described by a stochastic process $X(t) \in \mathcal{X}$, where $t \geq 0$ is a discrete or continuous time index. The system is subject to a potential $u(x; \lambda)$, with $x \in \mathcal{X}$ and with λ a parameter that can be controlled by an external agent. Furthermore, the system is in contact with an external load that can

do work on the system or, conversely, the system can do work on the load. For clarity, we first focus on isothermal systems in contact with a thermal reservoir at inverse temperature β . In Sec. VIII we generalize this setup to systems interacting with two thermal reservoirs.

We assume that at the initial time $t = 0$ the system is in equilibrium with the thermal reservoir, and thus the initial distribution of $X(0)$ takes the form

$$p_{X(0)}(x) = \exp(-\beta[u(x; \lambda_-) - f_-]), \quad (1)$$

where f_- is the free energy.

Subsequently, a nonautonomous demon implements a protocol $\lambda(s)$ with $s \in [0, \tau]$ that is initiated (terminated) at a time T ($T + \tau$) that is dependent on the trajectory of the process X . Specifically, the demon implements the protocol

$$\lambda_T(t) = \begin{cases} \lambda_- & \text{if } t \in [0, T], \\ \lambda(t - T) & \text{if } t \in [T, T + \tau], \\ \lambda_+ & \text{if } t \geq T + \tau. \end{cases} \quad (2)$$

Note that in standard setups in information thermodynamics the demon chooses one protocol out of two or more options based on the measurement of X at a fixed time [27,33,34]. On contrary, here the protocol is always the same, but the time T when the protocol starts differs from one realization to the other.

The random time T is a function of the trajectory X_0^∞ ; we use the notation $X_s^t = \{X(u) : u \in [s, t]\}$ for trajectories. Due to causality, we assume that T is statistically independent of X_T^∞ (i.e., T is a stopping time [1,35,36]).

The work done by the external load on the system is given by [37,38]

$$W(t) = \int_0^t \left. \frac{\partial u(X(s); \lambda)}{\partial \lambda} \right|_{\lambda=\lambda_T(s)} \frac{d\lambda_T(s)}{ds} ds, \quad (3)$$

with $W(t) > 0$ when the load does work on the system and $W(t) < 0$ when the system does work on the load. By the first law of thermodynamics, the heat exchanged with the thermal reservoir is given by

$$Q(t) = W(t) - u(X(t), \lambda_T(t)) + u(X(0), \lambda_T(0)), \quad (4)$$

with $Q(t)$ the heat released into the reservoir.

In what follows, we denote averages over multiple realizations of the process X by $\langle \cdot \rangle$.

III. MEMORY STORAGE OF THE RANDOM TIME T

We argue that a nonautonomous Maxwell demon needs to store the random time T in a memory to run the protocol λ_T . The underlying reason is the same as why Maxwell's demon in Szilard's engine requires a memory [27,28,39]: If an external protocol depends on a random variable, then this random variable must be stored in a memory, as otherwise it is not possible for the demon to evaluate the protocol. The only conceptual difference between Maxwell's demon in Szilard's engine and nonautonomous demons in random-time protocols is that in the former the random variable is a spatial coordinate, while in the latter it is a temporal coordinate. Since there is no justification for treating time variables differently from spatial variables, we require that demons also need to store random times T in a memory.

Let us slightly elaborate on the above argument that makes an analogy between protocols that depend on spatial information and those that depend on temporal information. In Szilard's engine, a nonautonomous Maxwell demon measures at a fixed time, say, $t = \tau_m$, the position $X(\tau_m)$ of a particle diffusing in a bounded domain [27,28,33]. As in Szilard's engine the purpose of the measurement is to determine whether the particle is located to the left or right of the center, we consider the coarser observable $Y \in \{R, L\}$, with $Y = R$ if $X(\tau_m)$ is to the right of the center, and $Y = L$ if $X(\tau_m)$ is to the left of the center. The corresponding protocol takes the general form

$$\lambda_Y(t) = \begin{cases} \lambda_- & \text{for } t \in [0, \tau_m], \\ \lambda_L(t) & \text{for } t \in [\tau_m, \tau + \tau_m] \text{ if } Y = L, \\ \lambda_R(t) & \text{for } t \in [\tau_m, \tau + \tau_m] \text{ if } Y = R. \end{cases} \quad (5)$$

Comparing the protocols (5) and (2), we note several similarities. Notably, in both cases the demon must know the outcome of a random variable to evaluate the protocol: In (2), the demon needs to know T to evaluate $\lambda_T(t)$ for times $t > T$, while in (5), the demon needs to know Y to evaluate $\lambda_Y(t)$ for $t > \tau_m$. In the latter case, it has been argued that the demon requires a memory to store the outcome of Y in order to evaluate the protocol [27,28,33]. Since there is no apparent reason to treat temporal information differently from spatial information, we conclude that T should likewise be stored in a memory.

IV. SECOND LAW OF THERMODYNAMICS FOR NONAUTONOMOUS DEMONS THAT USE RANDOM TIMES

Using an analogy with Szilard's engine, we have argued that spatial and temporal information should be treated on equal footing, and thus the outcome of T must be stored in a memory. The question is now whether this is sufficient to uphold the second law of thermodynamics? In this section, we show that this is indeed the case: nonautonomous demons that use protocols with random times satisfy the second law of thermodynamics, provided that the random time T is stored in a memory device.

For cyclic processes in isothermal systems, the Kelvin version of the second law of thermodynamics states that [40]

$$\langle W_{\text{useful}} \rangle \geq 0, \quad (6)$$

where $\langle W_{\text{useful}} \rangle$ is the average work that has been done on the system. Equation (6) states that the useful work cannot be on average negative, or in other words, it is not possible to absorb heat from a thermal reservoir and convert it into useful work.

To determine the useful work, let us first consider the work done by the external load on the system. Applying stochastic thermodynamics arguments akin to those in the works of Sagawa and Ueda [30–32] to setups with randomly timed protocols, we derive in the Appendix A the inequality

$$\beta(\langle W \rangle - \Delta f) \geq -H(T), \quad (7)$$

where $\langle W \rangle = \langle W(T + \tau) \rangle$ is the total work done by the load on the system, $\Delta f = f_+ - f_-$ is the free energy difference between the end state ($\lambda = \lambda_+$) and the initial state ($\lambda = \lambda_-$),

and where

$$H(T) = - \sum_{t \geq 0} p_T(t) \ln p_T(t) \quad (8)$$

is the Shannon entropy of the stopping time T measured in nats; here p_T is the probability distribution of T . The inequality (7) states that random-time protocols can thermodynamically perform work on an external load, i.e., $\langle W \rangle < 0$. For cyclic protocols with $\Delta f = 0$ the work is bounded by

$$\langle W \rangle \geq -H(T). \quad (9)$$

Thus, the larger the information contained in the random time T , the more work can, in principle, be extracted from the thermal reservoir.

We note that the Eq. (7) is a stochastic thermodynamics result. It can be derived in the setup of Markov processes that satisfy local detailed balance (see Appendix A). An alternative derivation of the inequality (7) that is based on continuously monitoring the system is presented in Ref. [10]. This is because at the level of mathematical formulation, both the sleeping demon and the conventional continuous demon can be described in the unified setup of error-free nonautonomous Maxwell demon with a path-dependent protocol $\lambda(t; X_0^t)$ [30–32]. Furthermore, following Ref. [41], the inequality (7) can be further refined by taking into account the unavailable information. However, our purpose is to derive the second law of thermodynamics for randomly timed protocols, and for this purpose inequality (7) provides the most parsimonious explanation.

To derive the Kelvin statement of the second law, (6), we further consider that the demon needs a memory to store the random time T . If the process is cyclic, then the memory should also be restored to its initial state. According to Landauer's erasure principle, the average work $\langle W_{\text{erase}} \rangle$ needed to erase $H(T)$ nats of information is lower bounded by [27–29]

$$\beta \langle W_{\text{erase}} \rangle \geq H(T) = - \sum_{t \geq 0} p_T(t) \ln p_T(t). \quad (10)$$

Combining the information theoretical bound (7) with Landauer's erasure principle (10), we find

$$\langle W_{\text{useful}} \rangle = \langle W \rangle + \langle W_{\text{erase}} \rangle \geq 0, \quad (11)$$

and thus we have recovered the second law of thermodynamics.

V. RANDOM-TIME SZILARD ENGINE: DISCRETE VERSION

To clearly illustrate the concepts of temporal information dynamics, we analyze an information engine that uses temporal information to exchange heat into work. This temporal information engine is a random-time version of Szilard's engine [27,33] (see Fig. 1).

The random-time Szilard engine consists of a particle diffusing in a box, a random-time detector, a sleeping demon, and a memory device. The sleeping demon is awoken by the detector at the first moment when the particle collides with the right boundary. At that moment, the demon inserts a movable wall near the right boundary, attaches a load and a pulley on

the right side of the movable wall, and quasistatically moves the wall to the left boundary. Importantly, the information that the demon gains in this protocol is purely time based. This is different from Maxwell's demon in the Szilard engine, which places the load on the right or left side of the movable wall, depending on the outcome of the measurement [27,33].

Note that sleeping demons simplify setups based on continuous Maxwell demons. Continuous Maxwell demons measure the full trajectory X_0^T through continuous monitoring [7–10]. Instead, the sleeping demon directly measures T without continuous monitoring. This is possible within classical physics as we show in Sec. VII.

In discrete time, the working substance is a particle that performs an unbiased random walk on a one-dimensional lattice of length ℓ with reflecting boundary conditions. The position $X(t)$ of the particle at time $t \in \mathbb{N}$ takes values $X(t) \in \{1, 2, \dots, \ell\}$. At every time step, the particle moves to one of its nearest neighbors with probability $p_0/2$ (two neighbors for bulk sites, $X(t) \notin \{1, \ell\}$, and one neighbor for boundary sites, $X(t) \in \{1, \ell\}$), and otherwise the particle does not move. Note that the particle diffuses because it is in contact with a thermal reservoir at inverse temperature β .

