

# Phase boundaries and desynchronization in the single-defect Olami-Feder-Christensen model: Toward understanding earthquake power laws

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We investigate the emergence of self-organized criticality (SOC) in the single-defect Olami-Feder-Christensen (OFC) model by quantifying the ordered phase and deriving theoretical phase boundaries. Using both square and honeycomb lattices, we analytically obtain expressions for the ordered-SOC transition boundary and the characteristic period, validating them against numerical simulations. The results show that while synchronization alone governs the transition to the ordered phase, the onset of SOC requires desynchronization triggered by supercritical toppling energies exceeding a certain threshold. These findings characterize the threshold-dependent nature of desynchronization under minimal inhomogeneity, providing a fundamental perspective that clarifies the mechanisms observed in the original OFC model.

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## I. INTRODUCTION

Power-law distributions characterize many natural and engineered systems. These distributions are widely believed to arise from self-organized criticality (SOC), a concept introduced by Bak, Tang, and Wiesenfeld [1]. SOC refers to systems that spontaneously evolve into a critical state without fine-tuning external parameters such as temperature. In this critical state, systems display scale invariance, i.e., avalanche-like events exhibit power-law size-frequency distributions. SOC has been observed across various areas, including solar flares [2], forest fires [3], and nervous systems [4,5]. Among these phenomena, earthquakes represent a prominent example of SOC [6].

The Olami-Feder-Christensen (OFC) earthquake model [7], a simplified version of the Burridge-Knopoff spring-block model [8], reproduces the Gutenberg-Richter power-law distribution [9] and serves as a representative SOC model. In this model, the emergence of SOC is considered to arise from the interaction between partial synchronization, which occurs due to inhomogeneity and spatial structure, and desynchronization, which represents the fragmentation of synchronization [10–13]. However, the mechanism of the emergence of SOC has not been fully elucidated.

Recently, we proposed a single-defect model, a modified version of the OFC model, to clarify the specific conditions required for the emergence of the power law [14]. The system displays SOC when  $\beta$  is sufficiently small compared to  $\alpha$ , where  $\beta$  is the transmission parameter representing the degree

of energy conservation at the defect site, and  $\alpha$  is the corresponding parameter for bulk sites (see Sec. II for details). As  $\beta$  gradually increases, the system undergoes phase transitions: from the SOC phase to the ordered phase, and subsequently to the asynchronous phase (Fig. 1). In particular, the lower boundary offers insights into the essential mechanisms for the emergence of SOC in the single-defect model.

In this study, we formulate the behavior of the ordered phase in the single-defect model and derive analytical expressions for the phase boundaries. We specifically examine the lower boundary separating the SOC and ordered phases, showing that the transition from the ordered phase to the SOC phase is driven by desynchronization triggered by sufficiently large supercritical toppling energies. The theoretical boundary and associated avalanche period are then verified by direct numerical simulations on square and honeycomb lattices, demonstrating consistency between theory and simulations. These results identify the conditions facilitating the emergence of SOC in the OFC model and provide insights into the origin of power-law behavior in earthquake statistics.

## II. SINGLE-DEFECT MODEL

The single-defect model is defined on an  $L \times L$  two-dimensional square lattice with periodic boundary conditions. Each site  $i$  carries a local energy  $z_i$ . Initially, each site is assigned a random value between 0 and the threshold value  $z_{th}$ . The system then evolves deterministically according to loading and toppling rules as described below.

The loading rule increases all local energies uniformly at a constant rate until one site reaches  $z_{th}$ , mimicking tectonic loading. This rule is implemented in two steps: first, the maximum value  $z_{max}$  among all local energies is identified, and then a value  $\Delta z = z_{th} - z_{max}$  is added to all local energies. Unit time is defined as the interval required for energy to increase from 0 to  $z_{th}$ .

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When one of the local energies reaches  $z_{\text{th}}$ , it becomes unstable and is called supercritical. The supercritical site  $i$  then redistributes its local energy to its four nearest-neighbor sites according to the toppling rule:

$$\begin{aligned} z_i &\rightarrow 0, \\ z_j &\rightarrow z_j + \alpha_i z_i \quad \text{for } \forall j \in N_4(i), \end{aligned}$$

where  $N_4(i)$  denotes the set of the four nearest-neighbor sites of site  $i$ , and  $\alpha_i$  is a transmission parameter of site  $i$ . The transmission parameter, ranging from 0 to 0.25, specifies the degree of local conservation at site  $i$ . A single defect site has a transmission parameter  $\alpha_i = \beta$ , while all other bulk sites have  $\alpha_i = \alpha$ .

If this toppling rule causes one or more nearest-neighbor sites to become supercritical, the toppling rule is applied repeatedly until all local energies fall below  $z_{\text{th}}$ . The toppling process is non-Abelian because the order of toppling affects the final configuration. Consequently, when multiple sites simultaneously become supercritical, these sites must either be toppled in parallel [10,15] or updated using the pseudosequential update [16]. We employ the parallel update in this study. A chain reaction of topplings constitutes an avalanche, which is interpreted as an “earthquake.” Avalanche size is defined as the total number of toppled sites. Time does not advance during avalanches, as the propagation speed of toppling is assumed to be much greater than the loading rate [7,15,17,18]. Upon completion of the avalanche, uniform loading resumes. Without loss of generality, we set  $z_{\text{th}} = 1$  in the following sections.

### III. PROPERTIES OF ORDERED PHASE

As noted in Sec. I, particular attention is given to the lower phase boundary between the SOC and ordered phases. Understanding the ordered phase is essential because its breakdown provides the key mechanism that drives the transition into SOC. Before deriving the phase boundaries theoretically, we first summarize four robust properties observed in numerical experiments of the ordered phase. These properties, concerning epicenter localization, system-size independence, symmetry of supercritical toppling energies, and their monotonic increase, form the empirical basis for the theoretical formulation developed in Sec. IV.

#### A. Epicenter

The first property is that only the nearest-neighbor sites of the defect can become epicenters. In this phase, the system exhibits periodic behavior, cycling through one to four different configurations. Consequently, the system generates at most four distinct avalanches, with the number of avalanches being equal to the number of epicenters. For example, when four avalanches occur, each epicenter is located at four nearest-neighbor sites of the defect, forming a cycle length of four. In events with a cycle length of three, epicenters occupy three of the four nearest-neighbor sites of the defect. A similar correspondence holds for a cycle length of one or two: the number of avalanches matches the number of epicenters, while the remaining sites that are not epicenters participate in the other avalanches [14].

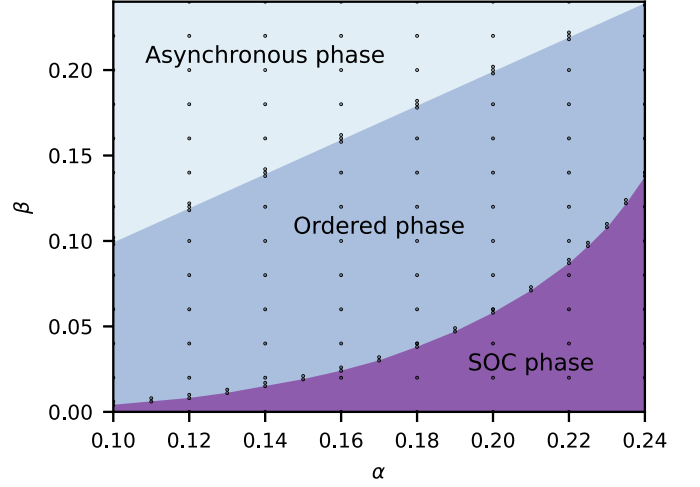


FIG. 1. Phase diagram of the single-defect model obtained in [14]. Each dot indicates an investigated pair of transmission parameters. The system exhibits phase transitions at the phase boundary  $\beta = \beta_c^{\text{asy-ord}}$  between the asynchronous and ordered phases, and at the phase boundary  $\beta = \beta_c^{\text{ord-soc}}$  between the ordered and SOC phases. The phase boundaries are confirmed to be independent of system size. Detailed characteristics of each phase are provided in Appendix A.

#### B. System-size independence

The second property is that the ordered phase behavior in the steady state is independent of system size and periodic boundary conditions. This is demonstrated in Fig. 2, which shows snapshots of the energy configuration during the initial transient for a defect located at the center of the system. The system begins from a random configuration across all sites, as seen in the outer region in Fig. 2. Spatial order originates from the defect, where a “synchronized pattern,” i.e., an aggregation of clusters of sites with similar local energies, propagates radially outward.

Within this ordered region, avalanche epicenters localize at the nearest-neighbor sites of the defect. Once an avalanche is formed, the avalanche border remains unchanged, and at most four types of avalanches repeat periodically, while the rim of the ordered region expands outward.

In contrast, in the outer region, each site topples independently, with single-site avalanches being predominant. This

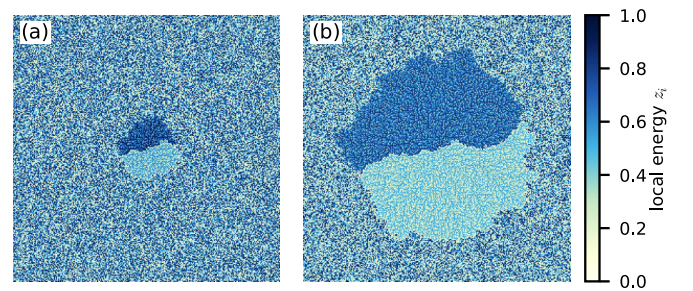


FIG. 2. Snapshots of system evolution for a lattice size  $256 \times 256$  with parameters  $(\alpha, \beta) = (0.14, 0.02)$ , showing configuration (a) after  $4 \times 10^6$  steps and (b) after  $10^7$  steps. The darker sites represent a state of higher local energy.