We assume that the random-time detector is error-free, so that the sleeping demon receives a signal from the detector the first time $T = \min\{t \geq 0 : X(t) = \ell\}$ when the particle reaches the last site of the lattice, at which point the demon awakens. The demon stores the time T in its memory, inserts a movable wall between the lattice sites $\ell - 1$ and ℓ , and attaches a load to a pulley on the right side of this wall. The demon quasistatically moves the wall to the left until it reaches the left-hand side of the lattice at time $T + \tau$, with τ significantly larger than the relaxation time to equilibrium. At time $T + \tau$, the demon removes the wall, resets its memory to its initial state, and goes back to sleep.

If the demon moves the wall quasistatically, then the average work done by the particle on the load is given by $-\beta\langle W(T + \tau) \rangle = \ln \ell$. Indeed, by the first law of thermodynamics, the work performed on the load equals the heat exchanged with the thermal environment, and since the protocol is quasistatically slow, the heat exchanged equals the entropy difference times the temperature. Thus we can conclude that the average work grows logarithmically with ℓ [27]. On the other hand, the entropy $H(T) \approx \ln(\ell^2)$, as time $T \sim \ell^2$. Thus, the average work the temporal information engine exerts on the load is smaller than the average work needed to restore the memory to its original state, in correspondence with Eqs. (7) and (11). In Fig. 2 we verify that indeed $-\beta\langle W(T + \tau) \rangle / H(T) < 1$ for all lattice lengths ℓ .

VI. RANDOM-TIME SZILARD ENGINE: CONTINUOUS VERSION

We consider a continuous version of the random-time Szilard engine. In this case, the working substance is a Brownian particle with diffusion constant D that diffuses in a one-dimensional, bounded domain of length ℓ . This model is not only physically more realistic than its discrete version but also analytically solvable.

In continuous time, the entropy $H(T) = +\infty$, and hence the demon needs an infinite amount of memory space to

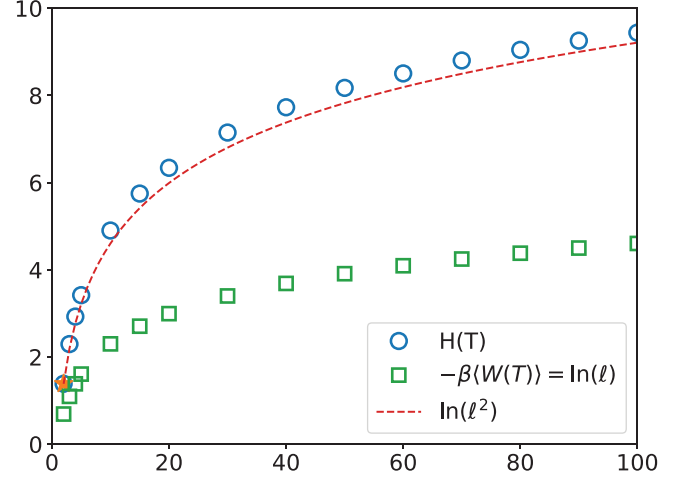


FIG. 2. Random-time Szilard engine in discrete time without measurement error (see Sec. V). Comparison between the work $-\beta\langle W \rangle = \ln(\ell)$ (blue circles) done by the Szilard engine on the external load and the entropy $H(T)$ (green squares) of the stored time T as a function of the length ℓ of the lattice; the hopping probability $p_0 = 1$. The (orange) star denotes the value $H(T) = 2 \ln 2$ for $\ell = 2$ (see Appendix B), and the red dotted line plots the function $\ln \ell^2$.

store T . Correspondingly, the demon can exchange an infinite amount of heat into work, as the diffusive particle can be localized arbitrarily close near the right boundary [7].

To make the model physically realistic, we assume that the demon has a finite memory device at its disposal. Consequently, the random time T when the movable wall is inserted into the system, and which is stored in the memory device, is different from the first-passage time,

$$T_{\text{fp}} = \min\{t \geq 0 : X(t) = \ell\}. \quad (12)$$

For the sake of simplicity, we consider a discretization of the form

$$T = \min\{m\theta : m \in \mathbb{N} \text{ and } m\theta \geq T_{\text{fp}}\}, \quad (13)$$

where $\theta > 0$ is a time interval. The entropy $H(T)$ is a function of the ratio

$$\zeta = \frac{\theta D}{\ell^2} \quad (14)$$

between the squared average distance θD traveled by the Brownian particle in a time-interval θ and the squared length ℓ^2 of the domain. For small values of ζ the entropy is well approximated by $-\ln \zeta$ (see Appendix C).

Discretization of time leads to loss of temporal information, and according to inequality (7), a corresponding loss in the amount of work that an information engine can do on an external load. Indeed, since T is different from T_{fp} , it holds in general that $X(T) \neq \ell$. To assure that the particle is located to the right of the movable wall, the sleeping demon inserts the wall at a distance $\tilde{\ell}$ from the right boundary of the domain. Analogously, the wall is removed at a distance $\tilde{\ell}$ from the left boundary of the domain. The average work takes thus the form

$$\beta\langle W(T + \tau) \rangle = p_+ \ln \frac{\tilde{\ell}}{\ell - \tilde{\ell}} + (1 - p_+) \ln \frac{\ell - \tilde{\ell}}{\tilde{\ell}}, \quad (15)$$

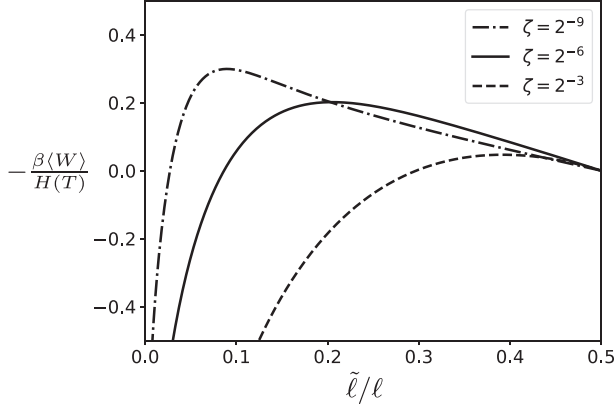


FIG. 3. Random-time Szilard engine in continuous time without measurement error (see Sec. VI). The ratio between the work $-\beta\langle W \rangle$ done by the system on the external load and the entropy $H(T)$ as a function of the length ratio $\tilde{\ell}/\ell$ for three given values of $\zeta = D\theta/\ell^2$. See Appendix C for explicit expressions of the plotted functions.

where

$$p_+ = \text{Prob}(X(T) \geq \ell - \tilde{\ell}) \quad (16)$$

is the probability that the particle is located at the right of the inserted wall. The probability p_+ is a function of the ratios ζ and $\tilde{\ell}/\ell$, with $p_+ = 0$ ($p_+ = 1$) if $\tilde{\ell} = 0$ ($\tilde{\ell} = \ell$), as we also show in Appendix C.

In Fig. 3 we plot the ratio between the average extracted work $-\beta\langle W \rangle$ and the amount of temporal information $H(T)$ as a function of the ratio $\tilde{\ell}/\ell$ for three values of ζ . The ratio is smaller than 1, in correspondence with the bound (7), expressing a trade-off between temporal information and extracted work. The average work reaches a maximum at an intermediate value of $\tilde{\ell}/\ell \in (0, 1)$, corresponding with the optimal placement of the inserted wall for a given value of ζ (see Appendix C for a more detailed analysis).

VII. RANDOM-TIME DETECTORS

The random-time Szilard engine of Fig. 1 contains a random-time detector, which so far we have assumed to be perfect. We develop now physical models for random-time detectors. We first consider the case of a discrete random-time Szilard engine.

We assume the state of the detector, $Y(t)$, takes two possible values, inactive $Y(t) = \downarrow$ and active $Y(t) = \uparrow$. These two levels have energies $\epsilon_\downarrow = 0$ and $\epsilon_\uparrow > 0$, respectively. We assume that the detector is in equilibrium with the thermal reservoir at inverse temperature β , just like diffusing particle. The sleeping demon measures the transitions from $\uparrow$ to the ground state $\downarrow$, i.e.,

$$T = \min\{t \geq 0 : \lim_{\epsilon \rightarrow 0} Y(t - \epsilon) = \uparrow, Y(t) = \downarrow\}. \quad (17)$$

For instance, we can imagine that during the transition a photon of a specific frequency is emitted, and the demon is highly responsive to this frequency.

The transition probabilities between the two detector states are assumed to take the form $\pi_\uparrow = \pi_0(1 - \delta_{X(t), \ell}) + \pi_\ell \delta_{X(t), \ell}$ and $\pi_\downarrow = [\pi_0(1 - \delta_{X(t), \ell}) + \pi_\ell \delta_{X(t), \ell}]e^{\beta\epsilon_\uparrow}$, for transitions

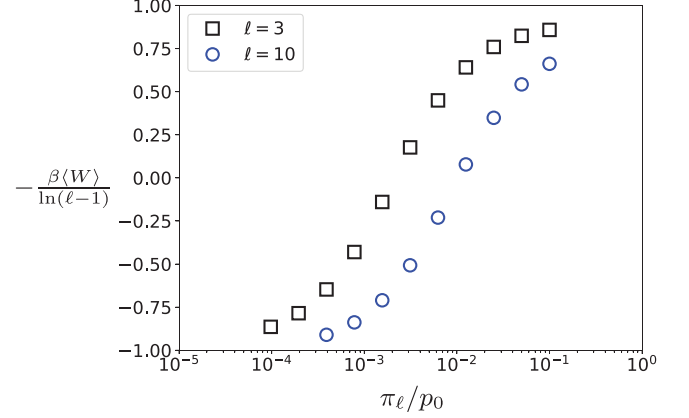


FIG. 4. Random-time Szilard engine with measurement error (see Sec. VII). Average work that a random-time Szilard engine exerts on an external load as a function of the parameter π_ℓ/p_0 that characterizes the quality of the detector. The other detector parameters are $\pi_0 = 0.0001$ and $\beta\epsilon_\uparrow = 4$. The hopping probability $p_0 = 0.1$ and ℓ is as given in the legend. The average work is obtained from the formula $-\beta\langle W \rangle = (2p_+ - 1)\ln(\ell - 1)$, where p_+ is the probability that $X(T) = \ell$; this is Eq. (15) with $\tilde{\ell} = 1$. Each marker is an average over $1e + 6$ simulated samples.

towards the $\uparrow$ and $\downarrow$ states, respectively. Here δ denotes the Kronecker delta function, and $\pi_0, \pi_\ell \in [0, \exp(-\beta\epsilon_\uparrow)]$. Note that the detector is coupled with the particle's state as the transition probabilities $\pi_\downarrow$ and $\pi_\uparrow$ depend on $X(t)$.

An effective detector requires tuning π_0 , π_ℓ , and $\epsilon_\uparrow$ to meet three conditions. First, we set $\exp(-\beta\epsilon_\uparrow) \ll 1/\ell$, so that the detector occupies mostly the ground state $\downarrow$. Second, we set $\pi_0 \ll p_0/\ell^2$, so that the detector is unlikely to activate when the particle is not on the ℓ th site. Third, we set $\pi_\ell \gg p_0$ so that the detector activates as soon as the particle reaches the ℓ th site. In Fig. 4 we confirm that under these conditions the detector functions well, similarly to the case of a perfect detector $\beta\langle W(T + \tau) \rangle \approx -\ln(\ell - 1)$. In fact, we observe that the random-time detector works already well for relatively small values of $\pi_\ell \gtrsim 0.1p_0$.

An alternative, possibly more accessible, way of measuring a first-passage time is illustrated in Fig. 5. In this case, a diffusing particle that is made of a material that conducts well electrical signals is immersed into an isolating and viscous environment. When the particle reaches the end of the segment, it switches on an electrical circuit, and the current in this circuit activates an alarm that awakens the sleeping demon.

Note that the function of the detector in both of the above examples is to awaken the demon. Once the demon is alerted, it measures the time T , stores T in a memory device, and initiates the protocol $\lambda_T(s)$. One caveat here is that the demon needs a clock to measure time. We elaborate on this in the discussion section of the paper.

VIII. CARNOT EFFICIENCY FOR CYCLIC HEAT ENGINES THAT USE RANDOMLY TIMED PROTOCOLS

We evaluate the efficiency of stochastic heat engines that rely on temporal information, see for example the model of Ref. [12]. Heat engines are systems in contact with two

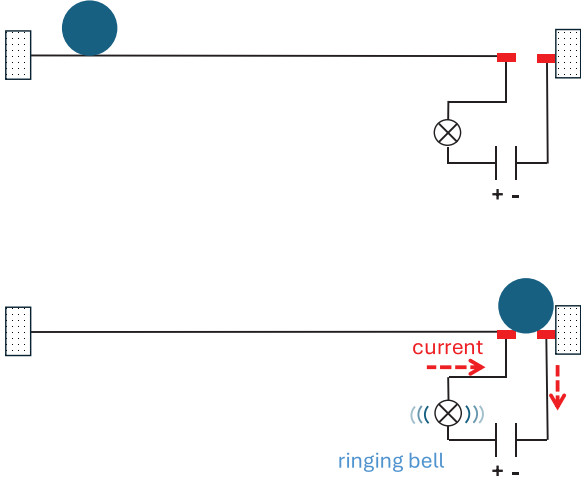


FIG. 5. Example of a setup that measures the first time when a particle reaches the end of a bounded domain without continuously monitoring the system. At the right end of the segment there is an electrical circuit that is usually turned off (see top panel). The diffusing colloidal particle is made of a material of high conductance. When the particle reaches the right end, it fills the gap between two electrodes and the electrical circuit is switched on (see bottom panel). Once the circuit is switched on, a bell rings that alerts the external observer (i.e., it “awakens” the sleeping demon).

or more thermal reservoirs at different temperatures. Unlike isothermal systems, cyclic processes coupled to two reservoirs at different temperatures can perform average work on an external load. However, the second law of thermodynamics limits the relative amount of useful work that can be done on an external load. For cyclic processes in contact with two thermal reservoirs, the average efficiency of a stochastic heat engine is bounded from above by the Carnot bound

$$\eta = \frac{\langle W_{\text{useful}} \rangle}{\langle Q_h \rangle} \leq 1 - \frac{\beta_h}{\beta_c}, \quad (18)$$

where Q_h is the heat released into the hot reservoir, β_h is the inverse temperature of the hot reservoir, and $\beta_c > \beta_h$ is the inverse temperature of the cold reservoir; note that in the typical operation of a heat engine, both $\langle W_{\text{useful}} \rangle$ and $\langle Q_h \rangle$ are negative.

Interestingly, for protocols that use random times, the efficiency can (apparently) exceed the Carnot bound, as shown in the recent paper [12]. Nevertheless, we demonstrate here that the Carnot bound is recovered when accounting for the cost associated with storing the random time T .

A. Information theoretical bound

We first derive the equivalent of Eq. (7) for heat engines. Specifically, if we define the efficiency

$$\eta_G = \frac{\langle W \rangle}{\langle Q_h \rangle}, \quad (19)$$

where $\langle W \rangle$ is the average work done by an external load on the system, then

$$\eta_G \leq \left(1 - \frac{\beta_h}{\beta_c}\right) - \frac{H(T)}{\beta_c \langle Q_h \rangle}. \quad (20)$$

Note that the right-hand side of (20) contains an information theoretical term corresponding with the additional work that we can extract by using the information contained in T . This term is positive in the normal operation of a heat engine ($\langle Q_h \rangle < 0$) and thus super-Carnot efficiencies are in principle allowed when $H(T)$ is large enough.

To derive (20), note that Eq. (7) generalizes into

$$\langle S_{\text{tot}}(T + \tau) \rangle \geq -H(T), \quad (21)$$

where $\langle S_{\text{tot}}(T + \tau) \rangle$ is the average entropy production. For a cyclic process in contact with two thermal reservoirs, we obtain

$$\beta_c \langle Q_c \rangle + \beta_h \langle Q_h \rangle \geq -H(T). \quad (22)$$

Here $\langle Q_c \rangle = \langle Q_c(T + \tau) \rangle$ and $\langle Q_h \rangle = \langle Q_h(T + \tau) \rangle$ quantify the heat released to the cold and the hot thermal reservoirs, respectively, after the completion of one cycle. Note that the cycle terminates at a random time $T + \tau$, as there is one step in the protocol that initiates at a random time T .

The first law of thermodynamics for this engine reads

$$\langle Q_c \rangle = \langle W \rangle - \langle Q_h \rangle, \quad (23)$$

where we have used the following sign conventions: $\langle W \rangle > 0$ implies that the external load does work on the system, and $\langle Q_c \rangle > 0$ (or $\langle Q_h \rangle > 0$) implies that heat flows from the system to the thermal reservoir. Using the first law (23) in Eq. (22) to eliminate $\langle Q_c \rangle$, we get that

$$-\langle W \rangle \leq -\left(1 - \frac{\beta_h}{\beta_c}\right) \langle Q_h \rangle + \frac{H(T)}{\beta_c}. \quad (24)$$

Assuming that $-\langle Q_h \rangle > 0$, i.e., the system takes heat from the hot thermal reservoir, we get that

$$\frac{\langle W \rangle}{\langle Q_h \rangle} \leq \left(1 - \frac{\beta_h}{\beta_c}\right) - \frac{H(T)}{\beta_c \langle Q_h \rangle}, \quad (25)$$

which we recognize as the inequality (20) that we were meant to derive.

B. Analysis with memory storage: Recovering the Carnot bound

As argued before, a nonautonomous demon needs a memory device to store the outcome of the random time T . By Landauer’s erasure principle, storage of T comes with an additional cost, and here we show that this cost is sufficient to recover the Carnot bound (18).

Using, as in Eq. (11) that the useful work is

$$\langle W_{\text{useful}} \rangle = \langle W \rangle + \langle W_{\text{erase}} \rangle, \quad (26)$$

where $\langle W_{\text{erase}} \rangle$ is the average work required to erase the memory, we find for the efficiency of the Carnot engine

$$\eta = \frac{\langle W \rangle + \langle W_{\text{erase}} \rangle}{\langle Q_h \rangle} = \eta_G + \frac{\langle W_{\text{erase}} \rangle}{\langle Q_h \rangle}. \quad (27)$$

In typical situations the signs are such that $\langle W \rangle < 0$ (the system performs work on an external load) and $\langle W_{\text{erase}} \rangle > 0$ [the demon (or system) does work on the memory device].

Using the information theoretical bound (20) in (27), we find

$$\eta \leq \left(1 - \frac{\beta_h}{\beta_c}\right) - \frac{H(T) - \beta_c \langle W_{\text{erase}} \rangle}{\beta_c \langle Q_h \rangle}. \quad (28)$$

By Landauer's erasure principle, the work required to erase the memory device is bounded from below by:

$$\langle W_{\text{erase}} \rangle \geq \frac{H(T)}{\beta_c}. \quad (29)$$

Here we have chosen to release the heat from the memory device into the colder reservoir, as $1/\beta_c \leq 1/\beta_h$.

Using (29) in (28) and the inequality $\langle Q_h \rangle < 0$, we recover Carnot's bound (18). In conclusion, a cyclic heat engine satisfies the Carnot bound, even if it uses a protocol with a random time T .