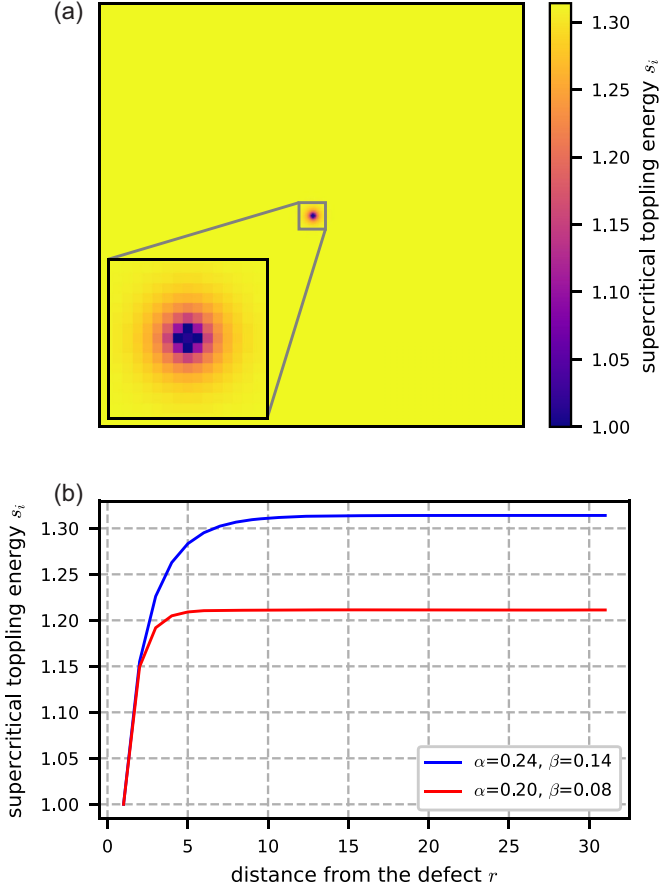


FIG. 3. (a) Distribution of  $s_i$  for  $(\alpha, \beta) = (0.24, 0.14)$  with the defect located at the center. The inset shows an enlarged view of the central region, where strict rotational symmetry is broken by the anisotropic toppling rule. (b) Distribution of  $s_i$  as a function of distance  $r$ , exhibiting monotonic increase and rapid convergence to a limiting value.

prevalence of single-site avalanches is characteristic of the asynchronous phase.

These findings demonstrate that periodic boundaries and system size do not affect the emergence of the ordered phase, justifying the subsequent analysis of sites sufficiently distant from the defect.

### C. Symmetry

The third property is the symmetry of supercritical toppling energies  $s_i$  around the defect. The supercritical toppling energy  $s_i$  is defined as the value of  $z_i$  at a supercritical site  $i$ . In other words,  $s_i$  represents the maximum value that  $z_i$  reaches immediately before toppling. At epicenter sites, toppling is triggered by uniform loading, yielding  $s_i = z_{th}$ . At nonepicenter sites, toppling is caused by redistributed local energy, resulting in  $s_i \geq z_{th}$ .

Measurements of  $s_i$  in the ordered phase reveal a radial distribution centered on the defect [Fig. 3(a)]. Importantly, this distribution is invariant: it remains unchanged regardless of initial conditions, avalanche shapes, or cycle lengths, for a given set of  $(\alpha, \beta)$  and system size.

### D. Monotonic increase

The fourth property is that the supercritical toppling energies increase monotonically with  $r$ , the horizontal distance from the defect [Fig. 3(b)]. This monotonic increase is consistently observed across different initial conditions, avalanche shapes, and cycle lengths, for a given set of  $(\alpha, \beta)$  and system size, in a manner similar to the observed radial symmetry. This is a nontrivial and unexpected property.

These four properties characterize the ordered phase and provide the foundation for the theoretical analysis in Sec. IV, where we formulate the ordered phase and derive the phase boundaries  $\beta = \beta_c^{\text{asy-ord}}(\alpha)$  and  $\beta = \beta_c^{\text{ord-soc}}(\alpha)$ .

## IV. FORMULATION OF DYNAMICS AND DERIVATION OF PHASE BOUNDARIES

Based on the preceding properties, we now formulate the conditions that each site must satisfy in the ordered phase.