IX. MEASURING FIRST-PASSAGE TIMES: COMPARISON BETWEEN CONTINUOUS MONITORING AND DIRECT METHODS

The sleeping demons in the present paper measure first-passage times directly. This approach is different from first-passage time measurements considered in the literature that measure first-passage times through continuous monitoring [7–11]. We discuss here how in these two approaches the cost of measurement scales with the extracted work.

A. Continuous monitoring

Let us first revisit the continuous monitoring formalism, see Refs. [7–11] and [31,41,42]. In the continuous monitoring framework, the demon measures X at regularly spaced time intervals $0, \delta t, 2\delta t, \dots$. The demon inserts a movable wall at position $\ell - \tilde{\ell}$ the first time when $X(n\delta t) \in [\ell - \tilde{\ell}, \ell]$ (notice that it is more efficient to measure a binary variable that indicates whether $X(n\delta t) \in [\ell - \tilde{\ell}, \ell]$). To avoid errors, we need to consider the limit of $\delta t \rightarrow 0$ which is the limit of continuous monitoring.

In continuous monitoring, the cost of measurement scales with the system size and with the work extracted. Indeed, let us denote by $\delta s_{\text{meas}} > 0$ the average amount of heat dissipated in a single measurement. Then, the average amount of dissipation due to measurements in continuous Maxwell demons equals, on average,

$$\langle S_{\text{meas}} \rangle = \delta s_{\text{meas}} \frac{\langle T \rangle}{\delta t}. \quad (30)$$

In the limit of $\delta t \rightarrow 0$, the measurement cost gets thus large, unless we can compensate with a small cost δs_{meas} for a single measurement. More troublesome is that the cost of measurement increases with the size of the domain, as $\langle T \rangle \sim \ell^2$. This implies that the cost of measurement grows faster with ℓ than the average extracted work $\langle W \rangle \sim \ln \ell$.

As discussed above, the proper operation of a continuous Maxwell demon relies on the assumption that $\delta s_{\text{meas}} > 0$ can be made arbitrary small. Contrarily to the dissipation associated with memory erasure, there is currently no known physical principle that limits the thermodynamic cost of measurements [27]. This is because the process of storing (erasing) information on a memory device is a physical operation that increases (decreases) the entropy of the device, while the operation of measurement does not have a clear connection with entropy. Nevertheless, physicists have investigated the cost of measurements with models of external protocols. Brillouin [43] demonstrated that light cannot be

used for this purpose, which led Bennett to develop a mechanical protocol to measure the position of the particle in Szilard's engine [44]. This protocol is quasistatically slow and is thus not compatible with the high-frequency measurements of continuous demons ($\delta t \rightarrow 0$). Therefore, it seems reasonable to associate a nonzero, albeit small, cost δs_{meas} to each single measurement of a continuous demon. According to (30) we can then expect a significant measurement cost in the limit of continuous monitoring.

B. Direct measurements

We discuss the measurement cost of setups that directly measure first-passage times T . As discussed in Sec. VII, a first-passage time detector is a device that sends out a signal as soon as the particle reaches the end of the bounded interval.

The measurements in the setup of Fig. 5 have a thermodynamic cost, as the current in the circuit dissipates heat. However, differently from continuous Maxwell demons, this cost is independent of ℓ , the size of the system. Hence, considering a large-enough system, the cost of measurement can be made negligible compared with the extracted work. This follows from the fact that in this setup first-passage times are measured with a single measurement. We leave a more detailed study of this setup for future work.

X. DISCUSSION

We have drawn attention to a class of perpetual mobile that use external protocols with a randomly timed step. Several recent papers have studied such protocols, see Refs. [1–12], demonstrating that the limits of thermodynamics can seemingly be overcome.

In the present paper, we have resolved these thermodynamic puzzles. The resolution, in a nutshell, is that temporal information must be treated on an equal footing with spatial information. Specifically, we have shown that the second law of thermodynamics is upheld once the random timing of events is recorded in a memory, as recording information on a tape increases the system entropy of the tape. This explanation is analogous to the way the second law of thermodynamics is resolved in Szilard's engine [27,45], the sole difference being in the nature of the random variables that is stored in the memory.

Consequently, using random times in external protocols incurs a thermodynamic cost that is informational in nature. This additional cost must also be accounted for in protocols not typically associated with Maxwell's demons. A pertinent example is stochastic resetting, a process that maintains a nonequilibrium steady state by returning a diffusing particle to the origin at a random time [13,14]. This process requires an input of work, as energy must be expended to bring the particle back to the origin [25]. However, there is an additional cost related to storing the random time T in memory. Neglecting this cost can lead to apparent violations of the second law of thermodynamics.

To illustrate the consequences of temporal information on nonequilibrium thermodynamics, we have introduced a nonautonomous Maxwell demon that directly measures the random time T and which we called the sleeping demon. We

have developed setups that measure first-passage times directly, unlike the conventional approach based on continuous monitoring that is typically employed in the literature [7–11]. In the direct approach, the cost of measurement is fixed and independent on the size of the system, while for continuous measurement the cost of measurement grows with the system size.

Sleeping demons need a clock in order to measure the time T at the moment when they are “awake.” The requirement of a clock is not a specific feature of the sleeping demon, as clocks are generally required for the operation of nonautonomous Maxwell demons; even Maxwell’s demon in Szilard’s engine needs a clock to operate. The requirement of a clock is evident from the dependency of $\lambda(t)$ on time t , and thus if a nonautonomous demon evaluates λ it must know the time t . Clocks are also required in certain measurement setups, such as in continuous measurements, as these setups measure the state of a system at evenly spaced time intervals. Thus the requirement of a clock in the measurement process is not specific for the sleeping demon model. As the clock operates at the same temperature as the system, maintaining its precision incurs a thermodynamic cost [46–50]. However, because the clock can serve a large number of parallel experiments, this cost does not scale with system size. This contrasts with the cost of resetting the memory that stores T , which does increase with system size.

Szilard engines have been implemented in electronic devices, such as single electron boxes [51], which raises the question on how to describe temporal information in quantum mechanics. We expect some complications due to the quantum Zeno effect [52–55]. There is also a possibility to define first-passage times in Bohmian mechanics [56,57], which may yield an alternative perspective on temporal information in quantum mechanics.

Temporal information is also relevant in models of computing, such as those based on finite-state machines [58], and hence the cost of temporal information should be considered in the thermodynamics of computation [39].

ACKNOWLEDGMENTS

The author thanks K. S. Olsen for pointing out a few relevant references. The author thanks S. Bo and M. Mitchison for insightful discussions.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF THE BOUND EQ. (7)

In this Appendix we derive the inequality

$$\langle \exp(-\beta[W(T + \tau) - \Delta f] + \ln p_T(T)) \rangle \leq 1. \quad (\text{A1})$$

Applying Jensen’s inequality to (A1), we recover (7).

Note that it is possible to refine the result (A1) into an equality following the approach in Ref. [41]. However, for

our purpose of deriving the second law for randomly timed protocols the result (A1) suffices.

We derive the bound (A1) for Markov chains in discrete time, even though the results also apply in continuous time. Our derivations are based on the framework of error-free nonautonomous Maxwell demons with a path-dependent protocol $\lambda(t; X_0^t)$, see Refs. [30–32]. For the analysis below, we require that T is a stopping time (i.e., T is independent of the values of X after T).

Let us consider a Markov chain $X(t) \in \mathcal{X}$ with $t \in \mathbb{N}$. We denote the transition probabilities by

$$\mathbf{q}_{xy}(\lambda_s(t-1)) := \text{Prob}\{X(t) = y | X(t-1) = x; \lambda_s(t-1)\}, \quad (\text{A2})$$

where

$$\lambda_s(t) = \begin{cases} \lambda^{(1)}(t) & t \in [0, s-1], \\ \lambda^{(2)}(t-s) & t \in [s, s+\tau-1], \\ \lambda_+ & t \in [s+\tau, \infty), \end{cases} \quad (\text{A3})$$

denotes an external time-dependent parameter and $\text{Prob}\{\cdot; \lambda_s(t-1)\}$ is a conditional probability that depends on $\lambda_s(t-1)$. For now, the time $s \in \mathbb{N}$ when the protocol changes from $\lambda^{(1)}$ to $\lambda^{(2)}$ is deterministic. We denote the probability distribution of the initial configuration by $p_{\text{init}}(x)$.

The probability that a sequence $X_0^{s+\tau} = (X(0), X(1), \dots, X(s+\tau))$ equals $x_0^{s+\tau} = (x(0), x(1), \dots, x(s+\tau))$ is thus

$$\begin{aligned} & \text{Prob}\{X_0^{s+\tau} = x_0^{s+\tau}; \lambda_{0;s}^{s+\tau-1}\} \\ &= p_{\text{init}}(x(0)) \mathbf{q}_{x(0)x(1)}(\lambda_s(0)) \mathbf{q}_{x(1)x(2)}(\lambda_s(1)) \\ & \dots \mathbf{q}_{x(s+\tau-1)x(s+\tau)}(\lambda_s(s+\tau-1)), \end{aligned} \quad (\text{A4})$$

where $\lambda_{0;s}^{s+\tau-1} = (\lambda_s(0), \lambda_s(1), \dots, \lambda_s(s+\tau-1))$. Let us introduce the notation

$$q(x_0^{s+\tau}; \lambda_{0;s}^{s+\tau-1}) := \text{Prob}\{X_0^{s+\tau} = x_0^{s+\tau}; \lambda_{0;s}^{s+\tau-1}\}$$

for path probabilities. Evaluating the path probability q at a stopping time, $s = T$, and on a random trajectory, $x_0^{T+\tau} = X_0^{T+\tau}$, we obtain

$$\begin{aligned} & q(X_0^{T+\tau}; \lambda_{0;T}^{T+\tau-1}) \\ &= p_{\text{init}}(X(0)) \mathbf{q}_{X(0)X(1)}(\lambda_T(0)) \mathbf{q}_{X(1)X(2)}(\lambda_T(1)) \\ & \dots \mathbf{q}_{X(T+\tau-1)X(T+\tau)}(\lambda_T(T+\tau-1)). \end{aligned} \quad (\text{A5})$$

Analogously, the time-reversed path probability is given by

$$\begin{aligned} & \tilde{q}(x_0^{s+\tau}; \lambda_{0;s}^{s+\tau-1}) \\ &= \tilde{p}_{\text{init}}(x(s+\tau)) \mathbf{q}_{x(s+\tau)x(s+\tau-1)}(\lambda_s(s+\tau-1)) \\ & \times \mathbf{q}_{x(s+\tau-1)x(s+\tau-2)}(\lambda_s(s+\tau-2)) \dots \mathbf{q}_{x(1)x(0)}(\lambda_s(0)), \end{aligned}$$

where $\tilde{p}_{\text{init}}$ is the distribution of $X(0)$ in the time-reversed dynamics. The ratio of $\tilde{q}$ and q takes the form

$$\begin{aligned} & \frac{\tilde{q}(X_0^{T+\tau-1}; \lambda_{0;T}^{T+\tau-1})}{q(X_0^{T+\tau-1}; \lambda_{0;T}^{T+\tau-1})} \\ &= \frac{\tilde{p}_{\text{init}}(X(T+\tau)) \mathbf{q}_{X(1)X(0)}(\lambda_T(0)) \mathbf{q}_{X(2)X(1)}(\lambda_T(1))}{p_{\text{init}}(X(0)) \mathbf{q}_{X(0)X(1)}(\lambda_T(0)) \mathbf{q}_{X(1)X(2)}(\lambda_T(1))} \\ & \dots \frac{\mathbf{q}_{X(T+\tau)X(T+\tau-1)}(\lambda_T(T+\tau-1))}{\mathbf{q}_{X(T+\tau-1)X(T+\tau)}(\lambda_T(T+\tau-1))}. \end{aligned} \quad (\text{A6})$$

Stochastic thermodynamics links the theory of Markov processes with nonequilibrium thermodynamics through the principle of local detailed balance [59,60]

$$\frac{q_{xy}(\lambda)}{q_{yx}(\lambda)} = \exp(S_{xy}^{\text{env}}(\lambda)), \quad (\text{A7})$$

with $S_{xy}^{\text{env}}(\lambda)$ the entropy change in the environment during the transition from x to y . Using the formula (A7) in Eq. (A6) for the ratio, we obtain

$$\frac{\tilde{q}(X_0^{T+\tau-1}; \lambda_{0:T}^{T+\tau-1})}{q(X_0^{T+\tau-1}; \lambda_{0:T}^{T+\tau-1})} = e^{-S_{\text{env}}(T+\tau)} \frac{\tilde{p}_{\text{init}}(X(T+\tau))}{p_{\text{init}}(X(0))}$$

with $S_{\text{env}}(T+\tau)$ the total change in the entropy in the environment along the trajectory $X_0^{T+\tau}$. If we consider an isothermal system, as in the main text, then $S_{\text{env}}(T+\tau) = \beta Q(T+\tau)$ with $Q(T+\tau)$ the heat released into the thermal reservoir. If, in addition, we set the initial conditions

$$p_{\text{init}}(x) = \exp(-\beta[u(x; \lambda_-) - f_-]) \quad (\text{A8})$$

and

$$\tilde{p}_{\text{init}}(x) = \exp(-\beta[u(x; \lambda_+) - f_+]), \quad (\text{A9})$$

and we use the first law of thermodynamics

$$u(X(T+\tau)) - u(X(0)) = W(T+\tau) - Q(T+\tau), \quad (\text{A10})$$

we get that

$$\frac{\tilde{q}(X_0^{T+\tau}; \lambda_{0:T}^{T+\tau-1})}{q(X_0^{T+\tau}; \lambda_{0:T}^{T+\tau-1})} = \exp(-\beta(W(T+\tau) - \Delta f)), \quad (\text{A11})$$

with $\Delta f = f_+ - f_-$; notice that in (A10) W is the work done on the system, while Q is the heat released into the environment, and hence the sign difference. Multiplying (A11) by $p_T(T)$ and taking an ensemble average over all trajectories x_0^∞ weighted by their path probability q , we obtain

$$\begin{aligned} & \langle \exp(-\beta(W - \Delta f) + \ln p_T(T)) \rangle \\ &= \int_{\mathcal{X}^\infty} dx_0^\infty q(x_0^\infty; \lambda_0^\infty) p_T(T(x_0^\infty)) \frac{\tilde{q}(x_0^{T+\tau}; \lambda_{0:T}^{T+\tau-1})}{q(x_0^{T+\tau}; \lambda_{0:T}^{T+\tau-1})} \\ &= \sum_{t=0}^{\infty} p_T(t) \int_{\mathcal{X}^{t+\tau}} dx_0^{t+\tau} \delta_{t,T(x_0^{t+\tau})} \tilde{q}(x_0^{t+\tau}; \lambda_{0:t}^{t+\tau-1}) \\ & \quad \times \int dx_{t+\tau+1}^\infty q(x_{t+\tau+1}^\infty | x(t+\tau); \lambda_+) \\ &\leq \sum_{t=0}^{\infty} p_T(t) \int_{\mathcal{X}^{t+\tau}} dx_0^{t+\tau} \tilde{q}(x_0^{t+\tau}; \lambda_{0:t}^{t+\tau-1}) \\ &= 1, \end{aligned} \quad (\text{A12})$$

which is the bound (A1) that we were meant to derive. Notice that in the second equality we introduced $\sum_{t=0}^{\infty} \delta_{t,T(x_0^\infty)} = 1$, we have used that T is a stopping time to write $T(x_0^\infty) = T(x_0^{T+\tau})$, and we have used the strong Markov property to write $q(x_0^\infty; \lambda_0^\infty) = q(x_0^{T+\tau}; \lambda_0^{T+\tau-1}) q(x_{T+\tau+1}^\infty | x(T+\tau); \lambda_+)$ [35].

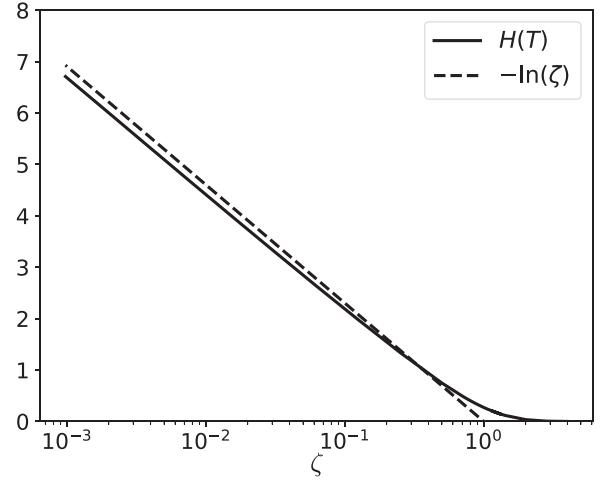


FIG. 6. Entropy $H(T)$ of the distribution $p_T(t)$ given by Eq. (C10) as a function of ζ . The dashed line denotes $-\ln(\zeta)$ for comparison.

APPENDIX B: ENTROPY $H(T)$ FOR THE RANDOM-TIME SZILARD ENGINE IN DISCRETE TIME AND FOR $\ell = 2$

We consider the discrete, random-time Szilard engine of Sec. V for $\ell = 2$ and without measurement error. In this case, the distribution of the initiation time of the protocol is given by

$$p_T(t) = \frac{1}{2} \delta_{t,0} + \frac{p_0}{4} \sum_{j=1}^{\infty} (1 - p_0/2)^{j-1} \delta_{t,j}, \quad (\text{B1})$$

and its entropy is

$$H(T) = -\frac{1}{2} \ln \frac{1}{2} - \frac{(1 - p_0/2)}{p_0} \ln(p_0[1 - p_0/2]/4). \quad (\text{B2})$$

For $p_0 = 1$ this yields $H(T) = \ln \ell^2$, and this value is annotated by a star in Fig. 6.

APPENDIX C: RANDOM-TIME SZILARD ENGINE IN CONTINUOUS TIME

We analyze the continuous version of the random-time Szilard engine, as introduced in Sec. VI. As in the main text, we assume the demon has a perfect random-time detector at its disposable that measures T without error, and we also assume that the memory of the demon is finite.

1. Model for the working substance

Let $X(t) \in [0, \ell]$ denote the position of a Brownian particle at time $t \in \mathbb{R}^+$ on a one-dimensional bounded domain of width ℓ . The evolution in time of the position of the Brownian particle is described by

$$\frac{dX}{dt} = \sqrt{2D} \frac{dZ}{dt}, \quad (\text{C1})$$

where D is the diffusivity constant, $Z(t)$ is a standard Wiener process, and we implement reflecting boundary conditions at $X(t) = 0$ and $X(t) = \ell$. We assume that the initial time $t = 0$,

the particle is equilibrated, so that it is uniformly distributed over the domain $[0, \ell]$.