Let  $P$  denote the set of the nearest-neighbor sites of the defect, and let  $Q$  denote the set of all other bulk sites. According to Sec. III A, the set  $P$  constitutes the collection of sites that can become epicenters. We define avalanche  $E_n$  as an avalanche generated at step  $n$ , where  $n$  is counted after the system has reached a limit cycle. In the ordered phase, identical avalanches repeat with cycle length  $l$ , where  $l$  takes integer values from 1 to 4. This implies that for any  $n$ , avalanche  $E_{n+l}$  ruptures the same region and has the same size as avalanche  $E_n$ .

Here, we consider  $z_i$  of site  $i \in Q$  that participates in an avalanche  $E_n$ . For simplicity, site  $i$  is assumed not to lie on the border of avalanche  $E_n$ , and  $l = 2$ . Immediately before the toppling chain reaction reaches the site  $i$ , one of its nearest-neighbor sites  $j_1$  topples and redistributes local energy to the site  $i$ . At this moment,  $z_i$  exceeds  $z_{th}$  and topples. The supercritical toppling energy  $s_i$  is defined as the value of  $z_i$  at this instant. After the toppling,  $z_i$  is reset to zero, and its energy is redistributed to the four nearest-neighbor sites. During the avalanche  $E_n$ , the remaining three nearest-neighbor sites of the site  $i$ , denoted as sites  $j_2$ ,  $j_3$ , and  $j_4$ , also topple. Hence, the site  $i$  receives a total redistributed energy of  $\alpha_{j_2}s_{j_2} + \alpha_{j_3}s_{j_3} + \alpha_{j_4}s_{j_4}$ . Here, we have used the assumption that the site  $i$  is not located on the avalanche border.

After the avalanche  $E_n$  terminates, subsequent loading resumes and triggers the next avalanche  $E_{n+1}$ . Denoting the energy accumulated during this loading phase as  $T_1$ ,  $z_i$  is updated as  $z_i \rightarrow z_i + T_1$ . By definition,  $T_1$  corresponds to the time interval between the avalanches  $E_n$  and  $E_{n+1}$ . The avalanche  $E_{n+1}$  occurs at a site other than  $i$ ,  $j_1$ ,  $j_2$ ,  $j_3$ , and  $j_4$ ; therefore,  $z_i$  remains unchanged during this avalanche. After the avalanche  $E_{n+1}$ , subsequent loading resumes again, and  $z_i$  is similarly updated as  $z_i \rightarrow z_i + T_2$ . At this moment (i.e., immediately prior to the onset of the avalanche  $E_{n+2}$ ), the value of  $z_i$  is  $z_i = \alpha_{j_2}s_{j_2} + \alpha_{j_3}s_{j_3} + \alpha_{j_4}s_{j_4} + T_1 + T_2$ . Since  $l = 2$ , the configuration at this time coincides with the state immediately prior to the avalanche  $E_n$ . Thus,  $T_1 + T_2 = T$ , where  $T$  denotes the period.

Based on the above considerations,  $z_i$  immediately prior to avalanche  $E_{n+1}$  is given by the sum of redistributed local energies from nearest-neighbor sites and the energy accumulated

through loading between avalanches  $E_n$  and  $E_{n+l}$  (which is equal to the period  $T$ ):

$$z_i^{\text{sum}} = \sum_{j \in R(i)} \alpha_j s_j + T$$

Here,  $R(i)$  denotes the set of sites that redistribute local energy to site  $i$  between avalanches  $E_n$  and  $E_{n+l}$ . This set typically consists of three nearest-neighbor sites as demonstrated in the example above. Note that when site  $i$  is an epicenter, this set contains four elements and when the toppling of site  $i$  is triggered by topplings of two or more of its nearest-neighbor sites during avalanche  $E_n$ , this set may contain two or one element. This formulation is valid irrespective of whether site  $i$  lies on the avalanche border and holds for any cycle length  $l$ .

From Sec. III A, we now derive the conditions that  $z_i$  at each site must satisfy immediately prior to the onset of avalanche  $E_{n+l}$  in the ordered phase. As previously noted, the toppling of an epicenter is always triggered by loading. Thus, in the ordered phase,  $z_i$  at the epicenter must exceed that at any other sites. Otherwise, sites other than the nearest-neighbor sites of the defect would become epicenters, which contradicts the property in Sec. III A. Therefore,  $z_i$  in both  $P$  and  $Q$  must satisfy the following inequality:

$$\sum_{j \in N_4(i)} \alpha_j s_j + T > \sum_{j \in R(k)} \alpha_j s_j + T \quad (1)$$

for  $\exists i \in P$  and  $\forall k \in Q$ ,

where  $R(k)$  is replaced with  $N_3(k)$ ,  $N_2(k)$ , or  $N_1(k)$ . It is sufficient to consider only the maximum value among all sites in  $Q$  on the right-hand side. Canceling  $T$  from both sides, Eq. (1) reduces to

$$\sum_{j \in N_4(i)} \alpha_j s_j > \sum_{j \in N_3(k)} \alpha_j s_j \quad (2)$$

for  $\exists i \in P$  and  $\forall k \in Q$ .

This inequality specifies the condition that  $s_i$  must satisfy in the ordered phase, although the exact values of  $s_i$  remain unknown.

To evaluate  $s_i$ , we derive recurrence relations among sites. Based on the property in Sec. III C [Fig. 3(a)],  $s_i$  exhibits approximate rotational symmetry around the defect. Therefore, we replace the notation  $s_i$  with  $S_r$  (denoted by a capital  $S$  with a subscript  $r$ ), which depends only on the distance  $r$  from the defect. In this notation,  $S_0$  represents the supercritical toppling energy of the defect, and  $S_1$  represents that of its nearest-neighbor sites, where  $r$  is defined in terms of the Manhattan distance.

Let us consider a site at distance  $r$  from the defect, which topples during avalanche  $E_n$ . The supercritical toppling energy  $S_r$  of this site is expressed by the sum of all energy received between its toppling in avalanche  $E_n$  and immediately prior to its toppling in avalanche  $E_{n+l}$ . Unlike  $z_i^{\text{sum}}$ ,  $S_r$  includes redistributed energy received during avalanche  $E_{n+l}$ , yielding

the sum of four redistributed energies:

$$S_r = \sum_j \alpha_j s_j + T$$

$$= \alpha_{j_1} s_{j_1} + \alpha_{j_2} s_{j_2} + \alpha_{j_3} s_{j_3} + \alpha_{j_4} s_{j_4} + T, \quad (3)$$

where  $j_1$ ,  $j_2$ ,  $j_3$ , and  $j_4$  denote the nearest-neighbor sites of the site under consideration.

For  $r = 0, 1, 2, \dots$ , Eq. (3) yields the following recurrence relations:

$$S_0 = 4\alpha S_1 + T, S_1 = 3\alpha S_2 + \beta S_0 + T = 1,$$

$$S_r = \begin{cases} 3\alpha S_{r+1} + \alpha S_{r-1} + T & \text{on the defect axis} \\ 2\alpha S_{r+1} + 2\alpha S_{r-1} + T & \text{off the defect axis} \end{cases} \quad (4)$$

The two forms in Eq. (4) are valid for  $r \geq 2$  for sites on and off the defect axis, respectively. As  $r$  increases, the sites located off the defect axis dominate, and Eq. (4) is approximated by the latter expression. This approximation yields the following solutions for  $S_r$  as functions of  $r$  and  $T$  (see Appendix B):

$$S_0 = 4\alpha + T,$$

$$S_r = 1 + \left( \frac{T}{1 - 4\alpha} - 1 \right) \times \left[ 1 - \left( \frac{1 - \sqrt{1 - 16\alpha^2}}{4\alpha} \right)^{r-1} \right] \quad \text{for } r \geq 1, \quad (5)$$

$$T = \frac{1 - 16\alpha\beta + 3\sqrt{1 - 16\alpha^2}}{1 - 4\alpha + 4\beta - 16\alpha\beta + 3\sqrt{1 - 16\alpha^2}} (1 - 4\alpha). \quad (6)$$

Since  $0 < (1 - \sqrt{1 - 16\alpha^2})/4\alpha \leq 1$ ,  $S_r$  is either monotonically increasing or decreasing depending on the sign of the coefficient  $T/(1 - 4\alpha) - 1$ . The property in Sec. III D indicates that  $S_r$  is monotonically increasing, which requires this coefficient to be positive. This condition yields the inequality  $T/(1 - 4\alpha) > 1$ , thereby determining the closed form of  $S_r$ . Applying this inequality to Eq. (6) gives  $\beta < \alpha$ , which specifies the upper phase boundary between the asynchronous and ordered phases:  $\beta_c^{\text{asy-ord}}(\alpha) = \alpha$ , a trivial result [14].

Next, we consider the lower phase boundary  $\beta_c^{\text{ord-soc}}(\alpha)$  by substituting  $S_r$  into Eq. (2). Since  $S_r$  is monotonically increasing, it approaches its maximum as  $r \rightarrow \infty$ . Accordingly, Eq. (2) can be rewritten as

$$\beta S_0 + 3\alpha S_2 > 3\alpha S_\infty,$$

where  $S_\infty \equiv \lim_{r \rightarrow \infty} S_r$  represents the supercritical toppling energy of sites sufficiently distant from the defect (here, we have used the property in Sec. III B). Substituting  $S_0$ ,  $S_2$ , and  $S_\infty$  from Eq. (5) yields

$$\beta > \frac{3\alpha}{4} \frac{1 - \sqrt{1 - 16\alpha^2}}{1 - 4\alpha^2},$$

which specifies the condition for the existence of the ordered phase. Thus, this condition suggests that the lower phase