2. Probability distribution of the first-passage time T_{fp}

The detector alerts the demon at the first-passage time T_{fp} when the Brownian particle hits the right boundary, see Eq. (12). The first-passage time distribution $p_{T_{\text{fp}}}$ can be obtained from the formula

$$p_{T_{\text{fp}}}(t) = -\frac{1}{\ell} \int_0^\ell dx_0 \partial_t \int_0^\ell dx G(x, t|x_0), \quad (\text{C2})$$

where the propagator $G(x, t|x_0)$ is the probability of finding the Brownian particle at $X(t) = x$ if $X(0) = x_0$ (with an absorbing boundary condition at $x = \ell$). The propagator solves [61]

$$\frac{\partial G(x, t|x_0)}{\partial t} = D \frac{\partial^2 G(x, t|x_0)}{(\partial x)^2} \quad (\text{C3})$$

with a reflecting boundary condition at $x = 0$,

$$\left. \frac{\partial G(x, t|x_0)}{\partial x} \right|_{x=0} = 0, \quad (\text{C4})$$

an absorbing boundary condition at $x = \ell$,

$$G(\ell, t|x_0) = 0, \quad (\text{C5})$$

and the initial condition,

$$G(x, 0|x_0) = \delta(x - x_0). \quad (\text{C6})$$

The solution to the Eqs. (C3)–(C6) is given by (see for example Ref. [62])

$$G(x, t|x_0) = \frac{2}{\ell} \sum_{n=0}^{\infty} \cos\left(\frac{(n+1/2)\pi x_0}{\ell}\right) \cos\left(\frac{(n+1/2)\pi x}{\ell}\right) \times \exp\left(-\frac{(n+1/2)^2 \pi^2 D t}{\ell^2}\right). \quad (\text{C7})$$

Substitution of (C7) in Eq. (C2), taking the derivative towards time, and evaluating the two integrals, we obtain

$$p_{T_{\text{fp}}}(t) = \frac{2D}{\ell^2} \sum_{n=0}^{\infty} \exp\left(-\frac{(n+1/2)^2 \pi^2 D t}{\ell^2}\right). \quad (\text{C8})$$

Notice that $p_{T_{\text{fp}}}$ is normalized and that T_{fp} scales as ℓ^2/D (see Appendix D).

3. The entropy $H(T)$ of the random control time T

The random time T when the demon starts its protocol is defined by Eq. (13) where $\theta > 0$ is a parameter denoting the time resolution of the control time. From the definition of T in terms of T_{fp} it follows that the probability mass function $p_T(t)$ of T is given by

$$p_T(t) = \int_{t\theta}^{(t+1)\theta} du p_{T_{\text{fp}}}(u). \quad (\text{C9})$$

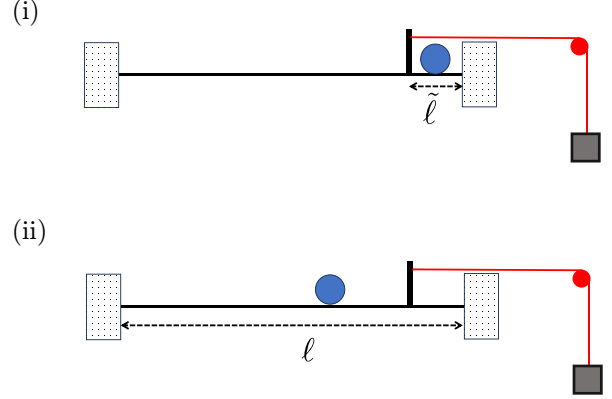


FIG. 7. Case (i): $X(T) \in [\ell - \tilde{\ell}, \ell]$, so that the particle does an amount $-\beta^{-1} \ln(\ell - \tilde{\ell})/\tilde{\ell}$ of work on the external load. Case (ii): $X(T) \in [0, \ell - \tilde{\ell}]$, so that the load does an amount $\beta^{-1} \ln(\ell - \tilde{\ell})/\tilde{\ell}$ of work on the system.

Using the expression (C8) for $p_{T_{\text{fp}}}$ we get

$$p_T(t) = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2} \exp(-t(n+1/2)^2 \pi^2 \zeta) \times [1 - \exp(-(n+1/2)^2 \pi^2 \zeta)], \quad (\text{C10})$$

where ζ is the ratio is the ratio between the average (squared) distance $D\theta$ traveled by a Brownian particle in a time interval θ and the squared interval width ℓ^2 [see Eq. (14)].

The entropy $H(T)$ of T , as defined by Eq. (8), is thus a function of the parameter ζ , and in Fig. 6 we plot the entropy as a function of ζ . Note that although $H(T)$ is a complicated function of ζ given by a double series, it is well approximated by $-\ln \zeta$ for values of $\zeta < 0.1$.

4. Average work $\langle W \rangle$

The average work done by the load on the particle is given by Eq. (15), with p_+ the probability that at time T the particle is located right of the movable wall. The first term is negative and the second term is positive, as work is done by the system on the external load if $X(T) \geq \ell - \tilde{\ell}$, while work is done by the external load on the system if $X(T) \leq \ell - \tilde{\ell}$. This is because the sleeping demon attaches the load to the right-hand side of the movable wall for every realization of the process (see Fig. 7 for an illustration of both cases).

As we show in Appendix E, the probability p_+ takes the expression

$$p_+ = \frac{\tilde{\ell}}{\ell} + \frac{4}{\pi^3} \sum_{k=0}^{\infty} \frac{\exp((k+1/2)^2 \pi^2 \zeta)}{\exp((k+1/2)^2 \pi^2 \zeta) - 1} \times \sum_{n=1}^{\infty} \frac{\sin(n\pi \tilde{\ell}/\ell)}{n} \times \frac{\exp(-n^2 \pi^2 \zeta) - \exp(-(k+1/2)^2 \pi^2 \zeta)}{(1/2+k)^2 - n^2}. \quad (\text{C11})$$

Note that p_+ is a function of two parameters, viz., the ratios $\zeta = D\theta/\ell^2$ and $\tilde{\ell}/\ell$. In the top panel of Fig. 8 we plot p_+ as a function of $\tilde{\ell}/\ell$ for several values of ζ . Even though the formal

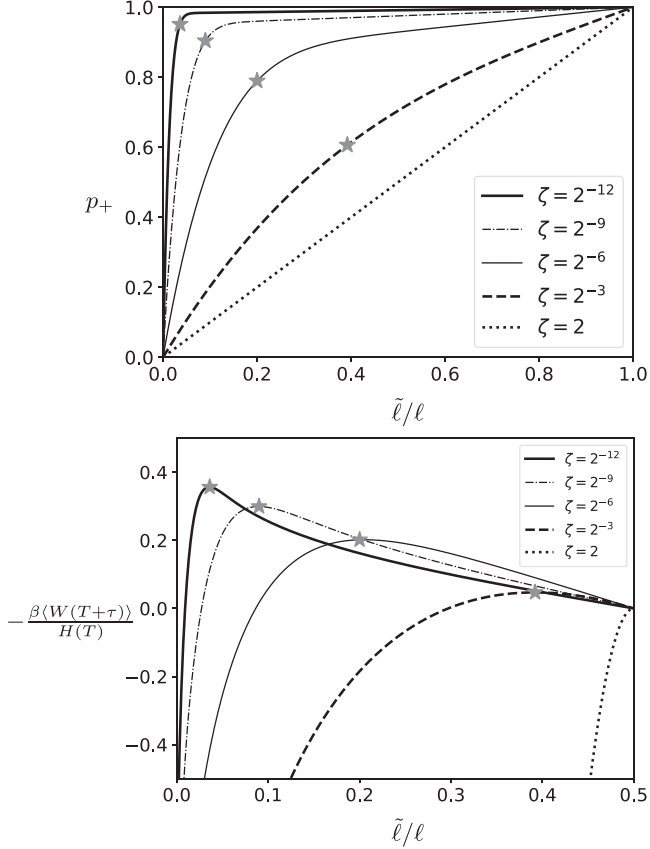


FIG. 8. Extended version of Fig. 3 in the main text. Top: The probability p_+ that $X(T) < \ell - \tilde{\ell}$ as a function of $\tilde{\ell}/\ell$ for different values of $\zeta = D\theta/\ell^2$ as indicated in the legend; the plot function is given by Eq. (C11). Bottom: The average work performed on the particle as a function of $\tilde{\ell}/\ell$ (extended version of Fig. 3 in the main text); for $\tilde{\ell} > \ell/2$ the average work is positive (not shown). The star indicates the points on the curves evaluated at the optimal placement of the movable wall, viz., $\tilde{\ell}/\ell = r^*$ with r^* the value of the ratio at which $-\beta\langle W \rangle/H(T)$ attains its maximum.

expression (C11) is a fairly complicated double series, Fig. 8 shows that p_+ is a rather simple monotonically increasing function of $\tilde{\ell}$ with $p_+ = 0$ if $\tilde{\ell} = 0$ and $p_+ = 1$ if $\tilde{\ell} = 1$. In the limit $\zeta \rightarrow 0$, p_+ is well approximated by (see top panel of Fig. 8)

$$p_+ \approx \begin{cases} 1 & \text{if } \frac{\tilde{\ell}}{\ell} > r^*, \\ \frac{1}{r^*} \frac{\tilde{\ell}}{\ell} & \text{if } \frac{\tilde{\ell}}{\ell} < r^*, \end{cases} \quad (\text{C12})$$

where r^* denotes the value of $\tilde{\ell}/\ell$ where the slope of p_+ changes from almost vertical to horizontal.

Using the expression (C11) for p_+ in Eq. (15), we obtain a formula for the average work $\langle W(T + \tau) \rangle$. The plot of $\langle W \rangle/(\beta H(T))$ in Fig. 3 of the main text is obtained from this formula together with the Eq. (C10) that yields the Shannon entropy $H(T)$. We provide a more detailed version of this figure in the bottom panel of Fig. 8. The stars in the bottom panel of Fig. 8 indicate the maximum value of $-\beta\langle W(T + \tau) \rangle/H(T)$, corresponding with the best placement of the movable wall. We observe that the maximum increases as a function of $1/\zeta$, which raises the question of what is the

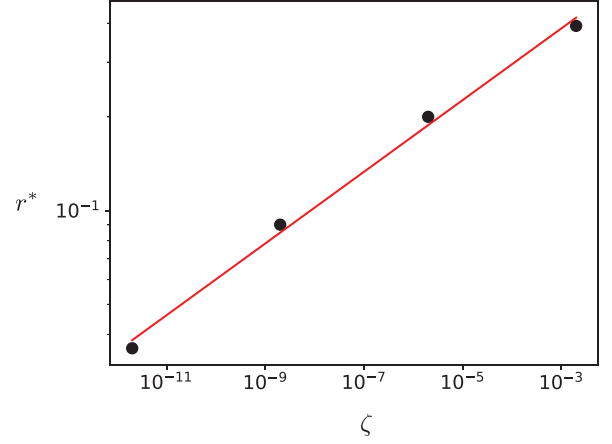


FIG. 9. A plot of r^* , the optimal position to place the movable wall [as defined in Eq. (C13)], as a function of $\zeta = D\theta/\ell^2$. The markers are obtained by numerically finding the maximum of the curves in the bottom panel of Fig. 8. The solid line is a fit to the power-law function βx^α ; the numerically obtained optimal fitting parameters are $\beta = 0.92$ and $\alpha = 0.38$.

maximum value of the ratio $-\beta\langle W(T + \tau) \rangle/H(T)$ in the limit $\zeta \rightarrow 0$?