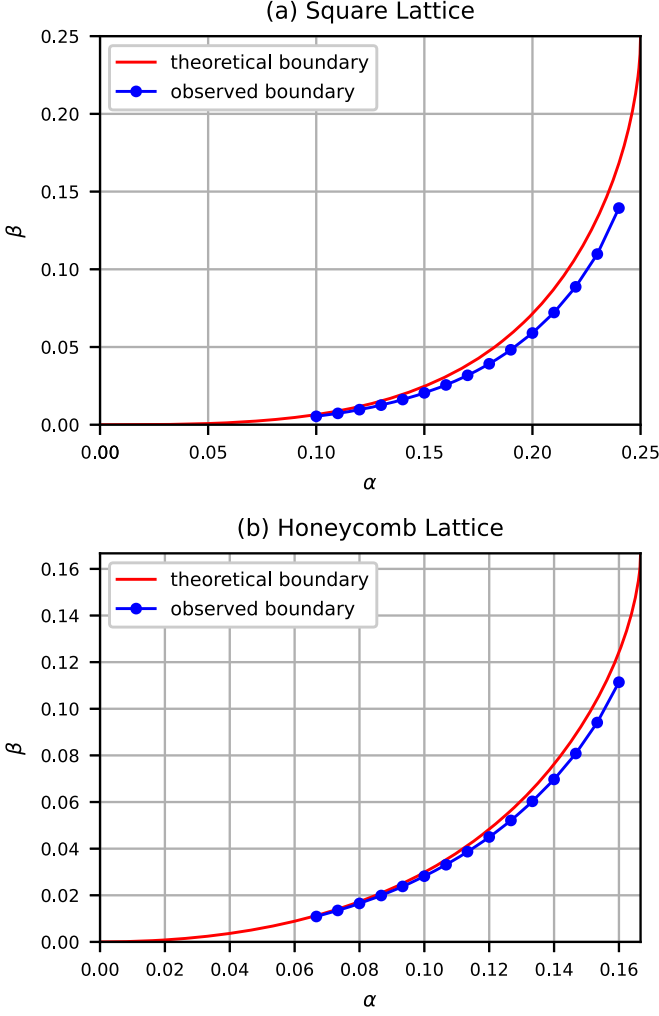


FIG. 4. (a) Phase boundary for the square lattice. (b) Phase boundary for the honeycomb lattice. In each figure, the red curve indicates the theoretical boundary, while the blue dots (connected by blue lines) indicate the observed boundary in simulations.

boundary is given by

$$\beta_c^{\text{ord-soc}}(\alpha) = \frac{3\alpha}{4} \frac{1 - \sqrt{1 - 16\alpha^2}}{1 - 4\alpha^2}. \quad (7)$$

In the following section, we compare the theoretically derived boundary with the boundary observed in simulations.

## V. VERIFICATION OF THEORETICAL PHASE BOUNDARY

Figure 4 presents a comparison between the theoretical [Eq. (7)] and observed boundaries. They show good agreement for small  $\alpha$ , while deviations become apparent as  $\alpha$  increases. These deviations likely originate from the anisotropy inherent in the square lattice geometry.

To examine this hypothesis, we perform analogous calculations using a honeycomb lattice. The numerical simulations were also carried out on a honeycomb lattice, and the results are compared with the theoretical phase boundary derived for a honeycomb lattice.

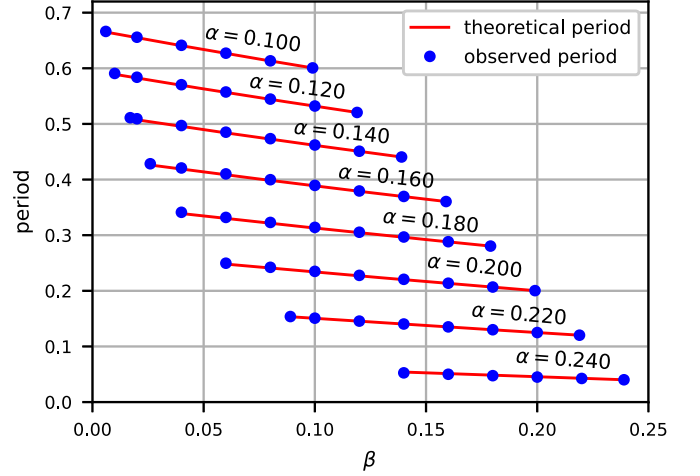


FIG. 5. Comparison of the period between the theory and observation both on a square lattice. The red curves indicate the theoretical periods, while the blue dots indicate the observed value. The plots correspond to parameter pairs  $(\alpha, \beta)$  within the ordered phase shown in Fig. 1.