To address this latter question, we define

$$r^* = \operatorname{argmin}_{\tilde{\ell}/\ell} \frac{\beta\langle W(T + \tau) \rangle}{H(T)}, \quad (\text{C13})$$

as the value of the ratio $\tilde{\ell}/\ell$ when $-\beta\langle W(T + \tau) \rangle/H(T)$ attains its maximum. In the top panel of Fig. 8 we plot $p_+(r^*)$ using the same star symbol as in the bottom panel of Fig. 8. This reveals that r^* , as defined in (C13), roughly equals the value of $\tilde{\ell}/\ell$ when the slope of p_+ changes from almost vertical to almost horizontal line (in the limit of small ζ); this justifies why we have used the same symbol of r^* in the Eqs. (C12) and (C13). In Fig. 9 we plot r^* as a function of ζ , and this reveals that

$$r^* \approx \gamma \zeta^\alpha, \quad (\text{C14})$$

with $\gamma = 0.92$ and $\alpha = 0.38$. Thus, using Eq. (15) to evaluate $\langle W \rangle$ for $\zeta \approx 0$ and $\tilde{\ell}/\ell = r^*$, and using that $p_+(r^*) \approx 1$ and $r^* \approx \gamma \zeta^\alpha$, we get

$$\beta\langle W(T + \tau) \rangle \approx \alpha \ln \zeta. \quad (\text{C15})$$

As the entropy $H(T) \approx -\ln(\zeta)$ in the limit $\zeta \approx 0$ (see Fig. 6), we get in this limit that for $\tilde{\ell}/\ell = r^*$ the average work equals

$$-\beta \frac{\langle W(T + \tau) \rangle}{H(T)} \approx \alpha = 0.38. \quad (\text{C16})$$

The value 0.38 for the maximum of the ratio $-\beta \frac{\langle W(T + \tau) \rangle}{H(T)}$ is in good correspondence with the numerics in the bottom panel of Fig. 8. Indeed, the figure shows that the maximum amount of extractable work converges to $0.38H(T)/\beta$ for $\zeta \rightarrow 0$.

APPENDIX D: MOMENTS OF THE DISTRIBUTION OF T_{fp}

We determine the moments of the first-passage time T_{fp} with a probability distribution given by (C8).

We rely on known result for Hurwitz ζ functions defined by (see pages 19–22 in Ref. [63])

$$\zeta(2k, x) = \sum_{n=0}^{\infty} (n+x)^{2k}. \quad (\text{D1})$$

As shown in Ref. [64], the Hurwitz ζ function satisfies

$$\zeta(2k, x) + \zeta(2k, 1-x) = -\frac{\pi}{(2k-1)!} \frac{d^{2k-1}}{(dx)^{2k-1}} \cot(\pi x), \quad (\text{D2})$$

where $\cot(y) = 1/\tan(y)$ is the cotangent. Setting $x = 1/2$ in (D2), we find that

$$\zeta(2k, 1/2) = -\frac{\pi}{2(2k-1)!} \frac{d^{2k-1}}{(dx)^{2k-1}} \cot(\pi x). \quad (\text{D3})$$

We will repeatedly use the formula (D3) at different values of $k \in \mathbb{N}$.

1. Normalization

That $p_{T_{\text{fp}}}$ is normalized follows from

$$\int_0^{\infty} dt p_{T_{\text{fp}}}(t) = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2} = \frac{2}{\pi^2} \zeta(2, 1/2). \quad (\text{D4})$$

Using (D3) for $k = 1$, we get

$$2\zeta(2, 1/2) = -\pi \frac{d}{dx} \cot(\pi x) \Big|_{x=1/2} = \pi^2, \quad (\text{D5})$$

which together with (D4) implies the normalization of $p_{T_{\text{fp}}}(t)$.

2. Mean first-passage time

We show that the mean first-passage time

$$\int_0^{\infty} dt t p_{T_{\text{fp}}}(t) = \frac{2\ell^2}{\pi^4 D} \sum_{n=0}^{\infty} \frac{1}{(n+1/2)^4} = \frac{1}{3} \frac{\ell^2}{D}. \quad (\text{D6})$$

Indeed, substituting (C8) in the integral on the left-hand side of (D6) and evaluating the integral in t we obtain the series

$$\int_0^{\infty} dt t p_{T_{\text{fp}}}(t) = \frac{2\ell^2}{\pi^4 D} \sum_{n=0}^{\infty} \frac{1}{(n+1/2)^4} = \frac{2\ell^2}{\pi^4 D} \zeta(4, 1/2), \quad (\text{D7})$$

where in the last equality we have used the definition (D1) of the Hurwitz ζ function. Using (D3) at $k = 2$ we obtain $\zeta(4, 1/2) = \pi^4/6$, and thus (D7) implies (D6).

APPENDIX E: PROBABILITY p_+ that $X(T) \geq \tilde{\ell}$

We derive the formula (C11) for p_+ . The probability p_+ can be expressed in terms of the propagator $F(x, t|\ell)$ that gives the probability of $X(t) = x$ if $X(0) = \ell$:

$$p_+ = \sum_{m=0}^{\infty} \int_0^{\theta} du p_{T_{\text{fp}}}((m+1)\theta - u) \int_{\ell-\tilde{\ell}}^{\ell} dx F(x, u|\ell). \quad (\text{E1})$$

The propagator F solves the equation

$$\frac{\partial F(x, t|\ell)}{\partial t} = D \frac{\partial^2 F(x, t|\ell)}{(\partial x)^2} \quad (\text{E2})$$

with reflecting boundary conditions

$$\frac{\partial F(x, t|\ell)}{\partial x} \Big|_{x=0} = 0 \quad \text{and} \quad \frac{\partial F(x, t|\ell)}{\partial x} \Big|_{x=\ell} = 0, \quad (\text{E3})$$

and initial condition

$$F(x, 0|\ell) = \delta(x - \ell). \quad (\text{E4})$$

The solution for F is given by [61]

$$F(x, t|\ell) = \frac{1}{\ell} + \frac{2}{\ell} \sum_{n=1}^{\infty} (-1)^n \cos\left(n\pi \frac{x}{\ell}\right) \exp\left(-\frac{n^2 \pi^2 D t}{\ell^2}\right). \quad (\text{E5})$$

Substitution of (E5) in (E1) gives

$$\begin{aligned} p_+ &= \sum_{m=0}^{\infty} \int_0^{\theta} du p_{T_{\text{fp}}}((m+1)\theta - u) \int_{\ell-\tilde{\ell}}^{\ell} dx F(x, u|\ell) \\ &= \frac{\tilde{\ell}}{\ell} + \frac{4D}{\ell^3} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \exp\left(-\frac{m\theta(k+1/2)^2 \pi^2 D}{\ell^2}\right) \\ &\quad \times \sum_{n=1}^{\infty} (-1)^n \int_{\ell-\tilde{\ell}}^{\ell} dx \cos\left(n\pi \frac{x}{\ell}\right) \int_0^{\theta} du \\ &\quad \times \exp\left(-\frac{(k+1/2)^2 \pi^2 D(\theta - u)}{\ell^2}\right) \exp\left(-\frac{n^2 \pi^2 D u}{\ell^2}\right). \end{aligned} \quad (\text{E6})$$

Evaluating the sums over m and the integrals over u and x we obtain the expression (C11).

-
- [1] I. Neri, É. Roldán, S. Pigolotti, and F. Jülicher, Integral fluctuation relations for entropy production at stopping times, *J. Stat. Mech.* (2019) 104006.
 [2] I. Neri, Second law of thermodynamics at stopping times, *Phys. Rev. Lett.* **124**, 040601 (2020).
 [3] H. Yang and H. Ge, Fluctuation theorems and thermodynamic inequalities for nonequilibrium processes stopped at stochastic times, *Phys. Rev. E* **108**, L052104 (2023).