The theoretical phase boundary can be obtained in the same way as for the square lattice case (see Appendix B) and is expressed as

$$\beta_c^{\text{ord-soc}}(\alpha) = \frac{2\alpha^2 + 3\alpha - 3\alpha\sqrt{1 - 4\alpha - 12\alpha^2}}{4(1 - 6\alpha^2)}. \quad (8)$$

Figure 4(b) shows the comparison between theoretical and observed boundaries on a honeycomb lattice. To quantitatively evaluate the agreement between theoretical and observed boundaries, we calculate the coefficient of determination  $R^2$ , where values closer to unity indicate higher accuracy. Our analysis yields  $R^2 \simeq 0.90$  for square lattices and  $R^2 \simeq 0.97$  for honeycomb lattices, thereby supporting the validity of our hypothesis.

Next, we verify the theoretical period  $T$  in Eq. (6) by comparing theoretical values with observed values (Fig. 5). For each value of  $\alpha$ , theoretical results show excellent agreement with observed periods across the entire range of  $\beta$ . This agreement is also maintained on the honeycomb lattice (see Appendix B).

## VI. DISCUSSION

In the OFC model, the system exhibits two processes: synchronization and desynchronization [11,13]. Synchronization is defined as the process by which multiple sites previously toppled independently in distinct avalanches begin to topple concurrently within a single avalanche. Conversely, desynchronization is defined as the process by which multiple sites that previously toppled simultaneously within a single avalanche begin to topple independently in separate avalanches.

It has been suggested that synchronization leads to SOC, whereas desynchronization generates aftershocks and foreshocks [13,19]. However, the original OFC model produces fewer aftershocks than observed in actual seismic activity [20]. To reproduce more realistic aftershock sequences, additional crust relaxation mechanisms have been incorporated,

such as transient creep [21], structural relaxation [22,23], or viscoelastic coupling [24], which successfully reproduced aftershock sequences obeying the Omori-Utsu law [25,26]. In addition to its role in generating aftershocks through these additional mechanisms, desynchronization is also considered a necessary element for the system to exhibit SOC [12].

Here, we apply the concepts of synchronization and desynchronization to the ordered phase. As seen in Sec. III, synchronization always occurs at the rim, transforming the random pattern into the ordered structure. This means that synchronization alone governs the transition to the ordered phase without desynchronization.

In contrast, the system evolution changes dramatically in the SOC phase. During the initial transient from a random configuration, synchronization likewise occurs at the rim, transforming the random pattern into synchronized patches. However, avalanche borders change over time, exhibiting both synchronization and desynchronization. Specifically, some avalanches in the synchronized pattern enlarge or merge due to synchronization, while others separate into smaller ones due to desynchronization. After the system reaches a SOC state, synchronization and desynchronization persist, thereby generating a power-law distribution.

As mentioned above, the system exhibits SOC when  $\beta < \beta_c^{\text{ord-soc}}$ , i.e., the local energy at a nonepicenter site exceeds that at an epicenter site. Under this condition, desynchronization also begins, i.e., sufficiently large supercritical toppling energies trigger desynchronization.

Previous studies have shown that in the OFC model the mere presence of supercritical toppling energy is sufficient to induce desynchronization [13]. In contrast, in the single-defect model, desynchronization is triggered only when supercritical toppling energy exceeds a certain threshold value. Given that open boundary conditions in the OFC model can be represented as defects arranged along the boundary, this difference may arise from the interaction of multiple defects. Since a single defect may represent the minimal inhomogeneity necessary for the emergence of SOC, this suggests that desynchronization fundamentally requires supercritical toppling energy exceeding a threshold.

Desynchronization also occurs when the condition  $\beta < \beta_c^{\text{asy-ord}}(\alpha)$  is violated. This type of desynchronization arises because the energy received from nearest-neighbor sites fails to reach the toppling threshold. However, this form of desynchronization is not crucial for the emergence of SOC.

This study contributes to the understanding of SOC by identifying the threshold-dependent dynamics of desynchronization in the single-defect model, though further work is necessary to comprehensively characterize the minimal ingredients required for SOC and to extend these findings to more complex systems with multiple or interacting defects.

## VII. CONCLUSION

We quantified the ordered phase in the single-defect model and theoretically derived the corresponding phase boundaries. We found that, in this minimal setting, the onset of

desynchronization is observed only when the supercritical toppling energy exceeds a certain threshold value, in contrast to the original OFC model where desynchronization can occur merely from the presence of supercritical toppling energy. By viewing a single defect as a minimal deviation from homogeneity, these results suggest that the magnitude of supercritical toppling energy plays an important role in triggering desynchronization under minimal inhomogeneity. While the generality of this interpretation requires further investigation, the present study provides a controlled reference case for understanding how inhomogeneity influences the balance between synchronization and desynchronization. These findings offer further insight into the microscopic processes underlying the emergence of SOC and may serve as a useful perspective for interpreting avalanche-like critical phenomena, including earthquake dynamics and related natural systems.

## ACKNOWLEDGMENTS

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

## APPENDIX A: CHARACTERISTICS OF THREE PHASES

The single-defect model exhibits three distinct phases controlled by the parameter pair  $(\alpha, \beta)$  as shown in Fig. 1. Here,  $\beta_c^{\text{ord-soc}}(\alpha)$  and  $\beta_c^{\text{asy-ord}}(\alpha)$  denote the critical points between the ordered and SOC phases, and between the asynchronous and ordered phases, respectively.

**SOC phase.**  $\beta \leq \beta_c^{\text{ord-soc}}(\alpha)$  The system exhibits an SOC state, and the avalanche size distribution follows a power law.

**Ordered phase.**  $\beta_c^{\text{ord-soc}}(\alpha) < \beta < \beta_c^{\text{asy-ord}}(\alpha)(= \alpha)$  The system dynamics converge to a limit cycle, i.e., the system periodically generates one to four distinct avalanches. Only the four nearest-neighbor sites of the defect can become epicenters (triggering sites).

**Asynchronous phase.**  $\beta_c^{\text{asy-ord}}(\alpha) \leq \beta$  At the critical point  $\beta = \beta_c^{\text{asy-ord}}(\alpha)$ , the system becomes homogeneous, effectively eliminating the defect. In this case, the dynamics are identical to the OFC model with periodic boundary conditions, exhibiting a periodic state with period  $1 - 4\alpha$  [10,11]. When  $\beta_c^{\text{asy-ord}}(\alpha) < \beta$ , the system does not reach a periodic state. In both cases, the avalanche size distribution is dominated by single-site avalanches, with large-size avalanches being notably absent.

APPENDIX B: DERIVATION FOR  $S_r$  AND  $T$ 

The solution for  $S_r$  is derived from the following recurrence relations:

$$S_0 = 4\alpha S_1 + T, \quad (\text{B1})$$

$$S_1 = 3\alpha S_2 + \beta S_0 + T = 1, \quad (\text{B2})$$

$$S_r = 2\alpha S_{r+1} + 2\alpha S_{r-1} + T \quad \text{for } r \geq 2. \quad (\text{B3})$$

From  $S_1 = 1$  on the epicenter sites, we obtain

$$\begin{aligned} S_0 &= 4\alpha + T, \\ S_2 &= \frac{1}{3\alpha}(1 - 4\alpha\beta - \beta T). \end{aligned} \quad (\text{B4})$$

The general solution of Eq. (B3) is given by

$$\begin{aligned} S_r = S_1 + \frac{2\alpha}{\sqrt{1-16\alpha^2}} \left( A - \frac{1 - \sqrt{1-16\alpha^2}}{4\alpha} B \right) \sum_{k=1}^{r-1} \left( \frac{1 + \sqrt{1-16\alpha^2}}{4\alpha} \right)^k \\ - \frac{2\alpha}{\sqrt{1-16\alpha^2}} \left( A - \frac{1 + \sqrt{1-16\alpha^2}}{4\alpha} B \right) \sum_{k=1}^{r-1} \left( \frac{1 - \sqrt{1-16\alpha^2}}{4\alpha} \right)^k, \end{aligned} \quad (\text{B5})$$

where  $A$  and  $B$  are constants. When we assume that  $S_r$  does not diverge as  $r \rightarrow \infty$ , the second term in Eq. (B5) must vanish, yielding the condition  $A = \frac{1 - \sqrt{1-16\alpha^2}}{4\alpha} B$ . Substituting this relation into Eq. (B5) and considering the limit  $r \rightarrow \infty$  in Eq. (B3), we obtain  $T/(1-4\alpha)$ . From these relationships, we finally obtain

$$S_r = 1 + \left( \frac{T}{1-4\alpha} - 1 \right) \left[ 1 - \left( \frac{1 - \sqrt{1-16\alpha^2}}{4\alpha} \right)^{r-1} \right]$$

for  $r \geq 1$ . Moreover, by combining Eq. (B4) and Eq. (B5), we derive the following expression for the period  $T$ :

$$T = \frac{1 - 16\alpha\beta + 3\sqrt{1-16\alpha^2}}{1 - 4\alpha + 4\beta - 16\alpha\beta + 3\sqrt{1-16\alpha^2}}(1 - 4\alpha).$$

For the honeycomb lattice, we consider six nearest-neighbor sites instead of four. Accordingly, the corresponding relations in Eqs. (B1)–(B3) are modified. The resulting phase boundary is shown in Eq. (8), and the theoretical period becomes

$$T = \frac{1 - 2\alpha - 24\alpha\beta + 3\sqrt{1-4\alpha-12\alpha^2}}{1 - 6\alpha + 4\beta - 24\alpha\beta + 3\sqrt{1-4\alpha-12\alpha^2}}(1 - 6\alpha).$$

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