- [4] G. Manzano, D. Subero, O. Maillet, R. Fazio, J. P. Pekola, and É. Roldán, Thermodynamics of gambling demons, *Phys. Rev. Lett.* **126**, 080603 (2021).
 [5] J. A. C. Albay, Y. Jun, and P.-Y. Lai, Winning strategies of a gambling demon in a Brownian particle under a squeezing potential, *Phys. Rev. Res.* **5**, 023115 (2023).
 [6] J. A. C. Albay, Y. Jun, and P.-Y. Lai, Gambling demon and stopping-time fluctuation relation of a Brownian particle under

- a time-dependent trapping potential, *Phys. Rev. E* **109**, 014124 (2024).
- [7] M. Ribezzi-Crivellari and F. Ritort, Large work extraction and the Landauer limit in a continuous Maxwell demon, *Nat. Phys.* **15**, 660 (2019).
 - [8] M. Ribezzi-Crivellari and F. Ritort, Work extraction, information-content and the Landauer bound in the continuous Maxwell demon, *J. Stat. Mech.* (2019) 084013.
 - [9] J. P. Garrahan and F. Ritort, Generalized continuous Maxwell demons, *Phys. Rev. E* **107**, 034101 (2023).
 - [10] R. K. Schmitt, P. P. Potts, H. Linke, J. Johansson, P. Samuelsson, M. Rico-Pasto, and F. Ritort, Information-to-work conversion in single-molecule experiments: From discrete to continuous feedback, *Phys. Rev. E* **107**, L052104 (2023).
 - [11] A. Archambault, C. Crauste-Thibierge, A. Imparato, C. Jarzynski, S. Ciliberto, and L. Bellon, Information engine fueled by first-passage times, *Phys. Rev. Lett.* **135**, 147101 (2025).
 - [12] T. Tohme, V. Bedoya, C. di Bello, L. Bresque, G. Manzano, and E. Roldán, Gambling carnot engine, *Phys. Rev. Lett.* **135**, 067101 (2025).
 - [13] M. R. Evans and S. N. Majumdar, Diffusion with stochastic resetting, *Phys. Rev. Lett.* **106**, 160601 (2011).
 - [14] M. R. Evans, S. N. Majumdar, and G. Schehr, Stochastic resetting and applications, *J. Phys. A: Math. Theor.* **53**, 193001 (2020).
 - [15] D. Gupta, C. A. Plata, A. Kundu, and A. Pal, Stochastic resetting with stochastic returns using external trap, *J. Phys. A: Math. Theor.* **54**, 025003 (2021).
 - [16] G. Mercado-Vásquez, D. Boyer, S. N. Majumdar, and G. Schehr, Intermittent resetting potentials, *J. Stat. Mech.* (2020) 113203.
 - [17] I. Santra, S. Das, and S. K. Nath, Brownian motion under intermittent harmonic potentials, *J. Phys. A: Math. Theor.* **54**, 334001 (2021).
 - [18] G. Mercado-Vásquez, D. Boyer, and S. N. Majumdar, Reducing mean first passage times with intermittent confining potentials: A realization of resetting processes, *J. Stat. Mech.* (2022) 093202.
 - [19] D. Gupta, A. Pal, and A. Kundu, Resetting with stochastic return through linear confining potential, *J. Stat. Mech.: Theory Exp.* (2021) 043202.
 - [20] H. Alston, L. Cocconi, and T. Bertrand, Non-equilibrium thermodynamics of diffusion in fluctuating potentials, *J. Phys. A: Math. Theor.* **55**, 274004 (2022).
 - [21] P. Xu, T. Zhou, R. Metzler, and W. Deng, Stochastic harmonic trapping of a Lévy walk: Transport and first-passage dynamics under soft resetting strategies, *New J. Phys.* **24**, 033003 (2022).
 - [22] B. Besga, A. Bovon, A. Petrosyan, S. N. Majumdar, and S. Ciliberto, Optimal mean first-passage time for a Brownian searcher subjected to resetting: Experimental and theoretical results, *Phys. Rev. Res.* **2**, 032029(R) (2020).
 - [23] R. Goerlich, M. Li, L. B. Pires, P.-A. Hervieux, G. Manfredi, and C. Genet, Taming a Maxwells' demon for experimental stochastic resetting, [arXiv:2306.09503](https://arxiv.org/abs/2306.09503).
 - [24] O. Tal-Friedman, A. Pal, A. Sekhon, S. Reuveni, and Y. Roichman, Experimental realization of diffusion with stochastic resetting, *J. Phys. Chem. Lett.* **11**, 7350 (2020).
 - [25] K. S. Olsen, D. Gupta, F. Mori, and S. Krishnamurthy, Thermodynamic cost of finite-time stochastic resetting, *Phys. Rev. Res.* **6**, 033343 (2024).
 - [26] S. Lahiri and S. Gupta, Efficiency of a microscopic heat engine subjected to stochastic resetting, *Phys. Rev. E* **109**, 014129 (2024).
 - [27] H. S. Leff and A. F. Rex, *Maxwell's Demon: Entropy, Information, Computing* (Princeton University Press, Princeton, NJ, 1990).
 - [28] K. Maruyama, F. Nori, and V. Vedral, Colloquium: The physics of Maxwell's demon and information, *Rev. Mod. Phys.* **81**, 1 (2009).
 - [29] R. Landauer, Irreversibility and heat generation in the computing process, *IBM J. Res. Dev.* **5**, 183 (1961).
 - [30] T. Sagawa and M. Ueda, Generalized Jarzynski equality under nonequilibrium feedback control, *Phys. Rev. Lett.* **104**, 090602 (2010).
 - [31] T. Sagawa and M. Ueda, Nonequilibrium thermodynamics of feedback control, *Phys. Rev. E* **85**, 021104 (2012).
 - [32] T. Sagawa and M. Ueda, Information thermodynamics: Maxwell's demon in nonequilibrium dynamics, in *Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond*, edited by R. Klages, W. Just, and C. Jarzynski (John Wiley & Sons, Ltd., 2013), Chap. 6, pp. 181–211.
 - [33] L. Szilard, Über die entropieverminderung in einem thermodynamischen system bei Eingriffen intelligenter wesen, *Z. Phys.* **53**, 840 (1929).
 - [34] J. M. Horowitz and J. M. Parrondo, Thermodynamic reversibility in feedback processes, *Europhys. Lett.* **95**, 10005 (2011).
 - [35] P. Brémaud, *Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues*, (Springer-Verlag, Berlin, 1999) Vol. 31.
 - [36] L. C. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales: Volume 1, Foundations* (Cambridge University Press, Cambridge, UK, 2000).
 - [37] K. Sekimoto, Langevin equation and thermodynamics, *Prog. Theor. Phys. Suppl.* **130**, 17 (1998).
 - [38] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, *Rep. Prog. Phys.* **75**, 126001 (2012).
 - [39] C. H. Bennett, The thermodynamics of computation—A review, *Int. J. Theor. Phys.* **21**, 905 (1982).
 - [40] E. Fermi, *Thermodynamics* (Dover, Garden City, NY, 1936).
 - [41] Y. Ashida, K. Funo, Y. Murashita, and M. Ueda, General achievable bound of extractable work under feedback control, *Phys. Rev. E* **90**, 052125 (2014).
 - [42] P. P. Potts and P. Samuelsson, Detailed fluctuation relation for arbitrary measurement and feedback schemes, *Phys. Rev. Lett.* **121**, 210603 (2018).
 - [43] L. Brillouin, Maxwell's demon cannot operate: Information and entropy. I, *J. Appl. Phys.* **22**, 334 (1951).
 - [44] C. H. Bennett, Demons, engines and the second law, *Sci. Am.* **257**, 108 (1987).
 - [45] R. P. Feynman, *Feynman Lectures on Computation* (CRC Press, Boca Raton, FL, 2001).
 - [46] A. C. Barato and U. Seifert, Cost and precision of Brownian clocks, *Phys. Rev. X* **6**, 041053 (2016).
 - [47] G. Milburn, The thermodynamics of clocks, *Contemp. Phys.* **61**, 69 (2020).

- [48] K. Prech, G. T. Landi, F. Meier, N. Nurgalieva, P. P. Potts, R. Silva, and M. T. Mitchison, Optimal time estimation and the clock uncertainty relation for stochastic processes, *Phys. Rev. X* **15**, 031068 (2025).
- [49] K. Macieszczak, Ultimate kinetic uncertainty relation and optimal performance of stochastic clocks, [arXiv:2407.09839](https://arxiv.org/abs/2407.09839).
- [50] V. Wadhia, F. Meier, F. Fedele, R. Silva, N. Nurgalieva, D. L. Craig, D. Jirovec, J. Saez-Mollejo, A. Ballabio, D. Chrastina *et al.*, Entropic costs of the quantum-to-classical transition in a microscopic clock, [arXiv:2502.00096](https://arxiv.org/abs/2502.00096).
- [51] J. V. Koski, V. F. Maisi, J. P. Pekola, and D. V. Averin, Experimental realization of a Szilard engine with a single electron, *Proc. Natl. Acad. Sci. USA* **111**, 13786 (2014).
- [52] B. Misra and E. G. Sudarshan, The Zeno's paradox in quantum theory, *J. Math. Phys.* **18**, 756 (1977).
- [53] K. Koshino and A. Shimizu, Quantum Zeno effect by general measurements, *Phys. Rep.* **412**, 191 (2005).
- [54] V. Dubey, C. Bernardin, and A. Dhar, Quantum dynamics under continuous projective measurements: Non-Hermitian description and the continuum-space limit, *Phys. Rev. A* **103**, 032221 (2021).
- [55] D. Das and S. Gupta, Quantum random walk and tight-binding model subject to projective measurements at random times, *J. Stat. Mech.* (2022) 033212.
- [56] H. Nitta and T. Kudo, Time of arrival of electrons in the double-slit experiment, *Phys. Rev. A* **77**, 014102 (2008).
- [57] S. Das, D.-A. Deckert, L. Kellers, and W. Struyve, Double-slit experiment remastered, *Ann. Phys.* **479**, 170054 (2025).
- [58] G. Manzano, G. Kardeş, É. Roldán, and D. H. Wolpert, Thermodynamics of computations with absolute irreversibility, unidirectional transitions, and stochastic computation times, *Phys. Rev. X* **14**, 021026 (2024).
- [59] P. G. Bergmann and J. L. Lebowitz, New approach to nonequilibrium processes, *Phys. Rev.* **99**, 578 (1955).
- [60] C. Maes, Local detailed balance, *SciPost Phys. Lect. Notes* **32** (2021).
- [61] S. Redner, *A Guide to First-passage Processes* (Cambridge University Press, Cambridge, UK, 2001).
- [62] F. Huang and H. Chen, Extremal statistics for a one-dimensional Brownian motion with a reflective boundary, *Physica A* **633**, 129389 (2024).
- [63] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, Berlin, 1966).
- [64] D. Cvijović and J. Klinowski, Values of the Legendre chi and Hurwitz zeta functions at rational arguments, *Math. Comput.* **68**, 1623 (1999).